

Time evolution of parity-odd cascades in homogeneous Abelian and non-Abelian media with chiral imbalance

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We study radiation by a fast particle in a medium with a finite chiral chemical potential. The medium's anomalous response encompasses the chiral magnetic effect, enabling novel chiral Cherenkov and pair production processes. The kinematics of the corresponding cascades is fundamentally different from conventional cascades. Notably, it is not dominated by the strong ordering of momenta. Employing the effective Chern-Simons extensions of QED and QCD to incorporate the chiral magnetic effect, we derive and solve the evolution equations describing the cascades in a chiral medium, presuming that the anomalous contributions are dominant.

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I. INTRODUCTION

Interaction of a fast particle with the medium induces a cascade. Such cascades are studied in many areas of particle and nuclear physics [1–11]. This paper studies cascades generated in a medium with chiral fermions. The chiral media are characterized by the anomalous response to the electromagnetic field which gives rise to the chiral magnetic [12–16] and anomalous Hall effects [15,17–20]. The anomalous response lifts the chiral degeneracy of the gauge boson dispersion relation, which acquires dependence on the gauge boson polarization. In particular, it exhibits two distinct branches: one is spacelike and the other is timelike as seen in Fig. 1. As a result the $1 \rightarrow 2$ and $2 \rightarrow 1$ processes, prohibited in the free space by the energy-momentum conservation, become allowed in the chiral medium.

The dynamics of the gauge theories at finite chiral chemical potential μ_5 can be conveniently described by adding the Chern-Simons term $-\theta F\tilde{F}/4$ to the Lagrangian [22,23] and requiring that the time derivative of the pseudoscalar field be constant: $\partial_0\theta = \mu_5 = b_0/c_A$, where c_A is the anomaly coefficient and b_0 is dubbed the chiral magnetic conductivity [16,24], and that its spacial gradients vanish: $\nabla\theta = 0$. The latter equation reflects the assumed uniformity of the chiral medium. We will refer to such an extension of QED and QCD as χ QED and χ QCD respectively. Unlike the original theories, the χ extensions are

parity-odd. As a result, the rates of processes involving the gauge bosons depend on their polarizations. The Chern-Simons term causes emergence of the effective photon mass as seen in Fig. 1. In particular, at photon momenta much larger than the plasma frequency the dispersion relation takes the form

$$\omega^2 = k^2 - \lambda b_0 |k|, \quad (1)$$

where $k = (\omega, \mathbf{k})$ and $\lambda = \pm 1$ indicate the right- or left-hand polarization respectively.

In the dilute regime, only $1 \rightarrow 2$ decays play essential role in the cascade. The decay $e \rightarrow e + \gamma$ in χ QED is known as the chiral Cherenkov radiation [25–28]¹ because like the conventional Cherenkov radiation it has infinite coherence length. Only one—right- or left-handed—circular photon polarization can be produced by the chiral Cherenkov radiation, namely, the polarization corresponding to the spacelike dispersion relation. On the other hand, the pair-production decay $\gamma \rightarrow e + \bar{e}$ proceeds only if the photon is timelike which corresponds to the opposite polarization. As a result, an electromagnetic cascade promptly develops 100% chirality. We will refer to all decays induced by the chiral anomaly collectively as *the chiral Cherenkov decays* and to the corresponding cascade as *the chiral Cherenkov cascade*.

The dynamics of the chiral Cherenkov cascade in χ QCD is more complicated because along with the quasi-Abelian processes $q \rightarrow q + g$ and $g \rightarrow q + \bar{q}$ it allows the genuine non-Abelian decay $g \rightarrow g + g$, which necessarily involves gluons of different polarizations [32].

¹It is closely related to the vacuum Cherenkov radiation [29–31].

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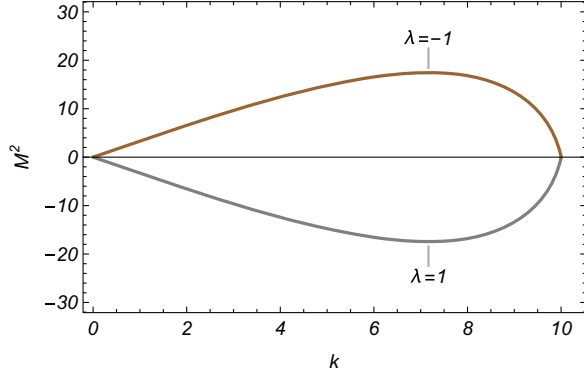


FIG. 1. The squared effective photon mass $M^2 = \omega^2 - k^2$ as a function of photon momentum $|k|$ in QED plasma at finite chiral chemical potential μ_5 for different polarizations λ . We used the dispersion relation derived in [21] with $\omega = \mu_5 = 100m_D$. Arbitrary units.

The mathematical description of the conventional cascades in QED and QCD significantly simplifies at high energies because the individual decays factor out thanks to the strongly ordered transverse momenta. This makes the decays quasiclassical, and thereby statistically independent at the leading logarithmic approximation. Unlike the conventional decays, the chiral Cherenkov decays are not strongly ordered in the transverse momentum. Rather the transverse momenta of the radiated particles are fixed by the kinematic constraints. Nevertheless, precisely this fixing of the transverse momenta makes the factorization and the associated quasiclassical interpretation possible for the chiral Cherenkov cascade as we argue in Sec. III.

Once the factorization property is established, the dynamics of the chiral Cherenkov cascade can be described by the kinetic equation. Consider the chiral Cherenkov cascade in χ QED. Introduce the electron, positron, right-handed photon and left-handed photon distributions $e(x, t)$, $\bar{e}(x, t)$, $\gamma^+(x, t)$, and $\gamma^-(x, t)$ respectively, where x denotes the longitudinal momentum fraction carried by the decay product and t is time. In realistic systems the chiral Cherenkov processes compete with the conventional ones. However, in this paper, we will ignore the conventional processes in order to obtain the most clear physical picture of the chiral Cherenkov cascade. Throughout the paper we assume for definitiveness that $b_0 > 0$; as a result the decay $e \rightarrow e + \gamma^+$ produces the right-handed photons, whereas the pair-production $\gamma^- \rightarrow e + \bar{e}$ destroys the left-handed ones. We note that the distributions e and $\bar{e}$ obey the same kinetic equation, although possibly with different initial conditions, given that the kinematic constraints due to the anomaly are unaffected by the sign of electron's electric charge. Therefore, the general structure of the kinetic equations for the chiral Cherenkov cascade in χ QED is

$$\begin{aligned} e(y, t + \Delta t) = & (1 - W_{\text{tot}}^{e \rightarrow e\gamma^+} \Delta t) e(y, t) \\ & + \Delta t \int_0^1 dz \int_0^1 dx \frac{dW^{e \rightarrow e\gamma^+}}{dx} e(z, t) \delta(y - xz) \\ & + \Delta t \int_0^1 dz \int_0^1 dx \frac{dW^{\gamma^- \rightarrow e\bar{e}}}{dx} \gamma^-(z, t) \delta(y - xz), \end{aligned} \quad (2a)$$

$$\begin{aligned} \bar{e}(y, t + \Delta t) = & (1 - W_{\text{tot}}^{\bar{e} \rightarrow \bar{e}\gamma^+} \Delta t) \bar{e}(y, t) \\ & + \Delta t \int_0^1 dz \int_0^1 dx \frac{dW^{\bar{e} \rightarrow \bar{e}\gamma^+}}{dx} \bar{e}(z, t) \delta(y - xz) \\ & + \Delta t \int_0^1 dz \int_0^1 dx \frac{dW^{\gamma^- \rightarrow e\bar{e}}}{dx} \gamma^-(z, t) \delta(y - xz), \end{aligned} \quad (2b)$$

$$\begin{aligned} \gamma^+(y, t + \Delta t) = & \gamma^+(y, t) + \Delta t \int_0^1 dz \int_0^1 dx \frac{dW^{e \rightarrow \gamma^+ e}}{dx} \\ & \times [e(z, t) + \bar{e}(z, t)] \delta(y - xz), \end{aligned} \quad (2c)$$

$$\gamma^-(y, t + \Delta t) = (1 - W_{\text{tot}}^{\gamma^- \rightarrow e\bar{e}} \Delta t) \gamma^-(y, t), \quad (2d)$$

where $dW^{a \rightarrow bc}/dx$ is the inclusive rate of the indicated process and $W_{\text{tot}}^{a \rightarrow bc}$ is the corresponding total rate. Unlike the inclusive rate, the total one includes the virtual contributions which are essential to maintain the charge and energy conservation and the gauge invariance. The terms in the right-hand side of (2a) have the following meaning: the expression in the parentheses in the first term is the probability that no photon is radiated during the time interval Δt , the second term describes the probability that the parent electron emits photons and ends up with the energy fraction y , and the third term is the pair production. Since the right-handed photon cannot decay, there is no term proportionate to γ^+ in the right-hand side of (2c). In contrast, in (2d) there are no gain terms because the left-handed photons can only decay, but not being produced. The differential equations are obtained by taking the limit $\Delta t \rightarrow 0$. The kinetic equations in χ QCD can be obtained in a similar way. The derivation and solution of these equations is the main subject of this paper.

