

Black holes in effective loop quantum gravity: Covariant holonomy modifications

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Emergent modified gravity provides a covariant, effective framework for obtaining spherically symmetric black hole solutions in models of loop quantum gravity with scale-dependent holonomy modifications. Exact solutions for vacuum black holes in the presence of a cosmological constant are derived here and analyzed in four different gauges, explicitly related to one another by standard coordinate transformations. The global structure is obtained by gluing space-time regions corresponding to the gauge choices, reconstructing a nonsingular wormhole space-time for an arbitrary scale-dependent holonomy parameter. This outcome demonstrates the robustness of black-hole models with covariant holonomy modifications under quantization ambiguities. Compared with previous constructions, full covariance of the resulting space-time models as derived here implies subtle new effects and leads to a novel understanding of the parameters in holonomy modifications, distinguishing a constant holonomy length from a possibly scale-dependent function that may change coefficients of holonomy terms. New physical results are obtained for instance in the context of a nontrivial zero-mass limit of holonomy-modified space-times. The existence of a consistent effective space-time structure implies various novel aspects of a net gravitational stress-energy and related thermodynamical properties.

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I. INTRODUCTION

Black holes are now everyday objects encountered in astronomical observations indicating that much of their exterior is well described by general relativity (GR). Yet, the region beyond the event horizon remains inaccessible to observations and we must rely on theoretical extrapolations to understand the interior physics. In particular, GR robustly predicts a singularity at the center of black holes [1], which has long been interpreted as the result of the breakdown of the theory. Hence, it has been argued that a quantum theory of gravity would be able to resolve such singularities in analogy to how the quantum theory of light resolves the ultraviolet catastrophe. One candidate background-independent and nonperturbative theory is loop

quantum gravity (LQG) [2,3]. We shall explore the fate of stationary, spherically symmetric (vacuum) solutions due to quantum corrections from LQG in an effective framework, making sure that full covariance is realized in any effective line element used to describe the physics of black holes.

The canonical quantization in LQG is based on the holonomy-flux algebra. This is so because the regularization scheme followed in LQG does not allow for quantum operators corresponding to the connection, but rather its parallel transport along one-dimensional curves. In effective black-hole models, one focuses on the spherically symmetric reduced theory and incorporates quantum corrections corresponding to this feature by modifying the Hamiltonian constraint from a simple polynomial dependence on the connection to a trigonometric one resulting from the real-valued combinations of holonomies. This procedure is often done by hand, rather than derived from first principles, and covariance of the resulting models is seldom checked or even acknowledged as a problem. (See [4,5] for detailed comparisons of different approaches to black-hole models within loop quantization.)

General covariance is harder to grasp in the canonical formulation than in its more common geometric or Lagrangian cousins. A modified Hamiltonian constraint will in general modify its Poisson brackets. One must

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therefore ensure that the brackets remain first class for the equations of motion to be gauge covariant. Even if this condition is fulfilled, there often remains a modified structure function in the Poisson bracket of two Hamiltonian constraints, whose proper interpretation is a modified kinematical dependence of the space-time metric on the phase-space variables [6,7]. Contrary to what is commonly believed, preserving the first-class nature of the constraint brackets does not imply that the space-time metric tensor is covariant, constructed in the usual way from phase-space variables, the structure function, as well as a lapse function and shift vector field: In a canonically modified theory, this object is not guaranteed to be subject to gauge transformations that are on-shell equivalent to the tensor transformation law. Rather, additional covariance conditions must be imposed on the structure function, which further restrict compatible modifications of the Hamiltonian constraint [8]. In order to implement these conditions, we will use emergent modified gravity (EMG) [7,8] as the underlying framework for effective LQG. (As discussed in these papers, the emergent nature of the space-time metric is essential for evading no-go theorems that had previously been shown for covariance in models of LQG. Special cases of covariant models that implicitly or explicitly use an emergent metric have been constructed in [9–13].) In particular, the spherically symmetric model of EMG can be solved for the most general, covariant, modified constraint in $(3+1)$ -dimensions with a dependence on the spatial derivatives of the phase space up to the second order, which contains terms that can be interpreted as holonomy modifications.

An earlier, special case of EMG in spherical symmetry was studied in [10,11] where the holonomy modifications were shown to lead to a nonsingular black-hole solution in vacuum. The global structure of such a solution is an interuniversal wormhole joining a black hole to a white hole through their interiors. The holonomy parameter μ in these works was assumed constant, $\mu = \mu_0$, a framework referred to as the μ_0 scheme in some of the LQG literature. This holonomy parameter is associated with the coordinate length of the links in the triangulation of space as required by LQG. More precisely, in the spherically symmetric effective framework where the space-time topology may be fixed to $\mathcal{M} = \mathbb{R}^2 \times S^2$, one only considers the angular discretization of the spheres (the submanifold $S^2 \subset \mathcal{M}$) and hence μ_0 corresponds to the angular coordinate length of these links. A constant μ_0 is required if the constraint operator is constructed from basic holonomy operators. However, holonomies depend on a canonical variable such as a spatial connection component or extrinsic curvature of the spheres, which unlike space-time curvature scalars are not guaranteed to be small in low-curvature regimes. Depending on specific properties of dynamical solutions, a constant μ_0 could therefore imply large holonomy modifications in regions that should remain close to the classical limit.

It is therefore necessary to introduce scale-dependent holonomy parameters μ that become smaller in

semiclassical regions in order to compensate for potentially growing extrinsic curvature. Heuristically, such a behavior can be motivated as including lattice-refinement effects of an underlying discrete spatial state. As spherical areas in a semiclassical region grow larger, a discrete lattice state would have to be refined more and more in order to avoid discreteness being magnified to macroscopic sizes. The coordinate length of lattice links correspondingly decreases, implying a scale-dependent μ as a function of the spherical area. The scale dependence is not unique and determined largely by phenomenological reasoning, making sure that the modified theory is consistent with classical behavior in the right regimes. In homogeneous cosmological models, there is only one relevant semiclassical regime given by large volumes. In spherical symmetry and the related black-hole models, however, there are different semiclassical regimes, including large radii as well as near-horizon geometries, which should also be semiclassical for astrophysical black holes. In the latter case, however, the conditions on a suitable scale dependence of μ are different from what is required for large radii because they refer to different components of the spatial metric. Moreover, these conditions are dependent on the space-time gauge or slicing used to express the near-horizon geometry. This observation reinforces the condition that holonomy modifications must be made compatible with general covariance of an effective line element, and that a large set of scale-dependent μ must be compatible with a given class of modified theories.

A simple *ad hoc* version of scale-dependent holonomy parameters is often referred to as a $\bar{\mu}$ -scheme [14], but it is not uniquely defined. The original motivation was to replace the coordinate length of the links of a triangulation with a more tangible length such as the Planck length ℓ_{Pl} , characteristic of quantum gravity. Since the holonomy around a closed loop of links is related to the curvature of the gauge connection, one may use the Planck area, or the smallest nonzero eigenvalue Δ in the area spectrum of loop quantum gravity, as an ingredient of the holonomy parameter μ . If the exponent of Δ in the holonomy parameter is chosen, which might be proportional to Δ itself as an area or $\sqrt{\Delta}$ as a side length, or perhaps some other power law depending on one's intuition, dimensional arguments require that this quantity be divided by a function of metric components with suitable length units. For this purpose, one can employ the component $q_{\theta\theta}$ that determines the areas of spheres. This procedure may then result in $\mu \propto q_{\theta\theta}^{-1}$ or $\mu \propto q_{\theta\theta}^{-1/2}$ or with some other exponent. (The square-root version is closest to the $\bar{\mu}$ -scheme in spatially flat cosmological models, where the relevant connection component is proportional to $\dot{a}$ in terms of the scale factor. This component is then multiplied by $\sqrt{\Delta}$ for dimensional reasons, and divided by a in order to make the expression independent of spatial rescalings. Here, we have a simple covariance argument, but, compared with

spherical symmetry, only referring to a small subset of possible coordinate transformations.) Such a construction does not lead to a unique form of a scale-dependent μ because the final result depends on the exponent chosen for Δ , and the radial metric component q_{xx} could also be used in such an ad-hoc argument [15].

A more systematic approach is therefore needed, which we construct here based on the condition of covariance. As we will see, this condition, while still not leading to a unique analog of μ , implies additional restrictions on possible phase-space dependencies of this function, which turn out to include a dependence on $q_{\theta\theta}$ but not on q_{xx} . Such a behavior is well-defined even when crossing the horizon because the physical length of the links remains spacelike everywhere and discretizes the submanifold $S^2 \subset \mathcal{M}$. In Schwarzschild coordinates, the spatial slices have the topology $\Sigma = \mathbb{R} \times S^2$, such that the exterior is foliated by spheres with squared areal radius $q_{\theta\theta}$, while the interior geometry is that of hypercylinders with squared radius $q_{\theta\theta}$. In the heuristic lattice picture, both cases can be interpreted as discrete submanifold S^2 with plaquettes of constant size if $\mu \propto q_{\theta\theta}^{-1/2}$ is used.

The purpose of this paper is to generalize the results of [10,11] to an arbitrary holonomy parameter as a function of the radius, $\mu(q_{\theta\theta})$, and hence obtaining a general triangulation scheme in its heuristic interpretation. We will note crucial conceptual differences in our new scheme as well, in particular its general covariance and its role in light of a complete discussion of canonical transformations that may be used to change a functional dependence such as $\mu(q_{\theta\theta})K_\phi$ where K_ϕ is one of the momenta, classically related to extrinsic curvature. While some properties of μ_0 and $\tilde{\mu}$ -type schemes can still be recognized in specific choices of canonical variables, our covariant construction (both in terms of space-time and phase-space transformations) provides a novel and unified picture of holonomy modifications. Since the relevant equations are obtained in the framework of emergent modified gravity, we follow the latter's notation and refer to the new holonomy parameter as $\lambda(q_{\theta\theta})$, highlighting its new origin and features.

The new conceptual understanding of holonomy modifications leads to novel physical insights. In particular, we extend previous constructions for black holes in LQG to asymptotically de Sitter space-times with a well-defined asymptotic infinity, rather than ending at a maximal radius. For this result, our covariant treatment of scale-dependent holonomy parameters is crucial. Covariance also allows us to apply standard space-time analysis in an unambiguous way, including the derivation of timelike and lightlike geodesics, effective stress-energy tensors, and thermodynamics.

After a general discussion of holonomy modifications, we apply our results to specific versions, including one that in its dynamical implications corresponds to the traditional choice of $\mu(q_{\theta\theta}) \propto q_{\theta\theta}^{-1/2}$. To this end we begin Sec. II with

a discussion of emergent modified gravity and describe how some modification functions can be interpreted as holonomy corrections and hence used as a covariant effective model for LQG. The black-hole solution with a general holonomy parameter is obtained in Sec. III. The procedure follows the solution of the equations of motion in four different gauges which we show are all related by standard coordinate transformations covering distinct but overlapping regions of the spacetime. These four patches are then sewed together to recover a global structure with the properties of a wormhole and present the maximal extension in null coordinates in Sec. IV. We further show that the physical global structure is nonunique, owing to ambiguities in the slicing and gluing of the maximal extension. Section V is devoted to an analysis of the interior structure of space-time with the black-hole horizon. We highlight the robust feature that the classical singularity is replaced by a nonsingular hypersurface of reflection symmetry, at which the initially collapsing interior can be glued to an expanding version. This outcome is largely independent of the specific scale dependence of holonomy parameters. Geodesic analysis can be used in order to relate this nonsingular behavior to heuristic effects such as repulsive modified gravity. In Sec. VI, we study several physical effects of holonomy modifications in the black-hole exterior including corrections to the precession frequency of nearly circular orbits, to the deflection angle of null rays, and to black hole thermodynamics. In Sec. VII, we revisit asymptotic problems in holonomy modifications that correspond to the traditional μ_0 -type scheme of scale-independent μ , including asymptotic corrections to the deflection angle, to the Brown-York quasilocal energy, and to the Bekenstein-Hawking entropy, as well as the existence of a maximum radius of the universe. In Sec. VIII we study the effects of a specific version of scale-dependent μ and confirm that it removes unexpected and potentially problematic features of the μ_0 -type scheme in asymptotic regimes.

II. EMERGENT MODIFIED GRAVITY IN SPHERICAL SYMMETRY

The framework of emergent modified gravity plays an important role in discussions of general covariance in a canonical setting and is therefore reviewed in this section. Its main feature is that the *space-time metric is emergent rather than effective*: It is not a corrected version of a preexisting classical metric tensor. Rather, it is the only metric tensor admissible in a modified canonical theory because the classical expression as a phase-space function fails to obey the tensor transformation law when gauge transformations are modified. The entire emergent metric is derived from covariance conditions, in contrast to what is usually referred to as an effective metric if perturbative correction terms are derived for the classical metric.

A. Spherically symmetric canonical gravity

Canonical gravity, originally derived by Dirac, is conveniently based on the Arnowitt-Deser-Misner (ADM) decomposition where the line element in spherical symmetry takes the form [16,17]

$$ds^2 = -N^2 dt^2 + q_{xx}(dx + N^x dt)^2 + q_{\theta\theta} d\Omega^2, \quad (1)$$

where q_{ab} is the three-dimensional metric induced on the spatial hypersurfaces foliating the manifold, while N and N^x are the lapse function and shift vector field associated to the observer frame with time coordinate t . The Hamiltonian is composed of the Hamiltonian constraint H and the diffeomorphism constraint H_x . As phase-space functions, depending on the spatial metric and its momenta, these expressions vanish on physical solutions, $H = 0$ and $H_x = 0$, bringing the system on-shell. The Hamiltonian that generates evolution with respect to t is given by $H[N, N^x] = H[N] + H_x[N^x]$ where the square brackets indicate that these terms are smeared (or spatially integrated) by the functions N and N^x . The constraints then generate time evolution via Poisson brackets, $\dot{\mathcal{O}} = \{\mathcal{O}, H[N, N^x]\}$ for any phase-space function $\mathcal{O}$.

The same constraints also generate gauge transformations by use of different smearing functions ϵ^0 and ϵ^x instead of the fixed N and N^x for a given frame, and hence the generator of a general gauge transformation is $H[\epsilon^0, \epsilon^x]$. The vanishing of the constraints must be consistent in different gauges, which means that the Poisson brackets of the constraints with themselves must vanish on-shell. The specific brackets found in canonical gravity are known as the hypersurface-deformation brackets, for spherical symmetry given by

$$\{H_x[N^x], H_x[\epsilon^x]\} = -H_x[\epsilon^x(N^x)' - N^x(\epsilon^x)'], \quad (2a)$$

$$\{H[N], H_x[\epsilon^x]\} = -H[\epsilon^x N'], \quad (2b)$$

$$\{H[N], H[\epsilon^0]\} = -H_x[q^{xx}(N'\epsilon^0 - N(\epsilon^0)'),] \quad (2c)$$

where the structure-function q^{xx} is the inverse of the radial metric component, $q^{xx} = 1/q_{xx}$. These brackets indeed vanish on-shell when the constraints equal zero. Furthermore, if the system is fully on-shell, also imposing equations of motion generated by the constraints, the gauge freedom of canonical gravity is equivalent to the freedom of choosing a space-time slicing or a coordinate system.

Therefore, the constraints are an essential ingredient of encoding general covariance. However, starting with gauge transformations of phase-space functions, two additional steps are needed for a full realization of general covariance, understood as gauge transformations being equivalent to coordinate changes, or as the existence of a tensor-transformation law for the space-time metric: First, we need gauge transformations not only for the spatial part q_{xx} ,

which is a basic phase-space degree of freedom, but also for the time-time and the mixed space-time components, depending on N and N^x . While the latter space-time functions do not have momenta and therefore are not included in the phase space (unless an extended version is used), their gauge transformations are uniquely determined by the constraints and their Poisson brackets through consistency conditions between evolution and gauge equations. Second, coordinate changes and the tensor transformation law require explicit time derivatives, rather than momentum components as they appear in the constraints and the gauge transformation they generate. Time derivatives are related to momenta by some of the canonical equations of motion generated by the constraints. No additional ingredient is therefore required in order to obtain full expressions of coordinate transformations, but this can be achieved only if the theory is taken fully on-shell by imposing not only the constraint equations but also some of the equations of motion.

We will include these ingredients in our present review in the more general setting of modified canonical gravity.

B. Modified canonical gravity

If one considers a modified Hamiltonian constraint $\tilde{H}$ while maintaining the diffeomorphism constraint without modifications, as implied for instance by the regularization procedure of LQG, then we must first ensure that the constraint brackets remain first class for the constraints to vanish consistently in all gauges. More specifically, we require that the general form of hypersurface-deformation brackets be preserved,

$$\{H_x[N^x], H_x[\epsilon^x]\} = -H_x[\epsilon^x(N^x)' - N^x(\epsilon^x)'], \quad (3a)$$

$$\{\tilde{H}[N], H_x[\epsilon^x]\} = -\tilde{H}[\epsilon^x N'], \quad (3b)$$

$$\{\tilde{H}[N], \tilde{H}[\epsilon^0]\} = -H_x[\tilde{q}^{xx}(N'\epsilon^0 - N(\epsilon^0)'),] \quad (3c)$$

where the structure-function $\tilde{q}^{xx}$ is determined by demanding that no additional terms appear on the right-hand sides. However, the specific structure function $\tilde{q}^{xx}$ need not equal the classical one, given by a specific function of one of the basic phase-space variables, q_{xx} , on which the modified $\tilde{H}$ depends (or an equivalent version in triad variables).

This condition is stronger than anomaly freedom, which would be compatible with, say, an additional H_x -term appearing in the Poisson bracket of two $\tilde{H}$. Both conditions, anomaly-freedom and the specific form of hypersurface-deformation brackets, restrict permissible modifications of the constraint. A comparison with the classical geometrical structure of space-time, together with an analysis of the transformation properties of lapse and shift to be discussed in the next section, suggests that the inverse of the modified structure function $\tilde{q}^{xx}$ plays the role of the radial component of an effective line element (in the stricter sense of

emergence) replacing q_{xx} in (1). However, anomaly freedom or brackets of hypersurface-deformation form do not imply that this effective line element is invariant. Rather, a further condition must be imposed on $\tilde{H}$, derived from the tensor transformation law of $(1/\tilde{q}^{xx}, N, N^x)$, such that the gauge transformations of the effective metric it generates correspond to infinitesimal coordinate transformations [8]. (The angular component $\tilde{q}_{\theta\theta}$ also plays a role in the covariance condition, but it is less direct because this component does not appear in the structure functions of spherically symmetric gravity.)

According to the classic analysis of [18,19], if one considers the classical spatial metric as a configuration variable of the phase space, then the Hamiltonian constraint is uniquely determined by the hypersurface-deformation brackets up to the choice of Newton's and the cosmological constant. These results therefore imply that H must be the classical constraint of GR, and generally covariant modifications are ruled out unless one includes higher-derivative terms that in a canonical formulation enlarge the classical phase space. However, in the recent formulation of emergent modified gravity [7,8] the assumption that the spatial metric q_{ab} is a configuration variable of the phase-space is not necessary in order to obtain a consistent field-theory description of space-time. Instead, emergent modified gravity assumes that the phase space is composed of certain fundamental fields different from the metric, and the metric is an emergent object to be regained by the imposition of anomaly freedom and covariance. In fact, the most general modified Hamiltonian constraint depending on spatial derivatives of the phase space up to second order and satisfying anomaly freedom and covariance in vacuum spherical symmetry has been obtained exactly in [8]. If LQG is to be treated in a covariant framework of effective line elements, without, as usually assumed in this context, an extension of the classical phase space in order to account for perturbative higher-curvature corrections or higher time derivatives, its effective Hamiltonian constraint must be included in the family of constraints allowed by emergent modified gravity, and its effective line element must be given by the corresponding emergent space-time.

C. Covariance conditions

The hypersurface-deformation brackets (3) together with general properties of a Poisson bracket, such as the Jacobi identity, imply that off-shell gauge transformations for the lapse function and shift vector are given by [6,20,21]

$$\delta_\epsilon N = \dot{\epsilon}^0 + \epsilon^x N' - N^x (\epsilon^0)', \quad (4a)$$

$$\delta_\epsilon N^x = \dot{\epsilon}^x + \epsilon^x (N^x)' - N^x (\epsilon^x)' + \tilde{q}^{xx} (\epsilon^0 N' - N (\epsilon^0)'). \quad (4b)$$

These equations play a central role in general covariance as explained below. They would fail to hold if the constraint brackets were modified beyond allowing for a new structure function $\tilde{q}^{xx}$, even if they remain first class.

Building on [6], the brackets (3) and the transformations (4) suggest the existence of the space-time metric

$$ds^2 = -N^2 dt^2 + \tilde{q}_{xx} (dx + N^x dt)^2 + \tilde{q}_{\theta\theta} d\Omega^2, \quad (5)$$

where $\tilde{q}_{xx}$ is the inverse of the (possibly nonclassical) structure-function, assuming for now that $\tilde{q}_{xx} > 0$, while $\tilde{q}_{\theta\theta}$ is not given by structure functions and must hence be chosen on different grounds. (This feature is a consequence of the symmetry-reduced theory under consideration and its space-time structure. Since the component is subject to canonical transformations, it is possible to prescribe a specific form of $\tilde{q}_{\theta\theta}$ as a phase-space function in order to reduce the freedom of performing canonical transformations.) Because the spatial metric is not directly determined by kinematical properties of the phase space but rather depends, at least in part, on the inverse of the structure-function, we refer to it as an emergent metric. This nomenclature will make more sense when dealing with nonclassical structure functions which arise from the requirement of anomaly-freedom of the modified constraints incorporating potential nonperturbative quantum corrections.

Anomaly-freedom of the brackets (3), even if they are of hypersurface-deformation form, does not ensure covariance of the resulting space-time (5), contrary to what is commonly stated. Instead, we have to *impose the stronger covariance condition* [8]

$$\delta_\epsilon \tilde{g}_{\mu\nu}|_{\text{o.s.}} = \mathcal{L}_\xi \tilde{g}_{\mu\nu}|_{\text{o.s.}}, \quad (6)$$

which ensures that canonical gauge transformations generate infinitesimal diffeomorphisms. This requirement contains (4) as a necessary condition, but it is also nontrivial when applied to the spatial part of the metric and imposes yet another set of restrictions on allowed modifications of the classical constraints. The anomaly freedom conditions (3) and the strong covariance condition (6) are independent of each other. Together, they provide a set of necessary and sufficient conditions for diffeomorphism-invariant theories with a Riemannian space-time structure on any given solution.

D. Covariant modified constraints

Following [22,23] we consider the vacuum spherically symmetric theory with canonical gravitational variables $\{K_\varphi(x), E^\varphi(y)\} = \delta(x-y)$ and $\{K_x(x), E^x(y)\} = \delta(x-y)$. (We use units such that Newton's constant equals one.) In the classical theory, the momenta E^x and E^φ are components of the densitized triad, while the configuration variables are directly related to the extrinsic-curvature components $\mathcal{K}_\varphi = K_\varphi$ and $\mathcal{K}_x = 2K_x$. Leaving the diffeomorphism constraint

$$H_x = E^\varphi K_\varphi' - K_x (E^x)' \quad (7)$$

unmodified and building on [24], we consider the most general ansatz for the Hamiltonian constraint up to second-order derivatives and quadratic first-order derivative terms [8]

$$\begin{aligned}\tilde{H} = & a_0 + ((E^x)')^2 a_{xx} + ((E^\varphi)')^2 a_{\varphi\varphi} + (E^x)'(E^\varphi)' a_{x\varphi} \\ & + (E^x)'' a_2 + (K_\varphi')^2 b_{\varphi\varphi} + (K_\varphi)'' b_2 + (E^x)' K_\varphi' c_{x\varphi} \\ & + (E^\varphi)' K_\varphi' c_{\varphi\varphi} + (E^\varphi)'' c_2.\end{aligned}\quad (8)$$

Here, $a_0, a_{xx}, a_{\varphi\varphi}, a_{x\varphi}, a_2, b_{\varphi\varphi}, b_2, c_2, c_{\varphi\varphi}, c_{x\varphi}$ are all functions of the phase-space variables, but not of their derivatives. We have not included terms quadratic in the radial derivatives of K_x in (8) because it can be readily shown that the covariance condition (6) does not allow radial derivatives of K_x of any order if $\tilde{q}_{\vartheta\vartheta} = \tilde{q}_{\vartheta\vartheta}(E^x)$ which is our case of interest [8].

Demanding that this ansatz satisfies the anomaly-free brackets of hypersurface deformation form and imposing the additional covariance conditions encoded in (6), the general Hamiltonian constraint reduces to [8]

$$\begin{aligned}\tilde{H} = & -\sqrt{E^x} \frac{g}{2} \left[E^\varphi \left(f_0 + \frac{K_x}{E^\varphi} f_1 \right) \right. \\ & + \frac{((E^x)')^2}{E^\varphi} \left(f_2 - \frac{K_x}{2E^\varphi} \left(\frac{\partial \ln g}{\partial K_\varphi} + C_{x\varphi} \right) \right) \\ & \left. + \frac{(E^x)'(E^\varphi)'}{(E^\varphi)^2} - \frac{(E^x)''}{E^\varphi} + \frac{(E^x)' K_\varphi'}{E^\varphi} C_{x\varphi} \right].\end{aligned}\quad (9)$$

Now, g, f_0, f_1, f_2 , and $C_{x\varphi}$ are functions of E^x and K_φ , and are related to one another by four differential equations [8]. Consequently, the structure function $\tilde{q}^{xx}$ is composed of these functions plus an additional direct dependence on the phase-space variables. More explicitly, it is given by [8]

$$\begin{aligned}\tilde{q}^{xx} = & \left(\frac{\partial f_1}{\partial K_\varphi} - f_1 C_{x\varphi} - \frac{1}{2} \left(\frac{\partial^2 \ln g}{\partial K_\varphi^2} + C_{x\varphi}^2 \right. \right. \\ & \left. \left. + \frac{\partial C_{x\varphi}}{\partial K_\varphi} + \frac{\partial \ln g}{\partial K_\varphi} C_{x\varphi} \right) \left(\frac{(E^x)'}{E^\varphi} \right)^2 \right) \frac{g^2}{4} \frac{E^x}{(E^\varphi)^2}.\end{aligned}\quad (10)$$

The constraint brackets (3) and the covariance condition (6), which can be formulated in terms of Poisson brackets as well, are both invariant under canonical transformations. We can therefore apply canonical transformations in order to further simplify the constraint and eliminate some of the free functions. (The structure function then transforms according to this transformation, but its role as a candidate inverse spatial metric remains unchanged.) An interesting subset of canonical transformations is given by those that preserve the diffeomorphism constraint (7)

$$\begin{aligned}K_\varphi = & f_c(E^x, \tilde{K}_\varphi), \quad E^\varphi = \tilde{E}^\varphi \left(\frac{\partial f_c}{\partial \tilde{K}_\varphi} \right)^{-1}, \\ K_x = & \frac{\partial(\alpha_c^2 E^x)}{\partial E^x} \tilde{K}_x + \tilde{E}^\varphi \frac{\partial f_c}{\partial E^x} \left(\frac{\partial f_c}{\partial \tilde{K}_\varphi} \right)^{-1}, \quad \tilde{E}^x = \alpha_c^2(E^x) E^x,\end{aligned}\quad (11)$$

where the new variables are written with a tilde. It can be shown [8] that there is a canonical transformation with a function $f_c = f_c(K_\varphi)$ and $\alpha_c = 1$, such that the transformed $C_{x\varphi}$ vanishes, at least locally in the phase space.

This canonical transformation allows us to include $C_{x\varphi} = 0$ among the conditions for covariance, resulting in five equations for the five functions in (9). The equations, which include partial differential ones, can be solved exactly, yielding [8]

$$g = \chi \cos^2(\lambda(K_\varphi + \lambda_\varphi)), \quad (12a)$$

$$gf_1 = 4\chi \left(c_f \frac{\sin(2\lambda(K_\varphi + \lambda_\varphi))}{2\lambda} + q \cos(2\lambda(K_\varphi + \lambda_\varphi)) \right), \quad (12b)$$

$$f_2 = -\frac{\alpha_2}{4E^x} + \frac{\sin(2\lambda(K_\varphi + \lambda_\varphi))}{2\lambda \cos^2(\lambda(K_\varphi + \lambda_\varphi))} \left(\lambda \frac{\partial(\lambda\lambda_\varphi)}{\partial E^x} + \lambda K_\varphi \frac{\partial \lambda}{\partial E^x} \right), \quad (12c)$$

$$\begin{aligned}gf_0 = & \chi \left(\frac{\alpha_0}{E^x} + 2 \frac{\sin^2(\lambda(K_\varphi + \lambda_\varphi))}{\lambda^2} \frac{\partial c_f}{\partial E^x} + 4 \frac{\sin(2\lambda(K_\varphi + \lambda_\varphi))}{2\lambda} \frac{\partial q}{\partial E^x} \right. \\ & + 4c_f \left(\frac{1}{\lambda} \frac{\partial(\lambda\lambda_\varphi)}{\partial E^x} \frac{\sin(2\lambda(K_\varphi + \lambda_\varphi))}{2\lambda} + \left(\frac{\alpha_2}{4E^x} - \frac{\partial \ln \lambda}{\partial E^x} \right) \frac{\sin^2(\lambda(K_\varphi + \lambda_\varphi))}{\lambda^2} \right) \\ & + 8q \left(-\lambda \frac{\partial(\lambda\lambda_\varphi)}{\partial E^x} \frac{\sin^2(\lambda(K_\varphi + \lambda_\varphi))}{\lambda^2} + \left(\frac{\alpha_2}{4E^x} - \frac{1}{2} \frac{\partial \ln \lambda}{\partial E^x} \right) \frac{\sin(2\lambda(K_\varphi + \lambda_\varphi))}{2\lambda} \right) \\ & \left. + 4K_\varphi \frac{\partial \ln \lambda}{\partial E^x} \left(c_f \frac{\sin(2\lambda(K_\varphi + \lambda_\varphi))}{2\lambda} + q \cos(2\lambda(K_\varphi + \lambda_\varphi)) \right) \right),\end{aligned}\quad (12d)$$

where $\chi, c_f, \alpha_0, \alpha_2, \lambda, q, \lambda_\varphi$ are undetermined functions of E^x , and we have dropped the tilde from the new variables for notational simplicity. The classical constraint is recovered in the limit $\chi, c_f, \alpha_0, \alpha_2 \rightarrow 1$ and $\lambda, q \rightarrow 0$. (The cosmological constant can be recovered by instead setting $\alpha_0 \rightarrow 1 - \Lambda E^x$, with $\Lambda > 0$ corresponding to asymptotically de Sitter space in the classical limit.)

