

Root- $T\bar{T}$ flows unify 4D duality-invariant electrodynamics and 2D integrable sigma models

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We present a single, dimension-independent framework that links four-dimensional duality-invariant nonlinear electrodynamics to two-dimensional integrable sigma models. Both sectors are shown to obey the same first-order Courant–Hilbert equation, solved by a common generating function and an auxiliary-potential formulation. Within this structure, a discrete φ parity acts as a selection rule, organizing deformation series into integer versus fractional powers. Two commuting deformations—an irrelevant parameter λ and a marginal parameter γ —admit universal flow representations that recover root- $T\bar{T}$ dynamics and extend them in a controlled way. The construction yields closed-form families (generalized Born-Infeld, logarithmic, and q deformed) and a new integrable model, all realized in 2D and 4D. These results replace case-by-case analyses with a unified route to solvable nonlinear theories, with immediate relevance to gauge dynamics, string-inspired effective actions, and integrable models.

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Introduction. Nonlinear field theories are tightly constrained by two organizing principles with broad reach across physics: *self-duality*, which fixes the form of admissible interactions in four-dimensional electromagnetism (capturing Born-Infeld and its string/D-brane realizations [1–3], as well as the conformal ModMax theory) [4–6], and *integrability*, which endows two-dimensional models with exact solvability and infinite conserved charges [7]. Both integrability and duality-invariant conditions manifest as partial differential equations (PDEs) [5,8–13], which constrain the admissible forms of nonlinear dynamics in two and four dimensions.

Notably, several case studies have shown that $T\bar{T}$ -type deformations can preserve both integrability in two-dimensional sigma models and electromagnetic self-duality in four-dimensional theories [14–18]. In two dimensions, the corresponding flow equations for scalar field theories with integrability conditions under $T\bar{T}$ deformations were derived in [14,15]. Analogously, for four-dimensional

duality-invariant electrodynamics [1,2,8], flow equations were first proposed in [16] and subsequently extended in [19–21]. Despite these advances, it remains an open question how far the classical 4D–2D correspondence extends across the full space of duality-invariant models.

This Letter shows that duality-invariant electrodynamics in 4D and integrable sigma models in 2D are governed by the *same* first-order Courant–Hilbert equation, generated by a single function $\ell(\tau)$ [22]. A compact auxiliary-field formulation makes the construction explicit and dimension independent [23,24]. Within this framework, two commuting deformations—an irrelevant coupling λ and a marginal coupling γ —admit universal flow rules: known flows (including root $T\bar{T}$) are recovered and precisely rescaled [12], while closed-form families (generalized Born-Infeld, logarithmic, and q deformed) arise uniformly [25,26]. The outcome is a concise, portable toolkit that unifies and extends solvable nonlinear models across dimensions, with immediate relevance to gauge dynamics, string-inspired effective actions, and integrable models.

Theory and method. Duality-invariant deformations of the Maxwell theory in four dimensions are encoded in a Lagrangian $\mathcal{L}(S, P)$, where

$$S = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad P = -\frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}. \quad (1)$$

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Consistency under electric-magnetic rotations imposes the self-duality condition [5,8,10,11]

$$(\mathcal{L}_S)^2 - 2\frac{S}{P}\mathcal{L}_S\mathcal{L}_P - (\mathcal{L}_P)^2 = 1, \quad (2)$$

with $\mathcal{L}_{S,P} \equiv \partial\mathcal{L}/\partial(S,P)$. Defining

$$U = \frac{1}{2}(\sqrt{S^2 + P^2} - S), \quad V = \frac{1}{2}(\sqrt{S^2 + P^2} + S), \quad (3)$$

one recasts the duality-invariant condition in the simple form [10],

$$\mathcal{L}_U\mathcal{L}_V = -1. \quad (4)$$

Two-dimensional integrable sigma models with group-valued fields $g(\sigma, \tau)$ are characterized by the flatness of a Lax connection. In light-cone coordinates, define the currents $j = g^{-1}dg$, $j_{\pm} = j_0 \pm j_1$ and the scalar invariants

$$\begin{aligned} P_1 &= -\text{tr}[j_+j_-], \\ P_2 &= \frac{1}{2}(\text{tr}[j_+j_+]\text{tr}[j_-j_-] + (\text{tr}[j_+j_-])^2). \end{aligned} \quad (5)$$