The paper is organized as follows. Sections II–IV deal with the chiral Cherenkov cascade in QED and in the remaining sections—in QCD. In Sec. II we quote the inclusive single particle rates for χ QED followed in Sec. III by the analysis of the double inclusive production where we prove the factorization of the cascade into the product of the single inclusive processes. In Sec. IV we derive the kinetic equations for χ QED—assuming that the chiral Cherenkov decays are dominant—and solve them numerically for special choices of the initial conditions. In Sec. V we derive and solve the kinetic equations of χ QCD. The

summary, conclusions and outlook are presented in Sec. VI where we also discuss how the chiral Cherenkov decays can be included into the existing models of cascades and jets.

II. SINGLE PARTON EMISSION IN χ QED

One of the two basic processes of the chiral Cherenkov cascade in χ QED is the photon production $e \rightarrow e\gamma$. Assume for simplicity that the plasma frequency is negligible $\omega_p = 0$ and denote $p = (E, \mathbf{p})$, $p' = (E', \mathbf{p}')$, $k = (\omega, \mathbf{k})$ the momenta of the initial and final fermion and photon respectively. The rate of the right-handed photon production is [25]

$$\frac{dW^{e \rightarrow \gamma^+ e}}{dx} = \frac{\alpha b_0}{4} \left(\frac{(1-x)^2 + 1}{x} - \frac{2m^2}{b_0 E} \right) H(x \leq x_0), \quad (3)$$

where $x = \omega/E$ is fraction of the incident energy carried away by the photon,

$$x_0 = \frac{1}{1 + \frac{m^2}{b_0 E}}, \quad (4)$$

and the step function $H(M)$ equals unity if the condition M is satisfied and vanishes otherwise. Let now x denote the energy fraction carried away by the final electron, i.e. $x = E'/E$. Then the electron production rate reads

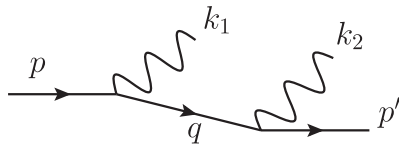
$$\frac{dW^{e \rightarrow e\gamma^+}}{dx} = \frac{\alpha b_0}{4} \left(\frac{x^2 + 1}{1-x} - \frac{2m^2}{b_0 E} \right) H(x \geq 1 - x_0). \quad (5)$$

Another basic process is the pair production $\gamma^- \rightarrow e\bar{e}$. Let k , p , and p' now refer to the momenta of the initial photon and final electron and positron respectively and $x = E/\omega$ be the fraction of the incident photon energy carried away by electron. The electron production rate is given by [25]

$$\frac{dW^{\gamma^- \rightarrow e\bar{e}}}{dx} = \frac{\alpha b_0}{4} \left(x^2 + (1-x)^2 + \frac{2m^2}{\omega b_0} \right) H(x_1 \leq x \leq x_2), \quad (6)$$

where

$$x_{1,2} = \frac{1}{2} \left(1 \mp \sqrt{1 - \frac{4m^2}{b_0 \omega}} \right). \quad (7)$$



It is convenient to introduce the following parameters: the characteristic timescale τ such that

$$\frac{1}{\tau} = \frac{\alpha b_0}{4}, \quad (8)$$

and the dimensionless mass parameter

$$\nu = \frac{m^2}{b_0 E_0}, \quad (9)$$

where E_0 is the total cascade energy.

III. DOUBLE PARTICLE EMISSION IN χ QED

We are now going to argue that the multiparticle production rate can be represented as a convolution of the single particle production rates. We first consider the double photon emission, represented by the diagrams in Fig. 2, as a specific example.

The general expression for the single photon emission rate reads

$$W^{e \rightarrow e\gamma} = \frac{1}{2E} \frac{1}{2} \sum_{\text{spins}} \sum_{\lambda} \int \frac{d^3 p'}{2E'(2\pi)^3} \times \int \frac{d^3 k}{2\omega(2\pi)^3} (2\pi)^4 \delta(p - k - p') |e \bar{u}' \not{\epsilon}^* u|^2. \quad (10)$$

We use the shorthand notation $u \equiv u(\mathbf{p}, s)$, $u' \equiv u(\mathbf{p}', s')$, $\epsilon \equiv \epsilon(\mathbf{k}, \lambda)$, with the bold letters referring to the spacial components of the corresponding four-vectors, s and s' label fermion spin states, and λ photon polarizations.

The rate for the two photon emissions by an electron reads

$$W^{e \rightarrow e\gamma\gamma} = \frac{1}{8E} \sum_{\text{spins}} \sum_{\lambda_1, \lambda_2} \int \frac{d^3 p'}{2E'(2\pi)^3} \int \frac{d^3 k_1}{2\omega_1(2\pi)^3} \times \int \frac{d^3 k_2}{2\omega_2(2\pi)^3} (2\pi)^4 \delta(p - k_1 - k_2 - p') \times \left| \frac{e^2 \bar{u}' \not{\epsilon}_2^* (\not{p} - \not{k}_1 + m) \not{\epsilon}_1^* u}{(p - k_1)^2 - m^2 + 2iE_q/\tau} + \frac{e^2 \bar{u}' \not{\epsilon}_1^* (\not{p} - \not{k}_2 + m) \not{\epsilon}_2^* u}{(p - k_2)^2 - m^2 + 2iE_q/\tau} \right|^2, \quad (11)$$

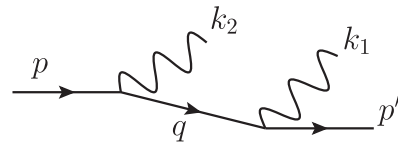


FIG. 2. The diagrams for double photon production.

where $E_q = E - \omega_1$, τ is the relaxation time controlling the resonance width [33,34] and the symmetry factor is $1/2$. In the ultrarelativistic approximation, the denominator of the propagator in the first term can be written as [34]

$$\frac{1}{-2p \cdot k_1 + k_1^2 + 2iE_q/\tau} = \frac{1}{\omega_1 E \left(\frac{m^2}{E^2 \omega_1} \frac{\omega_1 - x_0 E}{1 - x_0} + \vartheta^2 \right) + 2iE_q/\tau}. \quad (12)$$

Apparently, when $\omega_1 < x_0 E$ the propagator has a resonance at a finite photon emission angle ϑ . The second term in (11) has a similar resonance. Since the resonance contribution dominates the rate we can approximate

$$\left| \frac{1}{(p - k_1)^2 - m^2 + 2iE_q/\tau} \right|^2 \approx \frac{\pi\tau}{2E_q} \delta((p - k_1)^2 - m^2). \quad (13)$$

In other words, the intermediate fermion is effectively on the mass shell. A more detailed justification of the approximation (13) is presented in Appendix A. We argue there that the relaxation time must satisfy $\tau \gg 1/(2\pi b_0)$. Equation (13) allows us to write the rate of the double photon production as the product of two single photon rates. Writing

$$\delta(p - k_1 - k_2 - p') = \int d^4 q \delta(q - p + k_2) \delta(q - p' - k_2)$$

and $\not{p} - \not{k}_1 + m = \sum_s u(\mathbf{q}, s) \bar{u}(\mathbf{q}, s)$ we obtain employing (10)

$$W^{e \rightarrow e\gamma\gamma} = \tau \int_0^E d\omega_1 \frac{dW^{e(p) \rightarrow e(q)\gamma(k_1)}}{d\omega_1} \times \int_0^{E_q} d\omega_2 \frac{dW^{e(q) \rightarrow e(p')\gamma(k_2)}}{d\omega_2}. \quad (14)$$

The other branching process—the pair production—factors out in the same way.