Noting that the set of canonical transformations has not been exhausted, we can use the residual canonical transformation with $f_c = f_x K_\varphi - \tilde{\lambda}_\varphi$ and $\alpha_c = 1$ (where f_x and $\tilde{\lambda}_\varphi$ are allowed to depend on E^x), which preserves $C_{x\varphi} = 0$ [8], to further simplify the constraint. A convenient choice for such a simplification is given by $\tilde{\lambda}_\varphi = \lambda_\varphi$

$$g = \chi f_x \cos^2(\lambda f_x K_\varphi), \quad (13a)$$

$$gf_0 = \frac{\chi}{f_x} \left(\frac{\alpha_0}{E^x} + 2 \frac{\sin^2(\lambda f_x K_\varphi)}{\lambda^2} \frac{\partial c_f}{\partial E^x} + 4 \frac{\sin(2\lambda f_x K_\varphi)}{2\lambda} \frac{1}{\lambda} \frac{\partial(\lambda q)}{\partial E^x} + \left(\frac{\alpha_2}{E^x} - 2 \frac{\partial \ln \lambda^2}{\partial E^x} \right) \left(c_f \frac{\sin^2(\lambda f_x K_\varphi)}{\lambda^2} + 2q \frac{\sin(2\lambda f_x K_\varphi)}{2\lambda} \right) \right) + \frac{\partial \ln(\lambda f_x)}{\partial E^x} 4\chi_0 K_\varphi \left(c_f \frac{\sin(2\lambda f_x K_\varphi)}{2\lambda} + q \cos(2\lambda f_x K_\varphi) \right), \quad (13b)$$

$$gf_1 = 4\chi \left(c_f \frac{\sin(2\lambda f_x K_\varphi)}{2\lambda} + q \cos(2\lambda f_x K_\varphi) \right), \quad (13c)$$

$$gf_2 = \chi f_x \cos^2(\lambda f_x K_\varphi) \left(-\frac{\alpha_2}{4E^x} - \frac{\partial \ln f_x}{\partial E^x} + \frac{\sin(2\lambda f_x K_\varphi)}{2\lambda \cos^2(\lambda f_x K_\varphi)} \lambda^2 f_x K_\varphi \frac{\partial \ln(\lambda f_x)}{\partial E^x} \right). \quad (13d)$$

The associated structure function (10) becomes

$$\begin{aligned} \tilde{q}^{xx} &= \left(\frac{\partial f_1}{\partial K_\varphi} - \frac{1}{2} \frac{\partial^2 \ln g}{\partial K_\varphi^2} \left(\frac{(E^x)'}{E^\varphi} \right)^2 \right) \frac{g^2}{4} \frac{E^x}{(E^\varphi)^2} \\ &= \left(\left(c_f + \lambda^2 f_x^2 \left(\frac{(E^x)'}{2E^\varphi} \right)^2 \right) \cos^2(\lambda f_x K_\varphi) - 2q\lambda^2 \frac{\sin(2\lambda f_x K_\varphi)}{2\lambda} \right) \chi^2 f_x^2 \frac{E^x}{(E^\varphi)^2}. \end{aligned} \quad (14)$$

If we choose $f_x = 1$, then the resulting set of Hamiltonian constraints is given by

$$\begin{aligned} \tilde{H} &= -\sqrt{E^x} \frac{\chi}{2} \left[E^\varphi \left(\frac{\alpha_0}{E^x} + 2 \frac{\sin^2(\lambda K_\varphi)}{\lambda^2} \frac{\partial c_f}{\partial E^x} + 4 \frac{\sin(2\lambda K_\varphi)}{2\lambda} \frac{1}{\lambda} \frac{\partial(\lambda q)}{\partial E^x} \right) \right. \\ &\quad + \left(\frac{\alpha_2}{E^x} - 2 \frac{\partial \ln \lambda^2}{\partial E^x} \right) \left(c_f \frac{\sin^2(\lambda K_\varphi)}{\lambda^2} + 2q \frac{\sin(2\lambda K_\varphi)}{2\lambda} \right) \\ &\quad + 4 \left(\frac{K_x}{E^\varphi} + \frac{K_\varphi}{2} \frac{\partial \ln \lambda^2}{\partial E^x} \right) \left(c_f \frac{\sin(2\lambda K_\varphi)}{2\lambda} + q \cos(2\lambda K_\varphi) \right) \\ &\quad + \frac{((E^x)')^2}{E^\varphi} \left(-\frac{\alpha_2}{4E^x} \cos^2(\lambda K_\varphi) + \left(\frac{K_x}{E^\varphi} + \frac{K_\varphi}{2} \frac{\partial \ln \lambda^2}{\partial E^x} \right) \lambda^2 \frac{\sin(2\lambda K_\varphi)}{2\lambda} \right) \\ &\quad \left. + \left(\frac{(E^x)'(E^\varphi)'}{(E^\varphi)^2} - \frac{(E^x)''}{E^\varphi} \right) \cos^2(\lambda K_\varphi) \right], \end{aligned} \quad (15)$$

with the associated structure function

$$\tilde{q}^{xx} = \left(\left(c_f + \lambda^2 \left(\frac{(E^x)'}{2E^\varphi} \right)^2 \right) \cos^2(\lambda K_\varphi) - 2q\lambda^2 \frac{\sin(2\lambda K_\varphi)}{2\lambda} \right) \chi^2 \frac{E^x}{(E^\varphi)^2}. \quad (16)$$

If we instead choose $f_x = \tilde{\lambda}/\lambda$, where $\tilde{\lambda}$ is a constant reference value of $\lambda(E^x)$, the resulting set of Hamiltonian constraints is given by

$$\begin{aligned}
\tilde{H} = & -\frac{\tilde{\lambda}}{\lambda} \chi \frac{\sqrt{E^x}}{2} \left[E^\varphi \left(\frac{\lambda^2 \alpha_0}{\tilde{\lambda}^2 E^x} + 2 \frac{\sin^2(\tilde{\lambda} K_\varphi)}{\tilde{\lambda}^2} \frac{\partial c_f}{\partial E^x} + 4 \frac{\sin(2\tilde{\lambda} K_\varphi)}{2\tilde{\lambda}} \frac{\partial}{\partial E^x} \left(\frac{\lambda}{\tilde{\lambda}} q \right) + \left(\frac{\alpha_2}{E^x} - 2 \frac{\partial \ln \lambda^2}{\partial E^x} \right) \left(c_f \frac{\sin^2(\tilde{\lambda} K_\varphi)}{\tilde{\lambda}^2} + 2 \frac{\lambda}{\tilde{\lambda}} q \frac{\sin(2\tilde{\lambda} K_\varphi)}{2\tilde{\lambda}} \right) \right. \\
& + 4 K_x \left(c_f \frac{\sin(2\tilde{\lambda} K_\varphi)}{2\tilde{\lambda}} + \frac{\lambda}{\tilde{\lambda}} q \cos(2\tilde{\lambda} K_\varphi) \right) - \frac{((E^x)')^2}{E^\varphi} \left(\left(\frac{\alpha_2}{4E^x} - \frac{1}{2} \frac{\partial \ln \lambda^2}{\partial E^x} \right) \cos^2(\tilde{\lambda} K_\varphi) - \frac{K_x \tilde{\lambda}^2 \sin(2\tilde{\lambda} K_\varphi)}{E^\varphi 2\tilde{\lambda}} \right) \\
& \left. + \left(\frac{(E^x)'(E^\varphi)'}{(E^\varphi)^2} - \frac{(E^x)''}{E^\varphi} \right) \cos^2(\tilde{\lambda} K_\varphi) \right], \tag{17}
\end{aligned}$$

with the structure-function

$$\tilde{q}^{xx} = \left(\left(c_f + \left(\frac{(E^x)' \tilde{\lambda}}{2E^\varphi} \right)^2 \right) \cos^2(\tilde{\lambda} K_\varphi) - 2 \frac{\lambda}{\tilde{\lambda}} q \tilde{\lambda}^2 \frac{\sin(2\tilde{\lambda} K_\varphi)}{2\tilde{\lambda}} \right) \frac{\tilde{\lambda}^2}{\lambda^2} \chi^2 \frac{E^x}{(E^\varphi)^2}. \tag{18}$$

Note that the constraint (17), unlike (15), is bounded in K_φ and fully periodic with “frequency” $\tilde{\lambda}$. The two constraints, however, are related by a simple canonical transformation rescaling the component K_φ , and correspondingly transforming E^φ and K_φ according to (11). Therefore, they describe the same system, just in different phase-space coordinates. In the modified context, the phase-space variable K_φ has lost its direct classical relationship with extrinsic curvature, which is instead determined by the emergent space-time metric. Furthermore, it is not possible, except in the classical limit $\lambda \rightarrow 0$, to turn the emergent spatial metric component $\tilde{q}_{xx}$ or the emergent extrinsic curvature into canonical configuration or momentum variables [8]. Therefore, there is no canonical or geometrical preference in favor of one of the two versions. (However, the periodic nature of (17) may suggest that this version and the associated canonical variables are more relevant for a fundamental loop quantization in terms of basic holonomy operators.)

The classical constraint is recovered from (15) in the limit $\chi, c_f, \alpha_2 \rightarrow 1$, $\lambda \rightarrow 0$, and $\alpha_0 \rightarrow 1 - \Lambda E^x$ with cosmological constant Λ . The constraint (17) cannot reproduce the classical limit directly because the canonical transformation $K_\varphi \rightarrow (\tilde{\lambda}/\lambda) K_\varphi$ does not exist in the classical limit. Since the canonical transformation has introduced a new constant $\tilde{\lambda}$, its limit behavior must be specified separately. One possibility is the combination of $\lambda \rightarrow \tilde{\lambda}$ followed by $\tilde{\lambda} \rightarrow 0$.

If we redefine

$$\chi \rightarrow \tilde{\chi} \frac{\lambda}{\tilde{\lambda}}, \quad q \rightarrow \tilde{q} \frac{\tilde{\lambda}}{\lambda}, \quad \alpha_0 \rightarrow \frac{\tilde{\lambda}^2}{\lambda^2} \tilde{\alpha}_0, \quad \alpha_2 \rightarrow \tilde{\alpha}_2 + 4E^x \frac{\partial \ln \lambda}{\partial E^x}, \tag{19}$$

the free function λ disappears from the constraint (17) by being absorbed by the other parameters, and is completely replaced by the constant $\tilde{\lambda}$. A possible interpretation of $\tilde{\lambda}$ is given in the next subsection as the length of a reference

curve along which the holonomy is computed. Replacing the function λ with a constant $\tilde{\lambda}$ simplifies the constraint, but it obscures the effects of nonconstant λ by combining it with the other modification functions according to (19). Since we are specifically interested in nonconstant λ for the purpose of this paper, we will not make much use of this redefinition.

A modified angular metric component $\tilde{q}_{\vartheta\vartheta} = \tilde{q}_{\vartheta\vartheta}(E^x)$ may also be considered. However, through a residual canonical transformation (11) with $f_c = K_\varphi$ and the appropriate α_c , the angular metric component can always be rendered into its classical form $\tilde{q}_{\vartheta\vartheta} = E^x$. The appearance of α_c in the Hamiltonian constraint can thereby be absorbed in the undetermined functions of E^x , obtaining the same result (17). Any covariant Hamiltonian constraint of second order in spatial derivatives in the spherically symmetric sector is therefore contained in (17).

As a summary, the modified constraint (17) was derived by requiring that it be second order in spatial derivatives (such that it is tractable but still sufficiently general from the viewpoint of effective field theory) and imposing the hypersurface-deformation brackets (3) as well as the covariance condition (6) on the metric (5). The diffeomorphism constraint remains of classical form in order to ensure the well-understood structure of spatial geometry. This complete procedure, including the use of canonical transformations and the classification of physically inequivalent theories, was first shown in [8]. All these conditions are necessary for a systematic treatment of consistent space-time models.

Models that did not implement the above conditions lead to implications such as having a line-element that is not invariant under coordinate transformations [25]. It is important to note that canonical transformations cannot have an effect on the physical system because they merely represent a change of phase-space coordinates. The canonical transformations used here, and discussed in detail in [8], serve the purpose of simplifying the covariance conditions such that they can be solved exactly. Factoring out

canonical transformations is also important because a different choice of phase-space coordinates may lead one to misclassify a modified Hamiltonian constraint different from (17) as a new theory, even if it is in fact equivalent to a well-known one. For instance, [26] argued that the non-invertibility of a canonical transformation of the form $K_\varphi \rightarrow \sin(\lambda K_\varphi)/\lambda$ applied to the classical system could model holonomy modifications. This assertion is incorrect since the construction merely amounts to using the classical theory with phase-space coordinates that are valid in a bounded range. It is therefore not surprising that the results on critical collapse of scalar matter, reported in [26], did not show any deviation from GR. Similarly, [10] partially attributed modified gravitational effects to noninvertible canonical transformations, misidentifying the correct reason for singularity removal. The relevant modifications instead originate in a linear combination of constraints with phase-space dependent coefficients which result in additional terms for the new Hamiltonian constraint, as explained in [8].

E. Effective loop quantum gravity

The covariant Hamiltonian defined in (17) contains various independent modification functions in several of its terms. How does it relate to effective loop quantum gravity, which is usually considered with a smaller number of modifications? Phenomenologically, the parameter λ is special since it is responsible for the nonsingular black-hole solution obtained for constant $\lambda = \tilde{\lambda}$ in [10]. In the functional behavior of the constraint, this parameter has an appearance related to the holonomy length μ in LQG.

1. Traditional ingredients

The starting point of any loop quantization consists in replacing a direct quantization of the classical phase space by operators in the holonomy-flux representation. In spherically symmetric LQG [22,23,27], the basic holonomies are given by

$$h_e^x[K_x] = \exp\left(\int_e dx iK_x\right), \quad (20)$$

$$h_{v,\mu}^\varphi[K_\varphi] = \exp\left(\int_\mu d\vartheta iK_\varphi\right) = \exp(i\mu K_\varphi(v)), \quad (21)$$

where e stands for an arbitrary radial curve of finite coordinate length, while v stands for an arbitrary point in the radial line, and μ is the coordinate length of an arbitrary longitudinal curve on the 2-sphere intersecting the point v .

Holonomies in a canonical theory are defined for a spatial gauge connection. The components K_x and K_φ are used for this purpose in spherically symmetric models of loop quantum gravity because they mimic the feature of the

Ashtekar-Barbero connection as a basic canonical variable conjugate to the spatial metric or densitized triad [28–30]. However, the complete relationship between (K_x, K_φ) and connection components (A_x, A_φ) is nonlinear [27]. While A_x is linearly related to K_x (with an extra term that is pure gauge), the angular component is given by $A_\varphi = \sqrt{\Gamma_\varphi^2 + \gamma^2 K_\varphi^2}$ with the spin-connection component $\Gamma_\varphi = -(E^x)'/(2E^\varphi)$ and the Barbero-Immirzi parameter γ . The relationship between (21) and nonspherical holonomies remains unclear in the absence of a detailed symmetry-reduction procedure that could be directly applied to holonomies and their appearance in a Hamiltonian constraint. Nevertheless, using modifications based on (21) allows one to study potential implications of some features of holonomies, such as the appearance of nonpolynomial expressions in some of the basic phase-space variables.

In this context, it is important to note that a canonical theory also implies the freedom of performing canonical transformations. Given the indirect relationship of (21) with nonspherical holonomies, there is therefore no strong motivation to use a specific variable such as the original K_φ , rather than a canonically transformed version especially if this canonical transformation is linear in K_φ . We will make use of this freedom later on. Note that the classical theory does distinguish the original K_φ as a component of extrinsic curvature. However, this relationship is lost after modifications, which imply that extrinsic curvature has to be derived from the emergent metric. Moreover, being a component of extrinsic curvature is an on-shell feature because it requires equations of motion that relate K_φ to time derivatives of the triad components. However, modifications of the Hamiltonian constraint and their potential relationship to LQG can by definition be analyzed only off-shell. Since the notion of extrinsic curvature is not available at this level, it cannot be used to distinguish a specific choice of phase-space variables.

The explicit integration in the angular holonomy (21) is possible due to spherical symmetry, but the radial holonomy integration must remain formal. (The absence of a path ordering operator here is due to the spherically symmetric context, which implies a reduction of the original structure group $SU(2)$ to the Abelian $U(1)$.) Similarly, the fluxes are given by direct integration of the densitized triads E^x and E^φ over finite, 2-dimensional surfaces with normals in the radial and angular direction, respectively. Fluxes do not require exponentiation and are therefore closer to the original phase-space variables. For the following argument, we need only the expressions of holonomies.

Because loop quantization is based on the holonomy-flux variables, the Hamiltonian constraint must be rewritten as a function of holonomies rather than the bare (extrinsic) curvatures K_x and K_φ , leading to a modified constraint in the spirit of emergent modified gravity, as outlined in the

previous section. However, the radial holonomy (20) is essentially nonlocal, and cannot be studied within this framework, which has thus far only been formulated locally and up to second order in derivatives [8]. On the other hand, the angular holonomies (21) are indeed local when restricting the state space to spherically symmetric states. Therefore, it is natural to expect that quantum effects due to the angular holonomy components can be captured by the local equations of emergent modified gravity. Furthermore, since the angular holonomies can be integrated, they become simple complex exponentials of the angular extrinsic curvature (21). Because the Hamiltonian constraint must be Hermitian as an operator in the quantum theory, or simply real-valued prior to quantization, the holonomy modifications in the Hamiltonian constraint will appear as trigonometric functions of K_ϕ with frequency μ . This is indeed the case of the Hamiltonian constraint (17) derived in the previous section. Therefore, the parameter λ in the notation of emergent modified gravity can be given the interpretation of an angular holonomy length within LQG, and hence acquires the status of a quantum parameter under this interpretation. However, it also inherits new features from the enhanced symmetry properties of emergent modified gravity compared with traditional models of LQG.

In LQG, different triangulations of space are allowed, and a specific choice would determine the form of the quantum parameter $\mu(E^x)$ appearing in the holonomy modification function, using the phase-space variable E^x as the squared radius of our geometry. For example, we can choose a fine lattice such that the spheres are triangulated by small regular polygons with n sides of angular length μ . (It is common to choose tetrahedra for the three-dimensional lattice and hence triangles for the two-dimensional lattice of the spheres, setting $n = 3$ for this purpose.) Each polygon, also referred to as a plaquette, at radius $\sqrt{E^x}$ then covers a geometrical area of $nE^x\mu^2/(4\tan(\pi/n))$. Choosing the plaquettes at different radii to preserve the same geometrical area, we obtain the formula

$$\mu = \frac{\hat{r}}{\sqrt{E^x}} \hat{\mu}, \quad (22)$$

where $\hat{r}$ is a constant reference radius at which $\mu = \hat{\mu}$. This function satisfies $\mu \rightarrow 0$ as $E^x \rightarrow \infty$, as desired for recovering an asymptotically smooth geometry without triangulation effects at large radii.

2. Shortcomings

Several assumptions are used in arriving at this form. An additional, previously unrecognized issue is that the classical component E^x of a space-time metric is used in the geometrical construction, which in general may deviate from the correct geometrical meaning provided by the emergent space-time metric. As we have seen, in

spherically symmetric models it is always possible to choose phase-space variables such that the angular component of the metric is still given by E^x . The construction of holonomy length may therefore be applied as described here. However, strict proposals such as the $\bar{\mu}$ -scheme would require an application of these arguments to any surface, even if it extends into the radial direction which may still be homogeneous in the black-hole interior. In this case, an emergent space-time metric may have to be taken into account, but this is available only after the modified theory has been analyzed. In general, the $\bar{\mu}$ -scheme therefore does not provide a self-contained recipe for the construction of holonomy modifications.

One ingredient of a $\bar{\mu}$ -type scheme that may be applied here is a relationship between holonomy parameters and areas of interest, such as the smallest nonzero eigenvalue of an area operator in a loop quantization. Also this step is heuristic and not fully justified because holonomies of a gauge connection do not depend on a metric, and therefore need not have any relationship with geometrical areas. Nevertheless, such a relationship may be used in order to fix free parameters. (In practice, using area eigenvalues merely determines numerical factors multiplying suitable powers of the Planck length, whose form could easily be obtained from dimensional arguments.) To be specific, by choosing the plaquette size derived above to equal a certain distinguished area, $n\hat{r}^2\hat{\mu}^2/(4\tan(\pi/n)) = \Delta$, we arrive at the relation

$$\mu(E^x)^2 = \tilde{\Delta}/E^x \quad (23)$$

with $\tilde{\Delta} = 4n^{-1}\tan(\pi/n)\Delta$. The square of the quantum parameter (23) is proportional to the solid angle of a single plaquette on a sphere of radius $\sqrt{E^x}$. Since this expression equals $\tilde{\Delta}/E^x$, the number of plaquettes triangulating the sphere of radius $\sqrt{E^x}$ is given by

$$\mathcal{N}_\Delta(E^x) = 4\pi E^x/\tilde{\Delta}. \quad (24)$$

Loop quantization requires the Hamiltonian constraint to be explicitly periodic in K_ϕ . Having μ depend on E^x preserves periodicity in K_ϕ at any fixed value of E^x , but it turns the holonomy $\exp(i\mu(\hat{E}^x)\hat{K}_\phi)$ into a holonomy-flux hybrid operator upon quantization. Such expressions do not obey the basic holonomy-flux algebra, and they do not even provide a closed set of commutators because an expression such as

$$\{h_e^x[K_x], h_{v,\mu}^\phi[K_\phi]\} = -h_{e_1}^x[K_x] \frac{\partial\mu(\nu)}{\partial E^x(\nu)} K_\phi(\nu) h_{v,\mu}^\phi[K_\phi] h_{e_2}^x[K_x], \quad (25)$$

where $e = e_1 \cup e_2$, such that e_1 and e_2 join at $x = \nu$, violates the periodicity condition. In the basic holonomy-flux algebra, the right-hand side of the Poisson bracket

between two holonomy operators should be zero since the connection components Poisson commute. The factor of $K_\varphi(v)$ in (25) shows that trying to implement an E^x -dependent holonomy length is not compatible with a closed algebra of basic operators. Hence, the periodic version (17) of the Hamiltonian constraint is preferred over (15) when carrying out the loop quantization because it resolves two important problems: It depends on pure holonomies of the form $\exp(i\tilde{\lambda}\hat{K}_\varphi)$ with constant $\tilde{\lambda}$, and can therefore be expressed in terms of the generators of a closed basic algebra. At the same time, it allows for scale-dependent λ -effects because a nontrivial $\lambda(E^x)$ appears in the coefficients of (17) after it has been removed as a factor of K_φ in holonomies by a canonical transformation. It is then possible to choose the holonomy length that appears in trigonometric functions as being equal to the E^x -independent reference value $\tilde{\lambda} = \hat{\mu}$. At the same time, dynamical implications of nonconstant holonomy parameters are realized through modified coefficients of holonomy terms, as we demonstrate in detail below.

A common motivation for an E^x -dependent μ in models of loop quantum gravity is not based on properties of basic holonomy operators, but rather on phenomenological questions because a connection or extrinsic-curvature component such as K_φ may be large in classical regimes, depending on the space-time slicing. A holonomy length $\mu(E^x)$ might then be used to counteract growing holonomy modifications if it decreases at a suitable rate as E^x grows, implying large spherical areas. It is often argued that this reason requires an explicit appearance of a nonconstant $\mu(E^x)$ in holonomies. This reasoning seems to favor (15) with a corresponding $\lambda(E^x)$ in holonomy-like terms, even though covariance prevents this constraint from being strictly periodic in K_φ . Moreover, a quantum constraint operator for this version cannot be based on basic holonomies in a closed holonomy-flux algebra.

3. New features

Our new discussion of canonical transformations overturns this established understanding because it is possible to map any constraint of the form (15) into an expression (17) that is strictly periodic in K_φ . The canonical transformation implies that both versions generate exactly the same dynamics and physical observables. Effects of a nonconstant holonomy length in (15) are realized in (17) by a modified E^x dependence of coefficients of holonomy terms, indicated by the nonconstant ratios $\tilde{\lambda}/\lambda$ in (17), in particular in the K_x -term of the modified Hamiltonian constraint. This term is relevant for Hamilton's equation

$$\dot{E}^x = \frac{\tilde{\lambda}}{\lambda} \chi \frac{\sqrt{E^x}}{2} \left(4c_f + \tilde{\lambda}^2 \frac{((E^x)')^2}{(E^\varphi)^2} \right) \frac{\sin(2\tilde{\lambda}K_\varphi)}{2\tilde{\lambda}} \quad (26)$$

for the spherical area E^x (assuming vanishing N^x for the sake of simplicity). As an equation of motion generated by the periodic constraint, this result implies that the classical relationship between the canonical variable K_φ and the time derivative $\dot{E}^x$ receives an additional factor of $\lambda/\tilde{\lambda}$, making K_φ decrease as $\sqrt{E^x}$ grows according to the modified dynamics. (For small $\tilde{\lambda}$ and $\lambda/\tilde{\lambda}$ independent of $\tilde{\lambda}$, we have $K_\varphi \approx (\lambda/\tilde{\lambda})\sqrt{E^x}/(\chi c_f)$, where $c_f, \chi \rightarrow 1$ in the classical limit.) This behavior has the same effect of suppressing holonomy modifications in classical regimes as an explicitly E^x -dependent holonomy length would imply. It is only harder to see this outcome immediately because it requires an analysis of equations of motion.

By dismissing this option, the traditional arguments for holonomy modifications suggest an unjustified preference for one of the possible versions of canonically equivalent constraints, based on flawed heuristics that implicitly relate the canonical variable K_φ in a modified theory to a classical component of extrinsic curvature: Classically, K_φ is not necessarily small enough for $\sin(\tilde{\lambda}K_\varphi)/\tilde{\lambda}$ to be close to K_φ in regimes of small space-time curvature. It is therefore important to distinguish between small- λ regimes (as a formal limit of the constraint) and small- λK_φ regimes (as a dynamical feature of solutions for K_φ). In a modified theory, one cannot use classical heuristics in order to estimate the magnitude of K_φ , interpreted as a component of extrinsic curvature, in order to determine whether λK_φ is sufficiently small in a given semiclassical regime. Since λK_φ appears in a modified term of the Hamiltonian constraint when holonomies are used and since the emergent space-time metric is not the same as the classical metric, there is no kinematical relationship between the magnitude of K_φ in a modified theory and the magnitude of extrinsic curvature in a corresponding classical slicing. An evaluation of suitable holonomy modifications can therefore be made only if equations of motion as well as the emergent metric are considered. However, traditional arguments in favor of certain holonomy modifications are held entirely at a kinematical level.

The distinction between small- λ and small- λK_φ is relevant in these considerations even if one is interested only in leading-order corrections to the classical behavior. The leading order of a small- λ expansion of the theory given by (15) is simply the limit of $\lambda \rightarrow 0$, resulting in the classical dependence of the constraint on K_φ and an emergent metric identical with the classical metric. (The overall theory may still be modified in this limit if the remaining modification functions are nonclassical.) Holonomy terms can also be expanded on a subspace of the solution space where λK_φ is small, even if λ is not taken to zero in a limit. This expansion is relevant for physical properties of the semiclassical limit. In the low-curvature regime of small λK_φ , the quadratic order in this expansion

determines near-classical physics. The nonperiodic version of the Hamiltonian constraint, (15), then becomes

$$\begin{aligned} \tilde{H} \approx & -\sqrt{E^x} \chi^2 \left[E^\varphi \left(\frac{\alpha_0}{E^x} + 2K_\varphi^2 \frac{\partial c_f}{\partial E^x} + 4K_\varphi \frac{1}{\lambda} \frac{\partial(\lambda q)}{\partial E^x} \right) \right. \\ & + \left(\frac{\alpha_2}{E^x} - 2 \frac{\partial \ln \lambda^2}{\partial E^x} \right) (c_f K_\varphi^2 + 2q K_\varphi) \\ & + 4 \left(\frac{K_x}{E^\varphi} + \frac{K_\varphi}{2} \frac{\partial \ln \lambda^2}{\partial E^x} \right) (c_f K_\varphi + q(1 - 2\lambda^2 K_\varphi^2)) \\ & + \frac{((E^x)')^2}{E^\varphi} \left(-\frac{\alpha_2}{4E^x} (1 - \lambda^2 K_\varphi^2) + \left(\frac{K_x}{E^\varphi} + \frac{K_\varphi}{2} \frac{\partial \ln \lambda^2}{\partial E^x} \right) \lambda^2 K_\varphi \right) \\ & \left. + \left(\frac{(E^x)'(E^\varphi)'}{(E^\varphi)^2} - \frac{(E^x)''}{E^\varphi} \right) (1 - \lambda^2 K_\varphi^2) \right], \end{aligned} \quad (27)$$

with structure function

$$\tilde{q}^{xx} \approx \left(\left(c_f + \lambda^2 \left(\frac{(E^x)'}{2E^\varphi} \right)^2 \right) (1 - \lambda^2 K_\varphi^2) - 2q \lambda^2 K_\varphi \right) \chi^2 \frac{E^x}{(E^\varphi)^2}. \quad (28)$$

(A similar result follows for the periodic version.) Some holonomy effects therefore survive in the low-curvature regime and may lead to unexpected dynamical implications, unless the strict limit of $\lambda \rightarrow 0$ is taken in this regime too. This approximation of the Hamiltonian constraint is valid only in slices where λK_φ is sufficiently small. Physical implications can only be analyzed if equations of motion are studied, revealing through $\tilde{q}^{xx}$ the geometrical meaning and dynamical behavior of K_φ in suitable semiclassical regimes.

An important additional flaw in several discussions of holonomy modifications is the lack of covariance. It is then possible for $\sin(\tilde{\lambda} K_\varphi)/\tilde{\lambda}$ to be close to K_φ in some space-time slicings but deviate strongly in others, even in classical regimes. This problem is solved by our constructions, which explicitly realize slicing independence together with the correct geometrical meaning of K_φ . As already emphasized, covariance is also the main reason why nonperiodic terms are required in a Hamiltonian constraint with E^x -dependent holonomy length. Our constructions not only implement full covariance, they also relate different versions of holonomy modifications by carefully considering canonical transformations. For effective equations and their solutions, we may use either of the two, or perhaps others related by further canonical transformations. One may simply prefer an option that generates the simplest equations of motion for a given purpose which, however, will in general be gauge dependent.

Covariant holonomy modifications therefore cannot be expressed in the straitjacket of μ_0 and $\bar{\mu}$ -like schemes that have only a heuristic basis motivated by expectation about the holonomy length combined with a classical

understanding of canonical variables in a modified theory. The appearance of holonomy terms, classified in this way, depends on the canonical variables used, and a μ_0 -like scheme can be transformed into an equivalent $\bar{\mu}$ -like scheme if only the arguments of trigonometric functions are used in the definition of holonomy modifications. As an important implication of our new λ -scheme, dynamical properties of holonomy modifications and their suppression in classical regimes can be realized with strictly periodic holonomy terms, provided their coefficients are amended by suitable modification functions. (In traditional models of loop quantum gravity, such modifications are usually understood as inverse-triad corrections because these coefficients classically depend only on the densitized triad and its inverse but not on the momenta.)

4. Self-consistent treatment of holonomy effects

Taking these lessons into account, we arrive at a novel understanding of holonomy modifications in models of loop quantum gravity. The freedom of performing canonical transformations can be fixed by requiring that all terms in the Hamiltonian constraint depending on K_φ are strictly periodic in this variable. According to emergent modified gravity, covariance then implies that only the combination $\tilde{\lambda} K_\varphi$ with a constant $\tilde{\lambda}$ can appear in these periodic functions. At this point, the choice of canonical variables is unique if it is required to preserve the classical limit for $\tilde{\lambda} \rightarrow 0$, such that $\tilde{\lambda} K_\varphi$ cannot be used as a new phase-space variable conjugate to $E^\varphi/\tilde{\lambda}$. There is therefore an unambiguous interpretation of K_φ -dependent functions as an effective description of angular holonomies. (Any canonical transformation of K_φ will render the constraint nonperiodic in this variable.) Accordingly, we refer to the constant $\tilde{\lambda}$ as the holonomy length of the theory. By definition, this length is always constant and does not depend on E^x .

Different schemes are implemented by choosing specific functions for $\lambda(E^x)$, which does not appear in holonomy terms but rather in their coefficients in the Hamiltonian constraint. The choice of $\lambda(E^x)$ changes the geometrical meaning and the dynamical behavior of K_φ in holonomy terms according to (26). (The same observation would be correct in the classical Hamiltonian if one introduces a function $\lambda(E^x)$ just for the purpose of performing a canonical transformation without using holonomies. The geometrical meaning of all the phase-space variables then changes.) Our results demonstrate that once holonomy modifications are introduced, it is not possible for K_φ to have its classical correspondence with the extrinsic curvature. As already mentioned, this modification makes it difficult to assess the significance of holonomy modifications in different regimes because this task would require an estimate of the values taken by K_φ , which depends on the dynamics and the space-time gauge. In fact, the equation of

motion for E^x in a modified theory does not directly determine K_φ but rather the holonomy function

$$\frac{\sin(2\tilde{\lambda}K_\varphi)}{2\tilde{\lambda}} = \frac{\lambda}{\tilde{\lambda}\chi\sqrt{E^x}} \frac{2\dot{E}^x}{4c_f + \tilde{\lambda}^2((E^x)')^2/(E^\varphi)^2} \quad (29)$$

obtained by inverting (26) and still assuming vanishing N^x for the sake of convenience. For χ and c_f close to their classical values, the ratio $\lambda/\tilde{\lambda}$ determines the strength of holonomy modifications, defined as the deviation of $(2\tilde{\lambda})^{-1} \sin(2\tilde{\lambda}K_\varphi)$ from the classical expression $2\dot{E}^x/\sqrt{E^x}$ of K_φ . For the classical limit to be realized at $\tilde{\lambda} \rightarrow 0$, we need $\lambda(E^x) = \tilde{\lambda}h(E^x)$ with a function $h(E^x)$ independent of $\tilde{\lambda}$ (or finite and nonzero for $\tilde{\lambda} \rightarrow 0$). Since this new function $h(E^x) = \lambda(E^x)/\tilde{\lambda}$ describes the relative strength of holonomy modifications on different scales, we call it the holonomy function. Its appearance shows that it is independent of the holonomy length and instead describes the dynamical behavior of holonomy modifications or the variable K_φ .

Some choices of the holonomy function $h(E^x)$ resemble properties of the traditional μ_0 or $\bar{\mu}$ -type schemes, but in some cases the usual heuristics turns out to be misleading. For instance, a constant $h(E^x) = 1$, or $\lambda = \tilde{\lambda}$, could be related to a μ_0 -type scheme of constant holonomy parameter μ in the traditional formulation. A common argument against using such a constant is that the interpretation of $\tilde{\lambda}$ as a holonomy length derived from coordinates would imply a growing physical length $\tilde{\lambda}\sqrt{E^x}$ as measured by the relevant spatial geometry. The growth in E^x of this function then suggests that holonomy modifications increase in asymptotic regimes of large E^x , which are often expected to behave in a classical manner. However, Eq. (29) shows that holonomy modifications do not grow in this case but merely track the classical behavior of K_φ : The second term in the denominator on the right-hand side of (29) is subdominant in this regime, and thus the expression $(2\tilde{\lambda})^{-1} \sin(2\tilde{\lambda}K_\varphi)$ on the left is identical with the classical behavior $2\dot{E}^x/\sqrt{E^x}$ of K_φ for $\tilde{\lambda} = \lambda$. Modifications then appear only because one has to rewrite $\sin(2\tilde{\lambda}K_\varphi)$ in terms of $\sin(\tilde{\lambda}K_\varphi)$ for some terms of the constraint, using trigonometric identities. Holonomy modifications therefore do not grow in an unbounded manner in the case of a constant holonomy length, but they may not decrease quickly enough so as to become subdominant in asymptotic regimes. This behavior can be changed by a holonomy function $h(E^x)$ that decreases at a sufficient rate for growing E^x . We will see examples in our specific cases to be discussed later.

F. Gravitational observable

Consider a scalar function on the phase space $\mathcal{M}$. This function is a weak observable if $\delta_\epsilon \mathcal{M} = \mathcal{D}_H H + \mathcal{D}_x H_x$,

where $\mathcal{D}_H$ and $\mathcal{D}_x$ are functions on the phase space and ϵ is a gauge function. On-shell, the function $\mathcal{M}$ then remains invariant under time evolution or arbitrary gauge transformations. In this section, we will keep the most general form of our modification functions in the Hamiltonian constraint, before restricting them to take values as required in effective LQG in the next section. We consider the dependence $\mathcal{M} = \mathcal{M}(E^x, K_\varphi, (E^x)'/E^\varphi)$ as a generic expression of density weight zero with up to first-order on-shell derivatives. Using the periodic constraint (17), the weak observability condition $\delta_\epsilon \mathcal{M} = \mathcal{D}_H H + \mathcal{D}_x H_x$ determines the functional dependence [31]

$$\begin{aligned} \mathcal{M} = & d_0 + \frac{d_2}{2} \left(\exp \int dE^x \frac{\bar{\alpha}_2}{2E^x} \right) \left(c_f \frac{\sin^2(\tilde{\lambda}K_\varphi)}{\tilde{\lambda}^2} \right. \\ & + 2\tilde{q} \frac{\sin(2\tilde{\lambda}K_\varphi)}{2\tilde{\lambda}} - \cos^2(\tilde{\lambda}K_\varphi) \left(\frac{(E^x)'}{2E^\varphi} \right)^2 \Big) \\ & + \frac{d_2}{4} \int dE^x \left(\frac{\bar{\alpha}_0}{E^x} \exp \int dE^x \frac{\bar{\alpha}_2}{2E^x} \right), \end{aligned} \quad (30)$$

which is unique up to the constants d_0 and d_2 with classical limits $d_0 \rightarrow 0$ and $d_2 \rightarrow 1$. The uniqueness can be shown on assuming the dependence of the Dirac observable to be of the form $\mathcal{M} = \mathcal{M}(E^x, K_\varphi, (E^x)'/E^\varphi)$ [31]. (There exists an anomaly-free and covariant constraint more general than (17), but after imposing the existence of a weak observable, the extra terms are restricted to vanish and we recover (17) with (30) as the weak observable [31].) Here we use the barred functions as defined by (19) for the sake of conciseness.

