Soft theorems in Chern-Simons matter theories

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We describe tree-level soft theorems for Chern-Simons QED and QCD in $4+1$ dimensions. Soft theorems have been studied in various context in the past. The universal behavior of the leading and subleading soft theorems is understood as a consequence of the local on-shell gauge invariance of the scattering amplitudes. Chern-Simons terms are gauge invariant only up to a boundary term. Hence, they make for an interesting case to study the relationship between gauge invariance and soft theorems. In this paper, we show that the subleading soft theorems in QED and QCD get corrected when Chern-Simons terms are introduced and also derive those corrections. We start with a discussion on soft theorems as a consequence of gauge invariance. Then we move on to derive the Chern-Simons vertices. Finally, we calculate the correction to soft theorems due to the addition of Chern-Simons terms for both QED and QCD.

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I. INTRODUCTION

Soft theorems are statements about scattering amplitudes of processes involving external states with low energy (soft) gauge bosons. They relate the amplitudes with external soft gauge bosons to those without them via a soft operator. This soft operator is given as an expansion in the soft momenta of the external gauge bosons. The leading term in the expansion has a pole divergence in the soft momenta [1–6]. The leading and subleading terms in the soft operator have a universal form which is completely insensitive to the details of the process and only depends on the charge, momenta, and angular momentum of the hard particles. This universal behavior is understood to be a consequence of the local on-shell gauge invariance of the scattering amplitudes [2–5,7]. This naturally leads to the following question: does this statement apply to gauge theories with Chern-Simons terms as well? Unlike the usual gauge invariant terms in the QED or QCD action, the Chern-Simons terms are gauge invariant only up to a boundary term [8,9]. Therefore, they do not modify the Ward identity corresponding to small gauge transformations. Therefore, the gauge invariance of these terms is more subtle. Naturally then it is interesting to ask whether they modify the universal form of the leading or subleading soft theorems. In this paper, we answer this question. We calculate how the soft theorems get corrected at the tree level when we introduce a Chern-Simons term in QED and QCD.

Chern-Simons terms are only defined in odd space-time dimensions. They are usually studied in $2+1$ space-time dimensions. We are instead working in $4+1$ dimensions because we want to treat the Chern-Simons term perturbatively about the usual gauge theory of QED or QCD.[1] Hence, we consider a gauge field coupled to matter in $D=4+1$,

$$S_{\text{gauge}}=\int d^{4+1}x\left\{-\frac{1}{4g^2}F^2-\frac{1}{2g^2}(\partial.A)^2+\mathcal{L}_{\text{matter}}+\mathcal{L}_{\text{int}}\right\}, \tag{1}$$

where A_μ is the gauge field and $F_{\mu\nu}$ is the appropriate field strength tensor.

To this we can then add a deformation of the form

$$S_{\text{CS}}=\kappa\int \text{Tr}[\mathbb{A}(d\mathbb{A})^2+c_4\mathbb{A}^3d\mathbb{A}+c_5\mathbb{A}^5]. \tag{2}$$

In the case of QED, the gauge field is Abelian and hence the last two terms vanish. The first term by itself is then the Abelian Chern-Simons term. For QCD, the deformation becomes gauge invariant (up to a boundary term) when

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[1] In $D=2+1$, the Chern-Simons term goes like $\sim\mathbb{A}d\mathbb{A}+\mathbb{A}^3$ which dominates over the Yang-Mills action in the infrared. This is not the case in higher dimensions where the Chern-Simons term becomes less relevant in the infrared.

$\{c_4, c_5\} = \{\frac{3}{2}, \frac{3}{5}\}$. Then $S_{\rm CS}$ becomes the non-Abelian Chern-Simons term.[2]

The total action is then given by the sum of the two,

$$S = S_{\rm gauge} + S_{\rm CS}.$$

In the absence of the term $S_{\rm CS}$, the subleading soft theorem (for instance, for an external soft photon with momentum k_μ and polarization e_μ) takes the form

$$G_{n+1}(e,k;\varepsilon_i,p_i) = \sum_i \varepsilon_i^T q_i \frac{e.p_i}{p_i.k} G_n^{(i)}(p_i) - i\sum_i \varepsilon_i^T q_i \frac{e_\mu k_\nu}{p_i.k} (\hat{J}^{\mu\nu}_{(i)} G_n^{(i)})(p_i), \quad (3)$$

where ε_i, p_i refers to the polarization and energy of the external finite energy ("hard") particles and q_i refers to the charge of the matter coupling to the photon field. Similar expressions are there for gluons and gravitons as well. The leading term has a $1/k$ pole divergence while the subleading term is $\mathcal{O}(k^0)$. The soft photon theorem at leading order in k does not get infrared corrections at loop level [7,10–12]. In $D = 2+1$, the subleading term gets corrected due to infrared effects at loop level, but in $D = 4+1$ the subleading soft photon theorem is also valid to all loop orders. The soft gluon theorem in $D = 2+1$ admits infrared corrections at loop level even at the leading order but in $D = 4+1$, the theory becomes IR finite and corrections occur only at subleading order. In the case of multiple soft gauge bosons, each gauge boson brings a soft factor of divergence $1/k$ at leading order.

As mentioned before, the leading and subleading behavior in (3) is universal and can be understood as a consequence of local on-shell gauge invariance of the scattering amplitudes. This was shown for photons in [3] and for gluons in [7,13]. For gravitons, even the second subleading behavior (not stated here) is universal and can be obtained from gauge invariance of the amplitudes [5–7,13]. Sen's covaraintization procedure [14–18] utilizes gauge invariance to derive this universal behavior by giving a covariantization prescription to calculate the 3-point vertex [see $\Gamma^{(3)}(e,k;p,-p-k)$ in Fig. 1] for the leading order diagram. Here, one writes an action involving only hard particle fields and covariantizes it by minimally coupling the gauge field mode for the soft gauge boson. This is described in detail in Sec. III. Such a covariantization prescription does not apply to Chern-Simons terms because even though the small gauge Ward identity is still valid, the Chern-Simons terms are not minimally coupled to the matter. Hence, their contribution needs to be accounted for separately in the 3-point vertex, $\Gamma^{(3)}(e,k;p,-p-k)$.

Furthermore, we show that gauge invariance is a sufficient but not necessary condition for having the universal form of the leading soft theorems. For this, we will take c_4 and c_5 in (2) to be arbitrary (and hence make $S_{\rm CS}$ nongauge invariant) and show that the universal form of the leading soft theorem still holds and only the subleading term gets corrected.

Hence, ultimately the aim of this paper is to calculate how (3) gets corrected at tree level when a Chern-Simons term $S_{\rm CS}$ is added to the gauge theory.

The paper is organized as follows: In Sec. III, we review the soft photon and gluon theorems and derive them by covariantizing the hard particle action by minimally coupling the soft gauge field modes. References [14–18] demonstrate it for soft photons and gravitons. We generalize it to a generic non-Abelian Yang-Mills theory with SU(N) gauge invariance and derive the soft gluon theorem. Soft gluon theorems have been previously studied in various ways in [7,13,19–21]. Then in Sec. IV, we calculate the Feynman rules for the Chern-Simons vertices in $D = 4+1$. In Secs. V–VII we incorporate them to obtain the Chern-Simons correction to the soft theorems. We see that the leading soft theorems remain unchanged and the Chern-Simons terms only correct the subleading soft theorems.