This derivation can be readily generalized to an arbitrary multiparticle process. The systematic way to describe such a cascade is to employ the evolution equations which we develop in the following sections.

IV. CHIRAL CHERENKOV CASCADE IN χ QED

The distribution functions of electrons, positrons, and photons are $e(x)$, $\bar{e}(x)$, and $\gamma^\pm(x)$ respectively. They indicate the probability to find a particle with the energy fraction x of the total cascade energy and with any transverse momentum. We assume that the particle densities are small so that the linear approximation applies. We also neglect all conventional contributions to the cascades.

A. Evolution equations in chiral limit $m=0$

According to (3) and (6), in the chiral limit, the rates of various production processes are

$$\frac{dW^{e \rightarrow \gamma^+ e}}{dx} = \frac{1}{\tau} \frac{1 + (1-x)^2}{x}, \quad (15a)$$

$$\frac{dW^{e \rightarrow e \gamma^+}}{dx} = \frac{1}{\tau} \frac{1 + x^2}{1-x}, \quad (15b)$$

$$\frac{dW^{\gamma^- \rightarrow e \bar{e}}}{dx} = \frac{1}{\tau} [x^2 + (1-x)^2], \quad (15c)$$

where x is the fraction of energy carried by the outgoing particle listed first. The corresponding splitting functions including the virtual terms are

$$P_{\gamma^+ e}(x) = \frac{1 + (1-x)^2}{x}, \quad (16a)$$

$$P_{e\gamma^-}(x) = x^2 + (1-x)^2, \quad (16b)$$

$$P_{ee}(x) = \frac{1 + x^2}{(1-x)_+} + \frac{3}{2} \delta(1-x), \quad (16c)$$

$$P_{\gamma^- \gamma^-} = -\frac{2}{3} \delta(1-x). \quad (16d)$$

The distribution $1/(1-x)_+$ in (16c) is defined as usual:

$$\int_x^1 dz \frac{f(z)}{(1-z)_+} = \int_x^1 dz \frac{[f(z) - f(1)]}{1-z} + f(1) \ln(1-x). \quad (17)$$

The evolution equations (2) now read

$$\frac{de(y, t)}{dt} = \frac{1}{\tau} \int_y^1 \frac{dx}{x} \left\{ P_{ee}(x) e\left(\frac{y}{x}, t\right) + P_{e\gamma^-}(x) \gamma^-\left(\frac{y}{x}, t\right) \right\}, \quad (18a)$$

$$\frac{d\bar{e}(y, t)}{dt} = \frac{1}{\tau} \int_y^1 \frac{dx}{x} \left\{ P_{ee}(x) \bar{e}\left(\frac{y}{x}, t\right) + P_{e\gamma^-}(x) \gamma^-\left(\frac{y}{x}, t\right) \right\}, \quad (18b)$$

$$\frac{d\gamma^+(y, t)}{dt} = \frac{1}{\tau} \int_y^1 \frac{dx}{x} \left\{ P_{\gamma^+ e}(x) e\left(\frac{y}{x}, t\right) + P_{\gamma^+ e}(x) \bar{e}\left(\frac{y}{x}, t\right) \right\}, \quad (18c)$$

$$\frac{d\gamma^-(y, t)}{dt} = \frac{1}{\tau} \int_y^1 \frac{dx}{x} P_{\gamma^- \gamma^-}(x) \gamma^-\left(\frac{y}{x}, t\right), \quad (18d)$$

where τ is given by (8). These equations are similar to the evolution equations of QED. The main differences are (i) Eqs. (18) explicitly depend on the photon polarization,

i.e. γ^+ and γ^- evolve differently, and (ii) t is a linear rather than a logarithmic scale.

One can write these equations in terms of the moments (Mellin transform)

$$f_n(t) = \int_0^1 x^{n-1} f(x, t) dx, \quad (19)$$

as follows:

$$\frac{de_n(t)}{dt} = \frac{1}{\tau} \{A_n^{ee} e_n(t) + A_n^{e\gamma^-} \gamma_n^-(t)\}, \quad (20a)$$

$$\frac{d\bar{e}_n(t)}{dt} = \frac{1}{\tau} \{A_n^{ee} \bar{e}_n(t) + A_n^{e\gamma^-} \gamma_n^-(t)\}, \quad (20b)$$

$$\frac{d\gamma_n^+(t)}{dt} = \frac{1}{\tau} \{A_n^{\gamma^+e} e_n(t) + A_n^{\gamma^+e} \bar{e}_n(t)\}, \quad (20c)$$

$$\frac{d\gamma_n^-(t)}{dt} = \frac{1}{\tau} A_n^{\gamma^- \gamma^-} \gamma_n^-(t), \quad (20d)$$

where

$$A_n^{ab} = \int_0^1 x^{n-1} P_{ab}(x) dx. \quad (21)$$

In this form it is easy to check conservation of the electric charge and energy of the cascade. The electric charge in units of the electron charge is given by

$$\int_0^1 \{e(x, t) - \bar{e}(x, t)\} dx = e_1(t) - \bar{e}_1(t). \quad (22)$$

Using (20a) and (20b) we obtain

$$\frac{dQ}{dt} = \frac{1}{\tau} A_1^{ee} \{e_1(t) - \bar{e}_1(t)\} = 0 \quad (23)$$

because

$$A_1^{ee} = \int_0^1 P_{ee}(x) dx = 0. \quad (24)$$

Similarly the total energy of the cascade is

$$\begin{aligned} \int_0^1 x \{e(x, t) + \bar{e}(x, t) + \gamma^+(x, t) + \gamma^-(x, t)\} dx \\ = e_2(t) + \bar{e}_2(t) + \gamma_2^+(t) + \gamma_2^-(t). \end{aligned} \quad (25)$$

That the time derivative of the total energy vanishes is derived by substituting (20a)–(20d) and noting that

$$A_2^{ee} + A_2^{\gamma^+e} = \int_0^1 x (P^{ee}(x) + P^{\gamma^+e}(x)) dx = 0, \quad (26a)$$

$$2A_2^{e\gamma^-} + A_2^{\gamma^- \gamma^-} = \int_0^1 x (2P^{e\gamma^-}(x) + P^{\gamma^- \gamma^-}(x)) dx = 0. \quad (26b)$$

Let the initial conditions at $t = 0$ be

$$e(x, 0) = \delta(1-x), \quad \bar{e}(x, 0) = \gamma^+(x, 0) = \gamma^-(x, 0) = 0. \quad (27)$$

The general solution to (18d) is

$$\gamma^-(x, t) = \gamma^-(x, 0) e^{-2t/3\tau}. \quad (28)$$

In view of the initial conditions (27), the distribution of the negatively polarized photons vanishes $\gamma^-(x, t) = 0$. It follows from (20b) that $\bar{e}(x, t) = 0$. The solution to (20a) is

$$e_n(t) = \exp \frac{A_n^{ee} t}{\tau}, \quad (29)$$

where

$$A_n^{ee} = \frac{3}{2} - \frac{2n+1}{n(n+1)}. \quad (30)$$

To obtain the electron distribution one performs the inverse Mellin transform:

$$e(x, t) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} e_n(t) x^{-n} dn, \quad (31)$$

where the contour runs parallel to the imaginary axis to the right of the singularities of the integrand. Once $e(x, t)$ is known, it is easy to obtain $\gamma^+(x, t)$ from (18c). Unfortunately, the integral (31) cannot be done analytically.

The evolution equations can be solved numerically. We use the procedure developed in [35]. We first consider the initial conditions similar to (27) except that we replace the delta function by a smoother distribution:

$$e(x, 0) = 32 \left(x - \frac{3}{4} \right) H \left(x \geq \frac{3}{4} \right), \quad (32)$$

normalized such that $\int_0^1 e(x, 0) dx = 1$. The results are shown in Fig. 3. One can see rapid increase with time of the soft ($x \ll 1$) right-handed photons and the concurrent shift of the electron distribution towards smaller values of x . As expected, the cascade is 100% polarized.