The Lagrangian $\hat{\mathcal{L}}(P_1, P_2)$ must satisfy the PDE [27],

$$8(P_1^2 - P_2)\hat{\mathcal{L}}_{P_2}^2 + 8P_1\hat{\mathcal{L}}_{P_2}\hat{\mathcal{L}}_{P_1} + 4\hat{\mathcal{L}}_{P_1}^2 = 1. \quad (6)$$

Introducing

$$q_1 = \frac{1}{4}(\sqrt{2P_2 - P_1^2} - P_1), \quad q_2 = \frac{1}{4}(\sqrt{2P_2 - P_1^2} + P_1), \quad (7)$$

this reduces to a simple condition,

$$\hat{\mathcal{L}}_{q_1}\hat{\mathcal{L}}_{q_2} = -1. \quad (8)$$

Equations (4) and (8) are the same first-order nonlinear PDE [28], so four-dimensional self-duality and two-dimensional classical integrability are controlled by one universal characteristic equation. Hence, both sectors may be solved uniformly by a single generating Courant-Hilbert (CH)-function $\ell(\tau)$ introduced in the next section, which simultaneously encodes marginal (γ) and irrelevant (λ) stress-tensor flows in either dimension.

Courant-Hilbert framework. At the heart of the Courant-Hilbert framework [22] is a single generating function, $\ell(\tau)$, that solves both the four-dimensional duality-invariant and two-dimensional integrability conditions. For each pair in the positive quadrant, we need to define $\hat{\mathcal{L}}(q_1, q_2)$. Subject to $\hat{\mathcal{L}}(0, q_2) = \ell(q_2)$, is given by the characteristic ansatz,

$$\hat{\mathcal{L}} = \ell(\tau) - \frac{2q_1}{\ell'(\tau)}, \quad \tau = q_2 + \frac{q_1}{\ell'(\tau)^2}, \quad (9)$$

where $\ell(\tau)$ is an arbitrary generating function and $\ell'(\tau) = \frac{d\ell}{d\tau}$. Choosing $\ell(\tau) = \tau$ reproduces the free seed theories (Maxwell or principal chiral model), while other choices yield Born-Infeld, ModMax, logarithmic, and q -deformed interactions.

Moreover, one may construct root- $T\bar{T}$ and $T\bar{T}$ stress-tensor operators in d dimensions [16,29–33],

$$\begin{aligned} \mathcal{R}_\gamma &= \frac{1}{\sqrt{d}}\sqrt{T_{\mu\nu}T^{\mu\nu} - \frac{1}{d}T_\mu{}^\mu T_\nu{}^\nu}, \\ \mathcal{O}_\lambda &= \frac{1}{d}\left(T_{\mu\nu}T^{\mu\nu} - \frac{2}{d}T_\mu{}^\mu T_\nu{}^\nu\right), \end{aligned} \quad (10)$$

which, upon substitution of the general solution for $\hat{\mathcal{L}}(\alpha, \beta)$, collapse to the universal expressions

$$\mathcal{R}_\gamma = \tau\ell'(\tau), \quad \mathcal{O}_\lambda = -\ell(\tau)(\ell(\tau) - 2\tau\ell'(\tau)). \quad (11)$$

Thus, a single Courant-Hilbert framework provides a unified, dimension-independent classification of both duality-invariant electrodynamics and their two-dimensional integrable sigma-model counterparts in terms of the flow equations triggered by these deformations.