II. NOTATIONS AND CONVENTIONS

Throughout this paper, we work in natural units with $\hbar = c = 1$. We also use the "mostly minus" signature $(+---)$ for the space-time metric. $\epsilon^{\mu\nu\rho\sigma}$ refers to the Levi-Civita symbol normalized as $\epsilon^{0123} = 1$. μ, ν denote the Lorentz indices while α, β denote the indices of the representation of the gauge group. The latter is often omitted to tidy up the equations, in which case the order of multiplication is important. Lorentz indices are raised and lowered with the Minkowski metric $\eta_{\mu\nu}$. Whenever mentioned, the γ^μ denote the Dirac matrices satisfying the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. We use the Feynman slash notation where $\not{b} = b^\mu \gamma_\mu$ for any four vector b^μ. A_μ, a_μ are spin-1 gauge fields in Lorentz gauge with the latter (with small a) denoting a soft gauge mode. $F_{\mu\nu}$ is the field strength tensor. We work with SU(N) gauge theories for which a_μ etc. are matrices in the representation. They can be expanded in terms of the generators of the gauge group $a_\mu(x) = a_{a\mu}(x) T_a$. The following types of shorthand for index contractions will be present:

$$CD := C_\alpha D^\alpha,$$

$$C.D := C_\mu D^\mu.$$

[2]For a non-Abelian Chern-Simons term, under large gauge transformations the action is also shifted by a constant which is referred to as the winding number [9]. This does not affect the path integral since for appropriate discrete values of κ, the winding number is 2π times an integer and hence leaves e^{iS} invariant.

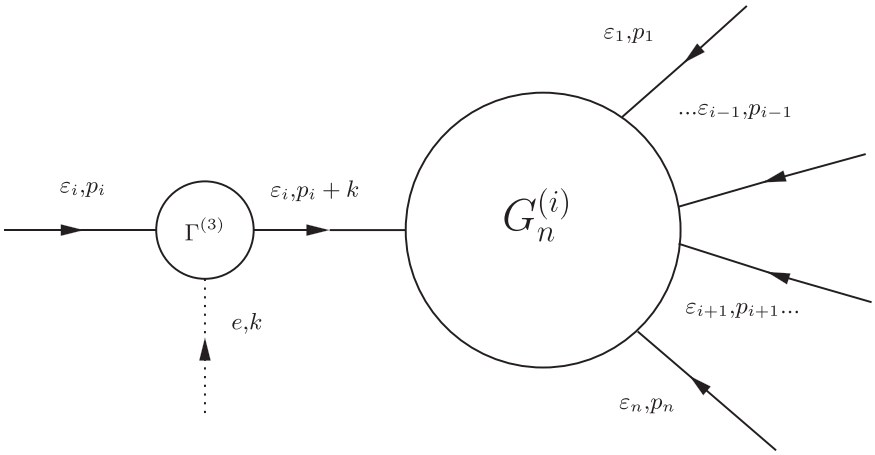


FIG. 1. $A1$: soft boson attaches to the external leg of a finite energy particle via a 1PI vertex $\Gamma^{(3)}$.

We also use the superscript T to denote the matrix transpose in the representation indices α, β. Einstein summation convention is used for all index contractions. Therefore,

$$C_a D_a \equiv \sum_a C_a D_a.$$

III. SUBLEADING SOFT THEOREM FROM SEN'S COVARIANTIZATION PROCEDURE

In this section we will summarize the subleading single soft theorem for photons, as given in [16] using Sen's covariantization procedure [14–18] and generalize it for gluons. For gluons, the amplitudes and the corresponding diagrams will be implicitly assumed to be color ordered. At loop level, the familiar soft theorems (3) get subleading infrared corrections in case of photons and leading infrared corrections in case of gluons. Therefore, we will restrict ourselves to only tree-level amplitudes even in this general covariantization procedure. We start by considering a scattering process for hard particles, with amplitude $G_n(p_i)$. We assume that the hard particles in the process are described by a single set of fields ϕ_α, belonging to some reducible representation of the Lorentz group. Let $S_{\text{hard}}[\phi]$ denote the corresponding action for the hard particle fields. We assume that this action is invariant under the global transformation,

$$\phi_\alpha \to [e^{i\theta_a T_a}]^\beta_\alpha \phi_\beta.$$

Here T_a are the generators of the Lie group of the gauge symmetry in the reducible representation and θ_a are constant parameters for the global symmetry. The generators T_a satisfy the Lie algebra,

$$[T_a, T_b] = i f_{abc} T_c,$$

where the structure constants f_{abc} are independent of the specific representation. The algebra is taken to be *a priori* non-Abelian but the final results can be rewritten for Abelian fields as a special case.

To find the amplitude $G_{n+1}(e,k;\varepsilon_i,p_i)$ for a process with one additional soft boson in the external state, we need to consider two kinds of Feynman diagrams, $A1$ and $A2$ as shown in Figs. 1 and 2.

Here $\Gamma^{(3)}$ is the tree-level 3-point vertex involving a soft boson as well as two hard particles in the external state. $G_n^{(i),\alpha_i}$ is the full n-point amplitude involving n external hard particles with ith particle polarization amputated. To evaluate these diagrams we need to find the vertices. These can be obtained by taking S_{hard} and covariantizing it with respect to the single mode soft gauge field $[a_\mu(x)]^\beta_\alpha = [e_\mu(k)]^\beta_\alpha \exp(ik.x)$ to get S_{soft}. This involves replacing all ordinary derivatives ∂_μ in the action with covariant derivatives D_μ as follows:

$$D_\mu \phi_\alpha = \partial_\mu \phi_\alpha - i[a_\mu]^\beta_\alpha \phi_\beta, \tag{4}$$

as well as all momentum conserving delta functions $\delta(p_1+\cdots+p_N)$ by $\delta(p_1+\cdots+p_N+k)$. The gauge field a_μ lies in the tangent space of the gauge group and hence can be expanded as

$$[a_\mu(x)]^\beta_\alpha = a_{a\mu}(x)[T_a]^\beta_\alpha,$$

and hence,

$$[e_\mu(k)]^\beta_\alpha = e_{a\mu}(k)[T_a]^\beta_\alpha.$$

We will often omit the representation indices for cleaner looking equations.

To be more explicit, suppose we want to evaluate $\Gamma^{(3)}$. For this, consider the quadratic part of S_{hard} which can be written as[3]

[3]Note that we use ϕ to denote the field in both position space and momentum space.