If we insert the classical values $\alpha_2 = 1$, and $\alpha_0 = 1 - \Lambda E^x$, and $d_0 = 0$, $d_2 = 1$ the weak observable simplifies to

$$\begin{aligned} \mathcal{M} = & \frac{\sqrt{E^x}}{2} \left(1 - \frac{\Lambda}{3} E^x + \frac{\tilde{\lambda}^2}{\lambda^2} \left(c_f \frac{\sin^2(\tilde{\lambda}K_\varphi)}{\tilde{\lambda}^2} \right. \right. \\ & \left. \left. + 2\frac{\lambda}{\tilde{\lambda}} \tilde{q} \frac{\sin(2\tilde{\lambda}K_\varphi)}{2\tilde{\lambda}} - \cos^2(\tilde{\lambda}K_\varphi) \left(\frac{(E^x)'}{2E^\varphi} \right)^2 \right) \right) \end{aligned} \quad (31)$$

where we have chosen the integration constant $\tilde{\lambda}^2$ for the exponential integrals. As already seen for the constraint, we must be careful in taking the classical limit. If we first invert the canonical transformation to the nonperiodic variables, the classical limit is given by $\lambda \rightarrow 0$. If this inversion is not performed, we must first take the limit $\lambda \rightarrow \tilde{\lambda}$, followed by $\tilde{\lambda} \rightarrow 0$. The classical limit of this observable is then the classical mass observable. The structure function (18) can be written in terms of this observable as

$$\tilde{q}^{xx} = \left(c_f + \lambda^2 \left(1 - \frac{2\mathcal{M}}{\sqrt{E^x}} - \frac{\Lambda}{3} E^x \right) \right) \frac{\tilde{\lambda}^2}{\lambda^2} \chi^2 \frac{E^x}{(E^\varphi)^2}. \quad (32)$$

In the nonperiodic version, the mass observable takes the form

$$\mathcal{M} = \frac{\sqrt{E^x}}{2} \left(1 - \frac{\Lambda}{3} E^x + c_f \frac{\sin^2(\lambda K_\varphi)}{\lambda^2} + 2q \frac{\sin(2\lambda K_\varphi)}{2\lambda} - \cos^2(\lambda K_\varphi) \left(\frac{(E^x)'}{2E^\varphi} \right)^2 \right), \quad (33)$$

and the structure-function is given by

$$\tilde{q}^{xx} = \left(c_f + \lambda^2 \left(1 - \frac{2\mathcal{M}}{\sqrt{E^x}} - \frac{\Lambda}{3} E^x \right) \right) \chi^2 \frac{E^x}{(E^\varphi)^2}. \quad (34)$$

Note that the mass observable is periodic in K_φ even in the nonperiodic version of the constraint.

G. Reflection symmetry of the transition surface

The Hamiltonian constraint (17), besides being periodic in K_φ has another interesting symmetry in the special case of $q = 0$. In this case, the structure function is symmetric around the surface $K_\varphi = \pi/(2\tilde{\lambda})$ in the sense that $\tilde{q}^{xx}$ is invariant under the reflection $\delta \rightarrow -\delta$ where δ is defined by $K_\varphi = \pm\pi/(2\tilde{\lambda}) + \delta$. The Hamiltonian constraint is invariant under the combined operation $\delta \rightarrow -\delta$, $K_x \rightarrow -K_x$. Thus, the two conditions $K_\varphi = \pm\pi/(2\tilde{\lambda})$ and $K_x = 0$ define a surface of reflection symmetry in the phase-space. This property can manifest itself in dynamical solutions as a reflection surface in space-time [32]. If $q \neq 0$, this reflection-symmetry is broken unless q , like K_x , changes sign in the dynamical solution, but this is unlikely to be the case in general. Physical implications of breaking the reflection-symmetry breaking effects of $q \neq 0$ will be studied elsewhere [33]. In the following we set $q = 0$, such that reflection symmetry is realized.

It will be useful in the following sections to evaluate the mass observable (31) at the reflection symmetry surface $K_\varphi = -\pi/(2\tilde{\lambda})$, or equivalently its incarnation in the non-periodic variables (33) at $K_\varphi = -\pi/(2\lambda)$. After some simplifications, we find that E^x , for a given value of the observable $\mathcal{M}$ obeys the equation

$$c_f + \lambda^2 \left(1 - \frac{2\mathcal{M}}{\sqrt{E^x}} - \frac{\Lambda}{3} E^x \right) = 0 \quad (35)$$

in any region of space-time where $K_\varphi = -\pi/(2\lambda)$. Since λ is an arbitrary function of E^x , this equation may have multiple solutions according to the chosen modification function. Multiple reflection surfaces may therefore exist in a space-time solution. As will be clear from dynamical solutions derived later on, this reflection symmetry happens around the hypersurface in space-time where the areal radius $\sqrt{E^x}$ reaches an extremum. For the specific solution $x_\lambda^{(-)}$, which we will use to denote the minimal value of $\sqrt{E^x}$ according to the dynamics defined by a given λ , this will correspond to a spacelike transition surface connecting a black hole to a white hole through a wormhole interior. In later sections, we will discuss the physical interpretation of these extremal values, including the case where there exists a maximal value at $x_\lambda^{(+)}$.

III. BLACK-HOLE SOLUTION FOR GENERAL TRIANGULATION SCHEMES

We continue to work with a general functional dependence of $\lambda = \lambda(E^x)$ and now demonstrate that, qualitatively, singularity resolution of black holes in models of loop quantum gravity based on holonomy modifications is a triangulation-independent feature. In order to isolate effects of the holonomy modification λ , we set the functions c_f , α_0 , α_2 , and q to their classical values, while keeping a non-classical, nontrivial $\chi(E^x)$ for rescaling freedom. For now, we keep a cosmological constant Λ and only set it to zero when we analyze suitable conditions for asymptotic flatness. As will become clear later on, it is necessary to scale the emergent metric using a suitable χ if we are interested in asymptotic flatness, depending on the chosen λ . For instance, a constant $\lambda = \tilde{\lambda}$ requires $\chi(\infty) = 1/\sqrt{1 + \tilde{\lambda}^2}$, while $\lambda = \sqrt{\Delta/E^x}$ requires $\chi(\infty) = 1$. The scaling of the spatial metric by a constant χ is not a simple coordinate transformation of the radial coordinate because the latter would change other components nontrivially if there is space-time curvature.

Given the classical choice for most of the modification functions, the nonperiodic constraint (15) simplifies to

$$\begin{aligned} \tilde{H} = & -\chi \frac{\sqrt{E^x}}{2} \left[E^\varphi \left(\frac{1}{E^x} - \Lambda + \frac{1}{E^x} \frac{\sin^2(\lambda K_\varphi)}{\lambda^2} + 4 \left(K_\varphi \frac{\sin(2\lambda K_\varphi)}{2\lambda} - \frac{\sin^2(\lambda K_\varphi)}{\lambda^2} \right) \frac{\partial \ln \lambda}{\partial E^x} \right) \right. \\ & + 4K_x \frac{\sin(2\lambda K_\varphi)}{2\lambda} - \frac{((E^x)')^2}{4E^\varphi} \left(\frac{1}{E^x} \cos^2(\lambda K_\varphi) - 4\lambda^2 \left(\frac{K_x}{E^\varphi} + K_\varphi \frac{\partial \ln \lambda}{\partial E^x} \right) \frac{\sin(2\lambda K_\varphi)}{2\lambda} \right) \\ & \left. + \cos^2(\lambda K_\varphi) \left(\frac{(E^x)'(E^\varphi)'}{(E^\varphi)^2} - \frac{(E^x)''}{E^\varphi} \right) \right]. \end{aligned} \quad (36)$$

with the structure function (16) becoming

$$\tilde{q}^{xx} = \left(1 + \lambda^2 \left(\frac{(E^x)'}{2E^\varphi}\right)^2\right) \cos^2(\lambda K_\varphi) \chi^2 \frac{E^x}{(E^\varphi)^2} = \left(1 + \lambda^2 \left(1 - \frac{2\mathcal{M}}{\sqrt{E^x}} - \frac{\Lambda}{3} E^x\right)\right) \chi^2 \frac{E^x}{(E^\varphi)^2}. \quad (37)$$

For comparison, the periodic version (17) becomes

$$\begin{aligned} \tilde{H} = & -\chi \frac{\tilde{\lambda} \sqrt{E^x}}{\lambda} \left[E^\varphi \left(\frac{\lambda^2}{\tilde{\lambda}^2} \left(\frac{1}{E^x} - \Lambda \right) + 4 \left(\frac{1}{4E^x} - \frac{\partial \ln \lambda}{\partial E^x} \right) \frac{\sin^2(\tilde{\lambda} K_\varphi)}{\tilde{\lambda}^2} \right) + 4K_x \frac{\sin(2\tilde{\lambda} K_\varphi)}{2\tilde{\lambda}} \right. \\ & + \frac{((E^x)')^2}{E^\varphi} \left(\cos^2(\tilde{\lambda} K_\varphi) \left(-\frac{1}{4E^x} + \frac{\partial \ln \lambda}{\partial E^x} \right) + \tilde{\lambda}^2 \frac{K_x}{E^\varphi} \frac{\sin(2\tilde{\lambda} K_\varphi)}{2\tilde{\lambda}} \right) \\ & \left. + \left(\frac{(E^x)'(E^\varphi)'}{(E^\varphi)^2} - \frac{(E^x)''}{E^\varphi} \right) \cos^2(\tilde{\lambda} K_\varphi) \right], \end{aligned} \quad (38)$$

with the structure function (18) becoming

$$\tilde{q}^{xx} = \left(1 + \tilde{\lambda}^2 \left(\frac{(E^x)'}{2E^\varphi}\right)^2\right) \cos^2(\tilde{\lambda} K_\varphi) \chi^2 \frac{\tilde{\lambda}^2}{\lambda^2} \frac{E^x}{(E^\varphi)^2} = \left(1 + \lambda^2 \left(1 - \frac{2\mathcal{M}}{\sqrt{E^x}} - \frac{\Lambda}{3} E^x\right)\right) \chi^2 \frac{\tilde{\lambda}^2}{\lambda^2} \frac{E^x}{(E^\varphi)^2}. \quad (39)$$

Since the effective line element can be written as

$$ds^2 = -N^2 dt^2 + \tilde{q}_{xx} (dx + N^x dt)^2 + E^x d\Omega^2, \quad (40)$$

we shall now solve for the phase-space variables, imposing different gauges, in order to explicitly obtain the resulting space-time geometry.

A. Nonperiodic phase-space coordinates

We will use the nonperiodic constraint (36) in this subsection to compute regions of a Schwarzschild-type static exterior and homogeneous regions in two different gauges. We show that a third gauge results in a modified Gullstrand-Painlevé space-time and is related to the two previous gauges by a standard coordinate transformation in regions of overlap.

These three gauges possess a second coordinate singularity beyond the horizons, but they all show regular Ricci and Kretschmann scalars at those points. In Sec. III B we will use the periodic version of the constraint and obtain another gauge that is free of such a coordinate singularity and related to the Schwarzschild-type homogeneous region by a standard coordinate transformation.

Because the overall factor χ always multiplies the lapse function in the equations of motion, it will be useful to define

$$\tilde{N} = \chi N. \quad (41)$$

We encourage readers to consult the Appendix for a detailed exposition of the equations of motion utilized in this section.

1. Schwarzschild gauge: Static region

We define the Schwarzschild gauge by

$$N^x = 0, \quad E^x = x^2. \quad (42)$$

The consistency condition $\dot{E}^x = 0$ implies the equation

$$\left(1 + \lambda^2 \left(\frac{(E^x)'}{2E^\varphi}\right)^2\right) \frac{\sin(2\lambda K_\varphi)}{2\lambda} = 0 \quad (43)$$

which, assuming consistency with the classical limit, is solved by imposing $K_\varphi = 0$. This result implies a second consistency equation $\dot{K}_\varphi = 0$, which we will address later on by solving it for the lapse function.

The vanishing of the diffeomorphism constraint further implies that $K_x = 0$. Finally, the vanishing of the Hamiltonian constraint implies the equation

$$0 = (1 - \Lambda x^2)(E^\varphi)^2 - 3x^2 + x^3(\ln(E^\varphi)^2)', \quad (44)$$

which is solved by

$$E^\varphi = \frac{x}{\sqrt{1 - c_\varphi/x - \Lambda x^2/3}}, \quad (45)$$

with constant c_φ . Substituting all these results into the mass observable (31), and rewriting it as $\mathcal{M} = M$ (interpreted as the mass parameter of a line element rather than a phase-space function), determines

$$c_\varphi = 2M. \quad (46)$$

Lastly, $\dot{K}_\varphi = 0$ implies that $\tilde{N}$ acquires the expression of the classical lapse function, up to a scaling constant α , since

$K_\varphi = 0$ eliminates all λ dependence in the constraint and the equations of motion it generates. Therefore,

$$N = \frac{\sqrt{1 - J(x)}}{\alpha\chi}, \quad (47)$$

where we have defined

$$J(x) = \frac{2M}{x} + \frac{\Lambda x^2}{3} \quad (48)$$

for brevity.

The structure function is then given by

$$\tilde{q}^{xx} = \chi^2(1 + \lambda^2(1 - J(x)))(1 - J(x)), \quad (49)$$

where λ is an arbitrary function of x through E^x . Space-time solutions therefore depend on λ through the emergent space-time metric, while traditional models of loop quantum gravity always imply λ -independent solutions in the static gauge. This property is one example of covariance problems in this setting because an unmodified static gauge makes it difficult to construct an equivalent nonstatic gauge for the same model. From (40), the space-time line element is given by

$$ds^2 = -(1 - J(x)) \frac{dt^2}{\alpha^2 \chi^2} + \frac{1}{(1 + \lambda^2(1 - J(x)))(1 - J(x))} \frac{dx^2}{\chi^2} + x^2 d\Omega^2, \quad (50)$$

for arbitrary $\lambda(x)$ and $\chi(x)$, and a constant α . For static solutions as constructed here, α may always be absorbed by a redefinition of the time coordinate.

The locations of the black-hole and cosmological horizons are still given by their classical expressions since they solve the same classical equation,

$$1 - \frac{2M}{x} - \frac{\Lambda x^2}{3} = 0 \quad (51)$$

implied by factors of $(1 - J(x))^{\pm 1}$ in (50). Therefore, using a small positive cosmological constant so as to match with observations for our Universe, the black-hole horizon appears at

$$x_H \approx 2M, \quad (52)$$

and the cosmological one at

$$x_\Lambda \approx \sqrt{\frac{3}{\Lambda}}. \quad (53)$$

Just as for a classical black hole in a de Sitter background, the static metric is valid only in the region $x_H < x < x_\Lambda$, beyond which the metric turns spatially homogeneous in an

extension of the same gauge, just flipping the roles of t and x as time and space coordinates.

However, now we have further coordinate singularities at the values of x that solve the equation

$$1 + \lambda^2 \left(1 - \frac{2M}{x} - \frac{\Lambda x^2}{3} \right) = 0 \quad (54)$$

implied by the new factor of dx^2 in (50). This equation can have multiple solutions depending on the chosen triangulation scheme $\lambda = \lambda(x)$, but we can still formulate several general results. These new coordinate singularities are closely related to nonclassical hypersurfaces of reflection symmetry implied by holonomy effects.

In particular, note that the term multiplying λ^2 is the expression on the left-hand side of the horizons equation (51) and hence it vanishes at x_H and x_Λ . Since $\lambda^2 > 0$, this term must turn negative for (54) to hold, and hence a coordinate singularity of this form cannot appear in the static region where this term is always positive. Thus, the solutions $x = x_\lambda^{(i)}$ of this equation, with i labeling the different solutions, must be either below the Schwarzschild horizon, $x_\lambda^{(-)} < x_H$ denoting the largest solution less than the Schwarzschild radius, or above the cosmological horizon, $x_\lambda^{(+)} > x_\Lambda$ denoting the smallest solution greater than the cosmological horizon. Therefore, solutions to (54) do not lie in the static region, but rather in the homogeneous ones.

Finally, we note that this equation corresponds precisely to (35) for the reflection-symmetry surface. Therefore, the solutions $x_\lambda^{(-)}$ and $x_\lambda^{(+)}$ correspond to extrema of the radius $\sqrt{E^x}$, around which the spacetime is symmetric, and will have particularly interesting consequences. Solutions of these two types are not guaranteed to exist for any triangulation scheme $\lambda = \lambda(E^x)$, and there could be more than two solutions in some cases. Nevertheless, we will show that there exists an $x_\lambda^{(-)}$, corresponding to the minimum of the areal radius, for any well-behaved holonomy function, introducing the quantum hypersurface where the geometry transitions from a black hole to a white hole through a wormhole interior. In later sections we will come back to specific solutions corresponding to the traditional μ_0 and $\bar{\mu}$ schemes. First, however, we must ensure that the homogeneous regions exist as solutions to the equations of motion in their corresponding gauges, and identify the coordinate transformations relating the different gauge choices in regions of overlap.

2. Schwarzschild gauge: Homogeneous regions

We now restrict the gauge by setting all spatial derivatives equal to zero. We will use the label t_h for the time coordinate and x_h for the radial coordinate in a homogeneous patch. The homogeneity condition implies the

partial gauge fixing

$$N^x = 0, \quad N' = 0. \quad (55)$$

The on-shell condition $\tilde{H} = 0$ can be solved to obtain

$$K_x = -\frac{E^\varphi}{4E^x} \frac{2\lambda}{\sin(2\lambda K_\varphi)} \left(1 - \Lambda E^x + \frac{\sin^2(\lambda K_\varphi)}{\lambda^2} \right) + 2 \left(K_\varphi \frac{\sin(2\lambda K_\varphi)}{2\lambda} - \frac{\sin^2(\lambda K_\varphi)}{\lambda^2} \right) \frac{\partial \ln \lambda^2}{\partial \ln E^x}. \quad (56)$$

Equations of motion for the remaining phase-space variables are

$$\dot{E}^x = 2\bar{N} \sqrt{E^x} \frac{\sin(2\lambda K_\varphi)}{2\lambda}, \quad (57)$$

as well as

$$\frac{\dot{E}^\varphi}{E^\varphi} = \frac{\bar{N}}{2\sqrt{E^x}} \frac{2\lambda}{\sin(2\lambda K_\varphi)} \left(\frac{\sin^2(\lambda K_\varphi)}{\lambda^2} - (1 - \Lambda E^x) \cos(2\lambda K_\varphi) - 2\lambda^2 \frac{\sin^4(\lambda K_\varphi)}{\lambda^4} \frac{\partial \ln \lambda^2}{\partial \ln E^x} \right). \quad (58)$$

and

$$\dot{K}_\varphi = -\frac{\bar{N}}{2\sqrt{E^x}} \left(1 - \Lambda E^x + \frac{\sin^2(\lambda K_\varphi)}{\lambda^2} + 2 \left(K_\varphi \frac{\sin(2\lambda K_\varphi)}{2\lambda} - \frac{\sin^2(\lambda K_\varphi)}{\lambda^2} \right) \frac{\partial \ln \lambda^2}{\partial \ln E^x} \right). \quad (59)$$

Combining these three time derivatives, we can pick E^x as an evolution parameter by using the chain rule, $dK_\varphi/dE^x = \dot{K}_\varphi/\dot{E}^x$, obtaining the equation

$$\frac{d}{d \ln E^x} \left(\frac{\sin^2(\lambda K_\varphi)}{\lambda^2} \right) = -\frac{1}{2} \left(1 - \Lambda E^x + \frac{\sin^2(\lambda K_\varphi)}{\lambda^2} \right), \quad (60)$$

which has the solution

$$\frac{\sin^2(\lambda K_\varphi)}{\lambda^2} = \frac{c_x}{\sqrt{E^x}} + \frac{\Lambda E^x}{3} - 1 \quad (61)$$

with constant c_x . Direct substitution into the mass observable (33) and relabeling $\mathcal{M} = M$ determines $c_x = 2M$. The equation of motion for the structure function is given by

$$\begin{aligned} \left(\ln \frac{\tilde{q}^{xx}}{\chi^2} \right)^{\cdot} &= \left(\ln \left(\cos^2(\lambda K_\varphi) \frac{E^x}{(E^\varphi)^2} \right) \right)^{\cdot} \\ &= 2 \left(-\lambda^2 \frac{\tan(\lambda K_\varphi)}{\lambda} \left(\dot{K}_\varphi + \frac{1}{2} K_\varphi \frac{\partial \ln \lambda^2}{\partial \ln E^x} \frac{\dot{E}^x}{E^x} \right) - \frac{\dot{E}^\varphi}{E^\varphi} \right) + \frac{\dot{E}^x}{E^x} \\ &= \frac{\bar{N}}{\sqrt{E^x} \tan(\lambda K_\varphi)} \frac{\lambda}{\sqrt{E^x}} \left(\frac{2M}{\sqrt{E^x}} + \frac{\Lambda}{3} E^x - \frac{2\lambda E^x}{3} \right). \end{aligned} \quad (62)$$

We now complete the gauge fixing by setting $E^x = t_h^2$, yielding

$$\frac{\sin^2(\lambda K_\varphi)}{\lambda^2} = \frac{2M}{t_h} + \frac{\Lambda}{3} t_h^2 - 1. \quad (63)$$

The consistency equation $\dot{E}^x = 2t_h$ can be used to solve for the lapse function as

$$\bar{N} = \frac{2\lambda}{\sin(2\lambda K_\varphi)} = (J(t_h) - 1)^{-1/2} (1 - \lambda^2 (J(t_h) - 1))^{-1/2} \quad (64)$$

with our previous definition of the function J , and the equation of motion for the structure function is solved by

$$\tilde{q}^{xx} = \alpha^2 \chi^2 \left(\frac{2M}{t_h} + \frac{\Lambda}{3} t_h^2 - 1 \right)^{-1} \quad (65)$$

with an integration constant α . We therefore obtain the space-time line element

$$\begin{aligned} ds^2 &= -\frac{1}{(1 - \lambda^2 (J(t_h) - 1))(J(t_h) - 1)} \frac{dt_h^2}{\chi^2} \\ &\quad + (J(t_h) - 1) \frac{dx_h^2}{\alpha^2 \chi^2} + t_h^2 d\Omega^2, \end{aligned} \quad (66)$$

for arbitrary λ and χ as functions of t_h , which matches the static Schwarzschild metric (50) up to the label swap $x \rightarrow t_h$, $t \rightarrow x_h$. This effective metric has a new (quantum) coordinate singularity at the time coordinates solving the equation

$$1 - \lambda^2 \left(\frac{2M}{t_h} + \frac{\Lambda t_h^2}{3} - 1 \right) = 0, \quad (67)$$

which is the same equation as the one obtained in the static region (54) up to the label change $x \rightarrow t_h$. As discussed in this context, solutions t_h represent hypersurfaces of reflection symmetry. (The name quantum coordinate singularity is motivated by the observation that this new singular hypersurface appears only in the presence of holonomy modifications. As mentioned earlier, on adopting suitable

coordinates, we shall demonstrate later that one of the solutions of (67) corresponds to the transition surface where $\sqrt{E^x}$ acquires a minimum value.)

Therefore, homogeneous Schwarzschild-type coordinates are valid only in the regions $x_\lambda^{(-)} < t_h < x_H$ where $x_\lambda^{(-)}$ is the largest solution of (67) smaller than x_H , and $x_\Lambda < t_h < x_\lambda^{(+)}$ where $x_\lambda^{(+)}$ is the smallest solution of (67) greater than x_Λ . If there is no $x_\lambda^{(-)}$ obeying the condition that defines it, we set $x_\lambda^{(-)} = 0$. If there is no $x_\lambda^{(+)}$ obeying the condition that defines it, we set $x_\lambda^{(+)} = \infty$. If (67) has more than two distinct solutions, additional regions can be introduced. However, we will show in Sec. III C that the solutions $x_\lambda^{(\pm)}$ closest to x_H and x_Λ , respectively, represent extremal radii, such that $x_\lambda^{(-)}$ is a minimum radius, or a “bounce,” and $x_\lambda^{(+)}$ is a maximum radius of a recollapsing homogeneous region. Solutions $x_\lambda^{(i)}$ less than $x_\lambda^{(-)}$ or greater than $x_\lambda^{(+)}$ are therefore never reached by dynamical solutions if we start in the static Schwarzschild region and extend it across coordinate singularities. Nevertheless, these alternative solutions might be used to define independent space-time models that could perhaps be connected with the static region by tunneling processes of a more complete quantum theory.

As already indicated, the homogenous solutions can formally be obtained from the static one by a simple flip of coordinates. However, the corresponding space-time regions are disjoint and separated by horizons. For a more precise connection between these regions, we should

construct a gauge that describes a space-time region overlapping with a portion of the static solution and a portion of a homogeneous one, crossing the horizon. Classically, an example for such a gauge choice is given by the Gullstrand-Painlevé coordinates, which we now construct for the modified equations.

3. From static Schwarzschild to Gullstrand-Painlevé

The metric (50) is static and thus has a timelike Killing vector $\xi = \partial_t$. The quantity $\varepsilon = -\xi \cdot \mathbf{u} = -u_t = -g_{t\nu}u^\nu$ is therefore conserved along geodesics, using the 4-velocity $\mathbf{u}$. Compared with the classical Gullstrand-Painlevé system, we are interested in geodesics for free fall starting at rest at a finite radius, $x = x_0$, because the asymptotic regime may be nonstatic if there is a cosmological horizon or nonclassical asymptotic behavior implied by certain types of holonomy modifications. The 4-velocity of such an object has a time component $u^t|_{x=x_0} = g^{tt}u_t|_{x=x_0} = -g^{tt}|_{x=x_0}\varepsilon$, using the diagonal nature of the static metric in the second step, while all spatial components vanish. Timelike normalization, $-1 = g_{\mu\nu}u^\mu u^\nu|_{x=x_0} = g_{tt}(u^t)^2|_{x=x_0}$, then implies

$$\varepsilon^2 = -\frac{1}{g^{tt}|_{x=x_0}} = \frac{1 - J(x_0)}{\alpha^2 \chi(x_0)^2}. \quad (68)$$

For $x \neq x_0$, u^x is nonzero and is related to u^t by the normalization condition. Since $u_t = -\varepsilon$ is conserved, it is convenient to evaluate normalization in the form $g^{\mu\nu}u_\mu u_\nu = -1$, such that

$$\begin{aligned} u_x &= s \sqrt{g_{xx}(x)(-g^{tt}(x)\varepsilon^2 - 1)} = s \sqrt{\frac{1}{(1 - J(x))(1 + \lambda(x)^2(1 - J(x)))\chi(x)^2} \left(\frac{\chi(x)^2}{\chi(x_0)^2} \frac{1 - J(x_0)}{1 - J(x)} - 1 \right)} \\ &= \frac{s}{\chi(x)\chi(x_0)} \frac{\sqrt{\chi(x)^2(1 - J(x_0)) - \chi(x_0)^2(1 - J(x))}}{|1 - J(x)|\sqrt{1 + \lambda(x)^2(1 - J(x))}} \end{aligned} \quad (69)$$

with a sign choice s that distinguishes outgoing from ingoing GP coordinates.

The normalization condition also implies that we can obtain the proper-time interval along our geodesics from the covelocity components

$$d\tau = -\frac{g^{\mu\nu}dx_\mu dx_\nu}{d\tau} = -u_\mu g^{\mu\nu}dx_\nu = -u_\mu dx^\nu. \quad (70)$$

Using proper time as the new GP coordinate, we obtain

$$dt_{\text{GP}} = -u_\mu dx^\mu = \varepsilon dt - \frac{s}{\chi(x)\chi(x_0)} \frac{\sqrt{\chi(x)^2(1 - J(x_0)) - \chi(x_0)^2(1 - J(x))}}{|1 - J(x)|\sqrt{1 + \lambda(x)^2(1 - J(x))}} dx. \quad (71)$$

The coordinate transformation $t_{\text{GP}}(t, x)$ is defined by an explicit integration of the 1-form (71), which depends on the choice of $\lambda(x)$ and $\chi(x)$ and therefore cannot be performed at a general level. However, since ε is constant and the coefficient of dx is independent of t , (71) defines a closed 1-form, such that an integral $t_{\text{GP}}(t, x)$ exists as a local coordinate for any choice of $\lambda(x)$ and $\chi(x)$.

Substituting $dt = \varepsilon^{-1} dt_{\text{GP}} + \varepsilon^{-1} u_x dx$ in (50), we obtain

$$\begin{aligned}
 ds_{\text{GP}}^2 &= -N^2 \varepsilon^{-2} (dt_{\text{GP}}^2 + 2u_x dt_{\text{GP}} dx + u_x^2 dx^2) + \tilde{q}_{xx} dx^2 + x^2 d\Omega^2 \\
 &= -\frac{N^2}{\varepsilon^2} dt_{\text{GP}}^2 - 2u_x \frac{N^2}{\varepsilon^2} dt_{\text{GP}} dx + \left(\tilde{q}_{xx} - \frac{N^2}{\varepsilon^2} u_x^2 \right) dx^2 + x^2 d\Omega^2 \\
 &= -dt_{\text{GP}}^2 + \frac{\tilde{q}_{xx} N^2}{\varepsilon^2} (dx - u_x \tilde{q}^{xx} dt_{\text{GP}})^2 + x^2 d\Omega^2 \\
 &= -dt_{\text{GP}}^2 + \frac{\chi^{-4}}{\alpha^2 \varepsilon^2} (1 + \lambda^2 (1 - J(x)))^{-1} \\
 &\quad \times \left(dx - s\chi \sqrt{J(x) - 1 - \frac{\chi^2}{\chi(x_0)^2} (J(x_0) - 1)} \sqrt{1 + \lambda^2 (1 - J(x))} dt_{\text{GP}} \right)^2 + x^2 d\Omega^2
 \end{aligned} \tag{72}$$

where we used $u_x^2 = \tilde{q}_{xx}(\varepsilon^2/N^2 - 1)$ twice in the third line, and then inserted the explicit u_x from (69). The metric remains regular at the horizon coordinates, but still diverges at the new (quantum) singular coordinates $x = x_\lambda^{(i)}$.

In the case of a vanishing cosmological constant, $\Lambda = 0$, it is useful to take the limit $x_0 \rightarrow \infty$, in which case $\alpha\varepsilon \rightarrow 1/\lim_{x \rightarrow \infty} \chi(x) =: 1/\chi_\infty$ from (68) and the metric simplifies to

$$ds_{\text{GP}}^2 = -dt_{\text{GP}}^2 + \frac{\chi_\infty^2}{\chi^4} \left(1 + \lambda^2 \left(1 - \frac{2M}{x} \right) \right)^{-1} \left(dx + s\chi \sqrt{\frac{2M}{x} + \frac{\chi^2}{\chi_\infty^2} - 1} \sqrt{1 + \lambda^2 \left(1 - \frac{2M}{x} \right)} dt_{\text{GP}} \right)^2 + x^2 d\Omega^2. \tag{73}$$

4. From homogeneous Schwarzschild to Gullstrand-Painlevé

We can now start with the GP metric (72) we just derived and extend it across the horizons. Because we expect the causal meaning of time and space coordinates to change in the interior of the black hole, let us define $T_{\text{GP}} = x$ and $X_{\text{GP}} = t_{\text{GP}}$ and rewrite the GP metric (72) as

$$\begin{aligned}
 ds_{\text{GP}}^2 &= -dX_{\text{GP}}^2 + \frac{\chi^{-4}}{\alpha^2 \varepsilon^2} (1 + \lambda^2 (1 - J(T_{\text{GP}})))^{-1} \left(dT_{\text{GP}} \right. \\
 &\quad \left. + s \frac{\chi}{\chi(x_0)} \sqrt{\chi^2 (1 - J(x_0)) - \chi(x_0)^2 (1 - J(T_{\text{GP}}))} \sqrt{1 + \lambda^2 (1 - J(T_{\text{GP}}))} dX_{\text{GP}} \right)^2 + T_{\text{GP}}^2 d\Omega^2 \\
 &= -(1 - \lambda^2 (J(T_{\text{GP}}) - 1))^{-1} (J(T_{\text{GP}}) - 1)^{-1} \frac{dT_{\text{GP}}^2}{\chi^2} \\
 &\quad + \frac{J(T_{\text{GP}}) - 1}{\alpha^2 \chi^2 \varepsilon^2} \left(dX_{\text{GP}} + \frac{s}{\chi} \sqrt{\frac{J(T_{\text{GP}}) - 1 + (\chi^2/\chi(x_0)^2)(1 - J(x_0))}{1 - \lambda^2 (J(T_{\text{GP}}) - 1)}} \frac{dT_{\text{GP}}}{J(T_{\text{GP}}) - 1} \right)^2 + T_{\text{GP}}^2 d\Omega^2.
 \end{aligned} \tag{74}$$

A comparison of (74) with (66) shows that the two are related by the coordinate transformation $t_h = T_{\text{GP}}$ and

$$dx_h = \frac{dX_{\text{GP}}}{\varepsilon} - \frac{s}{\varepsilon \chi} \sqrt{\frac{J(T_{\text{GP}}) - 1 + (\chi^2/\chi(x_0)^2)(1 - J(x_0))}{1 - \lambda^2 (J(T_{\text{GP}}) - 1)}} \frac{dT_{\text{GP}}}{J(T_{\text{GP}}) - 1} \tag{75}$$

for arbitrary λ and χ . The explicit integration can only be completed after specifying the choice of $\lambda(t_h)$ and $\chi(t_h)$, but such an integration always exists locally because dx_h is a closed 1-form.

The metric in (74) remains regular at the horizon coordinates if we choose $\alpha\varepsilon = \sqrt{1 - J(x_0)}/\chi(x_0)$, but it diverges at the reflection surfaces $T_{\text{GP}} = x_\lambda^{(i)}$ for any α . Therefore, the GP chart is valid only in the region $x_\lambda^{(-)} < x < x_\lambda^{(+)}$, where for more than two solutions to (54) the $\pm$ -versions of $x_\lambda^{(i)}$ are as defined at the end of Sec. III A 2.

In the case of a vanishing cosmological constant, we can take $x_0 \rightarrow \infty$ and $\alpha\varepsilon = 1/\lim_{x \rightarrow \infty} \chi(x) =: 1/\chi_\infty$, simplifying the metric to

$$ds_{\text{GP}}^2 = -(1 + \lambda^2(1 - J(T_{\text{GP}})))^{-1}(J(T_{\text{GP}}) - 1)^{-1} \frac{dT_{\text{GP}}^2}{\chi^2} + \frac{\chi_\infty^2}{\chi^2} (J(T_{\text{GP}}) - 1) \left(dX_{\text{GP}} - \frac{s}{\chi} \sqrt{\frac{J(T_{\text{GP}}) + \chi^2/\chi_\infty^2 - 1}{1 - \lambda^2(J(T_{\text{GP}}) - 1)}} \frac{dT_{\text{GP}}}{J(T_{\text{GP}}) - 1} \right)^2 + T_{\text{GP}}^2 d\Omega^2. \quad (76)$$

5. Gullstrand-Painlevé gauge

The GP metric (72) covers both the static and the homogeneous regions in a single coordinate chart. We now perform a consistency check by showing that the solution obtained above satisfies the canonical equations of motion in an appropriate GP gauge.

Based on the metric (72), we define the GP gauge as

$$N = 1, \quad E^x = x^2. \quad (77)$$

The consistency equation $\dot{E}^x = 0$ determines the shift vector

$$N^x = -\chi \left(1 + \lambda^2 \frac{x^2}{(E^\varphi)^2} \right) \frac{\sin(2\lambda K_\varphi)}{2\lambda}. \quad (78)$$

The on-shell conditions $H_x = 0$ and $\tilde{H} = 0$, respectively, determine $K_x = E^\varphi K'_\varphi / (2x)$ and

$$\frac{\sin^2(\lambda K_\varphi)}{\lambda^2} = \left(1 + \lambda^2 \frac{x^2}{(E^\varphi)^2} \right)^{-1} \left(\frac{c_\varphi}{x} + \frac{\Lambda x^2}{3} + \frac{x^2}{(E^\varphi)^2} - 1 \right), \quad (79)$$

for arbitrary λ , where c_φ is an integration constant.