New auxiliary field formalism. In [24], a new auxiliary field formalism is introduced for self-dual nonlinear electrodynamics, governed by an auxiliary-field potential, which is highly different from the conventional Ivanov–Zupnik formalism [34,35] and more recent advances [36,37]. The new auxiliary field approach is fully equivalent to the original formulation in terms of $\ell(\tau)$, being related by a simple change of variables. The relation is given by the Legendre transformation,

$$\ell(\tau) = \tau y - \Omega(y), \quad \tau = \Omega'(y), \quad y = \ell'(\tau), \quad (12)$$

where $y = e^\varphi$ and φ is auxiliary field. In this framework, causality guarantees a unique and consistent solution for the auxiliary field. A general solution for integrable models is encoded in the Lagrangian,

$$\hat{\mathcal{L}}(q_1, q_2) = -\frac{q_1}{y} + yq_2 - \Omega(y), \quad (13)$$

where $\Omega(y)$ is a master potential determined by dualization constraints. The variation with respect to y yields $\Omega'(y) = \frac{q_1}{y} + q_2$, ensuring compatibility with the duality-invariant condition (4) and the integrability condition (8). This formalism yields precise expressions for stress-energy tensors and their invariants, enabling the definition of marginal and irrelevant flow operators in terms of $\Omega(y)$.

Specifically, the root- $T\bar{T}$ and irrelevant $T\bar{T}$ operators are given by,

$$\mathcal{R}_\gamma = y\Omega'(y), \quad \mathcal{O}_\lambda = -\Omega^2(y) + y^2\Omega'^2(y). \quad (14)$$

These expressions confirm that both types of deformations originate from a single universal potential and are seamlessly captured within the auxiliary-field formalism. This structure not only supports exact solvability and integrability but also ensures applicability across dimensions, unifying two-dimensional sigma models with four-dimensional duality-invariant electrodynamics under a shared deformation paradigm.

Irrelevant $T\bar{T}$ -like flows and φ parity. Building on the unified flow structures described above, we now identify a structural feature that distinguishes classes of deformations beyond integrability or duality alone. Consider the parametrization $y = e^\varphi$, under which the Lagrangian (13) for duality-invariant nonlinear electrodynamics can be expressed as

$$\mathcal{L}(S, P; \varphi) = \cosh \varphi S + \sinh \varphi \sqrt{S^2 + P^2} - W(\varphi). \quad (15)$$

Theories invariant under $\varphi \rightarrow -\varphi$, which we refer to as possessing φ parity, form a special analytic class [24]. The potential describing the auxiliary field is represented by $W(\varphi)$, where $W(\varphi) = \Omega(y)$. This alternative formulation can significantly simplify certain calculations. A notable advantage is that it allows identification of a special class of theories whose weak-field expansion contains no square roots. These are precisely the theories for which the potential $W(\varphi)$ is an even function, satisfying $W(-\varphi) = W(\varphi)$. Born-Infeld theory, with $W(\varphi) = \frac{1}{2}\cosh \varphi$, is a prime example of this class. In the auxiliary potential formalism, this condition becomes $\Omega(y) = \Omega(1/y)$, corresponding to a potential symmetric under inversion.

Such parity has a direct diagnostic manifestation in the irrelevant deformation sector. For instance, the Born-Infeld theory respects this symmetry, whereas deformed models such as the q theory with $q = \frac{3}{4}$ do not. Remarkably, $T\bar{T}$ -like deformations provide a mechanism to differentiate these cases not only dynamically, but also structurally, through their deformation flow.

The general flow equation for these theories takes the form [21,38],

$$\partial_\lambda \hat{\mathcal{L}} = \sum_{m=0}^{\infty} C_m (T_{\mu\nu} T^{\mu\nu})^{1-\frac{m}{2}} (T_\mu{}^\mu T_\nu{}^\nu)^{\frac{m}{2}}. \quad (16)$$

Our analysis reveals that for analytic theories with φ parity, the coefficients C_m obey a selection rule: all odd-index terms vanish, i.e., $C_{2k+1} = 0$, and the flow consists solely of integer powers of the composite energy-momentum structures. In contrast, when φ parity is absent, fractional powers

naturally emerge in the expansion, indicating a deeper structural asymmetry in the theory's irrelevant sector.