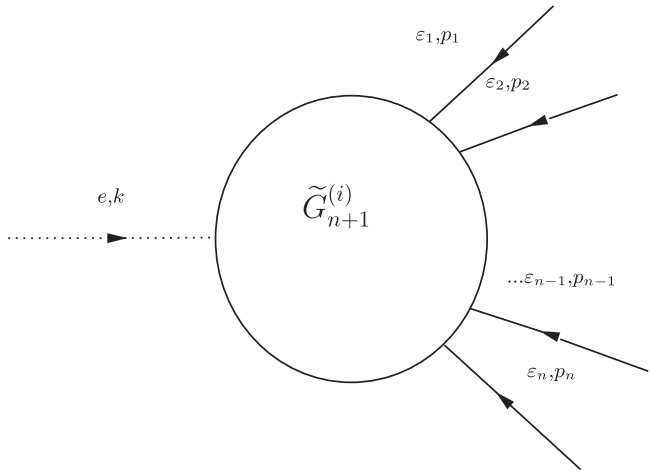


FIG. 2. $A2$: soft boson attaches to an internal line.

$$S^{(2)}_{\text{hard}} = \frac{1}{2}\int d^D x d^D y \phi_\alpha(x)\hat{K}^{\alpha\beta}(x,y)\phi_\beta(y) \tag{5}$$

$$= \frac{1}{2}\int \frac{d^D p_1}{(2\pi)^D}\frac{d^D p_2}{(2\pi)^D}\phi_\alpha(p_1)K^{\alpha\beta}(p_2) \times \phi_\beta(p_2)(2\pi)^D\delta^D(p_1+p_2), \tag{6}$$

with

$$\hat{K}^{\alpha\beta}(x,y) = \hat{K}^{\beta\alpha}(y,x), \quad K^{\alpha\beta}(p) = K^{\beta\alpha}(-p), \tag{7}$$

or equivalently,

$$K(p) = K^T(-p). \tag{8}$$

This gives us the Feynman propagator which takes the form

$$\Delta_{\alpha\beta}(p) = \mathrm{i}K^{-1}_{\alpha\beta}(p) = \frac{\Xi_{\alpha\beta}(p)}{p^2-m^2}. \tag{9}$$

Here $\Xi_{\alpha\beta}$ is the polarization sum (times i) for the hard particle fields. Invariance of (5) under global transformation (4) implies

$$T^T_a K + K T_a = 0; \tag{10}$$

multiplying by K^{-1} on both sides of (10) and using (9) gives

$$T_a \Xi + \Xi T^T_a = 0. \tag{11}$$

If p is an on-shell momentum of an external hard particle of polarization $\varepsilon(p)$, then

$$\varepsilon_\alpha(p)K^{\alpha\beta}(-p) = 0, \tag{12}$$

or equivalently

$$\varepsilon^T(p)K(-p) = 0. \tag{13}$$

Equation (9) then implies

$$\frac{\partial K(-p)}{\partial p_\mu}\Xi(-p) = 2\mathrm{i}p^\mu - K(-p)\frac{\partial \Xi(-p)}{\partial p_\mu}, \tag{14}$$

$$\frac{\partial \Xi(-p)}{\partial p_\mu}K(-p) = 2\mathrm{i}p^\mu - \Xi(-p)\frac{\partial K(-p)}{\partial p_\mu} \tag{15}$$

for the external hard particle. We have suppressed the representation indices.

Covariantizing of (5) with respect to a single soft mode gauge field $a_\mu(x) = e_\mu(k)\exp(\mathrm{i}k.x)$ using the procedure above gives the following terms in the action S_{soft} contributing to the vertex $\Gamma^{(3)}$:

$$S^{(3)}_{\text{soft}} = \frac{1}{2}\int \frac{d^D p_1}{(2\pi)^D}\frac{d^D p_2}{(2\pi)^D}(2\pi)^D\delta^{(D)}(p_1+p_2+k) \times \phi_\alpha(p_1)\left[-\frac{\partial K^{\alpha\gamma}}{\partial p_{2\mu}}(p)e^{\ \beta}_{\mu\gamma} - \frac{1}{2}\frac{\partial^2 K^{\alpha\gamma}}{\partial p_{2\mu}\partial p_{2\nu}}(p)k_\mu e^{\ \beta}_{\nu\gamma}\right] \times \phi_\beta(p_2), \tag{16}$$

or equivalently

$$S^{(3)}_{\text{soft}} = \frac{1}{2}\int \frac{d^D p_1}{(2\pi)^D}\frac{d^D p_2}{(2\pi)^D}(2\pi)^D\delta^{(D)}(p_1+p_2+k) \times \phi(p_1)\left[-\frac{\partial K}{\partial p_{2\mu}}(p)e_\mu - \frac{1}{2}\frac{\partial^2 K}{\partial p_{2\mu}\partial p_{2\nu}}(p)k_\mu e_\nu\right]\phi(p_2). \tag{17}$$

To see how these two terms come, assume that in position space $\hat{K}(x,y)$ has the general form

$$\hat{K}(x,y)\phi(y) = \delta^{(D)}(x-y)[C_1 + C^\mu_2\partial_\mu + C^{\mu\nu}_3\partial_\mu\partial_\nu]\phi(y),$$

or equivalently

$$K(p) = C_1 + \mathrm{i}C^\mu_2 p_\mu - C^{\mu\nu}_3 p_\mu p_\nu,$$

where we have suppressed the α, β indices on $\hat{K}$, ϕ, and $C_{1,2,3}$. This gives

$$\frac{\partial K}{\partial p_\mu}(p) = \mathrm{i}C^\mu_2 - 2C^{\mu\nu}_3 p_\nu, \qquad \frac{\partial^2 K}{\partial p_\mu \partial p_\nu}(p) = -2C^{\mu\nu}_3.$$

Then covariantizing such a term is achieved by replacing $\partial_\mu\phi \to D_\mu\phi$ and $\partial_\mu\partial_\nu\phi \to D_\mu D_\nu\phi$. In momentum space, the effect of the first is to make the replacement $q_\mu\phi_\alpha \to q_\mu\phi_\alpha + e^{\ \gamma}_{\mu\alpha}\phi_\gamma$ which gives the first term in (16). For the

second term we have

$$\begin{aligned} D_\mu D_\nu \phi_\alpha &= (\delta_\alpha{}^\gamma \partial_\mu - \mathrm{i} a_{\mu\alpha}^{\ \ \gamma})(\delta_\gamma{}^\beta \partial_\nu - \mathrm{i} a_{\nu\gamma}^{\ \ \beta})\phi_\beta \\ &= \partial_\mu \partial_\nu \phi_\alpha - \mathrm{i}(\partial_\mu a_\nu)_\alpha^\beta \phi_\beta - \mathrm{i}(a_\mu \partial_\nu + a_\nu \partial_\mu)_\alpha^\beta \phi_\beta + \mathcal{O}(a^2), \end{aligned}$$

which in momentum space gives the additional terms

$$\begin{aligned} [C_1 + C_2^\mu D_\mu + C_3^{\mu\nu} D_\mu D_\nu]\phi &\xrightarrow{\text{(momentum space)}} [K(p) - \mathrm{i} C_2^\mu e_\mu + 2C_3^{\mu\nu} p_\mu e_\nu + C_3^{\mu\nu} k_\mu e_\nu]\phi \\ &= \left[K(p) - \frac{\partial K}{\partial p_\mu}(p) e_\mu - \frac{1}{2}\frac{\partial^2 K}{\partial p_\mu \partial p_\nu}(p) k_\mu e_\nu\right]\phi, \end{aligned}$$

which is the same as (16). These terms in the action will give the vertex $\Gamma^{(3)}$ in Fig. 1, which is obtained by symmetrizing with respect to the external hard momenta,