Finally, the equation of motion for E^φ reads

$$\left(\ln \left(\chi^2 \frac{(E^\varphi)^2}{x^2} \right) \right)' = -\sqrt{\left(\frac{c_\varphi}{x} + \frac{\Lambda x^2}{3} + \frac{x^2}{(E^\varphi)^2} - 1 \right)} \left(1 + \lambda^2 \left(1 - \frac{c_\varphi}{x} - \frac{\Lambda x^2}{3} \right) \right) \left(\ln \left(\chi^2 \frac{(E^\varphi)^2}{x^2} \right) \right)' \quad (80)$$

and is solved by $E^\varphi = x/(\alpha\chi\varepsilon)$ with constant α and ε . (We introduce two integration constants for the sake of comparing with our previous line element, which will relate ε to the constant denoted earlier by the same letter. As a partial differential equation, (80) has additional solutions that depend on the specific choice of $\lambda(x)$ but will not be required here.) The resulting metric can then be derived from (40), which matches exactly with (72) upon the identification of $c_\varphi = 2M$ and $\varepsilon^2 = (1 - J(x_0))/(\alpha^2\chi(x_0)^2)$.

Therefore, the modified GP metric (74) is related by standard coordinate transformations to both the static and homogeneous solutions in regions of overlap. Furthermore, it is well-defined at the Schwarzschild horizon $x = x_H$ provided $x_0 > x_H$, and at the cosmological horizon $x = x_\Lambda$ provided $x_0 > x_\Lambda$. In the latter case, x_0 (defined as the reference position for the inertial observer) can no longer be interpreted as the coordinate where the observer is at rest, but the coordinate transformation still holds.

B. Internal-time gauge of homogeneous regions

As we have seen, the homogeneous Schwarzschild charts terminate at the quantum hypersurfaces of reflection symmetry at $x_\lambda^{(i)}$ solving (54). As we will show in Sec. V, the Ricci scalar remains regular at those surfaces, suggesting that another chart must exist for which the metric

remains regular there. Since the coordinates $x_\lambda^{(i)}$ always lie in the homogeneous regions, we will formulate a new homogeneous gauge where the time coordinate is not given by the radius in an extension of the Schwarzschild exterior, but is instead strategically chosen to track the angular momentum K_φ . To this end, it is useful to switch to the constraint (17) because it is periodic in the variable of interest, K_φ .

We set all spatial derivatives to zero and label t_φ as the time coordinate in our new gauge as well as x_h as the spatial coordinate, which agrees with the previous homogeneous coordinate. The partial gauge-fixing

$$N^x = 0, \quad N' = 0, \quad (81)$$

is used for the homogeneous gauge. The on-shell condition $\tilde{H} = 0$ can be solved for K_x :

$$K_x = -\frac{E^\varphi}{4E^x} \frac{2\tilde{\lambda}}{\sin(2\tilde{\lambda}K_\varphi)} \left(\frac{\lambda^2}{\tilde{\lambda}^2} (1 - \Lambda E^x) + \left(1 - 2 \frac{\partial \ln \lambda^2}{\partial \ln E^x} \right) \frac{\sin^2(\tilde{\lambda}K_\varphi)}{\tilde{\lambda}^2} \right). \quad (82)$$

The relevant equations of motion are

$$\frac{\dot{E}^x}{E^x} = \frac{\tilde{\lambda}}{\lambda} \frac{2\tilde{N}}{\sqrt{E^x}} \frac{\sin(2\tilde{\lambda}K_\varphi)}{2\tilde{\lambda}} \quad (83)$$

and

$$\frac{\dot{E}^\varphi}{E^\varphi} = \frac{\tilde{\lambda}}{\lambda} \frac{\tilde{N}}{2\sqrt{E^x}} \frac{2\tilde{\lambda}}{\sin(2\tilde{\lambda}K_\varphi)} \left(\left(1 - 2 \frac{\partial \ln \lambda^2}{\partial \ln E^x} \right) \frac{\sin^2(\tilde{\lambda}K_\varphi)}{\tilde{\lambda}^2} - \frac{\lambda^2}{\tilde{\lambda}^2} (1 - \Lambda E^x) \cos(2\tilde{\lambda}K_\varphi) \right), \quad (84)$$

as well as

$$\dot{K}_\varphi = -\frac{\tilde{\lambda}}{\lambda} \frac{\tilde{N}}{2\sqrt{E^x}} \left(\frac{\lambda^2}{\tilde{\lambda}^2} (1 - \Lambda E^x) + \left(1 - 2 \frac{\partial \ln \lambda^2}{\partial \ln E^x} \right) \frac{\sin^2(\tilde{\lambda}K_\varphi)}{\tilde{\lambda}^2} \right). \quad (85)$$

(Recall that $\tilde{N} = \chi N$.)

Combining these equations of motion we obtain

$$\frac{d}{d \ln E^x} \left(\frac{\tilde{\lambda}^2 \sin^2(\tilde{\lambda}K_\varphi)}{\lambda^2} \right) = -\frac{1}{2} \left(1 - \Lambda E^x + \frac{\tilde{\lambda}^2 \sin^2(\tilde{\lambda}K_\varphi)}{\lambda^2} \right), \quad (86)$$

which has a general solution

$$\frac{\tilde{\lambda}^2 \sin^2(\tilde{\lambda}K_\varphi)}{\lambda^2} = \frac{c_x}{\sqrt{E^x}} + \frac{\Lambda E^x}{3} - 1, \quad (87)$$

for arbitrary $\lambda(E^x)$. Inserting into the mass observable (31) and relabeling $\mathcal{M} = M$ a constant (rather than phase-space function) determines $c_x = 2M$. A different combination of the time derivatives gives us the equation of motion

$$\begin{aligned} \left(\ln \frac{\tilde{q}^{xx}}{\chi^2} \right)^{\cdot} &= -2\tilde{\lambda} \tan(\tilde{\lambda}K_\varphi) \dot{K}_\varphi - 2 \frac{\dot{E}^\varphi}{E^\varphi} + \left(1 - \frac{\partial \ln \lambda^2}{\partial \ln E^x} \right) \frac{\dot{E}^x}{E^x} \\ &= \frac{\tilde{\lambda}}{\lambda} \frac{\tilde{N}}{\sqrt{E^x}} \left[\frac{\sin(2\tilde{\lambda}K_\varphi)}{2\tilde{\lambda}} + \frac{\lambda^2}{\tilde{\lambda}^2} \left(\tilde{\lambda}^2 \frac{\tan(\tilde{\lambda}K_\varphi)}{\tilde{\lambda}} + \frac{2\tilde{\lambda}}{\tan(2\tilde{\lambda}K_\varphi)} \right) \right] \\ &= \frac{\tilde{N}}{\sqrt{E^x}} \frac{\lambda}{\tilde{\lambda}} \frac{\tilde{\lambda}}{\tan(\tilde{\lambda}K_\varphi)} \left[1 - \Lambda E^x + \frac{\tilde{\lambda}^2 \sin^2(\tilde{\lambda}K_\varphi)}{\lambda^2} \right] \end{aligned} \quad (88)$$

for the structure-function.

We now complete the gauge fixing by choosing $K_\varphi = -t_\varphi$. The consistency equation $\dot{K}_\varphi = -1$ determines the lapse function

$$N = \frac{\lambda}{\tilde{\lambda}} \frac{2\sqrt{E^x}}{\chi} \frac{\tilde{\lambda}^2}{\lambda^2} \left(1 - \Lambda E^x + \left(1 - 2 \frac{\partial \ln \lambda^2}{\partial \ln E^x} \right) \frac{\tilde{\lambda}^2 \sin^2(\tilde{\lambda}K_\varphi)}{\lambda^2} \right)^{-1}, \quad (89)$$

such that the equation of motion for the structure function becomes

$$\left(\ln \frac{\tilde{q}^{xx}}{\chi^2} \right)^{\cdot} = 2 \frac{\tilde{\lambda}}{\tan(\tilde{\lambda}K_\varphi)} \left(1 - \Lambda E^x + \frac{\tilde{\lambda}^2 \sin^2(\tilde{\lambda}K_\varphi)}{\lambda^2} \right) \left(1 - \Lambda E^x + \left(1 - 2 \frac{\partial \ln \lambda^2}{\partial \ln E^x} \right) \frac{\tilde{\lambda}^2 \sin^2(\tilde{\lambda}K_\varphi)}{\lambda^2} \right)^{-1}. \quad (90)$$

This equation is hard to solve directly, but we can perform the associated coordinate transformation from t_h to t_φ using (87) with $E^x = t_h^2$ and $K_\varphi = -t_\varphi$

$$\frac{\tilde{\lambda}^2 \sin^2(\tilde{\lambda}t_\varphi)}{\lambda(t_h^2)^2} = \frac{2M}{t_h} + \frac{\Lambda t_h^2}{3} - 1 \quad (91)$$

such that

$$\begin{aligned}
 \frac{\sin(2\tilde{\lambda}t_\varphi)}{2\tilde{\lambda}} dt_\varphi &= \frac{1}{2} \left(\frac{\lambda^2}{\tilde{\lambda}^2} \frac{\partial \ln \lambda^2}{\partial t_h} \left(\frac{2M}{t_h} + \frac{\Lambda t_h^2}{3} - 1 \right) - \frac{1}{t_h} \frac{\lambda^2}{\tilde{\lambda}^2} \left(\frac{2M}{t_h} - \frac{2\Lambda t_h^2}{3} \right) \right) dt_h \\
 &= \frac{1}{2\sqrt{E^x}} \frac{\lambda^2}{\tilde{\lambda}^2} \left(-\frac{2M}{\sqrt{E^x}} + \frac{2\Lambda E^x}{3} + 2 \frac{\partial \ln \lambda^2}{\partial \ln E^x} \frac{\tilde{\lambda}^2 \sin^2(\tilde{\lambda}K_\varphi)}{\tilde{\lambda}^2} \right) dt_h \\
 &= -\frac{1}{2\sqrt{E^x}} \frac{\lambda^2}{\tilde{\lambda}^2} \left(1 - \Lambda E^x + \left(1 - 2 \frac{\partial \ln \lambda^2}{\partial \ln E^x} \right) \frac{\tilde{\lambda}^2 \sin^2(\tilde{\lambda}K_\varphi)}{\tilde{\lambda}^2} \right) dt_h
 \end{aligned} \tag{92}$$

or

$$dt_h = -2\sqrt{E^x} \frac{\tilde{\lambda}^2 \sin(2\tilde{\lambda}t_\varphi)}{\lambda^2} \left(1 - \Lambda E^x + \left(1 - 2 \frac{\partial \ln \lambda^2}{\partial \ln E^x} \right) \frac{\tilde{\lambda}^2 \sin^2(\tilde{\lambda}t_\varphi)}{\tilde{\lambda}^2} \right)^{-1} dt_\varphi \tag{93}$$

The homogeneous Schwarzschild metric (66) then becomes

$$ds^2 = -4E^x \frac{\tilde{\lambda}^2}{\lambda^2} \left(1 - \Lambda E^x + \left(1 - 2 \frac{\partial \ln \lambda^2}{\partial \ln E^x} \right) \frac{\tilde{\lambda}^2 \sin^2(\tilde{\lambda}t_\varphi)}{\tilde{\lambda}^2} \right)^{-2} \frac{dt_\varphi^2}{\chi^2} + \frac{\tilde{\lambda}^2 \sin^2(\tilde{\lambda}t_\varphi)}{\lambda^2} \frac{dx_h^2}{\alpha^2 \chi^2} + E^x d\Omega^2, \tag{94}$$

for arbitrary functions λ and χ , where E^x is implicitly given in terms of t_φ by the solution to (87). Notice that the reference value $\tilde{\lambda}$, which is always constant, can be eliminated completely by scaling the time coordinate to $\tilde{t}_\varphi = \tilde{\lambda}t_\varphi$. One can now check that the lapse function of (94) matches the one obtained canonically, (89). Finally, we can take the inverse of the radial component of (94) as the structure function, take its time derivative and confirm that it indeed satisfies the canonical equation of motion (90) since

$$\partial_{t_\varphi} \ln \left(\frac{1}{\alpha^2 \chi^2} \frac{\tilde{\lambda}^2 \sin^2(\tilde{\lambda}t_\varphi)}{\lambda^2} \right) = 2 \frac{\tilde{\lambda}}{\tan(\tilde{\lambda}t_\varphi)} \left(1 - \Lambda E^x + \frac{\tilde{\lambda}^2 \sin^2(\tilde{\lambda}t_\varphi)}{\lambda^2} \right) \left(1 - \Lambda E^x + \left(1 - 2 \frac{\partial \ln \lambda^2}{\partial \ln E^x} \right) \frac{\tilde{\lambda}^2 \sin^2(\tilde{\lambda}t_\varphi)}{\lambda^2} \right)^{-1}. \tag{95}$$

C. Proof of extremality

We are now ready to prove our previous claim that any solution of (54) for the new quantum coordinate singularities in homogeneous regions of the modified Schwarzschild geometry (50) dynamically corresponds to an extremal radius (with some restrictions in case of higher multiplicities of the roots). Inserting $E^x = x^2$ in Eq. (54), we obtain the condition that the function defined by

$$\begin{aligned}
 b(E^x) &= 1 + \lambda^2 \left(1 - \frac{2M}{x} - \frac{\Lambda x^2}{3} \right) \\
 &= 1 + \lambda^2 \left(1 - \frac{2M}{\sqrt{E^x}} - \frac{\Lambda E^x}{3} \right)
 \end{aligned} \tag{96}$$

vanishes at $E^x = (x_\lambda^{(i)})^2$. With our identification of $c_x = 2M$ and Eq. (87), it then follows that

$$\sin^2(\tilde{\lambda}K_\varphi) = 1 - b(E^x) \tag{97}$$

and $\sin^2(\tilde{\lambda}K_\varphi) = 1$ at any solution of (54).

The function $b(E^x)$ is always positive for E^x between x_H^2 and x_Λ^2 . Outside of this range, its first zeros on the two sides are defined as $(x_\lambda^{(\pm)})^2$. If these solutions have odd multiplicity, $b(E^x)$ becomes negative for values of E^x less than $(x_\lambda^{(-)})^2$ or greater than $(x_\lambda^{(+)})^2$, such that there is no corresponding K_φ for which equation (97) could hold. The only dynamical option is for E^x to start increasing after it reaches $(x_\lambda^{(-)})^2$, or to start decreasing after it reaches $(x_\lambda^{(+)})^2$. These values are therefore dynamical extrema of E^x , as well as of $\sin^2(\tilde{\lambda}K_\varphi)$ as a measure of curvature. The extremum of $\sin^2(\tilde{\lambda}K_\varphi)$ also characterizes these extrema of E^x as hypersurfaces of reflection symmetry. If there are solutions of (54) other than $x_\lambda^{(\pm)}$, they will not play a role for dynamical solutions obtained by extending the static region across coordinate singularities. Unlike in Schwarzschild or GP coordinates, the metric (94) remains regular at any hypersurface of reflection symmetry, given by $t_\varphi = \pi/(2\tilde{\lambda})$ and can be used for this purpose. We discuss the global structure in Sec. IV and perform a detailed analysis of singularity resolution in Sec. V.

For completeness, we mention why the previous conclusions are restricted to $x_\lambda^{(\pm)}$ with odd multiplicity. If $x_\lambda^{(-)}$ or $x_\lambda^{(+)}$ has even multiplicity, the function $b(E^x)$ stays positive in a neighborhood around such a solution, and dynamical values of $\sqrt{E^x}$ beyond this solution are not ruled out by the condition (97). A detailed dynamical analysis would be required for a complete understanding, which we will not perform in this paper. Nevertheless, such cases, even though they are not generic, might be of interest as new types of extremal black holes, defined in the general sense that initially distinct coordinate singularities coincide for some parameter choices. For extremal black holes in classical general relativity, the inner and outer horizon coincide, implying a double root of the defining equation. For new extremal solutions in emergent modified gravity, the black-hole horizon retains its nonextremal form, but the internal structure changes because some of the interior coordinate singularities coincide. Such solutions could shed light on possible connections, such as tunneling processes, between space-time regions within $x_\lambda^{(\pm)}$ and values beyond these extrema that lie in disjoint space-time regions in the nonextremal case.

In these discussions, we assumed that the areal gauge is always available in a region around the extremal radius, which generically is the case for physically acceptable solutions: This gauge would not be available in a given region if and only if E^x is completely constant there in space and time. (If E^x is constant only spatially in a given slicing but depends on time, one can always deform the slicing locally such that $(E^x)' \neq 0$.) The diffeomorphism constraint then implies that $K'_\varphi = 0$ in this region because E^φ is nonzero around these new coordinate singularities. Maintaining $\dot{E}^x = 0$ in the region implies that K_φ cannot change in time according to the E^x -equation of motion, and stays at the value required for reflection symmetry on all slices in the region. With these conditions, the equation of motion (A7) for K_φ implies a first-order differential equation for λ which, for zero cosmological constant, has the solution $\lambda = (c/\sqrt{E^x} - 1)^{-1/2}$ with a constant c . This behavior is not of the form usually considered for holonomy modifications, and could only be realized in a finite range of E^x .

IV. GLOBAL STRUCTURE

The global structure of classical space-time solutions may change if singularities are resolved or if there are large-scale effects from holonomy modifications that could alter the asymptotic behavior. Different implications are possible, depending on the form of holonomy modifications as determined by the function $\lambda(E^x)$.

A. Asymptotic flatness and zero-mass limit

If we set $\Lambda = 0$, we may impose asymptotic flatness. For this purpose, the metric (50) for the modified

Schwarzschild exterior is the relevant one, which takes the asymptotic form

$$ds^2 \approx -\frac{dt^2}{\alpha^2 \chi(\infty)^2} + (1 + \lambda(\infty)^2)^{-1} \frac{dx^2}{\chi(\infty)^2} + x^2 d\Omega^2. \quad (98)$$

Therefore, asymptotic flatness partially determines the limiting behavior of the two modification functions by the relations

$$\alpha = \chi(\infty)^{-1}, \quad \chi(\infty) = 1/\sqrt{1 + \lambda(\infty)^2}, \quad (99)$$

which require χ , and hence λ , to be asymptotically constant because α is a constant, implied by one of our integrations. We will impose this condition in the following.

We may independently ask that flat space-time be recovered in the zero-mass limit $M \rightarrow 0$, in which case the line element takes the form

$$ds^2 \approx -\frac{dt^2}{\alpha^2 \chi^2} + (1 + \lambda^2)^{-1} \frac{dx^2}{\chi^2} + x^2 d\Omega^2. \quad (100)$$

The radial component is rendered flat by the choice

$$\chi = 1/\sqrt{1 + \lambda^2}, \quad (101)$$

which is stronger than the asymptotic-flatness condition (99), but compatible with it. On the other hand, the time component becomes flat only if

$$\chi = \text{constant} =: \chi_0, \quad \alpha = \chi_0^{-1}, \quad (102)$$

in which case the metric is asymptotically flat too according to (99). Therefore, the radial and time components can both be flat in the zero-mass limit only if λ is constant, or if λ depends not only on E^x but also on the mass (for instance via renormalization) such that it approaches a constant as $M \rightarrow 0$. Here, we will not consider the second possibility.

If one wishes to recover flat space as part of the classical geometry in the zero-mass limit, one needs a constant radial component. In this case, we impose the conditions (99) and (101) such that the static region is described by

$$ds^2 = -\left(1 - \frac{2M}{x}\right) \frac{1 + \lambda^2}{1 + \lambda_\infty^2} dt^2 + \left(1 - \frac{\lambda^2}{1 + \lambda^2} \frac{2M}{x}\right)^{-1} \times \left(1 - \frac{2M}{x}\right)^{-1} dx^2 + x^2 d\Omega^2, \quad (103)$$

where $\lambda_\infty = \lim_{x \rightarrow \infty} \lambda(x)$. The condition of having flat space in the zero-mass limit could therefore be used to determine one of the modification functions. However, the spatial quantum geometry realized in LQG on Planckian scales can be expected to leave traces even in an effective description, such that a non-Euclidean 3-dimensional space

could be acceptable at zero mass. Because the focus of this paper is on black holes in effective loop quantum gravity we will adopt this perspective and do not require (101).

Therefore, we impose only the asymptotic-flatness condition (99), such that the static region is described by

$$ds^2 = -\left(1 - \frac{2M}{x}\right) \frac{dt^2}{\chi^2(1 + \lambda_\infty^2)} + \left(1 + \lambda^2 \left(1 - \frac{2M}{x}\right)\right)^{-1} \times \left(1 - \frac{2M}{x}\right)^{-1} \frac{dx^2}{\chi^2} + x^2 d\Omega^2 \quad (104)$$

with the overall factor χ required to have the asymptotic limit $\chi(\infty) = 1/\sqrt{1 + \lambda_\infty^2} =: \chi_0$, but otherwise arbitrary. While several of the following analyses hold for arbitrary $\chi(x)$, we will, for simplicity of the notation, assume a constant overall factor, $\chi = \chi_0$. The metric then further simplifies to

$$ds^2 = -\left(1 - \frac{2M}{x}\right) dt^2 + \left(1 + \lambda^2 \left(1 - \frac{2M}{x}\right)\right)^{-1} \times \left(1 - \frac{2M}{x}\right)^{-1} \frac{dx^2}{\chi_0^2} + x^2 d\Omega^2, \quad (105)$$

This chart, and its homogeneous counterpart, will suffice for most of the following applications.

A second argument to prefer (105) over (103) is the realization that the time component dominates in the nonrelativistic regime: The proper-time interval for a timelike object moving along a radial curve with coordinate velocity $v = dx/dt$ is given by $d\tau = \sqrt{-ds^2} = \sqrt{-g_{tt} + g_{xx}v^2}dt = (\sqrt{-g_{tt}} + O(v^2))dt$. Therefore, (103) would imply a nonclassical gravitational potential even for slow-moving objects, whereas such effects are suppressed by a factor of v^2 if (105) is used in this regime.

B. The maximal extension

For any $\lambda(E^x)$ such that (54) has only two roots $x_\lambda^{(\pm)}$ we obtain a similar global structure to that of [10,34], where constant λ was assumed. To see this, we start with a static Schwarzschild region with metric (50). The GP metric (73) covers this region as well as two homogeneous Schwarzschild regions with metric (66), one for $x < x_H$ and another for $x > x_\Lambda$. Thus, we can sew all three charts together into a black-hole like space-time region that ends at the minimum radius $x_\lambda^{(-)}$, rather than $x = 0$, and at the maximum radius $x_\lambda^{(+)}$, rather than $x \rightarrow \infty$ [assuming that both $x_\lambda^{(\pm)}$ exist as finite positive solutions of (54)]. The additional homogeneous gauge based on K_φ as an internal time, implying the metric (94), covers two homogeneous Schwarzschild regions connected to each other through $x_\lambda^{(+)}$ or $x_\lambda^{(-)}$. This gauge can be used to sew together GP charts with two complete homogeneous charts at the extremal radii, forming a single wormhole-like space-time.

A transformation to null coordinates leads to the maximal extension as follows. Consider two null frames described by the 1-forms $du = v_\mu^{(u)} dx^\mu$ and $dv = v_\mu^{(v)} dx^\mu$. Replacing the time and the space coordinates with null coordinates u and v transforms a metric of the general form (40) into a Kruskal-Szekeres form. Following the general results from [32], we obtain the line element

$$ds^2 = -\frac{N^2 - \tilde{q}_{xx}(N^x)^2}{v_t^{(u)} v_t^{(v)}} du dv \quad (106)$$

(suppressing the angular part of the metric). Using the null condition $g^{\mu\nu} v_\mu^{(i)} v_\nu^{(i)} = 0$ for $i = u, v$, the components of the null frames can be related to one another by

$$\frac{v_t^{(i)}}{v_x^{(i)}} = s_{(i)} \sqrt{N^2 \tilde{q}^{xx}} + N^x, \quad (107)$$

where $s_{(u)} = -1$ and $s_{(v)} = +1$, such that one is an ingoing family of light rays and the other is outgoing.

Using the exterior metric (50), which is static, we can choose null geodesics such that the components $v_t^{(i)}$ are constant (being Killing conserved quantities) and can be absorbed into u and v . Using this, the null 1-forms can be written as

$$\begin{aligned} du &= dt + \alpha \left(1 - \frac{2M}{x} - \frac{\Lambda x^2}{3}\right)^{-1} \\ &\quad \times \left(1 + \lambda^2 \left(1 - \frac{2M}{x} - \frac{\Lambda x^2}{3}\right)\right)^{-1/2} dx, \\ dv &= dt - \alpha \left(1 - \frac{2M}{x} - \frac{\Lambda x^2}{3}\right)^{-1} \\ &\quad \times \left(1 + \lambda^2 \left(1 - \frac{2M}{x} - \frac{\Lambda x^2}{3}\right)\right)^{-1/2} dx. \end{aligned} \quad (108)$$

The null and Schwarzschild coordinates are then easily related by

$$u = t - x_*, \quad v = t + x_*, \quad (109)$$

where x_* is the direct integration of the x -component of the 1-forms (108), which can be done once λ is specified.

In these null coordinates, the metric is simply

$$ds^2 = -(\alpha\chi)^{-2} \left(1 - \frac{2M}{x} - \frac{\Lambda x^2}{3}\right) du dv. \quad (110)$$

In the limit $x \rightarrow x_H$ the null coordinates take the values $u \rightarrow +\infty$ and $v \rightarrow -\infty$, while $x_* \rightarrow -\infty$; on the other hand, in the limit $x \rightarrow x_\Lambda$, they take the values $u \rightarrow +\infty$ and $v \rightarrow +\infty$, and $x_* \rightarrow \infty$. We partially overcome these divergences by the usual coordinate transformation

$$U = -4Me^{-u/(4M)}, \quad V = 4Me^{v/(4M)}, \quad (111)$$

such that the region $x_H < x < x_\Lambda$ corresponds to $U \in (-\infty, 0)$, $V \in (0, +\infty)$, where $U, V = 0$ at the Schwarzschild horizon, and $U = -\infty, V = +\infty$ at the cosmological horizon. The metric becomes

$$ds^2 = -F_E(x)dUdV \quad (112)$$

with

$$F_E(x) = \frac{1}{\alpha^2 \chi^2} \left(1 - \frac{2M}{x} - \frac{\Lambda x^2}{3} \right) e^{-2x_*/(4M\alpha)}, \quad (113)$$

and $x = x(U, V)$ given implicitly by the solution to

$$UV = -16M^2 e^{2x_*/(4M)}. \quad (114)$$

In the limits $x \rightarrow x_H$ and $x \rightarrow x_\Lambda$ we obtain $F_E \rightarrow 0$ for finite and differentiable functions λ and χ , as expected from null surfaces. Therefore, we can extend the null coordinates to $x < x_H$ by taking negative values for U, V (which must be possible, since the existence of such a region is implied by the GP metric). Finally, performing the conformal transformation

$$\bar{U} = \arctan(U), \quad \bar{V} = \arctan(V), \quad (115)$$

we obtain the metric

$$ds^2 = -\bar{F}_E(x)d\bar{U}d\bar{V}. \quad (116)$$

In these new coordinates, the values $U \rightarrow -\infty$ and $V \rightarrow +\infty$, corresponding to the horizon $x = x_\Lambda$, are located at $\bar{U} = -\pi/2$ and $\bar{V} = +\pi/2$, respectively. We can therefore extend the chart past the cosmological horizon by including values of $\bar{U} < -\pi/2$ and $\bar{V} > +\pi/2$ in the ranges of the conformal coordinates.

The usual periodic Penrose diagram for the maximal extension therefore follows, see Figs. 1–3. This global structure was first obtained in [10,34] for constants $\lambda = \tilde{\lambda}$, $\chi = \chi_0$, and $\alpha = 1$, provided we still have exactly two finite solutions $x_\lambda^{(\pm)}$ for the extremal radii if $\Lambda \neq 0$, and only one solution $x_\lambda^{(-)}$ for $\Lambda = 0$. We therefore have a similar global structure with the exception that the metric is allowed to have arbitrary functions $\lambda(x)$ and $\chi(x)$, and a free constant α . An important result of this generalization is that the maximum radius can be avoided depending on the choice of λ . In particular, nonconstant $\lambda \propto 1/\sqrt{E^x}$, corresponding to the $\bar{\mu}$ -scheme in models of loop quantum gravity but now realized in a covariant fashion, does not develop such a maximum radius surface as we will see below.

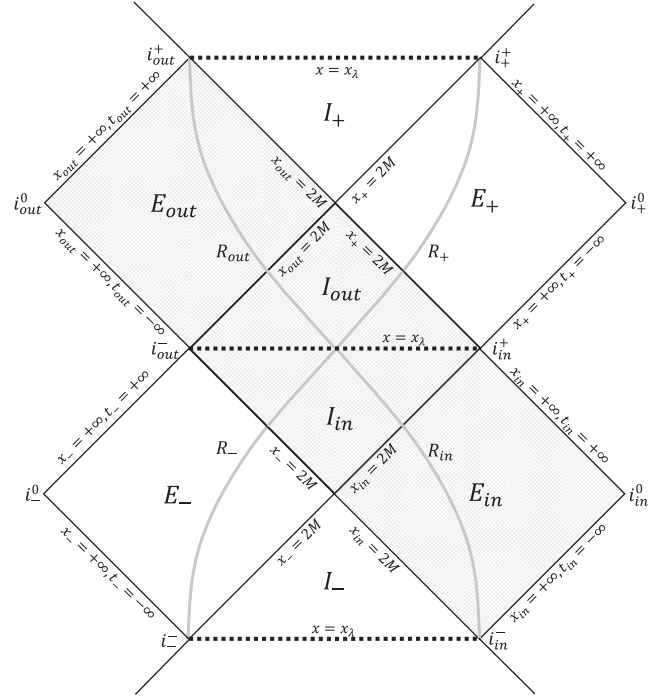


FIG. 1. Maximal extension of the wormhole solution in vacuum with vanishing cosmological constant. This case has only one solution for the coordinate x_λ of the hypersurface of reflection symmetry. The region $E_{in} \cup I_{in} \cup I_{out} \cup E_{out}$, corresponding to the shaded area, is the wormhole solution obtained by the sewing process. The gray lines R_i denote geodesics falling from (to) the remote past (far future).

C. Physical global structure

We emphasize that the global solution is not fully determined by solving the equations of motion alone. For simplicity, we will consider the case of a vanishing cosmological constant given by the diagram in Fig. 1. This is not necessarily the physical space-time as we show now with a simple explicit example.

Any sliced portion of the diagram Fig. 1 is a local vacuum solution. Therefore, if we take multiple slices of this diagram and glue them smoothly we arrive at a new global solution. As a simple example of such a procedure, consider the region $E_{in} \cup I_{in} \cup E_- \cup I_-$, sliced from the maximal extension at $x_{in} = x_\lambda$ and $x_- = x_\lambda$. We can then glue smoothly the two boundaries $x_{in} = x_\lambda$ and $x_- = x_\lambda$ because the two regions are locally identical. The result is shown in Fig. 4, which has the peculiarity that closed timelike curves exist.

The diagrams in Figs. 1 and 4 are both valid representations of the local solutions of the equations of motion, but they describe two different physical space-times as evidenced by their global properties such as the existence of closed timelike curves in the latter but not in the former. We conclude that the determination of the physical global solution requires extra input (such as the absence of closed

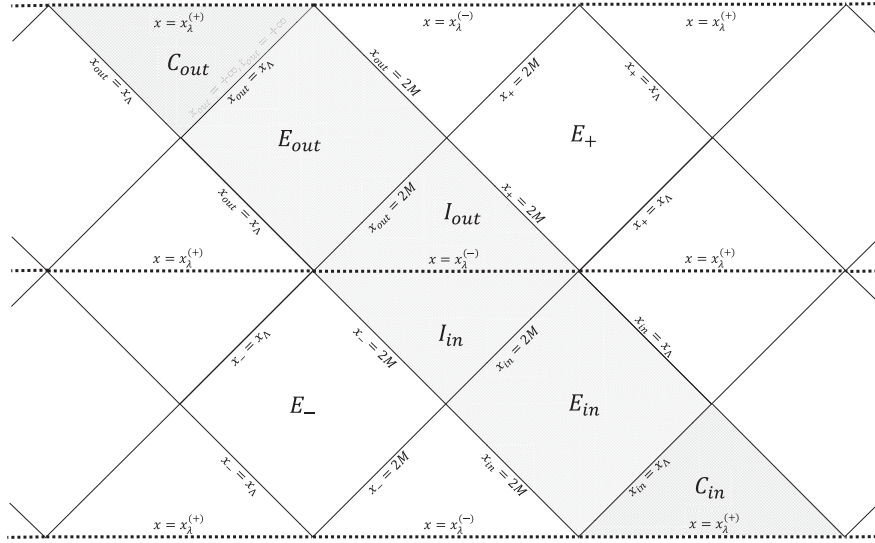


FIG. 2. Maximal extension of the wormhole solution in vacuum on a bounded de Sitter background. The region $E_{in} \cup I_{in} \cup I_{out} \cup E_{out}$, corresponding to the shaded area, is the wormhole solution obtained by the sewing process.

timelike curves) which may be seen as extended boundary conditions.

D. Presence of matter

The solutions obtained above are in a vacuum because no matter terms were considered in the constraints. Nevertheless, the solution contains the parameter M which, through the classical limit and its equivalence to the observable of the system, we may interpret as the mass of the black hole that produces the gravitational field and the curvature of space-time. In classical black holes, this mass can be interpreted as resting at the singularity, representing a trace of all the matter that collapsed to form

the black hole. However, the space-time implied by Fig. 1 has no such singularity and forces us to face an interpretational quandary as evidenced by the following thought experiment.

An observer may fall into the black hole from E_{in} , cross the hypersurface $x = x_\lambda^{(-)}$, and exit to the region E_{out} or even E_{+} without ever seeing any matter because the system is in vacuum. So, is there really a mass or not? The answer could be no, and in that case, the global space-time is akin to an eternal black hole or a wormhole. If the answer is that there must be some matter somewhere that accounts for the parameter M , then we need a model of emergent modified gravity coupled to matter such that the space-time solution

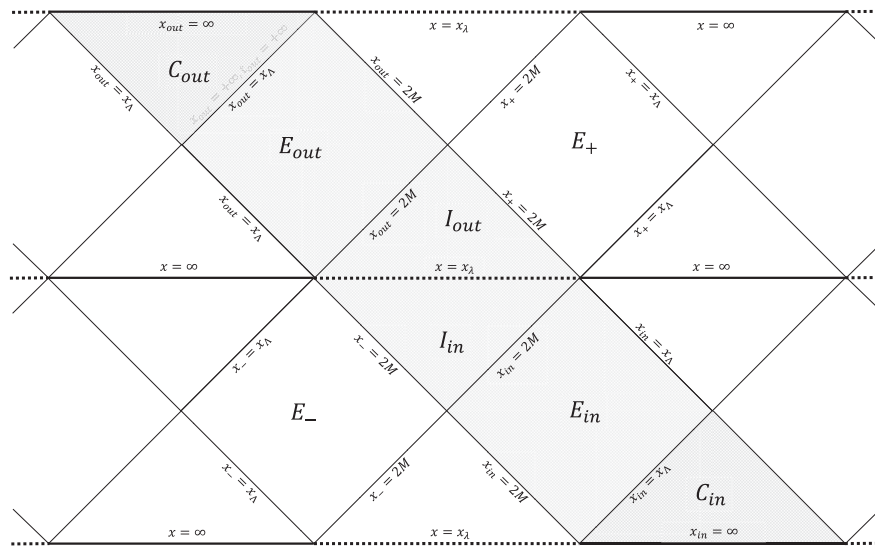


FIG. 3. Maximal extension of the wormhole solution in vacuum on an unbounded de Sitter background. The region $E_{in} \cup I_{in} \cup I_{out} \cup E_{out}$ is the wormhole solution obtained by the sewing process.