This result highlights a previously unnoticed symmetry-flow correspondence: φ parity, as a discrete duality property of the potential, directly governs the analytic structure of irrelevant $T\bar{T}$ -like flows. This discovery provides a new organizing principle for classifying solvable nonlinear theories in both 2D and 4D, extending beyond traditional criteria based solely on integrability or duality.

Extended coupling (λ) to pair space (λ, γ) and generating γ flows. In this section, we present a systematic framework for constructing deformed theories in the CH approach and the new auxiliary field formulation. Our approach introduces novel field transformations that generalize the conventional single-coupling framework in two significant ways:

- (1) Extension from a single coupling constant λ to a two-parameter space (λ, γ)
- (2) Preservation of the root- $T\bar{T}$ flow condition for all γ -coupled theories

The fundamental transformation rule takes the general form,

$$\mathcal{L}(\lambda) \rightarrow \tilde{\mathcal{L}}(\lambda, \gamma) = \mathcal{L}_{seed} + \lambda \mathcal{L}_1(\gamma) + \lambda^2 \mathcal{L}_2(\gamma) + \dots, \quad (17)$$

where $\mathcal{L}_{seed}$ is the Lagrangian for the seed theories in 2D or 4D and $\mathcal{L}_n(\gamma)$ is carefully constrained to maintain compatibility with the root- $T\bar{T}$ operator. The Lagrangian transformation $\mathcal{L}(\lambda) \rightarrow \tilde{\mathcal{L}}(\lambda, \gamma)$ is systematically implemented through coordinated transformations of the λ dependent function, $\ell(\tau)$, potential, $\Omega(y)$, and the coupling λ .

Courant-Hilbert approach. The most general integrable deformations follow from a Courant-Hilbert generating function admitting the weak-field expansion [17]

$$\ell^\lambda(\tau) = \tau + \lambda \mathcal{O}(\tau^2) + \lambda^2 \mathcal{O}(\tau^3) + \dots, \quad (18)$$

which in the limit $\lambda \rightarrow 0$ reproduces the 2D free principal chiral model (PCM) or 4D Maxwell seed and $\ell^\lambda(\tau)$ satisfies the most generic irrelevant flow equation associated with the stress tensor deformation Eq. (16). The marginal γ flow corresponds to the universal flow equations in (11) through rescalings of $\ell^\lambda(\tau)$ and the coupling λ , as,

$$\ell^{(\lambda, \gamma)}(\tau) = e^\gamma \ell^{\bar{\lambda}}(\tau), \quad \bar{\lambda} = e^\gamma \lambda, \quad (19)$$

defining a universal class of integrable theories where the deformation

$$\partial_\gamma \hat{\mathcal{L}}^{(\lambda, \gamma)} = \tau \ell'(\tau) \quad (20)$$

preserves integrability via consistent transformations of the Lax structure while maintaining the root- $T\bar{T}$ operator structure.

Uniform rescaling: A family arises from the simpler rescaling

$$\ell^{(\lambda,\gamma)}(\tau) = e^\gamma \ell^\lambda(\tau), \quad (21)$$

so that in perturbation theory $\ell^{(\lambda,\gamma)}(\tau) = e^\gamma \tau + \lambda e^\gamma \mathcal{O}(\tau^2) + \dots$. The corresponding Lagrangian of the CH function (21) satisfies the γ -flow equation,

$$\partial_\gamma \hat{\mathcal{L}}^{(\lambda,\gamma)} = \ell^{(\lambda,\gamma)}(\tau). \quad (22)$$

Single-trace flow: Finally, a single-trace deformation is implemented by

$$\ell^{(\lambda,\gamma)}(\tau) = \ell^{e^{\pm\gamma}\lambda}(\tau), \quad (23)$$

leading to the marginal flow,

$$\partial_\gamma \hat{\mathcal{L}}^{(\lambda,\gamma)} = \mp \frac{1}{d} T_\mu^\mu = \mp (\ell^{(\lambda,\gamma)}(\tau) - \tau \ell'^{(\lambda,\gamma)}(\tau)). \quad (24)$$

These exhaust all marginal flows compatible with the integrability condition and encompass perturbative and exact closed-form integrable theories. For the first time, we report a single-trace flow equation with respect to the γ coupling in the context of Courant-Hilbert approach [39].