Figure 4 exhibits evolution of the cascade initiated by the left-handed photons with a different initial distribution

$$\gamma^-(x, 0) = 32 \left(x - \frac{3}{4} \right) H \left(x \geq \frac{3}{4} \right), \quad (33a)$$

$$\gamma^+(x, 0) = e(x, 0) = \bar{e}(x, 0) = 0. \quad (33b)$$

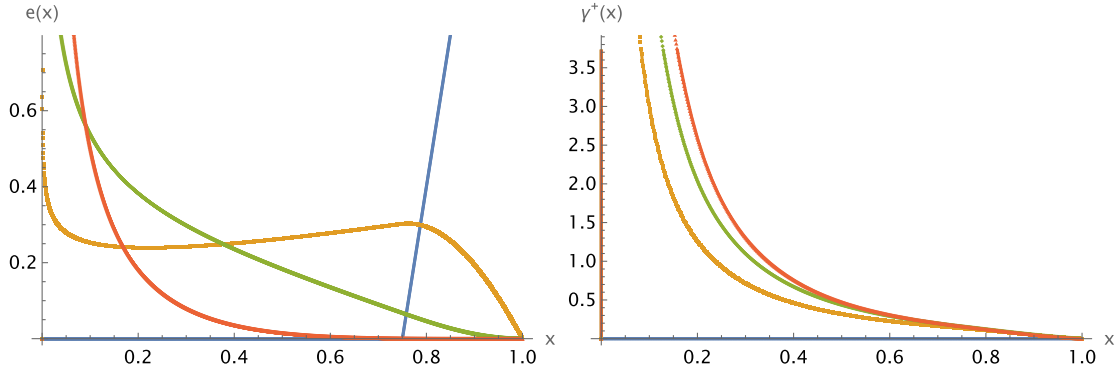


FIG. 3. Chiral Cherenkov cascade in χ QED with the initial conditions: $e(x, 0)$ given by (32) and $\bar{e}(x, 0) = \gamma^\pm(x, 0) = 0$. Blue line: $t = 0$; yellow line: $t = \tau/2$; green line: $t = \tau$; red line: $t = 2\tau$.

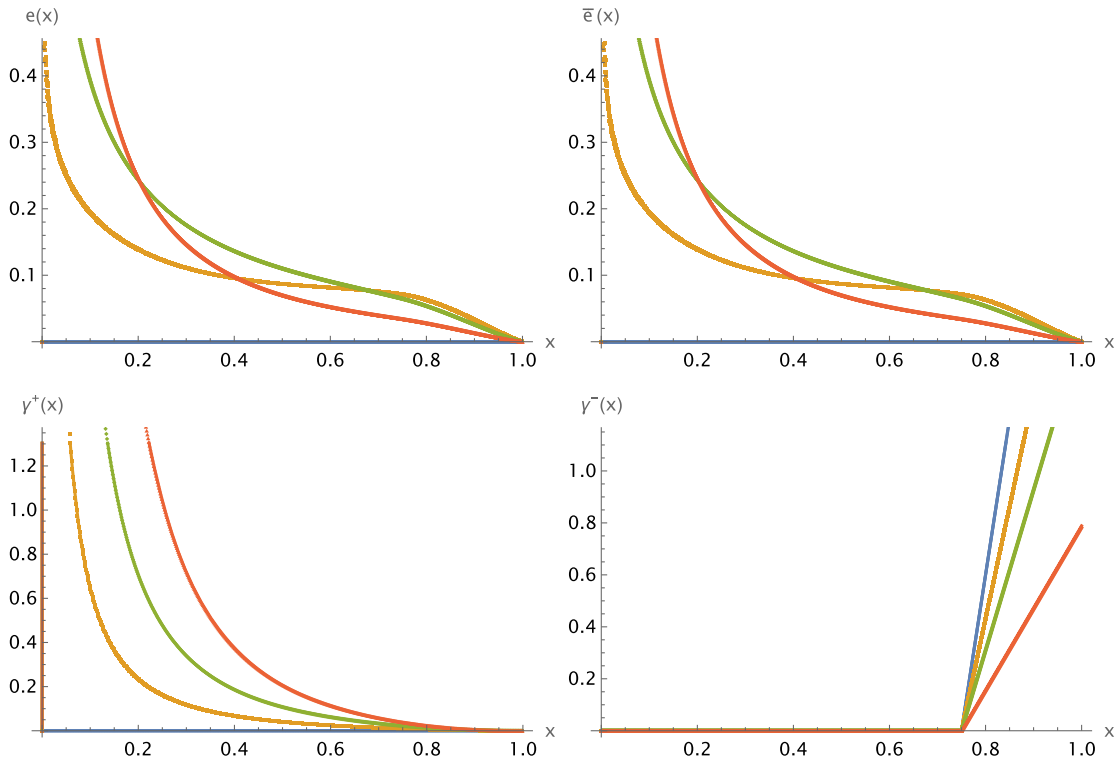


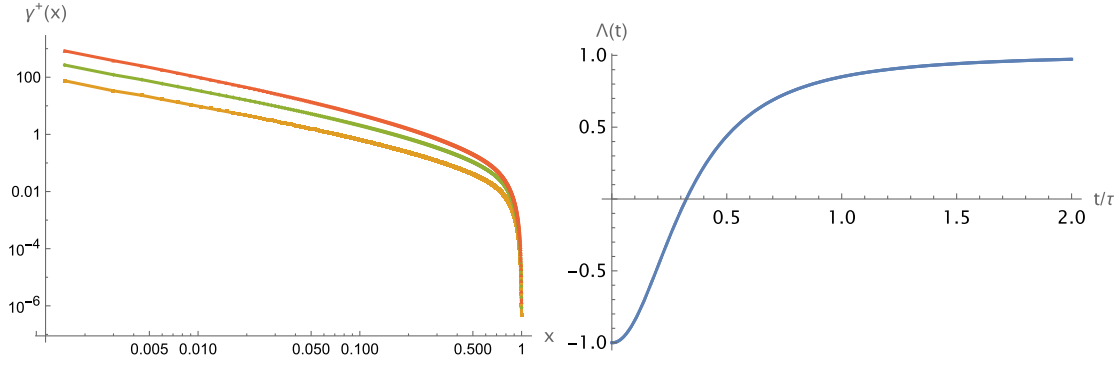
FIG. 4. Chiral Cherenkov cascade in χ QED with the initial conditions (33). Blue line: $t = 0$; yellow line: $t = \tau/2$; green line: $t = \tau$; red line: $t = 2\tau$.

It corresponds to the left-handed photon beam. The left-handed photons produce the electron-positron pairs which in turn radiate the right-handed photons. As a result the number of the right-handed photons increases, while the number of the left-handed ones decreases. The rapid increase of the number of the right-handed photons is especially clearly seen in the left panel of Fig. 5.

The chirality distribution of the cascade is proportional to $\gamma^+(x, t) - \gamma^-(x, t)$. To characterize the cascade polarization, define the *degree of the cascade polarization*:

$$\Lambda(t) = \frac{\int_0^1 (\gamma^+(x, t) - \gamma^-(x, t)) dx}{\int_0^1 (\gamma^+(x, t) + \gamma^-(x, t)) dx}. \quad (34)$$

For the cascade with the initial condition (33) it is displayed in the right panel of Fig. 5 which indicates gradual polarization of the cascade. The cascade starts as completely left-handed and by the time $t \sim \tau$ it is nearly entirely right-handed.

FIG. 5. Left: log plot for γ^+ . Right: The degree of polarization $\Lambda(t)$ of the cascade shown in Fig. 4.