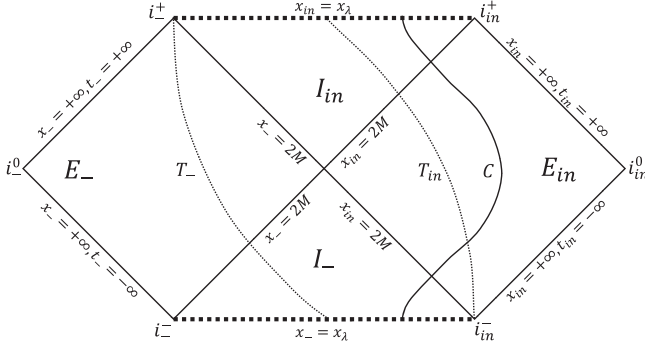


FIG. 4. Alternative wormhole solution in vacuum with vanishing cosmological constant, periodically identified at the dashed lines of x_λ . The solid line labeled by C is an example of a closed timelike curve. The dotted line given by $T = T_{in} \cup T_-$ is an example of a timelike curve falling from past timelike infinity of E_{in} , crossing the wormhole, and continuing all the way to future timelike infinity of E_- .

will differ from the vacuum case somewhere near the maximum-curvature hypersurface. Figure 1 then cannot give us the physical global structure because it cannot result from the gravitational collapse of matter. In particular, stability of singularity resolution in the presence of matter is not guaranteed [31,32,35–37]. Without knowing the nonvacuum space-time solution near the hypersurface of $x = x_\lambda^{(-)}$, we cannot extend the space-time through it in an obvious way. However, we may still use the existing vacuum solution to draw possible, though not definite, conclusions of what gravitational collapse would look like, proceeding as follows.

Consider a star with radius R_0 , such that the region $x > R_0$ is described by our vacuum solution but the region $x < R_0$ is not. Under gravitational collapse, the radius of the star will follow a timelike worldline R_{in} in Fig. 1 starting at i_{in}^- , crossing the horizon and arriving at the

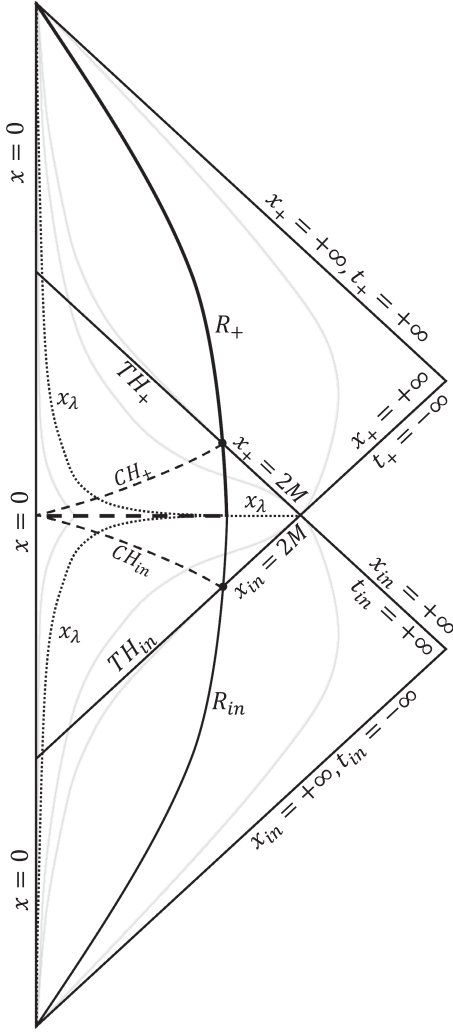


FIG. 5. Formation of an interuniversal wormhole. The dashed line at the center is the maximum-curvature hypersurface, which need not be $x = x_\lambda$ in the interior of a material star.

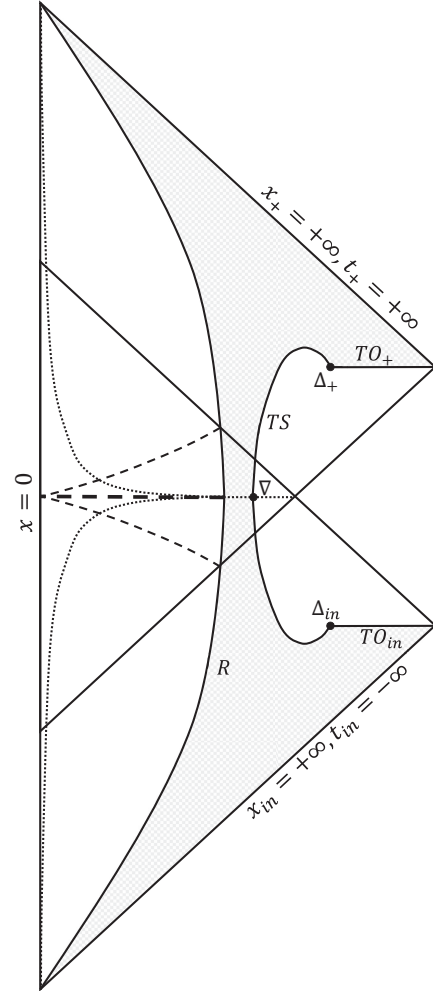


FIG. 6. Star-exterior of a black-hole to white-hole transition in a space-time diagram. The shaded region is a solution to the modified equations of the star exterior, with the two boundaries R and $S = TO_{in} \cup TS \cup TO_+$. The point where TS intersects the minimum-radius hypersurface is denoted by ∇ .

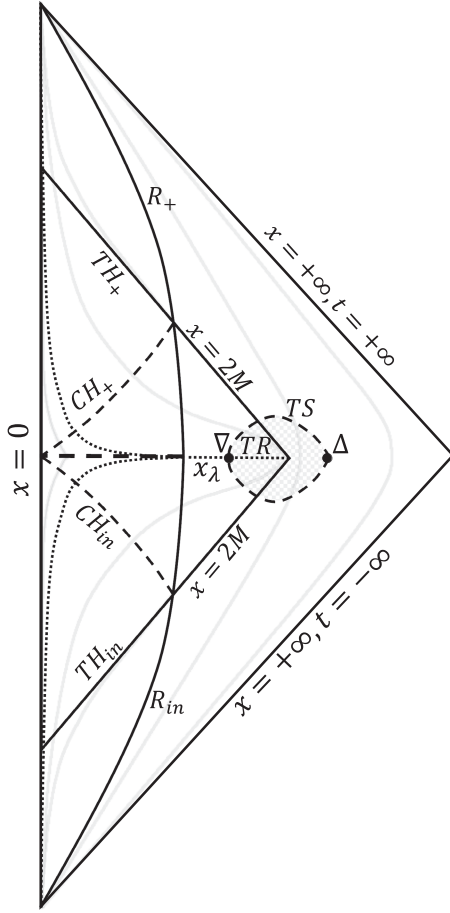


FIG. 7. Black-hole to white-hole transition in a conformal diagram. The dashed line ($t_{\text{GP}} = 0, x < x_\lambda$) at the center is the maximum-curvature hypersurface, enclosing the space-time singularity. We refer to the region TR as the transition region, bounded by the surface TS . The point Δ is chosen close to the horizon at the coordinates $(t_{\text{GP}} = 0, x = 2M + \delta)$, with small, positive δ , but otherwise undetermined. While the transition region has an undetermined geometry, one can schematically extrapolate the trajectories of constant- x curves. (Some of the ingredients in this diagram are similar to the construction shown in [38].).

reflection-symmetry hypersurface $x = x_\lambda^{(-)}$. After this, there are two possibilities: the worldline can either proceed outward all the way to i_{out}^+ , or matter coupling effects may make it bounce back and proceed to i_+^+ . We can now slice the vacuum portions at $R_1 = R_{\text{in}} \cup R_{\text{out}}$ or $R_2 = R_{\text{in}} \cup R_+$, arriving at different space-time solutions for the exterior of the star. In the latter case, for example, we obtain the diagram in Fig. 5, denoting the formation of a wormhole connecting two universes. We have simply

extrapolated the diagram to the interior of the star whose explicit solution requires specific matter couplings. See [32] for an exact solution for the collapse of dust where this procedure was implemented.

The diagram in Fig. 5 can be further sliced as shown in Fig. 6. The shaded area is a valid solution to the vacuum equations of motion with two boundaries, one given by the star's radius R , and the other given by S . The two surfaces TO_+ and TO_{in} can then be smoothly joined because the space-time surrounding them are locally identical. The resulting conformal diagram is shown in Fig. 7. We conclude that a detailed investigation of matter models for gravitational collapse is required before reliable conclusions can be drawn about potential astrophysical implications of nonsingular black holes.

V. INTERIOR PHYSICS

Unlike the homogeneous Schwarzschild or the GP coordinates, the internal time coordinates allow us to cross the reflection-symmetry hypersurfaces at $x_\lambda^{(\pm)}$ since the metric (94) is regular at any $x_\lambda^{(i)}$ solving (54). What we find then is that (94) connects two different regions on the two sides of the reflection surface, each with the same geometry (66). We therefore obtain a quantum extension of the modified Schwarzschild solution, connected by a reflection-symmetry hypersurface at a minimum radius, in which case we have two black/white-hole interiors with a range of $x_\lambda^{(-)} < x < x_H$, or at a maximum radius, in which case we have two de Sitter regions with a range $x_\lambda < x < x_\lambda^{(+)}$. In this section, we present a detailed analysis of the interior and the mechanism of singularity resolution implied by it.

A. Curvature around reflection-symmetry hypersurfaces

The classical Schwarzschild geometry is a vacuum solution and therefore has vanishing Ricci curvature. Even though we did not include matter in our new solutions, they solve modified field equations and there is no guarantee that the emergent space-time geometry remains Ricci flat. In particular, it could be possible that these geometries form new physical singularities where even the Ricci scalar becomes infinite, unlike at the classical Schwarzschild singularity. We first show that this possible outcome is not realized at a reflection-symmetry hypersurface, where Ricci curvature is not zero but finite.

The Ricci scalar of the metric (50) for arbitrary λ is given by

$$\mathcal{R} = \frac{2}{x^2} \left(1 - \chi^2 \left(1 + \lambda^2 \left(1 - \frac{3M^2}{x^2} + x \left(1 - \frac{2M}{x} \right) \left(1 - \frac{3M}{2x} \right) (\ln \lambda^2)' \right) \right) \right) + 4\chi^2 \Lambda \left(1 + \lambda^2 \left(\frac{3}{2} \left(1 - \frac{4M}{3x} \right) + \frac{5x}{12} \left(1 - \frac{9M}{5x} \right) (\ln \lambda^2)' \right) \right) - 2\chi^2 \lambda^2 \Lambda^2 x^2 \left(1 + \frac{x}{6} (\ln \lambda^2)' \right), \quad (117)$$

where $\lambda' = \partial\lambda/\partial x$. Using equation (54) for the location $x_\lambda^{(i)}$ of a reflection-symmetry hypersurface, we obtain the Ricci scalar

$$\mathcal{R}|_{x=x_\lambda^{(i)}} = \frac{2}{(x_\lambda^{(i)})^2} + \frac{2\chi^2}{(x_\lambda^{(i)})^2} \lambda^2 \left(\frac{M}{x_\lambda^{(i)}} - \frac{\Lambda(x_\lambda^{(i)})^2}{3} \right) \left(\frac{3M}{x_\lambda^{(i)}} + \Lambda(x_\lambda^{(i)})^2 - 2 \right) - \frac{2\chi^2}{x_\lambda^{(i)}} \left(\frac{3M}{x_\lambda^{(i)}} + \Lambda(x_\lambda^{(i)})^2 - 2 \right) (\ln \lambda)'|_{x=x_\lambda^{(i)}} \quad (118)$$

which is finite for any finite and differentiable $\lambda(x)$ as long as $x_\lambda^{(i)}$ is finite and nonzero. Therefore, we need a minimum radius $x_\lambda^{(-)} > 0$ or a maximum radius $x_\lambda^{(+)} \neq \infty$.

Finite negative values for $x_\lambda^{(-)}$ would imply finite Ricci curvature at this location, but the classical singularity at $x = 0$ would then be reached by an infalling observer before it could be resolved by holonomy-type effects. Thus, a finite and positive minimum radius must exist for the interior geometry to be regular at small scales. At large scales such that $\Lambda x^2 \gg 1$ and $M/x \ll 1$, the Ricci scalar is approximately given by

$$\mathcal{R} \approx 4\chi^2 \Lambda + \frac{2}{x^2} (1 - \chi^2 + \chi^2 \lambda^2 (1 + x(\ln \lambda^2)')) + 4\chi^2 \Lambda \lambda^2 \left(\frac{3}{2} + \frac{5x}{12} (\ln \lambda^2)' \right) - 2\chi^2 \lambda^2 \Lambda^2 x^2 \left(1 + \frac{x}{6} (\ln \lambda^2)' \right). \quad (119)$$

This is finite at $x \rightarrow \infty$ only if $\lambda(x)$ falls off at least as $1/x = 1/\sqrt{E^x}$, in which case no divergence occurs at large radii and no maximum radius is needed to tame it, though the latter might still exist based on this argument alone.

Because the Ricci scalar is an invariant object, its value (118) at a reflection-symmetry hypersurface, derived in static Schwarzschild coordinates, is the same as the Ricci scalar of the homogeneous metric (94) at the value $t_\varphi = \pm\pi/(2\tilde{\lambda})$ of our internal time. The advantage of the latter coordinate is that it can be extended past the reflection-symmetry hypersurface. There is therefore a robust sense in which our covariant holonomy modifications resolve the singularity inside a static Schwarzschild black hole, even without specifying the quantization ambiguity of the triangulation scheme $\lambda(E^x)$ chosen to regularize the curvature. We shall explore specific realizations of this reflection-symmetry hypersurface for a μ_0 (constant λ) and a $\bar{\mu}$ -type scheme in Secs. VII and VIII, respectively.

B. Energy conditions in an emergent space-time theory

The Einstein tensor of the emergent metric, defined as usual by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \quad (120)$$

can give us useful information about energy conditions and suitable interpretations related to singularity resolution. We may then define an effective stress energy tensor $T_{\mu\nu}^{\text{eff}} = (8\pi)^{-1} G_{\mu\nu}$ such that Einstein's equation $G_{\mu\nu} = 8\pi T_{\mu\nu}^{\text{eff}}$ formally holds. However, this equation does not play the role of an equation of motion because the space-time geometry is determined by emergence from the canonical equations. Moreover, there is no physical field with the effective stress-energy in our vacuum solutions, and in nonvacuum models the matter stress-energy in general differs from the effective stress energy as defined here. Nevertheless, the concept of $T_{\mu\nu}^{\text{eff}}$ is useful because of its geometrical relationship with space-time curvature. The usual energy conditions can then be analyzed, which, for reference purposes, are given by

- (i) Null-energy condition (NEC): For any null $v_{(i)}^\mu$
Geometric: $R_{\alpha\beta} v_{(i)}^\alpha v_{(i)}^\beta \geq 0$.
Physical: $T_{\alpha\beta}^{\text{eff}} v_{(i)}^\alpha v_{(i)}^\beta \geq 0$.
- (ii) Weak-energy condition (WEC): For any timelike $u_{(i)}^\mu$
Geometric: $G_{\alpha\beta} u_{(i)}^\alpha u_{(i)}^\beta \geq 0$.
Physical: $T_{\alpha\beta}^{\text{eff}} u_{(i)}^\alpha u_{(i)}^\beta \geq 0$.
- (iii) Strong energy condition (SEC): For any timelike $u_{(i)}^\mu$
Geometric: $R_{\alpha\beta} u_{(i)}^\alpha u_{(i)}^\beta \geq 0$.
Physical: $(T_{\alpha\beta}^{\text{eff}} - \frac{1}{2} T^{\text{eff}} g_{\alpha\beta}) u_{(i)}^\alpha u_{(i)}^\beta \geq 0$.

For any null $v_{(i)}^\mu$, we have $R_{\alpha\beta} v_{(i)}^\alpha v_{(i)}^\beta = G_{\alpha\beta} v_{(i)}^\alpha v_{(i)}^\beta = 8\pi T_{\alpha\beta}^{\text{eff}} v_{(i)}^\alpha v_{(i)}^\beta$ by definition of the Einstein tensor and the effective stress-energy tensor. For any timelike $u_{(i)}^\mu$, whenever $G_{\alpha\beta} u_{(i)}^\alpha u_{(i)}^\beta \geq 0$ we also have $T_{\alpha\beta}^{\text{eff}} u_{(i)}^\alpha u_{(i)}^\beta \geq 0$ by definition of the effective stress-energy tensor. Finally, $8\pi(T_{\alpha\beta}^{\text{eff}} - \frac{1}{2} T^{\text{eff}} g_{\alpha\beta}) = G_{\alpha\beta} + \frac{1}{2} R g_{\alpha\beta} = R_{\alpha\beta}$ by definition of the effective stress-energy tensor. Therefore, the geometric and physical versions of the energy conditions are equivalent to each other in any theory that can be formulated with effective Einstein equations.

For singularity resolution in a model, such as ours, in which solutions do not follow from effective Einstein equations, only the geometric versions are meaningful. They are based on focusing or defocusing properties of congruences of geodesics, geometrical properties captured by the Ricci tensor. Physical matter fields that propagate stress-energy are not required for this purpose. Importantly, such a theory still has to be fully covariant for the Ricci tensor to be well-defined, which is not the case for many models constructed in loop quantum gravity.

Taking these conclusions into account and for further clarity, we reduce the number of conditions and rename them as follows:

- (i) Null geometric condition (NGC)

$$R_{\alpha\beta}v_{(i)}^\alpha v_{(i)}^\beta \geq 0 \quad \text{for any null } v_{(i)}^\mu. \quad (121)$$

This condition corresponds to the NEC in GR.

- (ii) Timelike geometric condition (TGC)

$$R_{\alpha\beta}u_{(i)}^\alpha u_{(i)}^\beta \geq 0 \quad \text{for any timelike vector } u_{(i)}^\mu. \quad (122)$$

This condition corresponds to the SEC in GR.

(The WEC has no known geometric interpretation because the Einstein tensor itself does not have a direct geometric meaning; see [39] for a discussion.)

C. Congruences near a minimum-radius hypersurface

It is a common scheme in effective approaches to loop quantum gravity that black-hole singularities may be resolved by the appearance of a minimum positive radius, determined by the phase-space variable E^x . In most cases, this outcome is simply an extension of results from quantum cosmology to black holes, using homogeneity of the Schwarzschild interior in order to transfer cosmological models to this situation. Heuristically, the minimum radius can then be interpreted as an effect of gravity becoming repulsive at large curvature when a possibly discrete nature of space may change the dynamics.

However, the case of black holes is crucially different from cosmological models because it is more sensitive to coordinate or slicing changes going beyond time-reparametrizations: the homogeneous slicing of a Schwarzschild-type interior does not exist globally, and therefore cannot be interpreted as a preferred slicing of comoving observers. Any physical statements such as a possibly repulsive behavior of gravity at large curvature, as well as associated effects of singularity resolution, are meaningful only if they can be shown to be covariant when different slicings are allowed. For instance, it is important to show that the position of the surface which replaces the classical singularity is invariant with respect to coordinate choices, something that is true for our reflection-symmetry hypersurface. Most models of loop quantum gravity have been shown to fail this crucial condition, but emergent modified

gravity is in a position to present a detailed and consistent analysis.

Using this framework, we will demonstrate in this section that the geometry near the reflection-symmetry hypersurface is such that the dynamics of a collection of test particles is defocusing. We will carry out the analysis by studying the expansion rate θ , derived from the emergent space-time metric. As usual, this parameter describes infinitesimal changes in the cross-sectional area or volume for a congruence of null or timelike geodesics. Inside the horizon, both the ingoing and outgoing lightlike congruences are converging, $\theta < 0$, for any family of causally future-directed worldlines. Whether congruences are focusing or defocusing, depending on the geometry of space-time, is determined by the sign of $d\theta/d\psi$, where ψ is the affine parameter along lightlike geodesics or proper time along timelike geodesics.

1. Lightlike congruences

We start by studying null congruences. The relationship between components of ingoing and outgoing radial null worldlines has previously been used as (107) where v_t is constant along null geodesics. Here, we need a more detailed version that includes all possible sign choices. Starting with the null condition

$$-\frac{1}{N^2}v_t^2 + 2\frac{N^x}{N^2}v_tv_x + (\tilde{q}^{xx} - (N^x)^2/N^2)v_x^2 = 0 \quad (123)$$

and setting $v_t = -e$ equal to a constant, we obtain the radial component

$$v_x^{(s)} = e \frac{N^x + sN\sqrt{\tilde{q}^{xx}}}{N^2\tilde{q}^{xx} - (N^x)^2} = \frac{es}{N\sqrt{\tilde{q}^{xx}} - s(N^x)} \quad (124)$$

where $s = \pm 1$ is the sign choice obtained when solving the quadratic equation. The sign of e , which may be normalized to $e = \pm 1$ for null geodesics, also remains free at this point. We therefore have four different combinations of the signs of s and e , corresponding to the four directions of a light cone in two space-time dimensions.

The sign freedom should be partially fixed by restricting attention to future-pointing directions. This property is determined by the time component v_s^t with a raised index, given by

$$v_s^t = \frac{e}{N^2} + \frac{esN^x/N^2}{N\sqrt{\tilde{q}^{xx}} - s(N^x)} = \frac{e/N^2}{1 - s\sqrt{\tilde{q}^{xx}}N^x/N}. \quad (125)$$

The corresponding radial component with raised index equals

$$v_s^x = -\frac{eN^x}{N^2} + \frac{es(\tilde{q}^{xx} - (N^x)^2/N^2)}{N\sqrt{\tilde{q}^{xx}} - s(N^x)^2} = es\frac{\sqrt{\tilde{q}^{xx}}}{N}. \quad (126)$$

Our equations are based on the condition that $\tilde{q}^{xx}$ and N are always positive.

The expansion parameter of a congruence of null geodesics starting at the points of a sphere of some initial radius are obtained as

$$\theta_s = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} v_{(i)}^\mu) = \frac{es}{\sqrt{\tilde{q}_{xx}} N} (\ln q_{\theta\theta})' \quad (127)$$

with a radial derivative denoted by the prime. The second equality assumes a stationary coordinate system with time-independent metric components. In this case, the signs of the expansion parameters are determined by the product es , or by the sign of v_s^x .

For further evaluation, we express this result entirely in terms of phase-space coordinates and the lapse function, independently of any gauge choice, by using $q_{\theta\theta} = E^x$ and Eq. (37), in which we ignore the cosmological constant for the present analysis

$$\theta_s = es \frac{\chi(E^x)'}{NE^\varphi \sqrt{E^x}} \sqrt{1 + \lambda(E^x)^2 \left(1 - \frac{2M}{\sqrt{E^x}}\right)}. \quad (128)$$

However, a discussion of relevant sign combinations of e and s requires a specific gauge.

It is convenient to evaluate this expression in a gauge, such as GP, which is nondegenerate across the horizon. In this case, using (73), the sign of v_s^t is determined by

$$1 - s \sqrt{\tilde{q}_{xx}} N^x / N = 1 - s \frac{\chi_\infty}{\chi} \sqrt{\frac{2M}{x} + \frac{\chi^2}{\chi_\infty^2} - 1}. \quad (129)$$

This expression is always positive for $s = -1$, such that $e = 1$ implies future-directed null geodesics. Since $v_s^x < 0$ for any x , these are ingoing geodesics with negative expansion because $es < 0$.

For $s = +1$, (129) changes sign at the Schwarzschild radius, $x = 2M$, for any positive function $\chi(E^x)$. For $x > 2M$, the expression is still positive and we have $e = +1$ for future-directed null geodesics. For $x < 2M$, however, we need $e = -1$ for future direction, such that $es = -1$. Therefore, the expansion changes from positive values at $x > 2M$ to negative values at $x < 2M$. We conclude that the modification functions preserve the classical extent of the trapped region.

A complete expression for the expansion of interior null congruences in the GP gauge is given by

$$\theta_s = -\frac{2\chi_0}{x} \sqrt{1 + \lambda(x)^2 \left(1 - \frac{2M}{x}\right)}, \quad (130)$$

using the solutions for E^φ and N and assuming constant $\chi = \chi_0$ for the sake of simplicity. We can extract information about whether gravity focuses or defocuses these

congruences by deriving an evolution equation for this expansion parameter, obtained from

$$\frac{d\theta_s}{d\psi} = v_s^x \frac{d\theta_s}{dx} \quad (131)$$

where ψ is an affine parameter. The expression (131) is independent of the sign choice of s , which appears in both factors on the right. For instance, in the classical Schwarzschild space-time, still evaluated in a GP gauge, we have $d\theta_s/d\psi = -2/x^2 < 0$ which diverges as $x \rightarrow 0$. This means that both ingoing and outgoing congruences converge to the center if they are within the horizon.

Quantum gravity is often expected to have some sort of defocusing effect near the Planck scale, which can happen if the expression in Eq. (131) turns positive somewhere in the black-hole interior. (The defocusing effect may also be expressed as a repulsive behavior of modified gravity acting on test objects on our background vacuum solutions.) This outcome is indeed realized with our holonomy-modified metric. Without loss of generality, we express this evolution in the GP gauge

$$\frac{d\theta_s}{d\psi} = \frac{2\chi_0^2}{x^2} \left[-1 + \lambda^2(x) \left(\frac{3M}{x} - 1 \right) - \lambda(x) \lambda'(x) (2M - x) \right]. \quad (132)$$

The last two terms in the parenthesis can contribute to defocusing effects at sufficiently small x . Rewriting them as

$$\begin{aligned} & \frac{M}{x} (3\lambda^2 - x(\lambda^2)') - \frac{1}{2} (2\lambda^2 - x(\lambda^2)') \\ &= -x^3 \left(M \frac{d(\lambda^2/x^3)}{dx} - \frac{1}{2} \frac{d(\lambda^2/x^2)}{dx} \right), \end{aligned} \quad (133)$$

we see that the contribution proportional to M is most significant for small x . It can overcome the classical focusing behavior implied by the constant -1 in (132) provided λ^2/x^3 decreases at the rate of $1/x^2$, or faster. Therefore, $\lambda(x)$ can increase at most by $\lambda \propto \sqrt{x}$. The two cases of constant λ and $\lambda(x)$ falling off as $1/x$ are included in this range and will be discussed in detail in the following sections.

Any congruence of null radial geodesics in spherically symmetric space-time will maintain its shape, such that its shear vanishes, $\sigma_{\mu\nu} = 0$. It also crosses each hypersurface of constant radius in an orthogonal manner, such that $\omega_{\alpha\beta} = 0$. Therefore, in the Raychaudhuri equation, which through the geodesic deviation equation holds in any Riemannian geometry including an emergent one, only the self-coupling term θ^2 and curvature through the Ricci tensor contribute. The Ricci tensor contributes by its contraction with the components of v_s^μ ,

$$R_{\alpha\beta}v_s^\alpha v_s^\beta = -\frac{\chi_0^2 \lambda^2}{x^3} (2M + x(\ln \lambda^2)'(x - 2M)), \quad (134)$$

and therefore we obtain

$$-\frac{1}{2}\theta_s^2 - R_{\alpha\beta}v_s^\alpha v_s^\beta = \frac{2\chi_0^2}{x^2} \left[-1 + \lambda^2(x) \left(\frac{3M}{x} - 1 \right) + \lambda(x)\lambda'(x)(x - 2M) \right] \quad (135)$$

in agreement with (132). This result confirms that the Raychaudhuri equation indeed holds in unmodified form, emphasizing the importance of a reliable space-time geometry for solutions of a modified theory. Physically, there is an unambiguous conclusion that emergent Ricci curvature is responsible for defocusing congruences of light rays.

2. Timelike congruences and their implications

A similar computation can be performed for timelike congruences, with slightly longer expressions and fewer cancellations owing to an extra term from the normalization condition. Moreover, we keep e as a free real number in order to obtain a complete family of timelike geodesics rather than just two directions from $e = \pm 1$ used in the lightlike case.

Setting $u_t = -e$, normalization implies the radial component

$$u_x^{(s)} = e \frac{N^x + sN\sqrt{(1 - N^2/e^2)\tilde{q}^{xx} + (N^x)^2/e^2}}{N^2\tilde{q}^{xx} - (N^x)^2}. \quad (136)$$

Velocity components with raised indices are then given by

$$\begin{aligned} u_s^t &= \frac{e}{N^2 - \tilde{q}_{xx}(N^x)^2} \left(1 + s \frac{\sqrt{\tilde{q}_{xx}N^x}}{N} \sqrt{1 - \frac{N^2 - \tilde{q}_{xx}(N^x)^2}{e^2}} \right) \\ &= \frac{1}{N^2} \frac{e^2 + \tilde{q}_{xx}(N^x)^2}{e - s\sqrt{\tilde{q}_{xx}N^xN^{-1}}\sqrt{e^2 - N^2 + \tilde{q}_{xx}(N^x)^2}} \end{aligned} \quad (137)$$

and

$$u_s^x = se \frac{\sqrt{\tilde{q}^{xx}}}{N} \sqrt{1 - \frac{N^2 - \tilde{q}_{xx}(N^x)^2}{e^2}}. \quad (138)$$

For massive particles that reach infinity with zero initial velocity, we have $e = \pm 1$, where $e = 1$ corresponds to a future-pointing worldline falling in from infinity.

The sign of u_s^t is determined by the sign of

$$\begin{aligned} \frac{e}{N} - s\sqrt{\tilde{q}_{xx}N^xN^{-1}}\sqrt{\frac{e^2}{N^2} - 1 + \frac{\tilde{q}_{xx}(N^x)^2}{N^2}} \\ = X - sY\sqrt{X^2 + Y^2 - 1} \end{aligned} \quad (139)$$

where we defined $X = e/N$ and $Y = \sqrt{\tilde{q}_{xx}N^x}/N > 0$. It can easily be seen that this expression can vanish only if $Y^2 = 1$. If $Y < 1$, which corresponds to $x > 2M$ in the GP gauge, we have $X > Y\sqrt{X^2 + Y^2 - 1}$ for any $X > 0$, and therefore $u_s^t > 0$ if $e > 0$ for both choices of $s = \pm 1$. If $Y > 1$, or $x < 2M$ in the GP gauge, we have $X < Y\sqrt{X^2 + Y^2 - 1}$ for any X , and therefore u_s^t can be positive only for $e > 0$ and $s = -1$. There are only infalling timelike geodesics in this trapped region.

The definition (127) implies

$$\theta_s = \frac{se}{\sqrt{\tilde{q}_{xx}N}q_{\theta\theta}} \partial_x \left(q_{\theta\theta} \sqrt{1 + (\tilde{q}_{xx}(N^x)^2 - N^2)/e^2} \right) \quad (140)$$

or

$$\theta_s = 2\text{sgn}(e) s\chi_0 \frac{e^2 - 1 + 3M/(2x)}{x\sqrt{e^2 - 1 + 2M/x}} \sqrt{1 + \lambda(x)^2 \left(1 - \frac{2M}{x} \right)} \quad (141)$$

in the GP gauge. The rate of change relative to proper time τ is given by

$$\begin{aligned} \frac{d\theta_s}{d\tau} &= -\frac{\chi_0^2}{x^4(2M + (e^2 - 1)x)} (x(9M^2 + 8M(e^2 - 1)x + 2(e^2 - 1)^2x^2) \\ &\quad + \lambda(x)^2(-24M^3 + M^2(32 - 23e^2)x - 2M(3e^4 - 10e^2 + 7)x^2 + 2(e^2 - 1)^2x^3) \\ &\quad - x(x - 2M)\lambda(x)(2M + (e^2 - 1)x)(3M + 2(e^2 - 1)x)\lambda'(x)) \end{aligned} \quad (142)$$

The Ricci tensor contributes to the Raychaudhuri equation for timelike congruences by

$$R_{\mu\nu}u_s^\mu u_s^\nu = -\chi_0^2 \lambda^2(x)(2M + x(x - 2M)(\ln \lambda^2)') \frac{(6M^2 + (4e^2 - 7)Mx - 2(e^2 - 1)x^2)}{2(2M - x)x^4} \quad (143)$$

which provides defocusing effects in a collection of massive particles.

The long expressions above obscure the physical effects when considering a generic e . Hence, let us focus on the particular case where the collection of massive particles starts at rest from infinity ($e = 1$), such that the evolution of their geodesics takes the simpler form

$$\begin{aligned} \frac{d\theta_s}{d\tau} = & -\frac{9M\chi_0^2}{2x^3} \left(1 + \lambda^2(x) \left(1 - \frac{2M}{x} \right) \right) \\ & + \frac{3\chi_0^2 M \lambda^2(x)}{2x^4} (x(x-2M)(\ln \lambda^2)' + 2M) \end{aligned} \quad (144)$$

with

$$R_{\mu\nu} u_s^\mu u_s^\nu = -\frac{3M\chi_0^2 \lambda^2(x)}{2x^4} (x(x-2M)(\ln \lambda^2)' + 2M) \quad (145)$$

Unlike null geodesics, any cross-section of a timelike congruence is a 3-dimensional spatial hypersurface. Therefore, moving from one hypersurface to another may imply nonvanishing shear. Nonzero effects of shear can be computed in the GP-gauge to provide

$$\sigma_{\mu\nu} \sigma^{\mu\nu} = \frac{6M\chi_0^2}{2x^3} \left(1 + \lambda^2(x) \left(1 - \frac{2M}{x} \right) \right). \quad (146)$$

With this result, we can rewrite Eq. (144) as Raychaudhuri's equation,

$$\frac{d\theta_s}{d\tau} = -\frac{1}{3}\theta_s^2 - \sigma_{\mu\nu} \sigma^{\mu\nu} - R_{\mu\nu} u_s^\mu u_s^\nu \quad (147)$$

Both cases of geodesics show that the transition from focusing to defocusing is captured by space-time curvature through the emergent Ricci tensor. We can formalize this behavior by using our purely geometric conditions (121) and (122):

- (i) Null geometric condition:

Substituting the Ricci tensor contracted with tangent vectors to null geodesics, (134), into the NGC (121), we obtain

$$-\frac{\lambda^2 \chi_0^2}{x^3} (2M - x(\ln \lambda^2)'(2M - x)) \geq 0. \quad (148)$$

Our vacuum model clearly violates the condition everywhere inside the horizon $x < 2M$ for any constant or monotonically decreasing λ .