New auxiliary field approach. The transformed Lagrangian takes the following form of potential

$$\Omega(y) \rightarrow \tilde{\Omega}(\tilde{y}) = e^\gamma \Omega(e^{-\gamma} y), \quad \lambda \rightarrow e^\gamma \lambda, \quad (25)$$

where $\Omega(y)$ encodes the λ dependence and $\tilde{\Omega}(\tilde{y})$ is the modified potential with two parameters (λ, γ) . The potential transforms as $\Omega(y) \rightarrow \tilde{\Omega}(y, \gamma)$ with $\tilde{\Omega}(y, 0) = \Omega(y)$ as the boundary condition. Based on the above conversions, it can be shown that the derivative of the potential $\tilde{\Omega}(\tilde{y})$ satisfies the relation $\tilde{\Omega}'(\tilde{y}) = \frac{q_1}{\tilde{y}^2} + q_2$. Applying the transformation rules established in (25), we express the general modified Lagrangian with two coupling constants (λ, γ) in the form,

$$\tilde{\mathcal{L}}(q_1, q_2, \tilde{y}) = -\frac{q_1}{\tilde{y}} + \tilde{y} q_2 - \tilde{\Omega}(\tilde{y}). \quad (26)$$

Transformations introduced in Eq. (25) result in the formation of the root $T\bar{T}$ -flow equation within the theory's Lagrangian, expressed as,

$$\partial_\gamma \hat{\mathcal{L}}^{(\lambda,\gamma)} = \tilde{y} \tilde{\Omega}'(\tilde{y}). \quad (27)$$

Novel integrable theories. In this section, we implement two previously established approaches to construct distinct classes of integrable theories characterized by the couplings γ and λ . The CH approach generates both Born-Infeld and logarithmic theories [25,26,40], while the new auxiliary

field method produces q -deformed theories at $q = \frac{3}{4}$. This systematic framework yields novel families of closed-form integrable models, all of which remain consistent with the universal root $T\bar{T}$ -flow equations introduced in the previous section.

Theories from Courant-Hilbert approach. Generalized Born-Infeld theories: We examine a canonical example of integrable models and invariant electromagnetic theories possessing Born-Infeld duality, characterized by the CH-function $\ell^{(\lambda)}(\tau)$ with coupling constant λ . This function is defined by the following expression:

$$\ell^\lambda(\tau) = -\frac{1}{\lambda} \left(1 - \sqrt{1 + 2\lambda\tau} \right). \quad (28)$$

By applying the transformations in (19) to the CH-function $\ell^\lambda(\tau)$ defined in (28), we obtain a modified CH-function incorporating both λ and γ coupling parameters. This generalized function generates the following Lagrangian:

$$\hat{\mathcal{L}}_{GBI} = \frac{1}{\lambda} \left(-1 + \sqrt{1 - 2e^{-\gamma}\lambda q_1} \sqrt{1 + 2e^\gamma\lambda q_2} \right). \quad (29)$$

Furthermore, the Lagrangian in Eq. (29) represents the first type-I theory that admits a closed-form expression [41]. One can explicitly demonstrate that the root $T\bar{T}$ flow equation, given in Eq. (20), holds for the Lagrangian defined in Eq. (29). Applying the transformation (21) to the CH function in (28) defines a rescaled family of generalized Born-Infeld Lagrangians. The rescaled Lagrangian reveals that the corresponding flow equation is equivalent to the CH function,

$$\partial_\gamma \hat{\mathcal{L}}_{GBI}^{\text{res}} = \ell^{(\lambda,\gamma)}(\tau). \quad (30)$$

Remarkably, this flow equation is energy-momentum independent, a novel result, as all previously known flow equations for such theories depended explicitly on the energy-momentum tensor. This represents a significant departure from conventional formulations in the literature.