B. Evolution equations at finite m

At finite electron mass m the rates (3) and (6) read

$$\frac{dW^{e \rightarrow \gamma^+ e}}{dx} = \frac{1}{\tau} \Pi_{\gamma^+ e} \left(x, \frac{y}{x} \right), \quad (35a)$$

$$\frac{dW^{e \rightarrow e \gamma^+}}{dx} = \frac{1}{\tau} \Pi_{ee} \left(x, \frac{y}{x} \right), \quad (35b)$$

$$\frac{dW^{\gamma^- \rightarrow e \bar{e}}}{dx} = \frac{1}{\tau} \Pi_{e \gamma^-} \left(x, \frac{y}{x} \right). \quad (35c)$$

We defined the following generalizations of the splitting functions:

$$\Pi_{\gamma^+ e}(x, z) = \left[P_{\gamma^+ e}(x) - \frac{2\nu}{z} \right] H(x \leq x_0(z)), \quad (36a)$$

$$\begin{aligned} \Pi_{ee}(x, z) &= \left[\frac{1+x^2}{(1-x)_+} + C_e(z) \delta(1-x) - \frac{2\nu}{z} \right] \\ &\times H(x \geq 1 - x_0(z)), \end{aligned} \quad (36b)$$

$$\Pi_{e \gamma^-}(x, z) = \left[P_{e \gamma^-}(x) + \frac{2\nu}{z} \right] H(x_1(z) \leq x \leq x_2(z)), \quad (36c)$$

$$\Pi_{\gamma^- \gamma^-}(x, z) = C_\gamma(z) \delta(1-x), \quad (36d)$$

where ν is given by (9), $C_e(z)$, $C_\gamma(z)$ are given by (45), the step function $H(M)$ equals unity if the condition M is satisfied and vanishes otherwise, and

$$x_0(z) = \frac{1}{1 + \frac{z}{2}}, \quad (37)$$

$$x_{1,2}(z) = \frac{1}{2} \left(1 \mp \sqrt{1 - \frac{4\nu}{z}} \right). \quad (38)$$

The evolution equations read

$$\begin{aligned} \frac{de(y, t)}{dt} &= \frac{1}{\tau} \int_y^1 \frac{dx}{x} \left\{ \Pi_{ee} \left(x, \frac{y}{x} \right) e \left(\frac{y}{x}, t \right) \right. \\ &\quad \left. + \Pi_{e \gamma^-} \left(x, \frac{y}{x} \right) \gamma^- \left(\frac{y}{x}, t \right) \right\}, \end{aligned} \quad (39a)$$

$$\begin{aligned} \frac{d\bar{e}(y, t)}{dt} &= \frac{1}{\tau} \int_y^1 \frac{dx}{x} \left\{ \Pi_{ee} \left(x, \frac{y}{x} \right) \bar{e} \left(\frac{y}{x}, t \right) \right. \\ &\quad \left. + \Pi_{e \gamma^-} \left(x, \frac{y}{x} \right) \gamma^- \left(\frac{y}{x}, t \right) \right\}, \end{aligned} \quad (39b)$$

$$\begin{aligned} \frac{d\gamma^+(y, t)}{dt} &= \frac{1}{\tau} \int_y^1 \frac{dx}{x} \left\{ \Pi_{\gamma^+ e} \left(x, \frac{y}{x} \right) e \left(\frac{y}{x}, t \right) \right. \\ &\quad \left. + \Pi_{\gamma^+ e} \left(x, \frac{y}{x} \right) \bar{e} \left(\frac{y}{x}, t \right) \right\}, \end{aligned} \quad (39c)$$

$$\frac{d\gamma^-(y, t)}{dt} = \frac{1}{\tau} \int_y^1 \frac{dx}{x} \Pi_{\gamma^- \gamma^-} \left(x, \frac{y}{x} \right) \gamma^- \left(\frac{y}{x}, t \right). \quad (39d)$$

The functions Π_{ab} reduce to the splitting functions P_{ab} in the chiral limit $\nu \rightarrow 0$.

The coefficients C_e , C_γ are determined from the charge and energy conservation. To fix them we proceed along the same lines as in the chiral limit. Multiplying the difference of (39a) and (39b) by y^{n-1} and integrating over $0 \leq y \leq 1$ we obtain

$$\begin{aligned} &\frac{d}{dt} \int_0^1 dy y^{n-1} \{ e(y, t) - \bar{e}(y, t) \} \\ &= \int_0^1 dy y^{n-1} \int_y^1 \frac{dx}{x} \Pi_{ee} \left(x, \frac{y}{x} \right) \left\{ e \left(\frac{y}{x}, t \right) - \bar{e} \left(\frac{y}{x}, t \right) \right\}. \end{aligned} \quad (40)$$

Changing the order of integrations $\int_0^1 dy \int_y^1 dx = \int_0^1 dx \int_0^x dy$ and changing integration variables to x and $z = y/x$ we get

$$\begin{aligned} \frac{d}{dt} \int_0^1 dy y^{n-1} \{e(y, t) - \bar{e}(y, t)\} \\ = \int_0^1 dz z^{n-1} \mathcal{A}_n^{ee}(z) \{e(z, t) - \bar{e}(z, t)\}, \end{aligned} \quad (41)$$

where

$$\mathcal{A}_n^{ee}(z) = \int_0^1 dx x^{n-1} \Pi_{ee}(x, z). \quad (42)$$

When $\nu \rightarrow 0$, $\mathcal{A}_n^{ee}(z) \rightarrow A_n^{ee}$. However, at finite ν , $\mathcal{A}_n^{ee}(z)$ explicitly depends on z thus breaking the factorization in the Mellin space. In particular, the evolution equations for the moments do not reduce to the set of ordinary differential equations as was in the chiral case (20). Nevertheless, Eq. (41) with $n = 1$ indicates that $\dot{Q} = 0$ if

$$\mathcal{A}_1^{ee}(z) = 0 \quad (43)$$

for any z . The converse statement is not true, viz. (43) does not necessarily follow from the charge conservation, since the expression in the curly brackets in (41) is not positive-definite. Similarly, one can prove that the total energy is conserved if and only if

$$\mathcal{A}_2^{ee}(z) + \mathcal{A}_2^{\gamma^+ e}(z) = \int_0^1 x (\Pi^{ee}(x, z) + \Pi^{\gamma^+ e}(x, z)) dx = 0, \quad (44a)$$

$$2\mathcal{A}_2^{e\gamma^-}(z) + \mathcal{A}_2^{\gamma^- \gamma^-}(z) = \int_0^1 x (2\Pi^{e\gamma^-}(x, z) + \Pi^{\gamma^- \gamma^-}(x, z)) dx = 0 \quad (44b)$$

for any z . It follows from (43), (44a), and (44b) after a simple calculation using (17) that

$$C_e(z) = 2 - \frac{z^2}{2(z+\nu)^2} + 2 \ln \left(1 + \frac{\nu}{z} \right), \quad (45a)$$

$$C_\gamma(z) = -\frac{1}{3} \sqrt{1 - \frac{4\nu}{z}} \left(2 + \frac{4\nu}{z} \right). \quad (45b)$$

In the chiral limit $\nu = 0$: $C_e = 3/2$, $C_\gamma = -2/3$.

Equations (39) with (36) and (45) constitute the evolution equations of the chiral cascade in QED at any m .

V. CHIRAL CHERENKOV CASCADE IN χ QCD

We do not need to repeat all the arguments that lead us to the evolution equations in χ QED. Rather, we only need to list the appropriate splitting functions in the chiral limit. The corresponding amplitudes were computed in [32]. It should be noted that in addition to the conventional QCD vertices, the non-Abelian Chern-Simons term supplies an

additional vertex proportional to b_0 . However, in the high-energy limit its contribution to the scattering amplitudes is suppressed by the powers of energy.

The leading contributions to the production rates mediated by the chiral anomaly in QCD, in the high-energy limit, are given by

$$\frac{dW^{q \rightarrow g^+ q}}{dx} = \frac{C_F}{\tau} \frac{1 + (1-x)^2}{x}, \quad (46a)$$

$$\frac{dW^{q \rightarrow qg^+}}{dx} = \frac{C_F}{\tau} \frac{1 + x^2}{1-x}, \quad (46b)$$

$$\frac{dW^{g^- \rightarrow q\bar{q}}}{dx} = \frac{1}{2\tau} [x^2 + (1-x)^2], \quad (46c)$$

$$\frac{dW^{g^+ \rightarrow g^+ g^+}}{dx} = \frac{2N_c}{\tau} \frac{1}{x(1-x)}, \quad (46d)$$

$$\frac{dW^{g^- \rightarrow g^+ g^-}}{dx} = \frac{2N_c}{\tau} \frac{(1-x)^3}{x}, \quad (46e)$$

$$\frac{dW^{g^- \rightarrow g^- g^+}}{dx} = \frac{2N_c}{\tau} \frac{x^3}{(1-x)}, \quad (46f)$$

where $C_F = (N_c^2 - 1)/(2N_c) = 4/3$. All other channels are suppressed by the inverse powers of energy [32]. The total $g \rightarrow gg$ rate reads²

$$\frac{dW^{g \rightarrow gg}}{dx} = \frac{1}{2} \sum_{\lambda_1=\pm} \sum_{\lambda_2=\pm} \sum_{\lambda_3=\pm} \frac{dW^{g^{\lambda_1} \rightarrow g^{\lambda_2} g^{\lambda_3}}}{dx}, \quad (47)$$

where $1/2$ is the average over polarizations of the initial gluon. The only nonvanishing partial rates are (46c)–(46e). We define the real contributions (those with $x < 1$) to the polarized splitting functions as

$$\frac{dW^{g \rightarrow gg}}{dx} = \frac{1}{\tau} \tilde{P}_{gg}(x), \quad \frac{1}{2} \frac{dW^{g^{\lambda_1} \rightarrow g^{\lambda_2} g^{\lambda_3}}}{dx} = \frac{1}{\tau} \tilde{P}_{g^{\lambda_2} g^{\lambda_1}}(x). \quad (48)$$