- (ii) Timelike geometric condition:

Substituting the Ricci tensor contracted with tangent vectors to timelike geodesics falling in from some reference point, (148), into the TGC (122) we obtain

$$\begin{aligned} & -\chi_0^2 \lambda^2(x) \frac{(3M + 2(e^2 - 1)x)}{2x^4} \\ & \times (2M - x(\ln \lambda^2)'(2M - x)) \geq 0. \end{aligned} \quad (149)$$

This condition is violated everywhere in the interior for any constant or monotonically decreasing λ .

A similar analysis can be done near the hypersurface of maximum radius. However, a maximal radius does not appear for asymptotically vanishing holonomy functions $h(x) = \lambda(E^x)/\tilde{\lambda}$. It may therefore be interpreted as an undesired feature of the asymptotically constant (or increasing) case because it implies strong modifications in low-curvature regimes. The $\bar{\mu}$ -type scheme of $\lambda \propto 1/\sqrt{E^x}$, in particular, does not have a hypersurface of maximum radius. We will therefore forego details of an analysis of repulsive effects beyond a maximal radius.

D. The net stress-energy tensor and gravitational energy

We define the net stress-energy tensor

$$\bar{T}_{\mu\nu} \equiv T_{\mu\nu} - \frac{G_{\mu\nu} + \Lambda g_{\mu\nu}}{8\pi}, \quad (150)$$

where $T_{\mu\nu}$ is the matter stress-energy tensor (if present, and derived canonically from matter contributions to the constraints) and $G_{\mu\nu}$ is the Einstein tensor (120) of the emergent space-time metric. In the classical limit, where the Einstein equations hold, the net stress-energy is always zero. But in general it is nonzero in a modified gravity theory (not necessarily given by some effective version of LQG). In particular, in vacuum such that $T_{\mu\nu} = 0$, the net stress-energy tensor has purely gravitational contributions. It inherits the important property of a covariant conservation law, $\nabla^\mu \bar{T}_{\mu\nu} = -\nabla^\mu (G_{\mu\nu} + \Lambda g_{\mu\nu})/(8\pi) = 0$, from the Bianchi identities, provided the Einstein tensor is indeed well-defined and covariant. We may refer to this contribution as local gravitational stress-energy, which has only nonclassical contributions. This new concept is only defined relative to general relativity and, to the best of our knowledge, does not have an independent physical motivation. Nevertheless, we will show that it leads to several statements about modified gravity and new interpretations that are intuitively meaningful.

One possible application of this new concept is a reformulation of the energy conditions. However, as we will see, these conditions do not play an important role in singularity resolution when the classical Einstein equations do not hold. We will therefore not consider them as strict conditions on possible modifications, but various signs in these expressions are still of importance in interpretations related to energy. With this in mind, we make the following definitions:

(i) Null energy signature (NES)

$$\sigma_{\text{NES}} = \text{sgn}(\bar{T}_{\alpha\beta} v_{(i)}^\alpha v_{(i)}^\beta) \quad \text{for any null } v_{(i)}^\mu. \quad (151)$$

(ii) Timelike energy signature (TES)

$$\sigma_{\text{TES}} = \text{sgn}(\bar{T}_{\alpha\beta} u_{(i)}^\alpha u_{(i)}^\beta) \quad \text{for any timelike } u_{(i)}^\mu. \quad (152)$$

As mentioned earlier, the interpretation of the physical form of the SEC is that of gravity being attractive for the sign imposed by this condition. As an energy condition, this interpretation is available only if Einstein equations hold, and therefore we will not consider its analog as an energy condition in a modified theory.

The Ricci scalar in our models is given by

$$R = \frac{2\chi_0^2}{x^4} \left(\lambda_\infty^2 x^2 - \lambda^2 (x^2 - 3M^2) - \frac{x}{2} (\lambda^2)' (x - 2M)(2x - 3M) \right) \quad (153)$$

in an asymptotically flat space-time with a suitable value of the constant λ_∞ as discussed in Sec. IV. By definition of the net stress-energy tensor, the NES (151) has the opposite sign of the left-hand side of the NGC (121) in vacuum. Using a family of null geodesics, we obtain

$$\sigma_{\text{NES}} = \text{sgn}(M/x - (\ln \lambda)'(2M - x)) \quad (154)$$

as the NES by simply flipping the sign of (148) and removing positive factors. Using a family of timelike geodesics at rest at infinity for the TES (152) we obtain

$$\sigma_{\text{TES}} = \text{sgn}(1 - \lambda_\infty^2/\lambda^2 - 2(\ln \lambda)'(2M - x)) \quad (155)$$

using (149) and (153). Both signatures are positive in the interior $x < 2M$ for monotonically decreasing λ . For constant λ , (154) remains positive while (155) vanishes. Using the concept of a net stress-energy tensor, the NGC and TGC are violated, while the NES and TES are non-negative near the hypersurface of minimum radius.

If we do not restrict these expressions to the interior, we can obtain varying signs. For constant λ , we have $\sigma_{\text{NES}} = +1$ for any x , while $\sigma_{\text{TES}} = 0$. For $\lambda = \tilde{\lambda}/x$, we have $\sigma_{\text{NES}} = \text{sgn}(3M/x - 1)$ which is negative for $x > 3M$, and $\sigma_{\text{TES}} = \text{sgn}(4M/x - 1)$ which is negative for $x > 4M$. There is therefore negative net stress-energy in most of the exterior region in a $\tilde{\mu}$ -type scheme. We interpret this outcome as an additional gravitational binding energy compared with the classical behavior. This interpretation is compatible with the acceleration equations (132) and (144) for null and timelike congruences, where the λ -contributions strengthen the classical negative term for large $x \gg M$. In the interior, however, the defocusing behavior is dominant, turning the signs of the acceleration equations as well as the signatures of net stress-energy.

VI. EXTERIOR PHYSICS

Different holonomy schemes qualitatively result in a similar mechanism of singularity-resolution. A suitable regime for interesting physical effects that can distinguish between different choices of $\lambda(x)$ is therefore primarily given by the exterior geometry, described by the emergent metric (156). For now we consider exterior effects around a black hole, but far from any cosmological scales. We therefore continue to assume a vanishing cosmological constant. We will focus on the asymptotically flat modified Schwarzschild metric with a constant overall factor χ , given by (105), such that

$$ds^2 = -\left(1 - \frac{2M}{x}\right) dt^2 + \frac{dx^2}{\chi_0^2(1 + \lambda^2(1 - 2M/x))(1 - 2M/x)} + x^2 d\Omega^2 \quad (156)$$

with

$$\chi_0 = \frac{1}{\sqrt{1 + \lambda(\infty)^2}}. \quad (157)$$

A. Newtonian limit

Using the coordinate transformation $x = r(1 + M/(2r))^2$, which in the case of the classical solution corresponds to the isotropic radial coordinate, the metric (156) becomes

$$ds^2 = -\left(\frac{1 - M/2r}{1 + M/2r}\right)^2 dt^2 + \left(1 + \frac{M}{2r}\right)^4 \left(\left(1 + \lambda(x)^2 \left(\frac{1 - M/2r}{1 + M/2r}\right)^2\right)^{-1} \frac{dr^2}{\chi_0^2} + r^2 d\Omega^2 \right). \quad (158)$$

In this form, it is easier to derive the weak-field regime defined by $M \ll r$ and $\lambda^2 \ll 1$,

$$ds^2 \approx -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 + \frac{2M}{r}\right) \times \left(\left(1 - \lambda(x)^2\left(1 - \frac{2M}{r}\right)\right) \frac{dr^2}{\chi_0^2} + r^2 d\Omega^2\right), \quad (159)$$

where $x \approx r + M$ in the argument of $\lambda(x)$. Defining the new radial coordinate

$$\mathbf{r}(r) = r_0 \exp\left(\int_{r_0}^r \frac{d\tilde{r}}{\chi_0 \tilde{r}} \sqrt{1 - \lambda(\tilde{r} + M)^2 \left(1 - \frac{2M}{\tilde{r}}\right)}\right) \quad (160)$$

with a reference radius r_0 where $\mathbf{r}(r_0) = r_0$, the metric recovers its isotropic form in the weak-field regime,

$$ds^2 \approx -\left(1 - \frac{2M}{r(\mathbf{r})}\right)dt^2 + \left(1 + \frac{2M}{r(\mathbf{r})}\right)B(\mathbf{r})(d\mathbf{r}^2 + \mathbf{r}^2 d\Omega^2). \quad (161)$$

The function $B(\mathbf{r})$ results from the inversion of (160) to $r^2 = B(\mathbf{r})\mathbf{r}^2$. An explicit integration of (160) and hence $B(\mathbf{r})$ depends on the chosen holonomy function $\lambda(x)/\tilde{\lambda}$. For our approximations of weak-field gravity and small holonomy modifications, it can be written as

$$\begin{aligned} \mathbf{r} &\approx r_0 \exp\left(\frac{1}{\chi_0} \int_{r_0}^r \frac{1 - \frac{1}{2}\lambda(\tilde{r} + M)^2}{\tilde{r}} d\tilde{r}\right) \\ &\approx r_0 (r/r_0)^{1/\chi_0} \exp\left(-\frac{1}{2\chi_0} \int_{r_0}^r \frac{\lambda(\tilde{r})^2}{\tilde{r}} d\tilde{r}\right). \end{aligned} \quad (162)$$

The shift by M does not contribute because $\lambda(r + M)^2 \approx \lambda(r)^2 + 2M\lambda(r)\lambda'(r)$ has a subleading contribution from M for $M \ll r$ and $\lambda^2 \ll 1$. For constant $\lambda = \tilde{\lambda}$, we have

$$\mathbf{r}(r) \approx r_0 (r/r_0)^{(1 - \frac{1}{2}\tilde{\lambda}^2)/\chi_0}, \quad (163)$$

and for $\lambda(r) = \tilde{\lambda}/r$ with constant $\tilde{\lambda}$ we obtain

$$\mathbf{r}(r) \approx r_0 (r/r_0)^{1/\chi_0} \exp\left(-\frac{\tilde{\lambda}^2}{4r_0^2 \chi_0} \left(1 - \frac{r_0^2}{r^2}\right)\right). \quad (164)$$

Condition (157) for asymptotic flatness, applied for small λ , implies that $\chi_0 \approx 1 - \frac{1}{2}\tilde{\lambda}^2$ for constant λ , and $\chi_0 = 1$ for $\lambda(r) = \tilde{\lambda}/r$. Since the exponential function in (164) quickly approaches a constant for large r , $\mathbf{r}$ is close to a linear function, as in the classical solution.

B. Geodesic properties

The strict covariance conditions imposed on our effective space-time models make it possible to derive reliable relativistic effects from properties of timelike and lightlike geodesics, going beyond the Newtonian limit.

1. Timelike geodesics and effective potential

We first analyze worldlines of massive objects in the effective space-time. Because the effective space-time is static and spherically symmetric, we have corresponding Killing vectors and conserved quantities $\tilde{E}$ and $\tilde{L}$ related to the energy and angular momentum (per mass) of the particle. Using these conserved quantities and the lapse function in the space-time metric, we formulate the normalization condition $g_{\mu\nu}u^\mu u^\nu = -\kappa$ for the particle's 4-velocity u^μ . For the sake of convenience, we include both massive and massless options by the parameter $\kappa = 0, 1$

$$0 = \frac{1}{2} \left(\frac{dx}{d\tau}\right)^2 + \frac{\tilde{q}^{xx}}{2} \left(\kappa + \frac{\tilde{L}^2}{x^2} - \frac{\tilde{E}^2}{N^2}\right). \quad (165)$$

Here, τ is the particle's proper time (to be replaced by the affine parameter in the massless case).

This equation can be interpreted as an energy-balance law between kinetic and potential energy of classical 1-dimensional motion, resulting in the effective potential (per unit mass)

$$\begin{aligned} \tilde{V}_{\text{eff}} &= \frac{\tilde{q}^{xx}}{2} \left(\kappa - \frac{\tilde{E}^2}{N^2} + \frac{\tilde{L}^2}{x^2}\right) \\ &= \frac{\chi_0^2}{2} \left(1 + \lambda^2 \left(1 - \frac{2M}{x}\right)\right) \left(\left(\kappa + \frac{\tilde{L}^2}{x^2}\right) \left(1 - \frac{2M}{x}\right) - \tilde{E}^2\right). \end{aligned} \quad (166)$$

The relevant metric components are given by (50). We will also use the first derivative of the potential,

$$\begin{aligned} \frac{\partial \tilde{V}_{\text{eff}}}{\partial x} &= \frac{\chi_0^2}{x^5} \left[x(3\tilde{L}^2 M - \tilde{L}^2 x + \kappa M x^2) \right. \\ &\quad \left. - \lambda^2 (\tilde{L}^2 (8M^2 - 6Mx + x^2) + Mx^2 (4\kappa M + \tilde{E}^2 x - 2\kappa x)) \right. \\ &\quad \left. + (\lambda^2)' x (2M - x) (\tilde{L}^2 (2M - x) + x^2 (2\kappa M + \tilde{E}^2 x - \kappa x)) \right] \end{aligned} \quad (167)$$

where $\lambda' = d\lambda/dx$.

Circular orbits are characterized by $\partial \tilde{V}_{\text{eff}}/\partial x = 0$ and $dx/d\tau = 0$. Inserting these conditions into Eqs. (165) and (167), respectively, gives us two equations that relate the energy and angular momentum

$$\tilde{E}^2 = \left(1 - \frac{2M}{x}\right) \left(\frac{\tilde{L}^2}{x^2} + \kappa\right) \quad (168)$$

and

$$\tilde{E}^2 \lambda^2 \left(\frac{M}{x} + \left(1 - \frac{2M}{x}\right) \frac{d \ln \lambda}{d \ln x}\right) = \frac{3M \tilde{L}^2}{x^3} - \frac{\tilde{L}^2}{x^2} + \frac{\kappa M}{x} + \lambda^2 \left(1 - \frac{2M}{x}\right) \left(\left(\frac{4\tilde{L}^2 M}{x^3} - \frac{\tilde{L}^2}{x^2} + \frac{2\kappa M}{x^2}\right) + \left(1 - \frac{2M}{x}\right)^2 \left(\frac{\tilde{L}^2}{x^2} + \kappa\right) \frac{d \ln \lambda}{d \ln x}\right). \quad (169)$$

There is a qualitative difference in these equations compared with the classical case, in which case the left-hand side of (169) vanishes. The right-hand side then determines $\tilde{L}$, which then provides $\tilde{E}$ using (168). For nonzero λ , (169) depends on both $\tilde{L}$ and $\tilde{E}$, but $\tilde{E}$ can easily be eliminated by using (168). The terms with derivatives of λ then cancel out completely, and the resulting equation for $\tilde{L}$ reads

$$(x + \lambda^2(x - 2M))(3\tilde{L}^2 M - \tilde{L}^2 x + \kappa M x^2) = 0. \quad (170)$$

The first parenthesis by definition vanishes at an extremal radius $x_\lambda^{(i)}$ (solving (54) for $\Lambda = 0$), which never happens in

the static exterior region as we have shown. The second parenthesis must therefore vanish for circular orbits, which is independent of λ and implies the classical stationary radius

$$x_0^\pm = \frac{\tilde{L}^2}{2M\kappa} \left(1 \pm \sqrt{1 - \kappa \frac{12M^2}{\tilde{L}^2}}\right). \quad (171)$$

For the null case, $\kappa = 0$, only x_0^- is finite, given by the value $3M$ derived by taking the limit $\kappa \rightarrow 0$.

At this radial coordinate, we have

$$\begin{aligned} \left. \frac{\partial^2 \tilde{V}_{\text{eff}}}{\partial x^2} \right|_{x_0^\pm} &= \kappa^4 \chi_0^2 \frac{32M^4}{\tilde{L}^6} \left(1 \pm \sqrt{1 - \kappa \frac{12M^2}{\tilde{L}^2}}\right)^{-6} \left[-6\kappa \frac{M^2}{\tilde{L}^2} + \kappa \lambda^2 \frac{6M^2}{\tilde{L}^2} \left(\kappa \frac{4M^2}{\tilde{L}^2} - 1\right) \right. \\ &\quad \left. + \left(1 - \kappa \frac{6M^2}{\tilde{L}^2} + \lambda^2 \left(1 - \kappa \frac{8M^2}{\tilde{L}^2}\right)\right) \left(1 \pm \sqrt{1 - \kappa \frac{12M^2}{\tilde{L}^2}}\right) \right]. \end{aligned} \quad (172)$$

As in the classical case, only x_0^+ is a local minimum of the potential for a massive particle, while x_0^- is a local maximum for both massive and massless particles.

2. Nearly-circular orbits of massive objects

For nearly-circular orbits we consider only the case of massive objects and hence set $\kappa = 1$. It will be useful to invert the relation (171) for the angular momentum in terms of the equilibrium coordinate $x_0 = x_0^+$:

$$\tilde{L}^2 = \frac{M x_0^2}{x_0 - 3M}. \quad (173)$$

If the object is displaced slightly from its equilibrium radius, it oscillates around it with frequency

$$\omega_r^2 = \left. \frac{\partial^2 \tilde{V}_{\text{eff}}}{\partial x^2} \right|_{x=x_0, \kappa=1} = \frac{M}{x_0^3} \frac{x_0 - 6M}{x_0 - 3M} \chi_0^2 \left(1 + \lambda^2 \left(1 - \frac{2M}{x_0}\right)\right) = \omega_\varphi^2 \left(1 - \frac{6M}{x_0}\right) \chi_0^2 \left(1 + \lambda^2 \left(1 - \frac{2M}{x_0}\right)\right) \quad (174)$$

where ω_φ is the angular frequency of the orbit and given by

$$\omega_\varphi^2 = \frac{\tilde{L}^2}{x_0^4} = \frac{M}{x_0^2(x_0 - 3M)}. \quad (175)$$

Therefore, the precession rate for nearly-circular orbits equals

$$\omega_p = \omega_\varphi - \omega_r = \left(1 - \chi_0 \sqrt{1 - \frac{6M}{x_0}} \sqrt{1 + \lambda^2 \left(1 - \frac{2M}{x_0}\right)}\right) \omega_\varphi. \quad (176)$$

3. Null rays: Deflection angle and redshift

The effective potential (166) for a massless particle is

$$\tilde{V}_{\text{null}} = \frac{\lambda_0^2}{2} \left(1 + \lambda^2 \left(1 - \frac{2M}{x}\right)\right) \left(\frac{b^2}{x^2} \left(1 - \frac{2M}{x}\right) - 1\right) \tilde{E}^2, \quad (177)$$

where $b \equiv \tilde{L}/\tilde{E}$ is the impact parameter. Using $\tilde{L} = x^2 \dot{\phi}$ we can obtain $d\phi/dx = \dot{\phi}/\dot{x}$ from (165) with $\kappa = 0$,

$$\frac{d\phi}{dx} = -\frac{1}{\chi_0} \frac{1}{x^2} \left(1 + \lambda^2 \left(1 - \frac{2M}{x}\right)\right)^{-1/2} \left(\frac{1}{b^2} - \left(1 - \frac{2M}{x}\right) \frac{1}{x^2}\right)^{-1/2}. \quad (178)$$

For a null ray that is not captured by the black hole, its turning point is given by the largest solution x_{tp} to $V_{\text{null}}(x_{\text{tp}}) = 0$

$$\frac{x_{\text{tp}}^2}{b^2} = 1 - \frac{2M}{x_{\text{tp}}}. \quad (179)$$

The complete change in the angular coordinate can then be derived as

$$\begin{aligned} \Delta\phi &= \frac{2}{\chi_0} \int_{x_{\text{tp}}}^{\infty} \frac{dx}{x^2 \sqrt{(1 + \lambda^2(1 - 2M/x))(1/b^2 - (1 - 2M/x)/x^2)}} - \pi \\ &= \frac{2}{\chi_0} \int_0^{1/x_{\text{tp}}} \frac{du}{\sqrt{(1 + \lambda^2(1 - 2Mu))((1 - 2Mx_{\text{tp}}^{-1})x_{\text{tp}}^{-2} - (1 - 2Mu)u^2)}} - \pi, \end{aligned} \quad (180)$$

where we used the substitution $u = 1/x$ to obtain the second line. As usual, the subtraction of π expresses the result relative to the change of angle in flat space.

This integral is complicated in analytical form, even in the classical case $\lambda \rightarrow 0$. We therefore consider the expression at leading order in M/x_{tp} ,

$$\Delta\phi \approx \frac{2x_{\text{tp}}}{\chi_0} \int_0^{1/x_{\text{tp}}} du \frac{1}{\sqrt{1 - u^2 x_{\text{tp}}^2} \sqrt{1 + \lambda^2}} - \pi + \frac{2M}{\chi_0} \int_0^{1/x_{\text{tp}}} du \frac{1 + ux_{\text{tp}}(1 + ux_{\text{tp}}) + \lambda^2(1 + 2ux_{\text{tp}}(1 + ux_{\text{tp}}))}{(1 + ux_{\text{tp}})(1 - u^2 x_{\text{tp}}^2)^{1/2} (1 + \lambda^2)^{3/2}}. \quad (181)$$

Notice that this expression applies to a deflection where the coordinate origin lies to the right of the null ray's path, such that the latter undergoes a clockwise motion. An attractive force will result in a positive deflection angle, while a repulsive force results in a negative one. If the coordinate origin lies to the left of the null ray's path, then the deflection has the opposite direction.

The deflection angle, as well as the effective potential and the force it implies on massive objects, can be used for standard derivations of physical scenarios once we have chosen a holonomy function $h = \lambda/\tilde{\lambda}$. In the next two sections, we will do this in the cases of constant h and a decreasing $h(x) \propto 1/x$, respectively. In addition, one could also consider redshift effects, but we now show that there are no λ -modifications in this effect for radial null rays.

A stationary observer is described by a 4-velocity $u^\mu \partial_\mu = u^t \partial_t$. Timelike normalization implies

$$u^\mu \partial_\mu = \frac{\partial_t}{\sqrt{-g_{tt}}} = \left(1 - \frac{2M}{x}\right)^{-1/2} \partial_t. \quad (182)$$

Such an observer, located at a radial position x , observes the frequency

$$\omega(x) = -g_{\mu\nu} k^\mu u^\nu|_x = \sqrt{-g_{tt}} k^t = \sqrt{1 - \frac{2M}{x}} k^t, \quad (183)$$

of a light ray with wave 4-vector k^μ . Completing u^ν to a local inertial frame in the radial manifold by including the normalized spacelike vector

$$\zeta^\mu \partial_\mu = \frac{\partial_x}{\sqrt{\tilde{q}_{xx}}}, \quad (184)$$

the observed radial wave number is

$$k(x) = g_{\mu\nu} k^\mu \zeta^\nu|_x = \chi_0^{-1} \left(1 + \lambda(x)^2 \left(1 - \frac{2M}{x} \right) \right)^{-1/2} \times \left(1 - \frac{2M}{x} \right)^{-1/2} k^x. \quad (185)$$

The lightlike condition $g_{\mu\nu} k^\mu k^\nu = 0$ implies

$$k(x) = \pm \omega(x) \quad (186)$$

in the orthonormal frame, where the positive sign determines properties of outgoing light rays.

If the light ray was emitted near the horizon at $x = x_e \gtrsim 2M$ with frequency ω_e , as measured by a stationary observer at x_e , and follows a null geodesic, conservation of $k_t = g_{tt} k^t$ implies that it is simply subject to the classical redshift

$$\frac{\omega_o}{\omega_e} = \sqrt{\frac{1 - 2M/x_e}{1 - 2M/x_o}}, \quad (187)$$

where ω_o is the frequency observed by a stationary observer at $x = x_o$. We therefore see no holonomy modifications to gravitational redshift, for arbitrary λ .

The underlying reason for this result is that the component g_{tt} of the exterior metric (156) has the same form as the classical component in Schwarzschild space-time. Only the

spatial metric of the emergent line element is directly changed for a modified structure function, and the static gauge implies that λ -modifications do not affect conditions on the lapse function. For the same reason, gravitational time-dilation, for instance for an observer in a geostationary orbit at some distance R from the Earth with radius r_E , retains the classical expression

$$\Delta\tau = \Delta t \sqrt{\frac{1 - 2M/r_E}{1 - 2M/(r_E + R)}}, \quad (188)$$

devoid of any holonomy corrections for arbitrary λ .

C. Energy and thermodynamics

Physical properties of our space-time, rather than of objects moving on this background, are determined by curvature tensors. We will first derive relevant ones and then apply them to various considerations related to energy and thermodynamics.

1. Curvature tensors

The metric (156) is static and therefore has the Killing vector $\xi^\mu \partial_\mu = \partial_t$. The unit normal vector of the foliation given by constant t equals

$$n^\mu \partial_\mu = \left(1 - \frac{2M}{x} \right)^{-1/2} \partial_t = \left(1 - \frac{2M}{x} \right)^{-1/2} \xi^\mu \partial_\mu. \quad (189)$$

The Einstein tensor for an emergent metric of the form (156) with arbitrary λ takes the form

$$\begin{aligned} G_{\mu\nu} dx^\mu \otimes dx^\nu = & -\frac{\chi_0^2}{x^2} \left(1 - \frac{2M}{x} \right) \left(1 - \frac{\lambda_\infty^2}{\lambda^2} - \frac{4M^2}{x^2} + x(\ln \lambda^2)' \left(1 - \frac{2M}{x} \right)^2 \right) dt^2 \\ & + \frac{\lambda^2}{x^2} \left(1 - \frac{2M}{x} \right)^{-1} \left(1 + \lambda^2 \left(1 - \frac{2M}{x} \right) \right)^{-1} \left(1 - \frac{\lambda_\infty^2}{\lambda^2} - \frac{2M}{x} \right) dx^2 \\ & + \frac{\chi_0^2 \lambda^2}{2} \left(1 - \frac{M}{x} \right) \left(\frac{2M}{x} + x(\ln \lambda^2)' \left(1 - \frac{2M}{x} \right) \right) d\Omega^2, \end{aligned} \quad (190)$$

where $\lambda' = \partial\lambda/\partial x$.

Each symmetric 2-sphere has a normal vector

$$r^\mu \partial_\mu = \sqrt{\tilde{q}^{xx}} \partial_x, \quad (191)$$

in space, such that $\tilde{g}_{\mu\nu} r^\mu r^\nu = 1$. With this information, we can compute the extrinsic-curvature tensor of the spheres to find

$$\mathcal{K}_{\mu\nu}^{(S)} dx^\mu dx^\nu \equiv \left(\frac{1}{2} \mathcal{L}_r \tilde{q}_{\mu\nu} \right) dx^\mu dx^\nu = \sqrt{\tilde{q}^{xx}} x d\Omega^2 = \chi_0 x \sqrt{1 + \lambda^2 \left(1 - \frac{2M}{x} \right)} \sqrt{1 - \frac{2M}{x}} d\Omega^2, \quad (192)$$

with trace given by

$$\mathcal{K}^{(S)} = \frac{2\chi_0}{x} \sqrt{1 + \lambda^2 \left(1 - \frac{2M}{x}\right)} \sqrt{1 - \frac{2M}{x}}. \quad (193)$$

2. Geometric conditions, net stress-energy tensor, and gravitational energy

The null and timelike conditions, (148) and (149), are satisfied at sufficiently large distances for any $\lambda(x)$ that falls off at least as $1/\sqrt{x}$. In general, for any $\lambda(x) = ax^{-n}$, the NGC is satisfied for $x \geq 2M(n + 1/2)/n$, which lies outside the horizon and gets closer to it for large n . Violations

of the geometric conditions therefore generically start somewhere in the exterior. The corresponding coordinate value can be interpreted as the location where quantum effects become significant. The special case of constant λ may be included in this parametrization as the limit $n \rightarrow 0$, in which case the NGC is violated everywhere.

In Schwarzschild coordinates, the net stress-energy tensor, defined by (150) and using the Einstein tensor defined by (120) for the emergent space-time metric, takes the form

$$\begin{aligned} 8\pi \bar{T}_{\mu\nu} dx^\mu \otimes dx^\nu = & \frac{\chi_0^2 \lambda^2}{x^2} \left(1 - \frac{2M}{x}\right) \left(1 - \frac{\lambda_\infty^2}{\lambda^2} - \frac{4M^2}{x^2} + x(\ln \lambda^2)' \left(1 - \frac{2M}{x}\right)^2\right) dt^2 \\ & - \frac{\lambda^2}{x^2} \left(1 - \frac{2M}{x}\right)^{-1} \left(1 + \lambda^2 \left(1 - \frac{2M}{x}\right)\right)^{-1} \left(1 - \frac{\lambda_\infty^2}{\lambda^2} - \frac{2M}{x}\right) dx^2 \\ & - \frac{\chi_0^2 \lambda^2}{2} \left(1 - \frac{M}{x}\right) \left(\frac{2M}{x} + x(\ln \lambda^2)' \left(1 - \frac{2M}{x}\right)\right) d\Omega^2, \end{aligned} \quad (194)$$

using (190). In the exterior, $x > 2M$, and for monotonically decreasing λ , we obtain $\bar{T}_{xx} < 0$ everywhere, while $\bar{T}_{tt}$ and $\bar{T}_{\varphi\varphi}$ potentially change sign at some point, leading to violations of the energy conditions.

The energy density measured by a static observer with 4-velocity given by (182) equals

$$\rho \equiv \bar{T}_{\mu\nu} u^\mu u^\nu = \frac{\chi_0^2 \lambda^2}{x^2} \left(1 - \frac{\lambda_\infty^2}{\lambda^2} - \frac{4M^2}{x^2} + x(\ln \lambda^2)' \left(1 - \frac{2M}{x}\right)^2\right). \quad (195)$$

For monotonically decreasing λ , this expression is negative at sufficiently large values of x where the last term dominates. This shows that, unlike in the interior where the gravitational energy density is positive, in the exterior at large scales it does act as a binding energy for these observers. (The sign of the energy depends on the observers. In particular, for comoving observers the interior energy density is proportional to, and of the same sign as, the component $\bar{T}_{xx}$ of (194) which is negative.)

D. Black-hole thermodynamics

A useful energy quantity in general relativity is the (normalized) Brown-York quasilocal energy as seen by observers characterized by a vector $\hat{t}^\mu = \hat{N}n^\mu + \hat{N}^x s_x^\mu$,

$$E = -\frac{1}{8\pi} \int d^2z \hat{N} (\sqrt{\det \sigma} \mathcal{K}^{(S)} - \sqrt{\det \bar{\sigma}} \bar{\mathcal{K}}^{(S)}). \quad (196)$$

The integration is over a 2-dimensional surface with coordinates z and the induced 2-metric σ , and $\mathcal{K}^{(S)}$ is

the trace of extrinsic curvature on the 2-sphere. The barred quantities are evaluated in Minkowski space-time

The first steps of the derivation can be reproduced in a canonical setting, which we show here. However, it turns out that a crucial ingredient requires an extension of canonical variables into a timelike direction, which is not available in emergent modified gravity without using a specific solution. Brown-York quasilocal energy therefore cannot be defined by an independent derivation within emergent modified gravity. Instead, we will consider the classical geometrical expression (196) as a net quasilocal energy in an emergent space-time, just as we used the classical Einstein tensor in our definition of net stress-energy.

As for the derivation of the Brown-York expression, consider the classical Hamiltonian action over a space-time region $\mathcal{R}$,

$$S[N, \vec{N}, q, p] = \int_{\mathcal{R}} d^4x (p^{ab} \dot{q}_{ab} - HN - H_a N^a) \quad (197)$$

where $p^{ab} = (16\pi)^{-1} \sqrt{\det q} (K^{ab} - K q^{ab})$ is the momentum conjugate to the 3-metric q_{ab} , and K_{ab} is the extrinsic curvature with trace $K = K_{ab} q^{ab}$. The boundary contribution associated to the first term in the action is

$$S_{\partial\mathcal{R}} = \int_{\partial\mathcal{R}} d^3x p^{ab} q_{ab} = -\frac{1}{8\pi} \int_{\partial\mathcal{R}} d^3x \sqrt{|\det q^{(\partial\mathcal{R})}|} K \quad (198)$$

where $q^{(\partial\mathcal{R})}$ is the 3-metric of the boundary $\partial\mathcal{R}$, which in general may be spacelike or timelike. We obtain the Brown-York term by restricting $S_{\partial\mathcal{R}}$ to the boundary of the

(nonsmooth) boundary $\partial\partial\mathcal{R}$,

$$S_{\partial\partial\mathcal{R}} = -\frac{1}{8\pi} \int_{\partial\partial\mathcal{R}} d^2z \hat{N} \sqrt{|\det \sigma|} K. \quad (199)$$

Here, σ is the 2-metric and z are coordinates of the 2-dimensional surface $\partial\partial\mathcal{R}$, while $\hat{N}$ is the lapse function of the chosen observer which is normal to $\partial\mathcal{R}$. In triad variables we obtain the same boundary term since

$$\begin{aligned} S_{\partial\mathcal{R}} &= \int_{\partial\mathcal{R}} d^3x (E^\varphi K_\varphi + E^x K_x) \\ &= -\frac{1}{8\pi} \int_{\partial\mathcal{R}} d^3x \sqrt{|\det q^{(\partial\mathcal{R})}|} K. \end{aligned} \quad (200)$$

This derivation shows that K is directly derived from a linear combination of canonical variables. In emergent modified gravity, K is therefore not given by extrinsic curvature of the emergent space-time. The canonical variables K_x and K_φ are readily available on spacelike parts of the boundary $\partial\mathcal{R}$, but not on the timelike components if there is no fundamental space-time theory with a corresponding action principle. However, the time-like extension is precisely what is required for a derivation of the Brown-York quasilocal energy, making it impossible to derive a strict analog of (199). The final restriction to $\partial\partial\mathcal{R}$ is spacelike, and may be expressed in the same way as (199) in emergent modified gravity with the trace K of the canonical variables. However, the presence of a crucial gap in the derivation suggests that such a definition may be of limited value.

In what follows, we therefore proceed by using the classical result (196) as the net quasilocal energy, defined geometrically through extrinsic curvature in the emergent space-time. The barred functions are the corresponding quantities in the ground state which in classical general relativity is given by Minkowski space-time. But, as discussed in the previous subsection, this may no longer be the case in a modified theory. We therefore choose our ground state as the space-time obtained in the zero-mass limit.