The generalized Born-Infeld theories discussed here arise from applying the transformation $\lambda' = e^{\mp\gamma}\lambda$, [see (23)] to the CH-function (28), which we denote by $\hat{\mathcal{L}}_{GBI}^{ST}$. A key distinguishing feature of the resulting transformed theories is that, unlike the previous generalized Born-Infeld Lagrangians, their $\lambda \rightarrow 0$ limit reduces to either,

- (i) Maxwell's theory in four dimensions, or
- (ii) PCM in two dimensions

The energy-momentum tensor derived for the Lagrangian $\hat{\mathcal{L}}_{GBI}^{ST}$ sector yields the trace,

$$T_\mu^\mu = \frac{e^{\pm\gamma} \left(\sqrt{e^{\pm\gamma} - 2\lambda q_1} - \sqrt{e^{\pm\gamma} + 2\lambda q_2} \right)^2}{\lambda \sqrt{e^{\pm\gamma} - 2\lambda q_1} \sqrt{e^{\pm\gamma} + 2\lambda q_2}}. \quad (31)$$

One can explicitly verify that the γ -single trace flow equation in (23) holds for the Lagrangian $\hat{\mathcal{L}}_{\text{GBI}}^{ST}$.

Logarithmic theories. We consider a class of two-dimensional integrable logarithmic field theories characterized by two independent coupling constants, γ and λ . These models can be systematically derived through appropriate transformations of the fundamental equation (19) when applied to the CH function

$$\ell(\tau) = -\frac{1}{\lambda} \log(1 - \lambda\tau). \quad (32)$$

The transformation procedure preserves the integrability structure while introducing the additional coupling parameter λ , which governs the logarithmic sector of the theory. This construction generalizes previous results by allowing for more flexible interaction terms in the Lagrangian while maintaining exact solvability. The logarithmic Lagrangians obtained from transformations (19), are given by,

$$\begin{aligned} \hat{\mathcal{L}}_{\text{Log}} = & \frac{1}{\lambda} \left(-\sqrt{1 + 4\lambda e^{-\gamma} q_1 - 4\lambda^2 q_1 q_2} + 1 \right. \\ & \left. - \log \left(\frac{e^\gamma}{2\lambda q_1} \left(1 - \sqrt{1 + 4\lambda e^{-\gamma} q_1 - 4\lambda^2 q_1 q_2} \right) \right) \right). \end{aligned} \quad (33)$$

It can be explicitly shown that the above Lagrangian satisfies the root $T\bar{T}$ -flow equation

$$\partial_\gamma \hat{\mathcal{L}}_{\text{Log}} = \tau \ell^{(\lambda, \gamma)}(\tau). \quad (34)$$

Applying rescaling (21) to the CH-function (32), we obtain a logarithmic Lagrangian $\hat{\mathcal{L}}_{\text{Log}}^{\text{res}}$. This Lagrangian can be rescaled to recover the original form of (33). Furthermore, $\hat{\mathcal{L}}_{\text{Log}}^{\text{res}}$ satisfies a flow equation with respect to the γ -coupling, denoted by $\frac{\partial \hat{\mathcal{L}}_{\text{Log}}^{\text{res}}}{\partial \gamma} = \ell^{(\lambda, \gamma)}(\tau)$, which is proportional to the CH function. By applying (23) transformations to the CH function (32), one can generate a logarithmic theory $\hat{\mathcal{L}}_{\text{Log}}^{ST}$ that satisfies the single-trace flow equation, analogous to the general Born theory of $\hat{\mathcal{L}}_{\text{GBI}}^{ST}$. *q-deformed theories.* A large family of integrable q -deformed theories can be constructed for the CH function $\ell(\tau)$, subject to the initial condition $\ell(\lambda = 0, \tau) = \tau$.

$$\ell(\tau) = \frac{1}{\lambda} \left(1 - \left(1 - \frac{1}{\mathbf{q}} \lambda \tau \right)^{\mathbf{q}} \right). \quad (35)$$

These theories are deformed by the root- $T\bar{T}$ flow, as governed by the operator $\mathcal{R}_\gamma$ in the CH-function framework. For various values of the parameter q , the PDE in (8) can be solved by applying Eqs. (19) to (35) within the CH approach. This includes special cases such as $q = \frac{3}{2}$, $q = \frac{3}{4}$, and the

limiting configuration $q \rightarrow \infty$, which have been presented in Supplemental Material [42].