$$\Gamma^{(3)\alpha\beta}(e,k;p,-p-k) = \frac{\mathrm{i}}{2}\left[\frac{\partial K^{\alpha\gamma}}{\partial p_\mu}(-p-k) e_{\mu\gamma}^{\ \ \beta} - \frac{1}{2}\frac{\partial^2 K^{\alpha\gamma}}{\partial p_\mu \partial p_\nu}(-p-k) k_\mu e_{\nu\gamma}^{\ \ \beta} - \frac{\partial K^{\beta\gamma}}{\partial p_\mu}(p) e_{\mu\gamma}^{\ \ \alpha} - \frac{1}{2}\frac{\partial^2 K^{\beta\gamma}}{\partial p_\mu \partial p_\nu}(p) k_\mu e_{\nu\gamma}^{\ \ \alpha}\right],$$

where I have absorbed the soft field polarization e_μ into the definition of $\Gamma^{(3)}$. Notice that you need to be careful about the order of α and β when symmetrizing. We can simplify this to get

$$\begin{aligned} \Gamma^{(3)}(e,k;p,-p-k) &= \frac{\mathrm{i}}{2}\left[\frac{\partial K(-p-k)}{\partial p_\mu} e_\mu - \frac{1}{2}\frac{\partial^2 K(-p-k)}{\partial p_\mu \partial p_\nu} k_\mu e_\nu - \left(\frac{\partial K(p)}{\partial p_\mu} e_\mu\right)^T - \frac{1}{2}\left(\frac{\partial^2 K(p)}{\partial p_\mu \partial p_\nu} k_\mu e_\nu\right)^T\right] \\ &= \frac{\mathrm{i}}{2}\left[\frac{\partial K(-p)}{\partial p_\mu} e_\mu + \frac{1}{2}\frac{\partial^2 K(-p)}{\partial p_\mu \partial p_\nu} k_\mu e_\nu - \left(\frac{\partial K(p)}{\partial p_\mu} e_\mu\right)^T - \frac{1}{2}\left(\frac{\partial^2 K(p)}{\partial p_\mu \partial p_\nu} k_\mu e_\nu\right)^T\right] \\ &= \frac{\mathrm{i}}{2}\left[\frac{\partial K(-p)}{\partial p_\mu} e_\mu + \frac{1}{2}\frac{\partial^2 K(-p)}{\partial p_\mu \partial p_\nu} k_\mu e_\nu - e_\mu^T \frac{\partial K^T(-p)}{\partial p_\mu} - \frac{1}{2} k_\mu e_\nu^T \frac{\partial^2 K^T(-p)}{\partial p_\mu \partial p_\nu}\right] \\ &= \mathrm{i}\left[\frac{\partial K(-p)}{\partial p_\mu} e_\mu + \frac{1}{2}\frac{\partial^2 K(-p)}{\partial p_\mu \partial p_\nu} k_\mu e_\nu\right], \end{aligned} \tag{18}$$

where we have used (7) in the second last line of (18) and (10) in the last line of (18). If the soft boson line in $\Gamma^{(3)}$ is an internal line, then (18) needs to be modified by stripping of the polarization vector and contracting the index with the appropriate soft propagator. Using (18) and the relations (10), (12), (14), and (15), we can write

$$\begin{aligned} \Gamma^{(3)}(e,k;p,-p-k)\Xi(-p-k) &= \mathrm{i}\left[\frac{\partial K(-p)}{\partial p_\mu} e_\mu + \frac{1}{2}\frac{\partial^2 K(-p)}{\partial p_\mu \partial p_\nu} k_\mu e_\nu\right]\Xi(-p-k) \\ &= -\mathrm{i}\left[\frac{\partial K(-p)}{\partial p_\mu}\Xi(-p-k) + \frac{1}{2}\frac{\partial^2 K(-p)}{\partial p_\mu \partial p_\nu}\Xi(-p-k) k_\nu\right] e_\mu^T \\ &= -\mathrm{i}\left[2\mathrm{i} p^\mu - K(-p)\frac{\partial \Xi(-p)}{\partial p_\mu} + \frac{\partial K(-p)}{\partial p_\mu}\frac{\partial \Xi(-p)}{\partial p_\nu} k_\nu + \frac{1}{2}\frac{\partial^2 K(-p)}{\partial p_\mu \partial p_\nu}\Xi(-p) k_\nu\right] e_\mu^T \\ &= 2p^\mu e_\mu^T + \mathrm{i} K(-p)\frac{\partial \Xi(-p)}{\partial p_\mu} e_\mu^T + \frac{\mathrm{i}}{2}\frac{\partial K(-p)}{\partial p_\mu}\frac{\partial \Xi(-p)}{\partial p_\nu}(k_\mu e_\nu - k_\nu e_\mu)^T \\ &\quad + \frac{\mathrm{i}}{2} K(-p)\frac{\partial^2 \Xi(-p)}{\partial p_\mu \partial p_\nu} k_\nu e_\mu^T. \end{aligned} \tag{19}$$

The Feynman rules for $A1$ give

$$A1 = \varepsilon_{i;\alpha}\Gamma^{(3)\alpha\gamma}(e,k;p_i,-p_i-k)\frac{\Xi_{\gamma\beta_i}(-p_i-k)}{(-p_i-k)^2-m_i^2}G_n^{(i);\beta_i}(p_i+k)$$
$$= \frac{1}{p_i.k}\left[\varepsilon_{i;\alpha}p_i^\mu e^{T\alpha}_{\mu\beta_i} + \varepsilon_{i;\alpha}p_i^\mu e^{T\alpha}_{\mu\beta_i}k_\rho\frac{\partial}{\partial p_{i\rho}} + \frac{\mathrm{i}}{4}\varepsilon_{i;\alpha}\frac{\partial K^{\alpha\gamma}(-p_i)}{\partial p_{i\nu}}\frac{\partial\Xi_{\gamma\delta}(-p_i)}{\partial p_{i\mu}}(e_\mu^T k_\nu - e_\nu^T k_\mu)^\delta_{\beta_i}\right]G_n^{(i);\beta_i}(p_i)$$
$$= \frac{1}{p_i.k}\left[\varepsilon_i^T p_i^\mu e_\mu^T + k_\rho\varepsilon_i^T p_i^\mu e_\mu^T\frac{\partial}{\partial p_{i\rho}} + \frac{\mathrm{i}}{4}\varepsilon_i^T\frac{\partial K_i(-p_i)}{\partial p_{i\nu}}\frac{\partial\Xi_i(-p_i)}{\partial p_{i\mu}}(k_\nu e_\mu - k_\mu e_\nu)^T\right]G_n^{(i)}(p_i), \tag{20}$$

where in the last line we have suppressed the representation indices α, β etc. to tidy up the equation. As mentioned previously, the notation $G_n^{(i),\alpha_i}$ is shorthand for $\prod_{j\neq i}\varepsilon_{\alpha_j}G_n^{\alpha_1\ldots\alpha_i\ldots\alpha_n}$ which is the n point Green's function with the ith polarization vector amputated. The subscript i in various places indicates that the ith index is summed over in the implicit matrix product. Now we want to evaluate the contribution due to diagrams of type $A2$. For this we need the amputated Green's function $\tilde{G}_{n+1}(e,k;p_1,\ldots,p_n)$ of n finite energy fields and one soft gauge field which only includes diagrams in which the soft particle is not attached to any external leg. This is obtained by covariantizing the amputated Green's function $G_n^{\alpha_1\ldots\alpha_n}(p_1,\ldots,p_n)$ of n finite energy fields and taking only the leading order contribution (since the corresponding diagram is already subleading due to the absence of a pole in the propagator). Hence we have

$$\tilde{G}_{n+1}(e,k;p_1,\ldots,p_n) = -\sum_i\left(\prod_{j\neq i}\varepsilon_{\alpha_j}\right)\varepsilon_{\alpha_i}e_{\mu\beta_i}^{\alpha_i}\frac{\partial}{\partial p_{i\mu}}G_n^{\alpha_1\ldots\beta_i\ldots\alpha_n}.$$