Equations (47) and (48) imply that the total $g \rightarrow g$ splitting function can be computed as

$$P_{gg}(x) = \sum_{\lambda_1=\pm} \sum_{\lambda_2=\pm} P_{g^{\lambda_2} g^{\lambda_1}}(x). \quad (49)$$

Using (46) and the definition (48) we find

$$P_{g^+ q}(x) = C_F \frac{1 + (1-x)^2}{x}, \quad (50a)$$

$$P_{qq}(x) = C_F \left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2} \delta(1-x) \right], \quad (50b)$$

²A useful formula for computing the rates for a given circular photon polarization is derived in Appendix B.

$$P_{qg^-}(x) = \frac{1}{2}[x^2 + (1-x)^2], \quad (50c)$$

$$P_{g^+g^+} = \frac{N_c}{x} \left[\frac{1}{(1-x)_+} + C^+ \delta(1-x) \right], \quad (50d)$$

$$P_{g^+g^-} = \frac{N_c}{x} (1-x)^3, \quad (50e)$$

$$P_{g^-g^-} = N_c \left[\frac{x^3}{(1-x)_+} + C^- \delta(1-x) \right], \quad (50f)$$

and other splitting functions vanish. The evolution equations of the chiral Cherenkov cascade in QCD read

$$\frac{dq(y,t)}{dt} = \frac{1}{\tau} \int_y^1 \frac{dx}{x} \left\{ P_{qq}(x) q\left(\frac{y}{x}, t\right) + P_{qg^-}(x) g^-\left(\frac{y}{x}, t\right) \right\}, \quad (51a)$$

$$\frac{d\bar{q}(y,t)}{dt} = \frac{1}{\tau} \int_y^1 \frac{dx}{x} \left\{ P_{q\bar{q}}(x) \bar{q}\left(\frac{y}{x}, t\right) + P_{qg^-}(x) g^-\left(\frac{y}{x}, t\right) \right\}, \quad (51b)$$

$$\begin{aligned} \frac{dg^+(y,t)}{dt} = \frac{1}{\tau} \int_y^1 \frac{dx}{x} \left\{ N_f P_{g^+q}(x) q\left(\frac{y}{x}, t\right) \right. \\ \left. + N_f P_{g^+q}(x) \bar{q}\left(\frac{y}{x}, t\right) \right. \\ \left. + P_{g^+g^+}(x) g^+\left(\frac{y}{x}, t\right) + P_{g^+g^-}(x) g^-\left(\frac{y}{x}, t\right) \right\}, \end{aligned} \quad (51c)$$

$$\frac{dg^-(y,t)}{dt} = \frac{1}{\tau} \int_y^1 \frac{dx}{x} P_{g^-g^-}(x) g^-\left(\frac{y}{x}, t\right), \quad (51d)$$

where $q(x,t)$ and $\bar{q}(x,t)$ are quark and antiquark distributions for a single flavor. In the chiral limit all flavors have equal distributions.

To determine the virtual contributions at $x=1$, represented by the constants C^+ and C^- , we employ the charge and energy conservation as in the previous section. The charge conservation implies $A_1^{qq} = 0$ and is satisfied by (50b). The total energy of the system is the sum of the second moments of all distributions, see (25). Its conservation requires that

$$\frac{d}{dt} [N_f(q_2 + \bar{q}_2) + g_2^+ + g_2^-] = 0, \quad (52)$$

which implies that

$$A_2^{qq} + A_2^{g^+q} = 0, \quad (53a)$$

$$2N_f A_2^{qg^-} + A_2^{g^+g^-} + A_2^{g^-g^-} = 0, \quad (53b)$$

$$A_2^{g^+g^+} = 0. \quad (53c)$$

As in χ QED (53a) is satisfied automatically along with the charge conservation. The other two equations (53b) and (53c) yield

$$C^+ = 0, \quad N_c C^- = \frac{11N_c}{6} - \frac{N_f}{3}. \quad (54)$$

It is not surprising that C^+ vanishes because the only decay channel for the right-handed gluons is $g^+ \rightarrow g^+ + g^+$ and it clearly conserves their total energy.

Substituting (50d)–(50f) into (49) we obtain the familiar unpolarized gluon-gluon splitting function:

$$\begin{aligned} P_{gg}(x) = 2N_c \left[\frac{1-x}{x} + \frac{x}{(1-x)_+} + x(1-x) \right] \\ + \left(\frac{11}{6} N_c - \frac{N_f}{3} \right) \delta(1-x). \end{aligned} \quad (55)$$

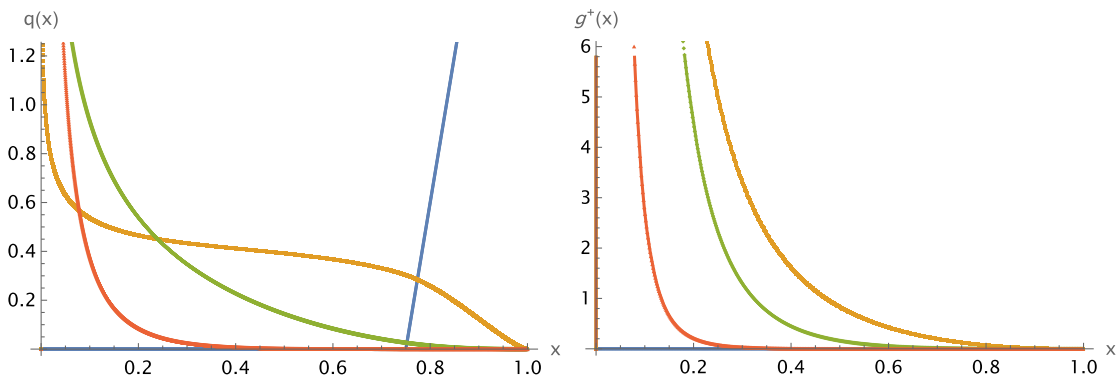


FIG. 6. Chiral Cherenkov cascade in χ QCD with the initial conditions: $q(x,0) = 32(x-3/4)H(x \geq 3/4)$, $\bar{q}(x,0) = g^\pm(x,0) = 0$. Blue line: $t=0$; yellow line: $t=\tau/2$; green line: $t=\tau$; red line: $t=2\tau$.