Theories from new auxiliary field formalism. In this section, we study flow equations derived from the new auxiliary field approach, including a q -deformed theory and a novel theory.

q-deformed theories. We demonstrate that an appropriate potential $\Omega(y)$ new auxiliary field formalism reproduces the q -deformed theory in the CH approach for $q = 3/4$ and yields the corresponding flow equations $\partial_\gamma \hat{\mathcal{L}}^{(\lambda, \gamma)} = \mathcal{R}_\gamma$ in (14). To achieve this, we consider the following potential:

$$\Omega(y) = \frac{1}{4\lambda} ((y^{-3} + 3y) - 4), \quad (36)$$

where y is determined by,

$$y = \sqrt{\frac{2\lambda q_1 + \sqrt{9 + 4\lambda(\lambda q_1^2 - 3q_2)}}{3 - 4\lambda q_2}}. \quad (37)$$

The Lagrangian takes the form [43],

$$\begin{aligned} \mathcal{L}_{q=\frac{3}{4}} = & \frac{1}{\lambda} - 2q_1 \left(1 - \frac{4\lambda}{9} (\Delta - 2\lambda q_1^2 + 3q_2) \right)^{1/4} \\ & - \frac{1}{\lambda} \left(1 - \frac{4\lambda}{9} (\Delta - 2\lambda q_1^2 + 3q_2) \right)^{3/4}, \end{aligned} \quad (38)$$

with Δ defined as,

$$\Delta = \sqrt{q_1^2(9 + 4\lambda^2 q_1^2 - 3\lambda q_2)}. \quad (39)$$

Using the approach presented in (25), we can introduce the γ coupling to the Lagrangian (38) such that the root flow equations take the form, $\partial_\gamma \hat{\mathcal{L}}^{(\lambda, \gamma)} = \tilde{y} \tilde{\Omega}'(\tilde{y})$. To achieve this, we apply (25) to the potential and redefine it as follows:

$$\tilde{\Omega}(\tilde{y}) = \frac{1}{4\lambda} \left(\frac{e^{3\gamma}}{\tilde{y}^3} + \frac{3\tilde{y}}{e^\gamma} - 4 \right), \quad (40)$$

with the auxiliary field,

$$\tilde{y} = \sqrt{\frac{2\lambda q_1 + \sqrt{9e^{2\gamma} + 4\lambda^2 q_1^2 - 12e^{3\gamma} \lambda q_2}}{3e^{-\gamma} - 4\lambda q_2}}. \quad (41)$$

The generalized Lagrangian is,

$$\hat{\mathcal{L}}_{q=\frac{3}{4}} = \frac{1}{\lambda} + \sqrt{3e^{-\gamma} - 4\lambda q_2} \left(\frac{e^{2\gamma}(4e^\gamma \lambda q_2 - 3)}{\Sigma^3 \lambda} - \frac{2q_1}{\Sigma} \right), \quad (42)$$

where,

$$\Sigma = \sqrt{2\lambda q_1 + \sqrt{9e^{2\gamma} + 4\lambda^2 q_1^2 - 12e^{3\gamma}\lambda q_2}}.$$

The root $T\bar{T}$ -flow equation in (27) remains satisfied.