We are interested in the net quasilocal energy defined by 2-spheres enclosing the black hole. Thus, the net quasilocal energy in the holonomy-modified model for observers with $\hat{N} = 1$ and $\hat{N}^x = 0$ is given by

$$E(x) = \chi_0 \left(\sqrt{1 + \lambda^2} - \sqrt{1 - \frac{2M}{x}} \sqrt{1 + \lambda^2 \left(1 - \frac{2M}{x}\right)} \right), \quad (201)$$

using (193), and $\chi_0 = 1/\sqrt{1 + \lambda_\infty^2}$ for asymptotically flat solutions. This expression is everywhere positive and monotonically decreasing for $\lambda' \leq 0$. Since we are considering vacuum solutions, we conclude that the gravitational

field itself has an energy contribution. In the asymptotic limit of $x \rightarrow \infty$, the energy acquires a correction to its classical value, given by the mass, unless the holonomy parameter is asymptotically vanishing,

$$\lim_{x \rightarrow \infty} E(x) = M \left(1 + \frac{\lambda_\infty^2}{1 + \lambda_\infty^2} \right). \quad (202)$$

A static observer has the 4-acceleration

$$a^\mu \partial_\mu = \chi_0^2 \frac{2M}{x^2} \left(1 + \lambda^2 \left(1 - \frac{2M}{x} \right) \right) \partial_x, \quad (203)$$

with norm

$$\begin{aligned} a &= \sqrt{\tilde{g}_{\mu\nu} a^\mu a^\nu} = \sqrt{\tilde{q}_{xx} (a^x)^2} \\ &= \chi_0 \frac{M}{x^2} \left(1 - \frac{2M}{x} \right)^{-1/2} \left(1 + \lambda^2 \left(1 - \frac{2M}{x} \right) \right)^{1/2}, \end{aligned} \quad (204)$$

subject to holonomy modifications. The surface gravity henceforth is given by

$$\kappa = (\sqrt{-g_{tt}} a)|_{x=2M} = \frac{\chi_0}{4M}$$

which gives the classical expression, up to a global factor χ_0 .

We perform a near-horizon expansion at $x = 2M + \rho^2/8M = 2M(1 + \bar{\rho}^2)$ for small $\bar{\rho}$

$$a = \frac{\chi_0}{4M\bar{\rho}} \left(1 - \frac{3}{2} \left(1 - \frac{\lambda_H^2}{3} \right) \bar{\rho}^2 + O(\bar{\rho}^4) \right) \quad (205)$$

where $\lambda_H = \lambda(2M)$.

The modified Schwarzschild metric (50) to leading order in $\bar{\rho}$ becomes

$$\begin{aligned} ds^2 &= -\bar{\rho}^2 (1 - \bar{\rho}^2) dt^2 + \frac{1 - \lambda_H^2 \bar{\rho}^2 + \lambda_H^4 \bar{\rho}^4}{1 - \bar{\rho}^2} (4M)^2 \frac{d\bar{\rho}^2}{\chi_0^2} + x^2 d\Omega^2 \\ &= -\bar{\rho}^2 (1 - \bar{\rho}^2) dt^2 + \frac{1 - \lambda_H^2 \bar{\rho}^2 + \lambda_H^4 \bar{\rho}^4}{1 - \bar{\rho}^2} \frac{(4M)^2}{\chi_0^2} d\bar{\rho}^2 + x^2 d\Omega^2 \\ &= \frac{(4M)^2}{\chi_0^2} (-\bar{\rho}^2 d\bar{t}^2 + d\bar{\rho}^2) + dX_\perp^2, \end{aligned} \quad (206)$$

where $\bar{t} = \chi_0 t / (4M)$, and $X_\perp$ are 2-dimensional local coordinates on the spheres. This is simply the Rindler metric. Thus, we do not see any major local modifications to leading order in $\bar{\rho}$, for arbitrary λ . The black-hole temperature is therefore classical up to a global factor χ_0 near the horizon,

$$T = \frac{\chi_0}{4\pi\sqrt{2Mx(1-2M/x)}}. \quad (207)$$

Since the space-time is static, it must be in thermal equilibrium, and the temperature away from the horizon is related to the horizon temperature through redshift. Thus, for some $\bar{x} > x$,

$$T(\bar{x}) = \frac{\chi_0}{4\pi\sqrt{2Mx(1-2M/\bar{x})}} \approx \frac{\chi_0}{8\pi M} \left(1 - \frac{2M}{\bar{x}}\right)^{-1/2}. \quad (208)$$

Taking $\bar{x} \rightarrow \infty$, we gain the Hawking temperature

$$T_H \equiv T(\infty) = \frac{\chi_0}{8\pi M} = \frac{\kappa}{2\pi}$$

where we have used the previous relation for surface gravity in order to arrive at the last relation. This shows that the zeroth law of black hole mechanics remains valid in our vacuum solution.

Let us now consider an observer at a fixed coordinate $\bar{x}$. A change δM in the mass will then change the net quasilocal energy (201) by

$$\begin{aligned} \delta E(\bar{x}) &= \chi_0 \left(1 - \frac{2M}{\bar{x}}\right)^{-1/2} \left(1 + 2\lambda^2 \left(1 - \frac{2M}{\bar{x}}\right)\right) \left(1 + \lambda^2 \left(1 - \frac{2M}{\bar{x}}\right)\right)^{-1/2} \delta M \\ &= T(\bar{x}) 8\pi M \left(1 + 2\lambda^2 \left(1 - \frac{2M}{\bar{x}}\right)\right) \left(1 + \lambda^2 \left(1 - \frac{2M}{\bar{x}}\right)\right)^{-1/2} \delta M, \end{aligned} \quad (209)$$

where we used the temperature (208) and have assumed that λ is independent of the mass. Using the thermodynamic analog $\delta E = T\delta S$, we find that the entropy seen by a static observer at $\bar{x}$ is given by

$$\begin{aligned} S(\bar{x}) &= \int_0^M 8\pi M' \left(1 + 2\lambda^2 \left(1 - \frac{2M'}{\bar{x}}\right)\right) \left(1 + \lambda^2 \left(1 - \frac{2M'}{\bar{x}}\right)\right)^{-1/2} dM' \\ &= \frac{8\pi\bar{x}^2}{15\lambda^4} \left(\sqrt{1 + \lambda^2 \left(1 - \frac{2M}{\bar{x}}\right)} \left(3 + \lambda^2 \left(1 + \frac{3M}{\bar{x}}\right) - 2\lambda^4 \left(1 + \frac{M}{\bar{x}} - \frac{6M^2}{\bar{x}^2}\right)\right) - \sqrt{1 + \lambda^2(3 + \lambda^2 - 2\lambda^4)} \right), \end{aligned} \quad (210)$$

for arbitrary λ independent of the mass. Notice that the entropy is independent of the reference ground state used in the definition of the quasilocal energy (201) because this ground state does not depend on the mass (as long as λ is mass independent) and hence does not enter the thermodynamic relation $\delta S = T\delta E$. In the zero-mass limit, the entropy is vanishing $S|_{M \rightarrow 0} = 0$ and is a minimum $\delta S/\delta M|_{M \rightarrow 0} = 0$. Therefore, no work can be extracted from the gravitational ground-state energy.

The entropy (210) has the correct classical limit,

$$\lim_{\lambda \rightarrow 0} S(\bar{x}) = \pi(2M)^2 = \frac{A_H}{4} = S_{\text{BH}}, \quad (211)$$

which is constant outside the event horizon. Therefore, all the information is contained inside the black hole. The general expression of the entropy, however, is not homogeneous. The asymptotic limit of the entropy in general depends on the asymptotic behavior of λ ,

$$S(\infty) = \left(1 + \frac{3}{2}\lambda_\infty^2 + O(\lambda_\infty^4)\right) S_{\text{BH}}, \quad (212)$$

and hence may not correspond to the Bekenstein-Hawking entropy unless λ has a vanishing asymptotic limit. Finally, entropy in general does not acquire the classical expression

at the horizon,

$$\begin{aligned} S(2M) &= \frac{8}{15\lambda_H^4} \left(3 + \frac{5}{2}\lambda_H^2 - \sqrt{1 + \lambda_H^2(3 + \lambda_H^2 - 2\lambda_H^4)}\right) S_{\text{BH}} \\ &= \left(1 + \frac{\lambda_H^2}{2} + O(\lambda_H^4)\right) S_{\text{BH}}, \end{aligned} \quad (213)$$

where $\lambda_H = \lambda(2M)$. In fact $S(2M) > S(\infty)$ if $\lambda_H \geq \lambda_\infty$. This is a sign that the gravitational field itself contains some information even in the vacuum exterior, a feature that is supported by the concept of net gravitational energy. The fact that the thermodynamic quasilocal entropy of the black hole is not a constant outside the horizon, in the presence of holonomy modifications, is a signal that the constant classical Bekenstein-Hawking entropy *does not* count the quantum degrees of freedom of the black hole. Using general arguments, it has previously been suggested that once black to white hole transitions are allowed, it can be shown that the Bekenstein-Hawking entropy is not a measure of the black-hole Hilbert space but rather of the degrees of freedom of the horizon through which the black hole interacts with its surroundings [40,41]. In the presence of holonomy terms, we find that this expectation is further extended in the sense that the gravitational field itself also contains information and hence the entropy expressions

computed by different external observers need not be the same. It will be interesting to check in the future whether information lost inside the black hole can indeed be recovered through the white hole once matter fields are added to the mix and the entropy stored in the gravitational field is taken into account.

VII. CONSTANT HOLONOMY FUNCTION

In this section we revisit the simplest, but nontrivial, case of constant $\lambda = \tilde{\lambda}$, such that $h(x) = 1$, which had been previously solved in [10]. In the LQG literature, a constant holonomy parameter is often referred to as the μ_0 -scheme. Since the angular lengths of holonomies on the symmetry spheres are constant, irrespective of their area, the geometrical length $\tilde{\lambda} \sqrt{E^x}$ or the area of a plaquette increase with the radius and is unbounded asymptotically. In the Schwarzschild gauge, for instance, the geometrical length equals $\tilde{\lambda} x$ and grows linearly with the radial coordinate. This increase has no effect on holonomy modifications in the static region of the Schwarzschild geometry because the increasing length is multiplied by $K_\varphi = 0$. However, it could be a problem in nonstatic gauges or when the static region is extended to a dynamical homogeneous region in the presence of a positive cosmological constant. For instance, if $E^x = t_h^2$, the geometrical holonomy length increases with time and is multiplied by K_φ which, according to its classical behavior, could be expected to be constant and nonzero: $K_\varphi \propto \dot{E}^x / \sqrt{E^x}$. However, the crucial point is that the dynamical behavior of K_φ is changed by holonomy modifications. The modified equation of motion (29) then shows that, for constant $\lambda = \tilde{\lambda}$ and $E^x = t_h^2$, holonomy modifications, quantified as the ratio $(2\tilde{\lambda})^{-1} \sin(2\tilde{\lambda} K_\varphi) / (2\dot{E}^2 / \sqrt{E^x})$ of the modified and classical terms representing K_φ , are in fact constant (for constant c_f) and do not show any potential implications of an increasing geometrical length of the holonomy.

In a covariant formulation of holonomy modifications based on emergent modified gravity, holonomy functions have implications not only for the Hamiltonian constraint and the canonical dynamics but also for the geometrical structure of space-time. Unlike the Hamiltonian constraint, the emergent metric contains a K_φ -independent λ -term that remains nonzero even in the static region of a Schwarzschild gauge. Imposing asymptotic flatness on the metric (50) with $\Lambda = 0$, which requires $\alpha = 1/\chi_0$ and $\chi_0 = 1/\sqrt{1 + \tilde{\lambda}^2}$, we obtain the line element

$$ds^2 = -\left(1 - \frac{2M}{x}\right) dt^2 + \left(1 - \frac{\tilde{\lambda}^2}{1 + \tilde{\lambda}^2} \frac{2M}{x}\right)^{-1} \times \left(1 - \frac{2M}{x}\right)^{-1} dx^2 + x^2 d\Omega^2. \quad (214)$$

The heuristic relationship between $\tilde{\lambda}$ and a holonomy length is lost in this expression, and $\tilde{\lambda}$ is not explicitly multiplied by factors of x . Nevertheless, this line element has been derived from a holonomy-modified Hamiltonian constraint before the gauge was fixed.

In spite of these cautionary remarks about customary statements concerning holonomy modifications with constant length $\lambda = \tilde{\lambda}$, we will demonstrate in this section that there are several unexpected features in this case, including:

- (1) The deflection angle of a light ray moving past a massive object receives a holonomy modification for arbitrarily large impact parameters, even in the zero-mass limit.
- (2) The black-hole entropy monotonically increases as the observer moves farther away from the horizon. The maximum, given by the asymptotic value does not correspond to the Bekenstein-Hawking entropy.
- (3) In the presence of a positive cosmological constant, the areal radius is bounded not only from below but also from above.

A. Minimum radius and geometric conditions

The minimum radius of space-times for constant $\lambda = \tilde{\lambda}$ is obtained by solving Eq. (54)

$$x_{\tilde{\lambda}} = \frac{2M\tilde{\lambda}^2}{1 + \tilde{\lambda}^2}. \quad (215)$$

At this minimum radius, the Ricci scalar (118) takes the value

$$\mathcal{R}|_{x=x_{\tilde{\lambda}}} = \frac{3M}{x_{\tilde{\lambda}}^3} = \frac{3}{8M^2} \left(\frac{1 + \tilde{\lambda}^2}{\tilde{\lambda}^2} \right)^3, \quad (216)$$

and the Kretschmann scalar equals

$$K|_{x=x_{\tilde{\lambda}}} \equiv R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}|_{x=x_{\tilde{\lambda}}} = \frac{1}{x_{\tilde{\lambda}}^4 \tilde{\lambda}^4} \left(\frac{9}{4} + \frac{\tilde{\lambda}^2}{2} + 4\tilde{\lambda}^4 \right). \quad (217)$$

Both expressions are finite but diverge in the classical limit, $\tilde{\lambda} \rightarrow 0$, in which $x_{\tilde{\lambda}} \rightarrow 0$. The parameter $\tilde{\lambda}$ is dimensionless and does not set a physical scale by itself. To ensure that holonomy corrections appear at the Planck scale, one can impose that curvature scalars become Planckian for a fixed mass scale M_0 . For example, fixing $\mathcal{R}|_{x=x_{\tilde{\lambda}}} = (M_0^2/M^2)/\Delta$, where $\Delta = \ell_{\text{Pl}}^2$ is the Planck area, determines the value of $\tilde{\lambda} = 1/\sqrt{(8M_0^2/3\Delta)^{1/3} - 1}$.

A comoving observer in the homogeneous interior passes through this region in a finite amount of proper time, given by

$$\tau_{\text{cross}} = \pi(2M + x_{\tilde{\lambda}}). \quad (218)$$

The expansion parameter for null geodesic congruences is

$$\theta_{\pm} = \pm \frac{2}{x} \sqrt{1 - \frac{x_{\tilde{\lambda}}}{x}} \quad (219)$$

and changes relative to the affine parameter ψ according to

$$\frac{d\theta_{\pm}}{d\psi} = -\frac{2}{x^2} \left(1 - \frac{3x_{\tilde{\lambda}}}{2x}\right). \quad (220)$$

In the classical limit, $\tilde{\lambda} \rightarrow 0$, we have

$$\lim_{\tilde{\lambda} \rightarrow 0} \frac{d\theta_{\pm}}{d\psi} = -\frac{2}{x^2} < 0 \quad (221)$$

which means that both ingoing and outgoing congruences converge toward the singularity, and therefore the geometry focuses null rays. For nonzero $\tilde{\lambda}$, however, light rays are defocused near the reflection symmetry surface in the region $x < 3x_{\tilde{\lambda}}/2$.

For timelike geodesics at rest at infinity, we have

$$\theta_{\pm} = -\frac{3}{2x} \sqrt{\frac{2M}{x}} \sqrt{1 - \frac{x_{\tilde{\lambda}}}{x}} \quad (222)$$

with proper-time rate of change

$$\frac{d\theta_{\pm}}{d\tau} = -\frac{9M}{2x^3} \left(1 - \frac{4x_{\tilde{\lambda}}}{3x}\right). \quad (223)$$

These geodesics are defocused in the region $x < 4x_{\tilde{\lambda}}/3$. Thus, photons start feeling the repulsive effects of the emergent space-time before massive particles do.

The geometric conditions, discussed in Sec. V C and now evaluated with $\lambda(x) = \tilde{\lambda}$, imply

(i) Null geometric condition: Using (134),

$$R_{\alpha\beta} v_{\pm}^{\alpha} v_{\pm}^{\beta} = -\frac{x_{\tilde{\lambda}}}{x^3} < 0 \quad (224)$$

and therefore the condition is violated everywhere in space-time, even at large scales in the exterior.

(ii) Timelike geometric condition: Using (143),

$$R_{\alpha\beta} u_{\pm}^{\alpha} u_{\pm}^{\beta} = -\frac{3Mx_{\tilde{\lambda}}}{2x^4} < 0 \quad (225)$$

and therefore the condition as well is violated everywhere in space-time.

The violation of geometric conditions everywhere for constant $\lambda = \tilde{\lambda}$ may be interpreted as an indication of large-scale problems since quantum effects such as defocusing are not expected to be dominant unless we are at large curvature. A quantitative assessment requires a detailed analysis of observational features.

B. Observational features

As in our general discussion, the covariant nature of our emergent space-time metric makes it possible to apply standard methods in order to derive modifications to weak-field gravity, geodesic motion, and thermodynamics.

1. Newtonian and zero-mass limits

The metric (214) can be written in terms of the isotropic coordinate

$$\mathbf{r}(x) = \frac{M - 2\sqrt{x - 2M}\sqrt{x - x_{\tilde{\lambda}}} + 2M - 2x + x_{\tilde{\lambda}}}{2k}, \quad (226)$$

where k is a free constant. Inverting this function,

$$x = k\mathbf{r} \left(\left(1 + \frac{M}{2k\mathbf{r}}\right)^2 + \frac{x_{\tilde{\lambda}}}{2k\mathbf{r}} \left(1 - \frac{M}{4k\mathbf{r}} - \frac{k\mathbf{r}}{M}\right) \right), \quad (227)$$

leads to the line element

$$\begin{aligned} ds^2 = & -\left(1 - \frac{2M}{k\mathbf{r}} \left(\left(1 + \frac{M}{2k\mathbf{r}}\right)^2 + \frac{x_{\tilde{\lambda}}}{2k\mathbf{r}} \left(1 - \frac{M}{4k\mathbf{r}} - \frac{k\mathbf{r}}{M}\right) \right)^{-1}\right) dt^2 \\ & + \left(\left(1 + \frac{M}{2k\mathbf{r}}\right)^2 + \frac{x_{\tilde{\lambda}}}{2k\mathbf{r}} \left(1 - \frac{M}{4k\mathbf{r}} - \frac{k\mathbf{r}}{M}\right) \right)^2 k^2 (d\mathbf{r}^2 + \mathbf{r}^2 d\Omega^2). \end{aligned} \quad (228)$$

In the weak-field regime, $M \ll \mathbf{r}$, the line element can be approximated by

$$ds^2 \approx -\left(1 - \frac{2M}{k\mathbf{r}} (1 + \tilde{\lambda}^2)\right) dt^2 + (1 + \tilde{\lambda}^2)^{-1} \left((1 + \tilde{\lambda}^2)^{-1} + \frac{2M + x_{\tilde{\lambda}}}{k\mathbf{r}} \right) k^2 (d\mathbf{r}^2 + \mathbf{r}^2 d\Omega^2). \quad (229)$$

Both k and $1 + \tilde{\lambda}^2$ can be eliminated by absorbing a factor of $k/(1 + \tilde{\lambda}^2)$ in the coordinate $\mathbf{r}$, or by simply choosing the free constant k to equal $1 + \tilde{\lambda}^2$. Asymptotically, $\mathbf{r} \rightarrow \infty$, we then obtain Minkowski space-time

$$ds^2 \approx -dt^2 + d\mathbf{r}^2 + \mathbf{r}^2 d\Omega^2. \quad (230)$$

Minkowski space-time is also recovered in the zero-mass limit at any value of x or r , as can be seen by simply inserting $M = 0$ in the Schwarzschild coordinates (214) or in isotropic coordinates (229), and using $x_{\tilde{\lambda}} = 2M\tilde{\lambda}^2/(1 + \tilde{\lambda}^2)$ in the latter case. As we saw previously, a constant holonomy function $h(x) = 1$, or $\lambda = \tilde{\lambda}$, is the only choice that recovers a Minkowski space-time in the zero-mass limit $M \rightarrow 0$ for the case $\Lambda = 0$.

2. Nearly circular orbits of massive objects

Massive objects in nearly circular orbits for constant $\lambda = \tilde{\lambda}$ oscillate about the equilibrium radius x_0 with frequency

$$\omega_r = \omega_\varphi \sqrt{1 - \frac{6M}{x_0}} \sqrt{1 - \frac{x_{\tilde{\lambda}}}{x_0}}. \quad (231)$$

The orbit has a precession rate

$$\omega_p = \left(1 - \sqrt{1 - \frac{6M}{x_0}} \sqrt{1 - \frac{x_{\tilde{\lambda}}}{x_0}}\right) \omega_\varphi. \quad (232)$$

3. Deflection angle

The deflection angle (181) for constant $\lambda = \tilde{\lambda}$, expanded to leading order in M/x_{tp} with the turning-point radius x_{tp} , is given by

$$\Delta\phi = \frac{4M}{x_{\text{tp}}} \frac{1 + 3\tilde{\lambda}^2/2}{1 + \tilde{\lambda}^2} + O(M^2/x_{\text{tp}}^2) \quad (233)$$

and has the correct classical limit as $\tilde{\lambda} \rightarrow 0$ or $M \rightarrow 0$. The limit for $x_{\text{tp}} \rightarrow \infty$ or $M \rightarrow 0$ of $\Delta\phi$ therefore has a vanishing deflection angle.

4. Net stress-energy tensor

For constant $\lambda = \tilde{\lambda}$, the Ricci scalar and the net stress-energy tensor, defined as the negative Einstein tensor, are given by

$$R^{(\tilde{\lambda})} = \frac{3Mx_{\tilde{\lambda}}}{x^4}, \quad (234)$$

$$\begin{aligned} \tilde{T}_{\mu\nu}^{(\tilde{\lambda})} dx^\mu \otimes dx^\nu = & -\frac{2Mx_{\tilde{\lambda}}}{x^4} \left(1 - \frac{2M}{x}\right) dt^2 \\ & + \frac{x_{\tilde{\lambda}}}{x^3} \left(1 - \frac{x_{\tilde{\lambda}}}{x}\right)^{-1} \left(1 - \frac{2M}{x}\right)^{-1} dx^2 \\ & - \frac{x_{\tilde{\lambda}}}{2x} \left(1 - \frac{M}{x}\right) d\Omega^2, \end{aligned} \quad (235)$$

using the metric (214). We have included the superscript $(\tilde{\lambda})$ in order to distinguish these quantities from the corresponding versions in a $\bar{\mu}$ -type scheme studied in the

next section. The energy densities according to a static observer in the exterior and a comoving observer in the homogeneous interior are given by

$$\rho^{(\tilde{\lambda})} \equiv \tilde{T}_{\mu\nu}^{(\tilde{\lambda})} \hat{t}^\mu \hat{t}^\nu = -\frac{2Mx_{\tilde{\lambda}}}{x^4}, \quad (236)$$

$$\rho_h^{(\tilde{\lambda})} \equiv \tilde{T}_{\mu\nu}^{(\tilde{\lambda})} \hat{t}_h^\mu \hat{t}_h^\nu = -\frac{x_{\tilde{\lambda}}}{t_h^3} \left(1 - \frac{x_{\tilde{\lambda}}}{t_h}\right)^{-2} \left(\frac{2M}{t_h} - 1\right)^{-2}, \quad (237)$$

respectively. Here, $\hat{t}^\mu \partial_\mu = N^{-1} \partial_t$ is the normalized time-like Killing vector field, and $\hat{t}_h^\mu \partial_\mu = N^{-1} \partial_{t_h}$ is the normalized comoving velocity associated to the metric (214) under the appropriate coordinate swap.

Let us list four important observations.

- (1) All the components of the Einstein tensor are asymptotically vanishing, with the angular pressure being the slowest to decay at a rate of x^{-1} and the energy density being the fastest at a rate of x^{-4} .
- (2) The Ricci scalar is positive everywhere, $R^{(\tilde{\lambda})} > 0$, even in vacuum.
- (3) Restricting to the exterior, the energy density is negative everywhere, $\rho^{(\tilde{\lambda})} < 0$, the radial pressure is positive everywhere, $\tilde{T}_{xx}^{(\tilde{\lambda})} > 0$, and the angular pressure is negative everywhere, $\tilde{T}_{\varphi\varphi}^{(\tilde{\lambda})} < 0$.
- (4) Restricting to the interior, the energy density for comoving observers is negative everywhere, the radial pressure is positive everywhere, while the angular pressure is negative for $t_h > M$ and positive for $t_h < M$.

5. Black-hole thermodynamics

The net quasilocal energy (201) for constant $\lambda = \tilde{\lambda}$ and with $\hat{N} = 1$ is given by

$$E_{\tilde{\lambda}}(x) = x \left(1 - \sqrt{1 - \frac{2M}{x}} \sqrt{1 - \frac{x_{\tilde{\lambda}}}{x}}\right), \quad (238)$$

implying the net quasilocal entropy

$$\begin{aligned} S_{\tilde{\lambda}}(x) = & \frac{8\pi x^2}{15\tilde{\lambda}^4} \sqrt{1 + \tilde{\lambda}^2} \left(\sqrt{1 - \frac{x_{\tilde{\lambda}}}{x}} \left(3 + \tilde{\lambda}^2 \left(1 + \frac{3M}{x}\right)\right) \right. \\ & \left. - 2\tilde{\lambda}^4 \left(1 + \frac{M}{x} - \frac{6M^2}{x^2}\right) \right) - (3 + \tilde{\lambda}^2 - 2\tilde{\lambda}^4) \end{aligned} \quad (239)$$

from (210). This expression does not recover the classical value asymptotically

$$\lim_{x \rightarrow \infty} S_{\tilde{\lambda}}(x) = S_{\text{BH}} \frac{1 + 2\tilde{\lambda}^2}{\sqrt{1 + \tilde{\lambda}^2}} > S_{\text{BH}}. \quad (240)$$

At the horizon, it takes the value

$$S_{\tilde{\lambda}}(2M) = S_{\text{BH}} \left(1 + \frac{\tilde{\lambda}^2}{2} + \mathcal{O}(\tilde{\lambda}^4) \right), \quad (241)$$

and therefore $\lim_{x \rightarrow \infty} S_{\tilde{\lambda}}(x) > S_{\tilde{\lambda}}(2M)$.

In fact, the expression (239) is a monotonically increasing function: the entropy increases as the observer moves farther from the horizon. One possible interpretation is that the gravitational field in the vacuum exterior contains entropy and reduces the information. The overall entropy is then maximal for asymptotic observers that have access to the whole gravitational field. This is contrary to what one would expect, namely, that if the gravitational field contains some information, then asymptotic observers, who unlike near-horizon observers have access to this information, would measure an entropy that would be the minimum of the above expression. Since the increase of entropy is implied by holonomy modifications, heuristically related to

discrete spatial structures, one might argue that our result for $S_{\tilde{\lambda}}$ can only take into account information accessible to measurements in a continuum model. Information related to spatial discreteness cannot be observed in this way, and therefore increases the entropy because it indirectly affects the geometry.

Whether this behavior is considered problematic depends on the physical significance attached to the net energy and entropy of a modified vacuum space-time.

C. Maximal radius in the presence of a cosmological constant

The presence of a cosmological constant can significantly alter the asymptotic and global structure of the modified space-time. Including a cosmological constant, the space-time line element for constant $\lambda = \tilde{\lambda}$ is given by

$$ds^2 = - \left(1 - \frac{2M}{x} - \frac{\Lambda x^2}{3} \right) dt^2 + \left(1 - \frac{\tilde{\lambda}^2}{1 + \tilde{\lambda}^2} \left(\frac{2M}{x} + \frac{\Lambda x^2}{3} \right) \right)^{-1} \left(1 - \frac{2M}{x} - \frac{\Lambda x^2}{3} \right)^{-1} dx^2 + x^2 d\Omega^2. \quad (242)$$

In the classical case, $\tilde{\lambda} \rightarrow 0$, a nonvanishing, positive Λ implies that a second horizon exists at

$$x_{\Lambda} \approx \sqrt{\frac{3}{\Lambda}} \quad (243)$$

(neglecting M/x for small Λ), beyond which the space-time is homogeneous.

With $\tilde{\lambda} \neq 0$, the coordinate singularity determined by Eq. (54) is a third-order polynomial and hence has three rather complicated solutions some of which may be complex in general. For small $x \ll 2M$, the Newtonian term dominates and the solution is approximately that of the $\Lambda = 0$ case,

$$x_{\tilde{\lambda}}^{(-)} \approx \frac{2M\tilde{\lambda}^2}{1 + \tilde{\lambda}^2}. \quad (244)$$

For large $x \gg 2M$, the cosmological constant dominates and the solution is approximately

$$x_{\tilde{\lambda}}^{(+)} \approx \sqrt{\frac{3}{\Lambda} \frac{1 + \tilde{\lambda}^2}{\tilde{\lambda}^2}} \quad (245)$$

if $\Lambda > 0$, which is the case we will focus on since it agrees with observations. This value is always outside the cosmological horizon and is a maximum-radius surface, beyond which the space-time starts collapsing; see Fig. 2.

The existence of a maximum radius due to the presence of a quantum parameter is unexpected since it occurs at macroscopic scales where quantum gravity is not expected

to play a significant role. This outcome of holonomy modifications in the traditional treatment is usually related to a growing geometrical length $\tilde{\lambda}t_h$ in a homogeneous region, which, unlike in the static region, does contribute to holonomy modifications in the Hamiltonian constraint because $K_{\varphi} \neq 0$. However, as already discussed, covariant holonomy modifications in the Hamiltonian constraint do not show an increasing holonomy length but rather imply a constant magnitude of holonomy modifications related to the classical form of extrinsic curvature via (29). The reason for unexpected effects on large scales is therefore not a growing holonomy length but rather the fact that holonomy modifications do not decrease quickly enough as t_h increases.

Although nonclassical behavior on large scales is unexpected as an effect of quantum gravity, it is not impossible. While local quantum-gravity effects should be small at low curvature, global properties in large regions are subject to a combination of a huge number of tiny quantum-gravity corrections, which could conceivably add up to sizeable contributions. Moreover, nonclassical behavior on large scales may well happen in the general context of modified gravity. Large-scale modifications of the classical behavior are not necessarily incompatible with observations if they happen sufficiently far outside of the cosmological horizon. Making $\tilde{\lambda}$ sufficiently small, all the unexpected effects found in this section may therefore be acceptable in an individual model. Nevertheless, the physical viability of a fundamental theory, such as the concept of holonomy modifications in general or of LQG, can easily be endangered if microscopic features have significant implications on macroscopic scales. Corrections to the classical theory

may then be hard to control when one goes beyond relatively simple models such as spherically symmetric ones. The cosmological constant itself, in its traditional explanation as vacuum energy, is an example of such UV-IR mixing. Its macroscopic problems seem to be exacerbated by holonomy modifications with constant length parameter. However, while there is potential observational evidence for the presence of a cosmological constant, there is no indication for the existence of a maximal radius. From this perspective, it is therefore best to avoid models that place an upper bound on the range of the radial coordinate.

Models with constant holonomy length may still be relevant if they are used in a finite range of radii. For instance, the complete function $\lambda(x)$ may not follow a strict power law, but be nearly constant in a bounded region of space-time outside of which it decreases toward zero asymptotically. It is then important to show that non-constant $\lambda(E^x)$ can indeed suppress holonomy-type effects in classical regimes and restrict implications of holonomy modifications to the expected quantum-gravity regime, as we will do in the next section. We also reiterate the importance of a covariant formulation based on an emergent space-time metric, which makes it possible to have consistent effects in the static gauge, in which traditional holonomy modifications would disappear, and in nonstatic homogeneous regions of space-time.

VIII. DECREASING HOLONOMY FUNCTION

In this section, we use a holonomy function (23) $h(E^x) = \lambda(E^x)/\tilde{\lambda} = \hat{r}/\sqrt{E^x}$ analogous to a $\bar{\mu}$ -type scheme, and derive the particular physical effects of this choice. As in Eq. (22), $\hat{r}$ is a reference radius where $h(\hat{r}) = 1$. In models of loop quantum gravity, a specific coefficient is often chosen in the parametrization

$$\lambda^2 = \tilde{\lambda}^2 \hat{r}^2 / E^x, \quad (246)$$

where the constant $\tilde{\lambda} = \lambda\sqrt{E^x}/\hat{r}$ can be interpreted as the side length of plaquettes in a uniform triangulation of spheres. If one uses regular polygons with n sides and assigns the area $\Delta = 8\pi\gamma\ell_{\text{Pl}}^2$ to each of them, suggested for instance by the area gap of LQG, it follows that

$$\tilde{\lambda}^2 \hat{r}^2 = 4n^{-1} \tan(\pi/n) \Delta =: \tilde{\Delta}. \quad (247)$$

(See Sec. II E for the full details of these relations.) Note, however, that holonomies of a gauge connection do not depend on the metric, and therefore it is not required to have a strict relationship between the constant $\tilde{\lambda}$ in (246) and the area spectrum, even in a fundamentally discrete theory such as loop quantum gravity.

Using this parametrization of $\lambda(E^x)$, we have $\lambda_\infty = 0$ and, for $\Lambda = 0$, the asymptotically flat line element (156) for the black hole exterior is simplified to

$$ds^2 = -\left(1 - \frac{2M}{x}\right) dt^2 + \left(1 + \frac{\tilde{\Delta}}{x^2} \left(1 - \frac{2M}{x}\right)\right)^{-1} \times \left(1 - \frac{2M}{x}\right)^{-1} dx^2 + x^2 d\Omega^2. \quad (248)$$

We will reincorporate the cosmological constant in the final subsection and revisit the status of a maximum radius that appeared for constant $\lambda = \tilde{\lambda}$. Our detailed derivations show that the decreasing parametrization of $\lambda(E^x)$ removes all the unexpected features of constant λ in low-curvature regimes.

A. Minimum radius and geometric conditions

The minimum radius of space-times for $\lambda \propto 1/\sqrt{E^x}$, obtained by solving Eq. (54), is given by

$$x_{\tilde{\Delta}} = (\tilde{\Delta}M)^{1/3} \frac{(1 + \sqrt{1 + \tilde{\Delta}/(27M^2)})^{2/3} - (\tilde{\Delta}/(27M^2))^{1/3}}{(1 + \sqrt{1 + \tilde{\Delta}/(27M^2)})^{1/3}}. \quad (249)$$

At this minimum radius, the Ricci scalar (118) takes the value

$$\mathcal{R}|_{x=x_{\tilde{\Delta}}} = \frac{2\tilde{\Delta}}{x_{\tilde{\Delta}}^4} \left(1 - \frac{7M}{x_{\tilde{\Delta}}} + \frac{9M^2}{x_{\tilde{\Delta}}^2}\right), \quad (250)$$

and the Kretschmann scalar equals

$$K|_{x=x_{\tilde{\Delta}}} = \frac{4\tilde{\Delta}^2}{x_{\tilde{\Delta}}^8} \left(\frac{81M^4}{x_{\tilde{\Delta}}^4} - \frac{126M^3}{x_{\tilde{\Delta}}^3} + \frac{M^2}{x_{\tilde{\Delta}}^2} \left(79 + \frac{48\tilde{\Delta}}{x_{\tilde{\Delta}}^2}\right) - 30 \frac{\tilde{\Delta}M}{x_{\tilde{\Delta}}^3} + \frac{3\tilde{\Delta}}{x_{\tilde{\Delta}}^2} \right). \quad (251)$$

Both expressions are finite but diverge in the classical limit, $\tilde{\Delta} \rightarrow 0$, in which $x_{\tilde{\Delta}} \rightarrow 0$.