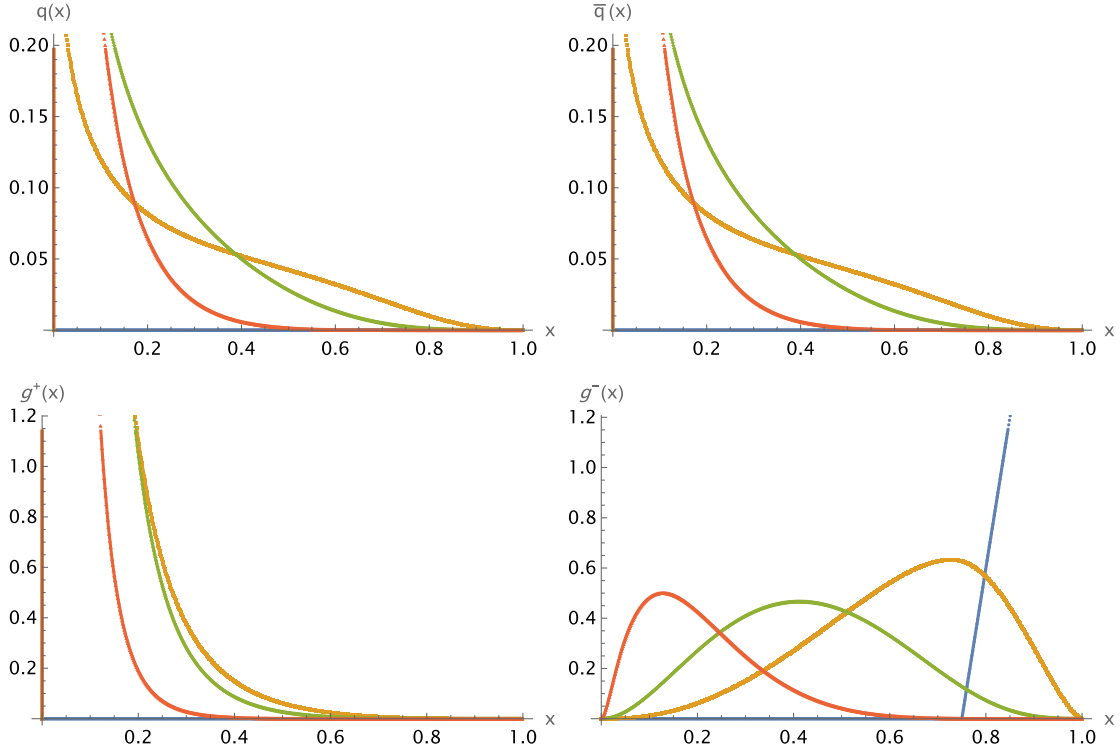


FIG. 7. Chiral Cherenkov cascade in χ QCD with the initial conditions: $g^-(x, 0) = 32(x - 3/4)H(x \geq 3/4)$, $q(x, 0) = \bar{q}(x, 0) = g^+(x, 0) = 0$. Blue line: $t = 0$; yellow line: $t = \tau/2$; green line: $t = \tau$; red line: $t = 2\tau$.

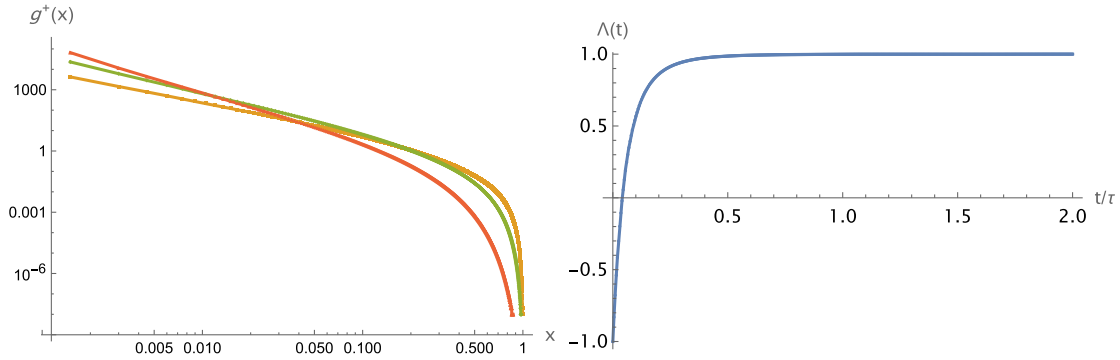


FIG. 8. Left: log plot for g^+ . Right: The degree of polarization $\Lambda(t)$ of the cascade shown in Fig. 7.

The deviations from the standard QCD splitting functions appear when the mass terms are taken into account as in (36). However, we do not consider them in the present work.

Figures 6 and 7 show the time evolution of the distributions functions in χ QCD with two different initial conditions similar to those of Figs. 3 and 4 in χ QED. The main difference between them is in the time evolution of gluon and photon distributions. The additional gluon decay channel facilitates production of the low- x right- and left-handed gluons. As a result, the number of right-handed gluons by far exceeds the number of the right-handed photons at the same t/τ . Also, the left-handed gluons are distributed in the entire interval $0 < x < 1$ whereas the

left-handed photons are found only with $x \geq 3/4$ fixed by the initial condition. As in the case of photons, the total number of the right-handed gluons increases, whereas the number of left-handed ones decreases. Thus, the degree of the cascade polarization Λ rapidly approaches unity, which happens to gluons an order of magnitude earlier than to photons as seen in the right panel of Fig. 8.

VI. SUMMARY AND DISCUSSION

We studied the time evolution of cascades using the Chern-Simons extensions of QED and QCD, which allows systematic incorporation of the chiral magnetic current

$j = b_0 \mathbf{B}$ in the perturbation theory. Our main result is the time-evolution equations (18), (39), (51) for the distribution functions. These equations omit the contribution of the conventional processes which has been extensively studied in the literature [1–11]. In [33,34] we analyzed in detail the contribution of the chiral anomaly to the Bethe-Heitler process and determined the conditions when such a contribution becomes essential.

The interplay of the conventional bremsstrahlung and pair production with their chiral Cherenkov counterparts is complex because they arise from different kinematic regions. For example, in contrast to the conventional QED/QCD bremsstrahlung which is dominated by soft photon/gluon emission, the chiral Cherenkov emission is determined by the resonance in the intermediate fermion states [34]. Unlike the former process that generates large logarithms of transverse momentum, the latter one generates large powers of the resonance relaxation time. The fact that the conventional and the chiral Cherenkov processes come from different parts of the phase space makes it challenging to combine them in a single framework. One possible approach could involve the kinetic equation with the conventional and chiral Cherenkov processes contributing to different collision integrals.

The evolution equations (18) and (51) closely resemble the Gribov-Lipatov equations of QED [36] and the DGLAP equations of QCD [37–39] in the leading logarithmic approximation. This is because the splitting functions are exactly the same in the high-energy approximation and in the chiral limit. Nevertheless, they are fundamentally different as the evolution equations of the chiral Cherenkov cascade (18), (51) involve the two photon and gluon polarizations in highly asymmetric way. As a result the cascade quickly becomes strongly polarized in violation of parity.

In [32] we computed the rate of energy loss $-dE/dz$ by a fast parton in the quark-gluon plasma due to the chiral Cherenkov radiation of a single gluon. We found that for a reasonable value of b_0 , this mechanism can compete with the conventional QCD process. Moreover, we observed that its energy dependence is similar to that of jet quenching in the Bethe-Heitler regime. Also, quantum interference effects are known to suppress the energy dependence of the conventional mechanism, whereas the chiral processes we considered in this paper are already coherent throughout the plasma volume. Thus, we anticipate that the relative contribution of the anomalous processes to the energy loss will increase with energy. An accurate estimation of the impact of the chiral anomaly on jet quenching necessitates the incorporation of the chiral Cherenkov processes into the phenomenological frameworks.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: DOUBLE PHOTON EMISSION

In Sec. III we argued that the intermediate states in the multiple-emission processes can be put on mass shell at large relaxation times τ . In this appendix we develop a more detailed argument considering as an example the double photon emission $e \rightarrow e\gamma\gamma$. We argue that the radiation rate $W^{e \rightarrow e\gamma\gamma}$ is dominated by a single pole induced by the chiral anomaly.

Let p, p', k_1, k_2 be the 4-momenta of the initial and final electrons and the two photons respectively. At high energy they read

$$p = \left(E, 0, 0, E - \sum_{i=0}^1 \frac{k_{i\perp}^2 + x_i(1-x_i)\mu^2(E)}{2x_i(1-x_i)E} \right), \quad (\text{A1a})$$

$$k_i = \left(z_i E, k_{i\perp}, 0, x_i E - \frac{k_{i\perp}^2 + \mu^2(x_i)}{2x_i E} \right), \quad (\text{A1b})$$

$$p' = \left(E', -k_{0\perp} - k_{1\perp}, 0, E' - \sum_{i=0}^1 \frac{k_{i\perp}^2 + \mu^2(E')}{2E'} \right). \quad (\text{A1c})$$

The matrix element can be written as

$$\begin{aligned} i\mathcal{M}^{e \rightarrow e\gamma\gamma} &\approx -g\bar{u}_{p's'} \not{\epsilon}_{k_1\lambda_1}^* u_{p's'} \frac{i}{(p' + k_1)^2 - m^2} \bar{u}_{ps} \not{\epsilon}_{k_0\lambda_0}^* u_{ps} \\ &= ig\bar{u}_{p's'} \not{\epsilon}_{k_1\lambda_1}^* u_{p's'} \frac{i}{(p' + k_1)^2 - m^2} i\mathcal{M}^{e \rightarrow e\gamma}. \end{aligned} \quad (\text{A2})$$