New integrable theory. We present here a novel integrable theory characterized by the potential $\Omega(y)$ and auxiliary variable y ,

$$\Omega(y) = \frac{1}{\lambda} \left(\frac{2}{3y} + \frac{1}{3}y^2 - 1 \right), \quad y = -\frac{\lambda}{2} \left(\Lambda^{\frac{1}{3}} - q_2 + \frac{q_2^2}{\Lambda^{\frac{1}{3}}} \right), \quad (43)$$

where Λ is given by,

$$\Lambda = \frac{1}{\lambda^3} \left(2\sqrt{2 + 3\lambda q_1}(2 + 3\lambda q_1 + \lambda^3 q_2^3) - 4 - 6\lambda q_1 - \lambda^3 q_2^3 \right).$$

The Lagrangian for this new theory takes the form,

$$\begin{aligned} \hat{\mathcal{L}}_{\text{new}} = & \frac{1}{\lambda} \left(1 - \frac{2q_1}{q_2 - \Lambda^{\frac{1}{3}} - \frac{q_2^2}{\Lambda^{\frac{1}{3}}}} + \frac{\lambda^2}{2} q_2 \left(q_2 - \Lambda^{\frac{1}{3}} - \frac{q_2^2}{\Lambda^{\frac{1}{3}}} \right) \right. \\ & \left. - \frac{2}{3\lambda \left(q_2 - \Lambda^{\frac{1}{3}} - \frac{q_2^2}{\Lambda^{\frac{1}{3}}} \right)} \left(2 + \frac{1}{8}\lambda^3 \left(q_2 - \Lambda^{\frac{1}{3}} - \frac{q_2^2}{\Lambda^{\frac{1}{3}}} \right)^3 \right) \right). \end{aligned} \quad (44)$$

By applying the transformation (25), we introduce the γ coupling into the new Lagrangian (44), allowing us to express the root $T\bar{T}$ flow as a (PDE) in terms of the auxiliary field $\tilde{y}$. Supplemental Material [42] extends the coupling (λ) to a two-parameter space (λ, γ) , enriching the new theory's structure.

Conclusions and outlook. This Letter isolates a single structural principle that unifies two seemingly distinct arenas of solvable dynamics. Duality-invariant nonlinear electrodynamics in four dimensions and integrable sigma models in two dimensions are both governed by one Courant-Hilbert characteristic function, reducing their dynamics to the same first-order PDE. A compact auxiliary-field potential makes this construction explicit, transparent, and dimension independent. Within this framework, a discrete φ parity acts as a structural selector: when preserved, only integer composites appear; when broken,

fractional structures emerge. Two commuting deformations—an irrelevant λ and a marginal γ —admit universal flow representations: the familiar root- $T\bar{T}$ and $T\bar{T}$ -like flows are recovered, while three distinct γ flows (generalized Born-Infeld, logarithmic, q deformed) and a new solvable model are organized into closed families. In this way, disparate case-by-case constructions are replaced by a unified, dimension-independent toolkit, broadening the catalog of exactly solvable nonlinear theories of direct relevance to gauge dynamics, string-inspired effective actions, and integrable models.

Several concrete directions follow from our construction.

(i) *Quantum robustness:* derive renormalization-group equations for the generators $\mathcal{L}(\tau)$ and $\Omega(y)$ and test whether the universal identities $\partial_\gamma \mathcal{L} = \mathcal{R}_\gamma$ and $\partial_\lambda \mathcal{L} = \mathcal{O}_\lambda$ persist at one loop; confront these with factorizable S -matrices and solvable $T\bar{T}$ dynamics in 2D, and with dispersion/positivity bounds in 4D effective electrodynamics [14,15,44]. (ii) *Structure and geometry:* construct an explicit Lax pair whose zero-curvature condition is equivalent to commutativity of the (γ, λ) flows and identify the spectral parameter in terms of the Courant-Hilbert generator, with checks in the Born-Infeld, logarithmic, and q -deformed families [7]. (iii) *Generalizations:* promote the correspondence to non-Abelian sectors (Yang-Mills/WZW-type traces), include matter and curved backgrounds, and relate the auxiliary potential to D-brane worldvolume actions and other string-motivated systems [2,45–47]. (iv) *Applications:* use the closed-form models reported here as benchmarks for non-perturbative methods and as building blocks for duality-based effective theories and holographic constructions. Further examples and derivations appear in Supplemental Material [42].

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