The contribution due to the Feynman diagram of type $A2$ will be

$$A2 = -\sum_i\varepsilon_i^T e_\mu\frac{\partial G_n^{(i)}}{\partial p_{j\mu}}.$$

Summing over the soft boson and attaching to all distinct external legs for $A1$ and adding to $A2$ we get

$$G_{n+1}(e,k;\varepsilon_i,p_i) = \sum_i\varepsilon_i^T\frac{e^T.p_i}{p_i.k}G_n^{(i)}(p_i) + \sum_i\varepsilon_i^T\frac{e_\mu^T k_\nu}{p_i.k}\left\{p_i^\mu\frac{\partial}{\partial p_{i\nu}} - p_i^\nu\frac{\partial}{\partial p_{i\mu}}\right\}G_n^{(i)}(p_i) + \sum_i\frac{1}{p_i.k}\frac{\mathrm{i}}{4}\varepsilon_i^T\frac{\partial K_i(-p_i)}{\partial p_{i\nu}}\frac{\partial\Xi_i(-p_i)}{\partial p_{i\mu}} \times (k_\nu e_\mu - k_\mu e_\nu)^T G_n^{(i)}(p_i). \tag{21}$$

Notice that the first two terms in (21) are independent of K and Ξ, whereas the last term depends on their derivatives and hence depends explicitly on the action.

A. Soft photon theorem in QED

In the case of soft photons, the gauge group is simply U(1) and the gauge field is Abelian. There is only one generator for the lie algebra which is denoted by Q_α^β and is sometimes called the charge matrix since its matrix elements are the electric charges of the various fields ϕ_α in the external state. Therefore we have

$$G_{n+1}(e,k;\varepsilon_i,p_i) = \sum_i\varepsilon_i^T\frac{e.p_i}{p_i.k}Q^T G_n^{(i)}(p_i) + \sum_i\varepsilon_i^T Q^T\frac{e_\mu k_\nu}{p_i.k}\left\{p_i^\mu\frac{\partial}{\partial p_{i\nu}} - p_i^\nu\frac{\partial}{\partial p_{i\mu}}\right\}G_n^{(i)}(p_i) + \sum_i\frac{1}{p_i.k}\frac{\mathrm{i}}{4}\varepsilon_i^T\frac{\partial K_i(-p_i)}{\partial p_{i\nu}}\frac{\partial\Xi_i(-p_i)}{\partial p_{i\mu}} \times Q^T(k_\nu e_\mu - k_\mu e_\nu)G_n^{(i)}(p_i).$$

Now for the ith external particle of charge q_i

$$Q\varepsilon_i = q_i\varepsilon_i.$$

Therefore,

$$G_{n+1}(e,k;\varepsilon_i,p_i) = \sum_i\varepsilon_i^T q_i\frac{e.p_i}{p_i.k}G_n^{(i)}(p_i) + \sum_i\varepsilon_i^T q_i\frac{e_\mu k_\nu}{p_i.k}\left\{p_i^\mu\frac{\partial}{\partial p_{i\nu}} - p_i^\nu\frac{\partial}{\partial p_{i\mu}}\right\}G_n^{(i)}(p_i) + \sum_i\frac{1}{p_i.k}\frac{\mathrm{i}}{4}\varepsilon_i^T q_i\frac{\partial K_i(-p_i)}{\partial p_{i\nu}}\frac{\partial\Xi_i(-p_i)}{\partial p_{i\mu}} \times (k_\nu e_\mu - k_\mu e_\nu)G_n^{(i)}(p_i). \tag{22}$$

If the ith external particle to which photon is attached is a Dirac fermion,

$$K_i(p_i) = \not{p}_i - m_i,$$

$$\Xi_i(p_i) = \mathrm{i}(\not{p}_i + m_i),$$

where K_i, Ξ_i denote the ith blocks in K and Ξ corresponding to the ith particle. We have used the Feynman slash notation to define $\not{p} = \gamma^\mu p_\mu$. For this case, the theory dependent term is

$$\begin{aligned}
&\frac{\mathrm{i}}{4}(e_\mu k_\nu - e_\nu k_\mu) q_i \varepsilon_i^T \frac{\partial K_i(-p_i)}{\partial p_{i\nu}} \frac{\partial \Xi_i(-p_i)}{\partial p_{i\mu}} \\
&= -\frac{\mathrm{i}}{4} q_i \varepsilon_i^T (e_\mu k_\nu - e_\nu k_\mu) i\gamma^\nu \gamma^\mu \\
&= \frac{1}{4} q_i \varepsilon_i^T (\not{k}\not{e} - \not{e}\not{k}) \\
&= q_i \varepsilon_i^T \frac{1}{4} [\gamma^\mu, \gamma^\nu] k_\mu e_\nu \\
&= -\mathrm{i} q_i \varepsilon_i^T \frac{k_\mu e_\nu}{p_i.k} S^{\mu\nu}_{(i)}.
\end{aligned}$$

Lastly, for photons $\Gamma^{(3)}$ simply vanishes due to charge conjugation invariance of the QED action (Furry's theorem). Therefore the soft theorem in QED takes the form

$$\begin{aligned}
G_{n+1}(e, k; \varepsilon_i, p_i) = &\sum_i \varepsilon_i^T q_i \frac{e.p_i}{p_i.k} G_n^{(i)}(p_i) \\
&- \mathrm{i} \sum_i \varepsilon_i^T q_i \frac{e_\mu k_\nu}{p_i.k} (\hat{J}^{\mu\nu}_{(i)} G_n^{(i)})(p_i),
\end{aligned} \tag{23}$$

where we have again identified the angular momentum operator $\hat{J}^{\mu\nu}$ acting on the Green's function $G_n^{(i)}$ as

$$\begin{aligned}
(\hat{J}^{\mu\nu}_{(i)} G_n^{(i)})(p_i) = &\mathrm{i}\left\{ p_i^\mu \frac{\partial}{\partial p_{i\nu}} - p_i^\nu \frac{\partial}{\partial p_{i\mu}} \right\} G_n^{(i)}(p_i) \\
&+ \frac{\mathrm{i}}{4} [\gamma^\mu, \gamma^\nu] G_n^{(i)}(p_i).
\end{aligned} \tag{24}$$