To analyze the effect for small masses, we define $\delta = 4M^2/\tilde{\Delta}$ and expand around $\delta = 0$

$$\frac{x_{\tilde{\Delta}}}{2M} = 1 - \delta + 3\delta^2 + O(\delta^3). \quad (252)$$

Therefore, to leading order, the horizon of a small mass corresponds to the minimum radius such that the wormhole interior will be very short, determined as a timelike distance since x is a time coordinate in the interior. This cannot happen in the case of constant $\lambda = \tilde{\lambda}$, where $x_{\tilde{\lambda}} = 2M\tilde{\lambda}^2/(1 + \tilde{\lambda}^2)$ and the horizon and minimal radius do not match to leading order unless $M = 0$. On the other hand, for large M black-holes, we define $\tilde{\delta} = \tilde{\Delta}/(27M^2)$, and by expanding around $\tilde{\delta} = 0$, we obtain

$$\frac{x_{\tilde{\Delta}}}{2M} = \frac{3}{4^{1/3}} \left(\tilde{\delta}^{1/3} - \frac{1}{4^{1/3}} \tilde{\delta}^{2/3} + O(\tilde{\delta}^{4/3}) \right) \quad (253)$$

The results indicate that the minimum radius is relatively small compared to the horizon, suggesting that the wormhole interior will be significantly large, similar to the constant $\lambda = \tilde{\lambda}$ case.

The expansion parameter for null geodesic congruences is

$$\theta_{\pm} = -\frac{2}{x} \sqrt{1 + \frac{\tilde{\Delta}}{x^2} \left(1 - \frac{2M}{x} \right)} \quad (254)$$

and changes according to

$$\frac{d\theta_{\pm}}{d\psi} = -\frac{2}{x^2} \left[1 + \frac{\tilde{\Delta}}{x^2} \left(2 - \frac{5M}{x} \right) \right] \quad (255)$$

The correction term is dominant near the surface of reflection symmetry, and the sign changes at some surface $\tilde{x} = f(M, \tilde{\Delta})x_{\tilde{\Delta}}$ with $f(M, \tilde{\Delta}) > 1$. Similarly, for timelike congruences, we have

$$\theta_{\pm} = -\frac{3}{2x} \sqrt{\frac{2M}{x}} \sqrt{1 + \frac{\tilde{\Delta}}{x^2} \left(1 - \frac{2M}{x} \right)} \quad (256)$$

with the rate of change

$$\frac{d\theta_{\pm}}{d\tau} = -\frac{9M}{2x^3} \left[1 + \left(\frac{5}{3} - \frac{4M}{x} \right) \frac{\tilde{\Delta}}{x^2} \right]. \quad (257)$$

The correction terms are dominant near the hypersurface of reflection symmetry, changing sign at $\tilde{x} = g(M, \tilde{\Delta})x_{\tilde{\Delta}}$,

where once more $g(M, \tilde{\Delta}) > 1$. It can be demonstrated that $g(M, \tilde{\Delta}) < f(M, \tilde{\Delta})$. Thus, as with constant $\lambda = \tilde{\Delta}$, free-falling photons start being defocused at larger scales compared with massive particles.

The geometric conditions for $\lambda \propto 1/\sqrt{E^x}$ are

(i) Null geometric condition

$$R_{\mu\nu} v_{\pm}^{\mu} v_{\pm}^{\nu} = \frac{2}{x^2} \frac{\tilde{\Delta}}{x^2} \left(1 - \frac{3M}{x} \right) \geq 0, \quad (258)$$

This condition is violated only in the region $x < 3M$.

(ii) Timelike geometric condition

$$R_{\mu\nu} u_{\pm}^{\mu} u_{\pm}^{\nu} = \frac{3M}{x^3} \frac{\tilde{\Delta}}{x^2} \left(1 - \frac{3M}{x} \right) \geq 0, \quad (259)$$

Also this condition is violated only in the region $x < 3M$.

Unlike for constant $\lambda = \tilde{\lambda}$, here the geometric conditions are satisfied at large scales and therefore the modified gravitational force focuses congruences far from the black hole, similar to its classical effect.

B. Observational features

As before, the weak-field behavior and properties of geodesic motion can help to distinguish physical implications of different modification functions.

1. Newtonian limit

For a decreasing holonomy function $h(E^x) = \hat{r}/\sqrt{E^x}$ and using the coordinate transformation $x = r(1 + M/(2r))^2$, the emergent space-time line element is given by

$$\begin{aligned} ds^2 &= -\left(\frac{1 - \frac{M}{2r}}{1 + \frac{M}{2r}} \right)^2 dt^2 + \left(1 + \frac{M}{2r} \right)^4 \left(\left(1 + \frac{\tilde{\Delta}}{r^2} \frac{\left(1 - \frac{M}{2r} \right)^2}{\left(1 + \frac{M}{2r} \right)^4} \right)^{-1} \frac{dr^2}{\chi_0^2} + r^2 d\Omega^2 \right) \\ &\approx -\left(1 - \frac{2M}{r} \right) dt^2 + \left(1 + \frac{2M}{r} \right) \left(\left(1 + \frac{\tilde{\Delta}}{r^2} \left(1 - \frac{3M}{r} \right) \right)^{-1} \frac{dr^2}{\chi_0^2} + r^2 d\Omega^2 \right). \end{aligned} \quad (260)$$

where we assumed $M/r \ll 1$ in the second line. The isotropic coordinate (160) is now given by

$$\mathbf{r} = k \exp \int \frac{dr}{r} \left(1 - \frac{\tilde{\Delta}}{r^2} \left(1 - \frac{3M}{r} \right) \right)^{1/2} \approx k \exp \int \frac{dr}{r} \sqrt{1 - \tilde{\Delta}/r^2} \left(1 + \frac{3\tilde{\Delta}M}{2r(r^2 - \tilde{\Delta})} \right). \quad (261)$$

The integration can be done in closed form, but the resulting expressions are lengthy.

In most cases of interest, the weak-field limit $M \ll r$ also implies $\tilde{\Delta} \ll r^2$, which can be used to simplify (261). To leading order in $\tilde{\Delta}/r^2$ and M/r , the integral is independent of M and therefore agrees with its value in the zero-mass limit.

2. Zero-mass limit

The decreasing holonomy function implies a metric (50) that is not flat in the zero mass limit $M \rightarrow 0$ in the case of $\Lambda = 0$, and instead approaches the line element

$$ds^2 = -dt^2 + \left(1 + \frac{\tilde{\Delta}}{x^2}\right)^{-1} dx^2 + x^2 d\Omega^2. \quad (262)$$

We interpret this result as suggesting that the spatial geometry is not smooth at the Planck scale $x \sim \sqrt{\tilde{\Delta}}$. The 3-dimensional volume in a slice of constant t and enclosed by $x < r$ is given by

$$\begin{aligned} V_{\Delta}(r) &= 4\pi \int_0^r dx x^2 \left(1 + \frac{\tilde{\Delta}}{x^2}\right)^{-1/2} \\ &= \frac{4\pi r^3}{3} \left(\left(1 - 2\frac{\tilde{\Delta}}{r^2}\right) \sqrt{1 + \frac{\tilde{\Delta}}{r^2}} + 2\left(\frac{\tilde{\Delta}}{r^2}\right)^{3/2} \right) \\ &= \frac{4\pi r^3}{3} \left(1 - \frac{3}{2}\frac{\tilde{\Delta}}{r^2} + 2\frac{\tilde{\Delta}^{3/2}}{r^3} + O\left(\frac{\tilde{\Delta}^2}{r^4}\right) \right). \end{aligned} \quad (263)$$

The volume of a given region is therefore always smaller compared with flat space.

The zero-mass metric (262) can be expressed in the isotropic coordinate

$$\mathbf{r} = \sqrt{\frac{\tilde{\Delta}}{4}} \sqrt{\frac{\sqrt{1 + \tilde{\Delta}/x^2} + 1}{\sqrt{1 + \tilde{\Delta}/x^2} - 1}}, \quad (264)$$

such that

$$ds^2 = -dt^2 + \left(1 - \frac{\tilde{\Delta}}{4\mathbf{r}^2}\right)^2 (d\mathbf{r}^2 + \mathbf{r}^2 d\Omega^2). \quad (265)$$

3. Nearly-circular orbits of massive objects

Massive objects in nearly circular orbits, using a decreasing holonomy function $h(E^x) = \hat{r}/\sqrt{E^x}$, oscillate around the equilibrium radius x_0 with frequency

$$\omega_r = \omega_{\varphi} \sqrt{1 - \frac{6M}{x_0}} \sqrt{1 + \frac{\tilde{\Delta}}{x_0^2} \left(1 - \frac{2M}{x_0}\right)}. \quad (266)$$

Their orbits have the precession rate

$$\omega_p = \left(1 - \sqrt{1 - \frac{6M}{x_0}} \sqrt{1 + \frac{\tilde{\Delta}}{x_0^2} \left(1 - \frac{2M}{x_0}\right)}\right) \omega_{\varphi}. \quad (267)$$

4. Deflection angle

The integration to obtain a deflection angle such as (181) for $\lambda^2 = \tilde{\Delta}/x^2$ is too complicated. However, assuming

$\tilde{\Delta}/x_{\text{tp}}^2 \ll 1$, we integrate the expression to leading order in M/x_{tp} and $\tilde{\Delta}/x_{\text{tp}}^2$, given by

$$\begin{aligned} \Delta\phi &= -\frac{\pi}{4x_{\text{tp}}^2} + \frac{4M}{x_{\text{tp}}} \left(1 + \frac{3\tilde{\Delta}}{2x_{\text{tp}}^2}\right) \left(1 + \frac{\tilde{\Delta}}{x_{\text{tp}}^2}\right)^{-3/2} + O(M^2/x_{\text{tp}}^2) \\ &= -\frac{\pi}{4x_{\text{tp}}^2} + \frac{4M}{x_{\text{tp}}} \left(1 - \frac{3}{8}\left(\frac{\tilde{\Delta}}{x_{\text{tp}}^2}\right)^2 + O\left(\frac{\tilde{\Delta}^3}{x_{\text{tp}}^6}\right)\right) + O(M^2/x_{\text{tp}}^2). \end{aligned} \quad (268)$$

While a repulsive correction survives the zero-mass limit, it decreases quickly for $x_{\text{tp}}^2 \gg \tilde{\Delta}$. For finite x_{tp} , however, there is a nonzero remnant deflection angle even in the absence of any mass. This result is surprising, but it is not necessarily problematic in an observational context. The traditional, Eddington-type measurement of a deflection angle does not determine a single $\Delta\phi$ but rather the difference of two $\Delta\phi$, one with a heavy mass close to the light ray ($M \neq 0$) and one with a light ray far from any heavy mass ($M \rightarrow 0$). Since the unexpected $\tilde{\Delta}$ -term in $\Delta\phi$ is mass-independent, it cancels out in any difference of two deflection angles.

5. Net stress-energy tensor

The Ricci scalar for $h(x) = \hat{r}/x$ is given by

$$\mathcal{R} = \frac{2\tilde{\Delta}}{x^4} \left(1 - \frac{7M}{x} + \frac{9M^2}{x^2}\right), \quad (269)$$

and, using (190), the net stress-energy tensor equals

$$\begin{aligned} \tilde{T}_{\mu\nu}^{(\tilde{\Delta})} dx^{\mu} \otimes dx^{\nu} &= -\frac{\tilde{\Delta}}{x^4} \left(1 - \frac{6M}{x}\right) \left(1 - \frac{2M}{x}\right)^2 dt^2 \\ &\quad - \frac{\tilde{\Delta}}{x^4} \left(1 + \frac{\tilde{\Delta}}{x^2} \left(1 - \frac{2M}{x}\right)\right)^{-1} dx^2 \\ &\quad + \frac{\tilde{\Delta}}{x^4} \left(1 - \frac{3M}{x}\right) \left(1 - \frac{M}{x}\right) x^2 d\Omega^2. \end{aligned} \quad (270)$$

The gravitational energy densities in the exterior and interior are given by

$$\rho \equiv \tilde{T}_{\mu\nu}^{(\tilde{\Delta})} \hat{t}^{\mu} \hat{t}^{\nu} = -\frac{\tilde{\Delta}}{x^4} \left(1 - \frac{6M}{x}\right) \left(1 - \frac{2M}{x}\right), \quad (271)$$

$$\rho_h \equiv \tilde{T}_{\mu\nu}^{(\tilde{\Delta})} \hat{t}_h^{\mu} \hat{t}_h^{\nu} = -\frac{\tilde{\Delta}}{t_h^4} \left(1 - \frac{\tilde{\Delta}}{t_h^2} \left(\frac{2M}{t_h} - 1\right)\right)^{-2} \left(\frac{2M}{t_h} - 1\right)^{-1}, \quad (272)$$

respectively, where $\hat{t}^{\mu} \partial_{\mu} = N^{-1} \partial_t$ is the normalized stationary velocity, while $\hat{t}_h^{\mu} \partial_{\mu} = N^{-1} \partial_{t_h}$ is the normalized

comoving velocity associated to the metric (248) under the appropriate coordinate swap.

The four main differences between a decreasing holonomy function $h(x) = \hat{r}/x$ and a constant $h(x) = 1$ are

- (1) All the components of the Einstein tensor decay faster with increasing x for a decreasing holonomy function, with the exception of G_{tt} which has a similar decay in both schemes. For $h(x) = \hat{r}/x$, all components decrease at the same rate to leading order in $1/x$.
- (2) For $h(x) = \hat{r}/x$, the Ricci curvature (269) vanishes at the coordinates

$$\frac{x_0^\pm}{2M} = \frac{7 \pm \sqrt{13}}{4}, \quad (273)$$

where $x_0^+ \approx 5.3M > 2M$ and $x_0^- \approx 1.69M < 2M$. Therefore, unlike for a constant holonomy function, the Ricci scalar is not positive everywhere, becoming negative outside of but near the horizon,

$$R \xrightarrow{x \rightarrow 2M(1+\delta)} -\frac{\tilde{\Delta}}{32M^4} + O(\delta^2/(2M)^2). \quad (274)$$

- (3) Restricted to the exterior, the energy density (271) has different signs in different regions, $\rho < 0$ for $x > 6M$ and $\rho > 0$ for $2M < x < 6M$; the radial pressure is negative everywhere, $\bar{T}_{xx} < 0$, and the angular pressure changes sign, such that $\bar{T}_{\varphi\varphi} > 0$ for $x > 3M$ and $\bar{T}_{\varphi\varphi} < 0$ for $2M < x < 3M$.

- (4) Restricted to comoving observers in the interior, the energy density is negative everywhere, the radial pressure is positive everywhere, $\bar{T}_{tt} > 0$, and the angular pressure changes sign at $t_h = M$ such that it is negative for $t_h > M$ and positive for $t_h < M$.

Finally, we note that in the zero-mass limit, $M \rightarrow 0$, the net stress-energy tensor (270) does not vanish but reduces to

$$\begin{aligned} \bar{T}_{\mu\nu}^{(\tilde{\Delta})} dx^\mu \otimes dx^\nu|_{M \rightarrow 0} = & -\frac{\tilde{\Delta}}{x^4} dt^2 - \frac{\tilde{\Delta}}{x^4} \left(1 + \frac{\tilde{\Delta}}{x^2}\right)^{-1} dx^2 \\ & + \frac{\tilde{\Delta}}{x^4} x^2 d\Omega^2, \end{aligned} \quad (275)$$

while the energy density is simply given by $\rho|_{M \rightarrow 0} = \bar{T}_{tt}|_{M \rightarrow 0}$. Some gravitational stress-energy therefore survives in the vacuum but quickly decays as $\tilde{\Delta}/x^4$ and may therefore be observable only at the Planck scale.

6. Black-hole thermodynamics

The net quasilocal energy (201) for $h(x) = \hat{r}/x$ with $\hat{N} = 1$ is given by

$$E_{\tilde{\Delta}}(x) = -x \left(\sqrt{1 - \frac{2M}{x}} \sqrt{1 + \frac{\tilde{\Delta}}{x^2} \left(1 - \frac{2M}{x}\right)} - \sqrt{1 + \frac{\tilde{\Delta}}{x^2}} \right), \quad (276)$$

and the net quasilocal entropy (210) by

$$S_{\tilde{\Delta}}(\bar{x}) = \frac{8\pi\bar{x}^6}{15\tilde{\Delta}^2} \left(\sqrt{1 + \frac{\tilde{\Delta}}{\bar{x}^2} \left(1 - \frac{2M}{\bar{x}}\right)} \left(3 + \frac{\tilde{\Delta}}{\bar{x}^2} \left(1 + \frac{3M}{\bar{x}}\right) - 2\frac{\tilde{\Delta}^2}{\bar{x}^2} \left(1 + \frac{M}{\bar{x}} - \frac{6M^2}{\bar{x}^2}\right) \right) - \sqrt{1 + \frac{\tilde{\Delta}}{\bar{x}^2}} \left(3 + \frac{\tilde{\Delta}}{\bar{x}^2} - 2\frac{\tilde{\Delta}^2}{\bar{x}^4} \right) \right). \quad (277)$$

This entropy recovers its classical value asymptotically, $\lim_{x \rightarrow \infty} S_{\tilde{\Delta}}(x) = S_{\text{BH}}$. At the horizon, however, it has the nonclassical value

$$S_{\tilde{\Delta}}(2M) = \frac{2(4\pi)^4 A_{\text{H}}^3}{15 \tilde{\Delta}^2} \left(3 + 10\pi \frac{\tilde{\Delta}}{A_{\text{H}}} - \sqrt{1 + 4\pi \frac{\tilde{\Delta}}{A_{\text{H}}}} \left(3 + 4\pi \frac{\tilde{\Delta}}{A_{\text{H}}} \left(1 - 8\pi \frac{\tilde{\Delta}}{A_{\text{H}}} \right) \right) \right) = S_{\text{BH}} \left(1 + 2\pi \frac{\tilde{\Delta}}{A_{\text{H}}} + O\left(\frac{\tilde{\Delta}^2}{A_{\text{H}}^2}\right) \right), \quad (278)$$

such that $S_{\tilde{\Delta}}(2M) > S_{\text{BH}}$. In contrast to constant $h(x) = 1$, the entropy $S_{\tilde{\Delta}}$ is monotonically decreasing. This means that the gravitational field in the exterior vacuum has some information and hence the entropy is minimal for asymptotic observers who have access to the whole gravitational field.

C. Presence of a cosmological constant

After reincorporating the cosmological constant, the metric (248) for $h(x) = \hat{r}/x$ such that $\lambda(x) = \sqrt{\tilde{\Delta}}/x$ reads

$$ds^2 = -\left(1 - \frac{2M}{x} - \frac{\Lambda x^2}{3}\right) dt^2 + \left(1 + \frac{\tilde{\Delta}}{x^2} \left(1 - \frac{2M}{x} - \frac{\Lambda x^2}{3}\right)\right)^{-1} \times \left(1 - \frac{2M}{x} - \frac{\Lambda x^2}{3}\right)^{-1} dx^2 + x^2 d\Omega^2. \quad (279)$$

The presence of a cosmological constant can significantly alter the global structure of the modified space-time. Even in the classical case, $\tilde{\Delta} \rightarrow 0$, a nonvanishing, positive Λ implies that a second horizon exists at

$$x_\Lambda \approx \sqrt{\frac{3}{\Lambda}}, \quad (280)$$

beyond which the space-time is homogeneous.

If we consider a decreasing holonomy function, $h(x) = \hat{r}/x$, Eq. (54) for coordinate singularities, evaluated at small $x \ll 2M$, has the approximate solution (249) for a minimum radius similar to constant λ . However, for large $x \gg 2M$, the Newtonian term can be neglected, and the approximate solution

$$x_\Delta^{(+)} \approx \sqrt{\frac{3\tilde{\Delta}}{\tilde{\Delta}\Lambda - 3}} \quad (281)$$

is imaginary for $\tilde{\Delta}\Lambda < 3$. Hence, the radius is not bounded from above for a decreasing holonomy function $h(x) = \hat{r}/x$ in such a case. Based on the observed value of the cosmological constant, $\tilde{\Delta}\Lambda \sim 10^{-122}$ is negligible for any Planck-sized area $\tilde{\Delta}$, and we do not need to consider the case of $\tilde{\Delta}\Lambda > 3$ in which $x_\Delta^{(+)}$ is a real solution. This simple solution is exact only if $M = 0$.

With this in mind, we now write the only real, exact solution to the minimal radius for $h(x) = \hat{r}/x$, including the cosmological constant

$$x_\Delta = \left(-\tilde{\Delta}(3 - \tilde{\Delta}\Lambda) + (3\tilde{\Delta}M)^{2/3} \left((3 - \tilde{\Delta}\Lambda)^2 + \sqrt{(3 - \tilde{\Delta}\Lambda)^3(3 - \tilde{\Delta}\Lambda + \tilde{\Delta}/(9M^2))}\right)^{2/3}\right) \times (3 - \tilde{\Delta}\Lambda)^{-1} (3\tilde{\Delta}M)^{-1/3} \left((3 - \tilde{\Delta}\Lambda)^2 + \sqrt{(3 - \tilde{\Delta}\Lambda)^3(3 - \tilde{\Delta}\Lambda + \tilde{\Delta}/(9M^2))}\right)^{-1/3}. \quad (282)$$

The expression (54) is recovered in the limit $\Lambda \rightarrow 0$, and appears as the leading zeroth order in a $\tilde{\Delta}\Lambda$ expansion. For massive black holes where $\tilde{\Delta}/M^2 \ll 1$ we have, to leading order,

$$x_\Delta = \left(\frac{2M\tilde{\Delta}}{1 - \tilde{\Delta}\Lambda/3}\right)^{1/3}. \quad (283)$$

For small masses, defining $\delta \equiv 4M^2/\tilde{\Delta}$, we have

$$\frac{x_\Delta}{2M} = 1 - (1 - \tilde{\Delta}\Lambda/3)\delta \quad (284)$$

to leading order in δ . For $h(x) = \hat{r}/x$, the conformal diagram of the maximal extension of the vacuum solution in a de Sitter background is shown in Fig. 3 with $x_\lambda = x_\Delta$.

IX. CONCLUSIONS

Emergent modified gravity provides a classification of modified canonical theories compatible with a covariant space-time geometry for each of its solutions. So far, explicit realizations have been derived in spherically symmetric models [7,8] and for polarized Gowdy models [42], with up to second-order in derivatives. Even though this is the same as the classical derivative order, a large

number of modifications are possible that are not of higher-curvature form. The direct canonical formulation makes this formalism an ideal candidate for an analysis of holonomy or other modifications proposed in models of loop quantum gravity.

Our main result is the first covariant formulation of an effective theory of holonomy-modified gravity. The theory provides a consistent set of constraints that resemble what has been analyzed before in this context, going back to [43]. But an important new ingredient is the appearance of an emergent space-time metric that differs from the classical expression in terms of canonical fields. This metric is emergent rather than effective because, in this setting, it is the only metric object that obeys the tensor transformation law. An effective metric would instead be a correction to a classical metric, such that there would be at least two different metric objects within the same theory. Our explicit examples of space-time solutions in various gauges revealed the importance of the emergent space-time metric for a consistent interpretation of holonomy modifications. For instance, the old problem of how to reconcile, on one hand, zero holonomy modifications in a Schwarzschild gauge where extrinsic curvature vanishes, and nontrivial holonomy modifications in a nonstatic gauge such as Gullstrand-Painlevé on the other hand, is resolved by the observation that the emergent metric does receive

corrections at the kinematical level. These corrections survive even in the Schwarzschild gauge.

Our formulation of holonomy modifications is covariant in space-time because its solutions are compatible with a coordinate and slicing invariant space-time structure. It is also covariant in phase-space in that it provides an unambiguous interpretation of holonomy modifications and their specific ingredients, taking into account the option to apply canonical transformations. Our detailed discussion showed that the traditional distinction between different triangulation schemes that determine the behavior of holonomies in homogeneous reduced models, such as μ_0 or $\bar{\mu}$, cannot be maintained at the covariant level, in particular when canonical transformations are taken into account. Instead, the requirement that holonomy terms have a certain form, in particular periodicity in the relevant phase-space variable, distinguish a specific set of canonical variables in which the modification function λ that initially introduced holonomy modifications is split into two separate parts. This function, which may depend on the spherical area E^x and at first resembles the traditional holonomy length μ in models of loop quantum gravity, must be split into two contributions, $\lambda(E^x) = \tilde{\lambda}h(E^x)$ with a constant $\tilde{\lambda}$ and a function $h(E^x)$ that remains finite and nonzero for $\tilde{\lambda} \rightarrow 0$. Only the constant $\tilde{\lambda}$ then appears in strictly periodic holonomy modifications and therefore represents the holonomy length, while the holonomy function $h(E^x)$ provides nontrivial modifications of the triad-dependent coefficients of holonomy terms. In this way, $\tilde{\lambda}$ resembles a traditional constant holonomy parameter μ_0 , and $h(E^x)$ would usually be interpreted as an inverse-triad correction. For covariant holonomy corrections, however, both expressions originate in the same modification function λ of a generic modified theory. A nonconstant holonomy function $h(E^x)$, rather than a nonconstant holonomy length $\tilde{\lambda}$, can then be used to control the strength of holonomy modifications in various regimes. Such an approach also clearly elucidates how holonomy modifications can come from fundamental holonomies of a gauge connection that do not refer to a metric for their definition, thereby preserving the holonomy-flux algebra. In particular, it is possible to have suppressed holonomy effects at low curvature provided $h(E^x)$ decreases sufficiently quickly for large E^x , even if the holonomy length $\tilde{\lambda}$ remains constant.

Another significant advantage of a covariant formulation is that it allows a well-defined application of standard space-time concepts, such as the definition of black holes through horizons, an analysis of curvature singularities, the description of motion in curved space-time by geodesics, and the introduction of various physical quantities related to thermodynamical behavior. For the latter, we introduced several new concepts of net energy expressions, including a net stress-energy tensor and a net quasilocal energy, that quantify deviations of emergent space-time from classical

space-time in general relativity. Even in vacuum, these quantities are in general nonzero and may be positive or negative, allowing a generalization of energy conditions and conclusions about the focusing or defocusing behavior of space-time on different scales. Our covariant analysis fills a lacuna in the LQG literature by exhibiting an effective repulsive behavior of the emergent space-time, underlying the singularity resolution in this model, that is independent of any gauge or coordinate choice.

We have presented detailed derivations of such examples for two cases, a constant holonomy function $h(E^x) = 1$ and a decreasing one, $h(E^x) \propto 1/x$, respectively, confirming the generic resolution of black-hole singularities in spherically symmetric models. By and large, we did not encounter significant problems in these models, even for constant $h(E^x)$ which in the usual interpretation as a μ_0 -scheme is often considered problematic. However, there were several unexpected features in the latter case which are not as severe as often claimed in traditional formulations but still lead to modifications on small curvature scales. In particular, as already observed in [34], the presence of a positive cosmological constant implies the existence of a maximal radius for constant holonomy functions. This outcome represents a global nonclassical effect that may be interpreted as resulting from the accumulation of a large number of small quantum corrections. Such instances of UV/IR mixing need to be interpreted carefully, keeping in mind that they may be artefacts of symmetry-reduced models in which a large number of identical small corrections distributed over a strictly homogeneous sphere can only add up. The symmetry assumption implies that they do not average out to small values as may be expected in an inhomogeneous context.

Heuristic holonomy-modification schemes, motivated by phenomenological considerations, typically rule out such possibilities from the outset. However, reliably evaluating a fundamental theory requires careful considerations of all possible outcomes. Our constructions present a major step in this direction by working, for the most part, with general holonomy functions $h(E^x)$. A constant holonomy function may then be realized in bounded regions of space-time, provided it merges into a suitable fall-off behavior for large E^x . Our discussion of decreasing holonomy functions show that the appearance of a maximal radius can easily be avoided. But they also reveal additional subtleties, such as remnants of quantum corrections that make the zero-mass limit of black-hole solutions differ from classical Minkowski space-time. Nevertheless, our derivation of various potentially observable properties demonstrated the overall consistency of this framework.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: EQUATIONS OF MOTION

In this appendix, we provide the equation of motion for both nonperiodic and periodic phase-space variables, taking the classical values for all the modification functions except λ and χ_0 . It is worth mentioning the structure of the basic canonical bracket,

$$\begin{aligned}\{K_\varphi(x), E^\varphi(y)\} &= \delta(x-y) \\ \{K_x(x), E^x(y)\} &= \delta(x-y),\end{aligned}\quad (\text{A1})$$

and the fact that the diffeomorphism constraint remains classical

$$H_x = E^\varphi K'_\varphi - K_x (E^x)'. \quad (\text{A2})$$

The time evolution of any space function f depending on E^x , E^φ , K_x , K_φ , and their spatial derivatives can be written

as

$$\dot{f} = \{f, \tilde{H}[N] + H_x[N^x]\} \quad (\text{A3})$$

where we implicitly absorb the global factor χ in the lapse function

$$\tilde{N} \equiv \chi N. \quad (\text{A4})$$

We provide the explicit equation of motion for E^x , E^φ , and K_φ . The dynamics of K_x is not included for two reasons: first, it is quite lengthy compared to the other phase-space variables. Second, we can always use the vanishing of the constraints, either $\tilde{H} = 0$ or $H_x = 0$, to solve for K_x .

1. Nonperiodic variables

For nonperiodic phase-space variables, the Hamiltonian constraint is given by (36). Therefore, the equations of motion are

$$\dot{E}^x = N^x (E^x)' + \tilde{N} \frac{\sin(2\lambda K_\varphi)}{\lambda} \sqrt{E^x} \left(1 + \left(\frac{\lambda (E^x)'}{2E^\varphi} \right)^2 \right) \quad (\text{A5})$$

and

$$\begin{aligned}\dot{E}^\varphi &= (N^x E^\varphi)' + \tilde{N} \left[\frac{E_\varphi}{2\sqrt{E^x}} \frac{\sin(2\lambda K_\varphi)}{\lambda} + 2\sqrt{E^x} K_x \cos(2\lambda K_\varphi) \left(1 + \left(\frac{\lambda (E^x)'}{2E^\varphi} \right)^2 \right) \right. \\ &\quad \left. + \lambda \sin(2\lambda K_\varphi) \frac{\sqrt{E^x}}{2} \left(\frac{E^\varphi}{E^x} \left(\frac{(E^x)'}{2E^\varphi} \right)^2 + \left(\frac{(E^x)'}{2E^\varphi} \right)' \right) \right. \\ &\quad \left. + 2\sqrt{E^x} E^\varphi \frac{\partial \ln \lambda}{\partial E^x} \left(K_\varphi \cos(2\lambda K_\varphi) - \frac{\sin(2\lambda K_\varphi)}{2\lambda} + \left(\frac{\lambda (E^x)'}{2E^\varphi} \right)^2 \left(K_\varphi \cos(2\lambda K_\varphi) + \frac{\sin(2\lambda K_\varphi)}{2\lambda} \right) \right) \right],\end{aligned}\quad (\text{A6})$$

as well as

$$\begin{aligned}\dot{K}_\varphi &= N^x K'_\varphi + \tilde{N} \left[\frac{\sqrt{E^x}}{2} \left(\Lambda - \frac{\lambda^2 + \sin^2(\lambda K_\varphi)}{\lambda^2 E^x} \right) + \frac{1}{2\sqrt{E^x}} \left(\frac{(E^x)'}{2E^\varphi} \right)^2 \cos^2(\lambda K_\varphi) \right. \\ &\quad \left. + \frac{\sqrt{E^x}}{2} \frac{\partial \ln \lambda}{\partial E^x} \left(4 \frac{\sin^2(\lambda K_\varphi)}{\lambda^2} - 4K_\varphi \frac{\sin(2\lambda K_\varphi)}{2\lambda} - 2\lambda K_\varphi \left(\frac{(E^x)'}{2E^\varphi} \right)^2 \sin(2\lambda K_\varphi) \right) \right] \\ &\quad + \tilde{N}' \frac{\sqrt{E^x} (E^x)'}{2(E^\varphi)^2} \cos^2(\lambda K_\varphi)\end{aligned}\quad (\text{A7})$$

2. Periodic variables

For periodic phase-space variables (constant $\tilde{\lambda}$), the Hamiltonian constraint is given by (17). The equations of motion is

$$\dot{E}^x = N^x (E^x)' + \tilde{N} \frac{\tilde{\lambda} \sin(2\lambda K_\varphi)}{\lambda} \sqrt{E^x} \left(1 + \left(\frac{\tilde{\lambda} (E^x)'}{2E^\varphi} \right)^2 \right) \quad (\text{A8})$$

and

$$\begin{aligned} \dot{E}^\varphi = (N^x E^\varphi)' + \frac{\tilde{N} \tilde{\lambda} \sqrt{E^x}}{2\lambda} \left[4K_x \left(1 + \left(\frac{\tilde{\lambda} (E^x)'}{2E^\varphi} \right)^2 \right) \cos(2\tilde{\lambda} K_\varphi) \right. \\ \left. + \left(\left(\frac{4E^\varphi}{\tilde{\lambda}^2} + \frac{((E^x)')^2}{E^\varphi} \right) \left(\frac{1}{4E^x} - \frac{\partial \ln \lambda}{\partial E^x} \right) + \left(\frac{(E^x)'}{E^\varphi} \right)' \right) \tilde{\lambda} \sin(2\tilde{\lambda} K_\varphi) \right], \end{aligned} \quad (\text{A9})$$

as well as

$$\begin{aligned} \dot{K}_\varphi = K'_\varphi N^x + \frac{\tilde{\lambda} \tilde{N}' \sqrt{E^x} (E^x)' \cos(\tilde{\lambda} K_\varphi)}{2\lambda (E^\varphi)^2} + \frac{\tilde{\lambda} \tilde{N} ((E^x)')^2 \cos(\tilde{\lambda} K_\varphi)^2}{8\lambda \sqrt{E^x} (E^\varphi)^2} \\ - \tilde{N} \frac{\tilde{\lambda} \sqrt{E^x}}{\lambda} \left[\frac{\lambda^2}{\tilde{\lambda}^2} \left(\frac{1}{E^x} - \Lambda \right) + 4 \left(\frac{1}{4E^x} - \frac{\partial \ln \lambda}{\partial E^x} \right) \frac{\sin^2(\tilde{\lambda} K_\varphi)}{\tilde{\lambda}^2} \right] \end{aligned} \quad (\text{A10})$$

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