B. Soft gluon theorem in QCD

For gluons, the gauge group is SU(3) and the Lie algebra has eight generators. If the external states are quarks, they will be in the fundamental representation of SU(3) in which case the generators are given in terms of the eight Gell-Mann matrices λ_a,

$$T_a \varepsilon_i = g_i \frac{\lambda_{i;a}}{2} \varepsilon_i, \tag{25}$$

where $\lambda_{i;a}$ are such that the ith block is the corresponding Gell-Mann matrix while the other blocks are identity. If the external states are gluons then the generators are in the adjoint representation of SU(3),

$$T_a \varepsilon_i = g_i \tilde{T}_{i;a} \varepsilon_i, \tag{26}$$

where

$$[\tilde{T}_{i;a}]^{\mu\nu}_{bc} = f_{abc} \eta^{\mu\nu}.$$

In general we can write

$$T_a \varepsilon_i = g_i T_{i;a} \varepsilon_i,$$

In general, there can be some gluons and some quarks in the external state. So we write the general soft gluon theorem as

$$\begin{aligned}
G_{n+1}(e, k; \varepsilon_i, p_i) = &\sum_i \varepsilon_i^T g_i \frac{e_a.p_i}{p_i.k} T_{i;a} G_n^{(i)}(p_i) \\
&- \mathrm{i} \sum_i \varepsilon_i^T g_i \frac{e_{a\mu} k_\nu}{p_i.k} T_{i;a} (\hat{J}^{\mu\nu}_{(i)} G_n^{(i)})(p_i).
\end{aligned} \tag{27}$$

This is the single soft gluon theorem in QCD. Note that the sum will only contain terms that are in accordance with the appropriate color ordering.

IV. CHERN-SIMONS VERTICES IN $D = 4 + 1$

Now that we have the subleading soft theorems we can move to the Chern-Simons correction. For this we need to derive the tree-level Chern-Simons vertices. Let us look at the case of Abelian gauge field first and then the non-Abelian.

A. Abelian case: Chern-Simons QED

Consider the Abelian gauge field A_μ. The Chern-Simons term in this case is given by

$$S_{\mathrm{CS}} = \kappa \int \mathbb{A}(d\mathbb{A})^2 = \kappa \int d^{4+1}x \epsilon^{\mu\nu\rho\sigma\delta} A_\mu \partial_\nu A_\rho \partial_\sigma A_\delta. \tag{28}$$

We can calculate the Chern-Simons 3-point vertex. The vertex factor in the Feynman rules is again found by symmetrizing the Chern-Simons term in external momenta and factoring out the external polarization factors $\varepsilon_\mu(k)$,

$$\begin{aligned}
\mathrm{i}\mathcal{M} = &\mathrm{i}\kappa \epsilon^{\mu\nu\rho\sigma\delta} [\varepsilon_\mu(k_1) \mathrm{i}k_{2\nu} \varepsilon_\rho(k_2) \mathrm{i}k_{3\sigma} \varepsilon_\delta(k_3) + \varepsilon_\mu(k_1) \mathrm{i}k_{3\nu} \varepsilon_\rho(k_3) \mathrm{i}k_{2\sigma} \varepsilon_\delta(k_2) \\
&+ \varepsilon_\mu(k_2) \mathrm{i}k_{3\nu} \varepsilon_\rho(k_3) \mathrm{i}k_{1\sigma} \varepsilon_\delta(k_1) + \varepsilon_\mu(k_2) \mathrm{i}k_{1\nu} \varepsilon_\rho(k_1) \mathrm{i}k_{3\sigma} \varepsilon_\delta(k_3) \\
&+ \varepsilon_\mu(k_3) \mathrm{i}k_{1\nu} \varepsilon_\rho(k_1) \mathrm{i}k_{2\sigma} \varepsilon_\delta(k_2) + \varepsilon_\mu(k_3) \mathrm{i}k_{2\nu} \varepsilon_\rho(k_2) \mathrm{i}k_{1\sigma} \varepsilon_\delta(k_1)] \\
= &-2\mathrm{i}\kappa \epsilon^{\mu\nu\rho\sigma\delta} [k_{1\nu} k_{2\sigma} + k_{2\nu} k_{3\sigma} + k_{3\nu} k_{1\sigma}] \varepsilon_\mu(k_1) \varepsilon_\rho(k_2) \varepsilon_\delta(k_3).
\end{aligned} \tag{29}$$

Therefore, the vertex factor is

$$\begin{aligned}\Gamma^{\mu\rho\delta}_{\mathrm{CS}}(k_1,k_2,k_3) &= -2\mathrm{i}\kappa\epsilon^{\mu\nu\rho\sigma\delta}[k_{1\nu}k_{2\sigma}+k_{2\nu}k_{3\sigma}+k_{3\nu}k_{1\sigma}]\\ &= -6\mathrm{i}\kappa\epsilon^{\mu\nu\rho\sigma\delta}k_{1\nu}k_{2\sigma}.\end{aligned} \tag{30}$$

The rest of the Feynman rules are the same as those of usual QED.

B. Non-Abelian case: Chern-Simons QCD

Now let us consider a non-Abelian gauge field A_μ,

$$S_{\mathrm{CS}} = \kappa\int \mathrm{Tr}[\mathbb{A}(d\mathbb{A})^2 + c_4\mathbb{A}^3 d\mathbb{A} + c_5\mathbb{A}^5]. \tag{31}$$

As mentioned before, rational coefficients are fixed by gauge invariance.

Using S_{CS}, the 3-point vertex is given by

$$\Gamma^{(3)\mu\rho\delta}_{\mathrm{CS};abc}(k_1,k_2,k_3) = -\frac{3\mathrm{i}}{2}\kappa\epsilon^{\mu\nu\rho\sigma\delta}d_{abc}k_{1\nu}k_{2\sigma}, \tag{32}$$

where $\frac{\mathrm{i}}{2}d^{abc} = \mathrm{Tr}[T^a\{T^b,T^c\}]$. Similarly we can obtain the 4-point vertex to be

$$\begin{aligned}\Gamma^{(4)\mu\nu\rho\delta}_{\mathrm{CS};abcd}(k_1,k_2,k_3,k_4) = -c_4\kappa\epsilon^{\mu\nu\rho\sigma\delta}[&k_{1\sigma}\{f_{dce}\mathrm{Tr}(T_bT_eT_a)+f_{bde}\mathrm{Tr}(T_cT_eT_a)+f_{cbe}\mathrm{Tr}(T_dT_eT_a)\}\\ &+k_{2\sigma}\{f_{cde}\mathrm{Tr}(T_aT_eT_b)+f_{dae}\mathrm{Tr}(T_cT_eT_b)+f_{ace}\mathrm{Tr}(T_dT_eT_b)\}\\ &+k_{3\sigma}\{f_{dbe}\mathrm{Tr}(T_aT_eT_c)+f_{ade}\mathrm{Tr}(T_bT_eT_c)+f_{bae}\mathrm{Tr}(T_dT_eT_c)\}\\ &+k_{4\sigma}\{f_{bce}\mathrm{Tr}(T_aT_eT_d)+f_{cae}\mathrm{Tr}(T_bT_eT_d)+f_{abe}\mathrm{Tr}(T_cT_eT_d)\}].\end{aligned} \tag{33}$$

Using the definitions of f_{abc} and d_{abc} and simplifying using the Jacobi identity gives

$$\begin{aligned}\Gamma^{(4)\mu\nu\rho\delta}_{\mathrm{CS};abcd}(k_1,k_2,k_3,k_4) = -\frac{c_4}{4}\mathrm{i}\kappa\epsilon^{\mu\nu\rho\sigma\delta}[&(k_{1\sigma}-k_{2\sigma})f_{dce}d_{bea}+(k_{1\sigma}-k_{3\sigma})f_{bde}d_{cea}+(k_{1\sigma}-k_{4\sigma})f_{cbe}d_{dea}\\ &+(k_{2\sigma}-k_{3\sigma})f_{dae}d_{ceb}+(k_{2\sigma}-k_{4\sigma})f_{ace}d_{deb}+(k_{3\sigma}-k_{4\sigma})f_{bae}d_{dec}].\end{aligned} \tag{34}$$

We can also calculate the 5-point vertex $\Gamma^{(5)}_{\mathrm{CS}}$ but for our purpose it will suffice to just note that it will have no momentum dependence at tree level.

V. CHERN-SIMONS CORRECTION TO SINGLE SOFT PHOTON THEOREM IN QED

Now we add the Chern-Simons term to the action and see how it modifies the soft photon theorem. We start with considering a single soft photon in the external state. Then the relevant diagrams are the ones of type $A1$ and $A2$. From (30), we see that the Chern-Simons vertex modifies $\Gamma^{(3)}$ only when the external hard particle is the massless $\mathrm{U}(1)$ gauge boson. In that case we have

$$\begin{aligned}\delta\Gamma^{(3)\alpha\beta}(e,k;p,-p-k) &= e_\mu\Gamma^{\mu\alpha\beta}_{\mathrm{CS}}(k,-p-k,p)\\ &= 6\mathrm{i}\kappa\epsilon^{\mu\nu\alpha\sigma\beta}e_\mu k_\nu p_\sigma,\end{aligned} \tag{35}$$

where in this case the representation indices α,β,γ are the same as the Lorentz indices since the external line to which the soft photon attaches is also a photon line (with finite energy). For a finite energy photon field,

$$K^{\mu\nu}_{\mathrm{photon}}(p) = -p^2\eta^{\mu\nu},$$

$$\Xi^{\mathrm{photon}}_{\mu\nu}(p) = -\mathrm{i}\eta_{\mu\nu}.$$

Hence, $A1$ will receive a correction (at the subleading level) of the form

$$\begin{aligned}\delta A1 &= \varepsilon^T(p_i)\delta\Gamma^{(3)}(e,k;p_i,-p_i-k)\frac{\Xi_i^{\text{photon}}(-p_i-k)}{2p_i.k}G_n^{(i)}(p_i+k)\\ &= 3\mathrm{i}\kappa\frac{1}{p_i.k}\varepsilon_{i;\alpha}\epsilon^{\mu\nu\alpha\sigma\beta}e_\mu k_\nu p_{i\sigma}\Xi^{\text{photon}}_{\beta\gamma}(-p_i)G_n^{(i);\gamma}(p_i)\\ &= 3\kappa\epsilon^{\mu\nu\alpha\sigma\beta}\frac{e_\mu k_\nu}{p_i.k}\varepsilon_{i;\alpha}p_{i\sigma}\eta_{\beta\gamma}G_n^{(i);\gamma}(p_i).\end{aligned}$$

Since, the Chern-Simons vertex is already subleading, the Chern-Simons correction to $A2$ is higher order. Hence for single soft photon emission, the Chern-Simons modifies (21) to give

$$\begin{aligned}G_{n+1}(e,k;\varepsilon_i,p_i) &= \sum_i\frac{1}{p_i.k}\varepsilon_i^T e_\mu p_i^\mu Q_i^T G_n^{(i)}(p_i)+\sum_i\frac{1}{p_i.k}e_\mu k_\nu\varepsilon_i^T Q_i^T\left\{p_i^\mu\frac{\partial}{\partial p_{i\nu}}-p_i^\nu\frac{\partial}{\partial p_{i\mu}}\right\}G_n^{(i)}(p_i)\\ &+\sum_i\frac{1}{p_i.k}\frac{\mathrm{i}}{4}(e_\mu k_\nu-e_\nu k_\mu)\varepsilon_i^T\frac{\partial K_i(-p_i)}{\partial p_{i\nu}}\frac{\partial\Xi_i(-p_i)}{\partial p_{i\mu}}Q_i^T G_n^{(i)}(p_i)+\widetilde{\sum_j}3\kappa\epsilon^{\mu\nu\alpha\sigma\beta}\frac{e_\mu k_\nu}{p_j.k}\varepsilon_{j;\alpha}p_{j\sigma}G^{(j)}_{n;\beta}(p_j),\end{aligned}\tag{36}$$

where $\tilde{\Sigma}$ only sums over external photon legs.

VI. CHERN-SIMONS CORRECTION TO SINGLE SOFT GLUON THEOREM IN QCD

For single soft gluon emission, the story does not change much. Again $A2$ is subleading and the correction to $A1$ is given by

$$\begin{aligned}\delta\Gamma^{(3)\alpha\beta}_{ab}(e,k;p,-p-k) &= e_{c\mu}\Gamma^{\mu\alpha\beta}_{\text{CS};cab}(k,-p-k,p)\\ &= \frac{3\mathrm{i}\kappa}{2}\epsilon^{\mu\nu\alpha\sigma\beta}d_{cab}e_{c\mu}k_\nu p_\sigma,\end{aligned}\tag{37}$$

where in this case the representation indices are the pair $\{a,\alpha\}$ of color indices and Lorentz indices since the external line to which the soft gluon attaches is also a gluon line (with finite energy). Also,

$$K^{\text{gluon};\mu\nu}_{ab}(p) = -p^2\eta^{\mu\nu}\delta_{ab},\tag{38}$$

$$\Xi^{\text{gluon}}_{\mu\nu ab}(p) = -\mathrm{i}\eta_{\mu\nu}\delta_{ab}.\tag{39}$$

Hence, $A1$ will receive a correction (at the subleading level) of the form

$$\begin{aligned}\delta A1 &= \varepsilon^T(p_i)\delta\Gamma^{(3)}(e,k;p_i,-p_i-k)\frac{\Xi_i(-p_i-k)}{2p_i.k}\\ &\quad\times G_n^{(i)}(p_i+k)\\ &= \frac{3\mathrm{i}\kappa}{4}\frac{1}{p_i.k}\varepsilon_{i;a\alpha}\epsilon^{\mu\nu\alpha\sigma\beta}d_{cab}e_{c\mu}k_\nu p_{i\sigma}\Xi^{\text{gluon}}_{\beta\gamma bd}(-p_i)G^{(i);\gamma}_{n;d}(p_i)\\ &= \frac{3\kappa}{4}\epsilon^{\mu\nu\alpha\sigma\beta}d_{cab}\frac{e_{c\mu}k_\nu}{p_i.k}\varepsilon_{i;a\alpha}p_{i\sigma}\eta_{\beta\gamma}G^{(i);\gamma}_{n;b}(p_i).\end{aligned}$$

Since the Chern-Simons vertex is already subleading, the Chern-Simons correction to $A2$ is higher order. Hence for single soft gluon emission, the Chern-Simons modifies (21) to give

$$\begin{aligned}&G_{n+1}(e,k;\varepsilon_i,p_i)\\ &= \sum_i\varepsilon_i^T g_i\frac{e_a.p_i}{p_i.k}T_{i;a}G_n^{(i)}(p_i)\\ &\quad-\mathrm{i}\sum_i\varepsilon_i^T g_i\frac{e_{a\mu}k_\nu}{p_i.k}T_{i;a}(\hat{J}^{\mu\nu}_{(i)}G_n^{(i)})(p_i)\\ &\quad+\widetilde{\sum_j}\frac{3\kappa}{4}\epsilon^{\mu\nu\alpha\sigma\beta}d_{cab}\frac{e_{c\mu}k_\nu}{p_j.k}\varepsilon_{j;a\alpha}p_{j\sigma}G^{(j)}_{n;b\beta}(p_j),\end{aligned}\tag{40}$$

where $\tilde{\Sigma}$ only sums over external gluon legs. Again, we assume that the sums only contain terms that are in accordance with the color ordering of the amplitude.

VII. CHERN-SIMONS CORRECTION TO SOFT THEOREMS WITH MULTIPLE SIMULTANEOUS SOFT EMISSIONS

We now want to consider multiple soft boson emissions where the energies of the emitted soft particles are of the same order, i.e. the soft limits of amplitudes are taken simultaneously for all soft bosons. One important thing to note is that all the Chern-Simons vertices have one extra power of momentum and therefore are subleading in the small k limit to their usual gauge theory counterparts,

$$\Gamma^{(3)}_{\text{CS}}\sim k\Gamma^{(3)}_{\text{gauge}},$$

$$\Gamma^{(4)}_{\text{CS}}\sim k\Gamma^{(4)}_{\text{gauge}}.$$

This implies that there are no diagrams involving these vertices that contribute to the leading soft theorem. This can be seen by realizing that if there were any such diagrams we could have replaced the Chern-Simons vertices with their

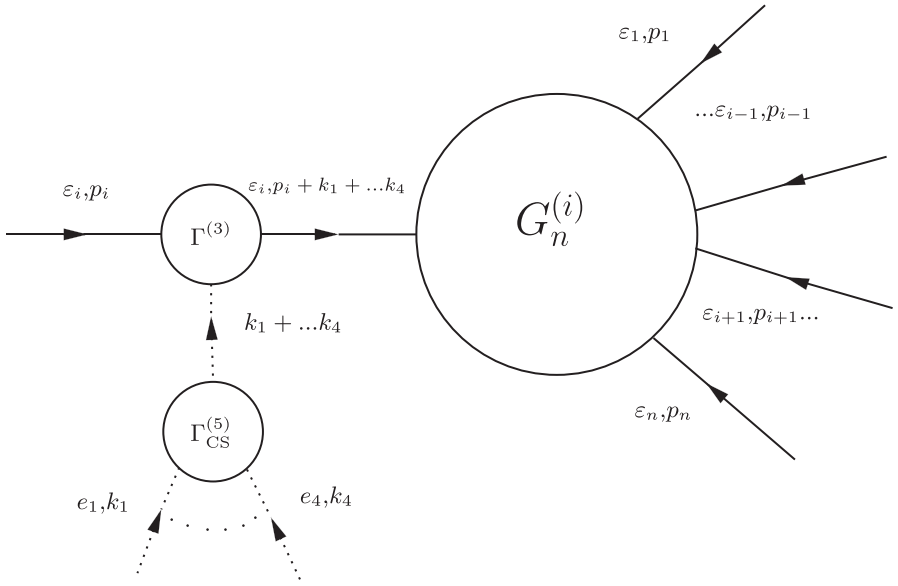


FIG. 3. Diagram containing a 5-point Chern-Simons vertex contributing to the subleading soft theorem with four soft emissions.

pure gauge counterparts and would have gotten an even more dominant contribution. This also implies that the Chern-Simons corrections to subleading soft theorems come largely from adding the Chern-Simons correction to the gauge vertices in the diagrams contributing to the leading soft theorem, as was seen in the case of a single soft emission. The only exception are diagrams that have the 5-point Chern-Simons vertex as they do not have a corresponding more dominant gauge vertex. But we can argue that these also lead to only subleading contributions. For instance, consider Fig. 3 which contributes to the Chern-Simons correction for the subleading soft theorem with four soft emissions. This is the only diagram containing the 5-point vertex which contributes at the subleading order for this process, i.e. at $\mathcal{O}(1/k^3)$. It is easy to see that there is no diagram containing the 5-point vertex which is more dominant than this since it maximizes the negative k powers in the propagator while having the 5-point Chern-Simons vertex. A similar argument holds for soft theorems with even more soft emissions. Hence, we can safely conclude that even for multiple soft emissions, the Chern-Simons corrections to the soft theorem for tree-level amplitudes are subleading to the gauge theory.

VIII. CHERN-SIMONS TERMS IN HIGHER DIMENSIONS

Finally let us comment on what happens if you add a perturbative Chern-Simons term to the gauge theory in space-time dimensions other than five. As we stated earlier, in $D = 2 + 1$ the Chern-Simons term is more dominant than the gauge term in the infrared. Therefore, we cannot treat the Chern-Simons term as a perturbation over the gauge theory in three dimensions. In $D > 4 + 1$, the 3-point Chern-Simons vertex is absent.[4] Therefore, it is expected that the Chern-Simons term will not correct the subleading soft theorem at the subleading order in $D > 4 + 1$. Therefore, one concludes that the universality of the subleading soft theorem is still maintained in $D > 4 + 1$.

IX. CONCLUSION

We have shown in (36) and (40) that the Chern-Simons corrections to the soft photon and soft gluon theorems are subleading in powers of soft momenta k. The leading soft theorems remain unchanged even in the presence of a Chern-Simons term. This is interesting for two reasons:

Firstly, the subleading term gets modified when Chern-Simons terms are added and therefore, it is no longer universal even though the amplitude is still gauge invariant under small gauge transformations. Also, such terms cannot be obtained from Sen's covariantization procedure and need to be added separately.

Secondly, we can in principle take arbitrary values of $\{c_4, c_5\}$ which would leave the deformation S_{CS} completely nongauge invariant. Even under such nongauge-invariant deformations, the amplitudes still obey the "universal" leading soft theorem, at least at the tree level. Therefore, we have an example of a nongauge-invariant action that still satisfies the usual universal leading soft theorem at the tree level. This means that gauge invariance of amplitudes is a sufficient but not necessary condition for the universal form of the tree-level leading soft theorem.

The analysis in this paper is limited to tree-level amplitudes. Calculations at higher loop orders may modify the

[4]For instance, in $D = 6 + 1$ the lowest vertex is the 4-point vertex which goes as $\sim \int \mathbb{A} d\mathbb{A}^3$.

second statement since the loss of gauge invariance should imply fewer constraints on the amplitudes. Therefore, one would expect that to show up in the IR behaviour of the amplitudes at higher loop orders.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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