Using explicit expressions for $u_{p's'}$ and $\not{\epsilon}_{k_1\lambda_1}^*$ we can obtain an expression for the square of the matrix element

$$\begin{aligned} |\mathcal{M}^{e \rightarrow e\gamma\gamma}|^2 &\approx g^2 \frac{2(1-x_1)k_{1\perp}^2}{(k_{1\perp}^2 + \mu^2(x_1)(1-x_1) + m^2x_1^2)^2} \\ &\times \left[\frac{1 + (1-x_1)^2}{x_1} + \frac{m^2x_1^3}{k_{1\perp}^2} \right] |\mathcal{M}^{e \rightarrow e\gamma}|^2. \end{aligned} \quad (\text{A3})$$

The rate is therefore

$$\begin{aligned} W^{e \rightarrow e\gamma\gamma}(y) &\approx 2g^2 \int \frac{dx_1 dk_{1\perp}^2}{16\pi^2} \frac{k_{1\perp}^2}{(k_{1\perp}^2 + \mu^2(x_1)(1-x_1) + m^2x_1^2)^2} \\ &\times \left[\frac{1 + (1-x_1)^2}{x_1} + \frac{m^2x_1^3}{k_{1\perp}^2} \right] dW^{e \rightarrow e\gamma}, \end{aligned} \quad (\text{A4})$$

where x_1 is bounded from below y . The divergence is regulated by the relaxation time τ . The contribution from the pole becomes

$$W_{\text{pole}}^{e \rightarrow e\gamma\gamma}(y) \approx \frac{g^2 \tau}{4\pi E} \int \frac{dx_1 k_{1\perp}^2 dk_{1\perp}^2}{x_1(1-x_1)} \times \delta(k_{1\perp}^2 + \mu^2(x_1)(1-x_1) + m^2 x_1^2) \times \left[\frac{1 + (1-x_1)^2}{x} + \frac{m^2 x_1^3}{k_{1\perp}^2} \right] dW^{e \rightarrow e\gamma}. \quad (\text{A5})$$

Integrating over $k_{1\perp}$ yields

$$W_{e \rightarrow e\gamma\gamma}^{\text{pole}}(y) \approx \frac{g^2 \tau}{4\pi E} \int_y^{x_0} \frac{dx_1}{x_1^2} [-\mu^2(x_1)(1 + (1-x_1)^2) - 2m^2 x_1^2] dW_{e \rightarrow e\gamma}. \quad (\text{A6})$$

At this point, one needs to be careful when integrating over x_1 due to the fact that $dW^{e \rightarrow e\gamma}$ may depend on it. Luckily the parameter b_0 is believed to be small in comparison to the fermion's mass. If b_0 is small enough such that $b_0 E \ll m^2$. In this limit, $x_+^q \ll 1$, and $W^{e \rightarrow e\gamma}$ can be pulled out of the integral. After integrating over frequencies for $\lambda_1 = 1$ and $\mu^2(z_1) = -b_0 E x_1$ gives

$$W_{\text{pole}}^{e \rightarrow e\gamma\gamma}(y) \approx \frac{2g^2 W_{e \rightarrow e\gamma} \tau b_0}{4\pi} \left[\log \frac{x_0}{y} + \frac{(y-3x_0)(x_0-y)}{4x_0} \right]. \quad (\text{A7})$$

In the soft photon limit such that $y \ll x_0$ we have

$$W_{\text{pole}}^{e \rightarrow e\gamma\gamma}(y) \approx \frac{2g^2 W_{e \rightarrow e\gamma} \tau b_0}{4\pi} \log \frac{x_0}{y}, \quad (\text{A8})$$

which can be compared to the contribution away from the pole.

Now consider the behavior away from the pole such that $k_{1\perp}^2 + \mu^2(x_1)(1-x_1) + m^2 x_1^2 > \frac{E}{\tau}$. In this region (A5) needs no regulator, and the integrals may be taken directly. Integration over $k_{1\perp}^2$ gives

$$W_{\text{away}}^{e \rightarrow e\gamma\gamma}(y) \approx \frac{g^2 W_{q \rightarrow qg}}{8\pi^2} \int_y^{x_0} \frac{dx_1}{x_1} (1 + (1-x_1)^2) \log \frac{K_{\perp}^2 \tau}{E}, \quad (\text{A9})$$

where $K_{\perp}^2$ is the upper bound of $k_{1\perp}^2$. Integrating over frequencies gives

$$W_{\text{away}}^{e \rightarrow e\gamma\gamma}(y) \approx \frac{g^2 W^{e \rightarrow e\gamma} H(E \geq E_1)}{3\pi^2} \times \left[\log \frac{x_0}{y} + \frac{x_0^2 - y^2}{2} - 4(x_0 - y) \right] \log \frac{K_{\perp}^2 \tau}{E}. \quad (\text{A10})$$

For soft photons this simplifies to

$$W_{\text{away}}^{e \rightarrow e\gamma\gamma} \approx \frac{g^2 W^{e \rightarrow e\gamma} H(E \geq E_1)}{3\pi^2} \log \frac{K_{\perp}^2 \tau}{E} \log \frac{x_0}{y}. \quad (\text{A11})$$

This can be compared to the contribution from the pole:

$$\frac{W_{\text{away}}^{e \rightarrow e\gamma\gamma}}{W_{\text{pole}}^{e \rightarrow e\gamma\gamma}} \approx \frac{H(E \geq E_1)}{2\pi b_0 \tau} \log \frac{K_{\perp}^2 \tau}{E}. \quad (\text{A12})$$

Therefore if $\tau \gg \frac{1}{2\pi b_0} \log \frac{K_{\perp}^2 \tau}{E}$ then the pole gives the dominate contribution. For $\frac{1}{\tau} = \frac{ab_0}{4}$ this implies that $\frac{8\pi}{\alpha} \gg \log \frac{4K_{\perp}^2}{ab_0 E}$ which holds for nearly any E .

This procedure is not restricted to the double photon production. For any given process the anomaly introduces a new pole for every intermediate particle involved. As such, each of these particles can be approximated to be on mass shell. The justification in each case is very similar. The contributions to their corresponding radiation rates are dominated by such poles.

APPENDIX B: CIRCULAR POLARIZATION TENSOR

Define the photon polarization sum as

$$d_{\lambda}^{\mu\nu} = (\epsilon_{k,\lambda}^{\mu})^* \epsilon_{k,\lambda}^{\nu} \quad (\text{no summation}). \quad (\text{B1})$$

The polarization vectors describe the right- and left-hand circular polarizations and obey the identity $i\mathbf{k} \times \epsilon_{k,\lambda} = \lambda |\mathbf{k}| \epsilon_{k,\lambda}$. In the radiation gauge the polarization vectors satisfy $\epsilon_{k,\lambda}^0 = 0$, $\epsilon_{k,\lambda} \cdot \mathbf{k} = 0$, $(\epsilon_{k,\lambda})^* \cdot \epsilon_{k,\lambda'} = \delta_{\lambda,\lambda'}$. It follows that $d_{\lambda}^{00} = d_{\lambda}^{0i} = 0$, $d_{\lambda}^{ij} k^i = 0$, $\text{Tr} \sum_{\lambda} d_{\lambda}^{ij} = 2$, and $d_{\lambda}^{ij} \epsilon_{k,\lambda}^j = \epsilon_{k,\lambda}^i$ (no summation over λ). The space-space components of (B1) that satisfy these constraints read

$$d_{\lambda}^{ij} = (\epsilon_{k,\lambda}^i)^* \epsilon_{k,\lambda}^j = \frac{1}{2} \left(\delta^{ij} - \frac{k^i k^j}{k^2} \right) + \frac{i\lambda}{2} \epsilon^{ij\ell} \frac{k^{\ell}}{|\mathbf{k}|}. \quad (\text{B2})$$

See also the Appendix in [40].

In the collinear limit one can verify (B2) by using the explicit form of the polarization vector $\epsilon_{k,\lambda} = (0, \epsilon_{\perp}, \epsilon_z)$ with $\epsilon_{\perp} = (\hat{\mathbf{x}} + i\lambda\hat{\mathbf{y}})/\sqrt{2}$, $\epsilon_z = -\epsilon_{\perp} \cdot \mathbf{k}/k_z$, correct up to small $k_{\perp}^2/k_z^2$ terms:

$$d_{\lambda}^{ij} = \left(\frac{1}{\sqrt{2}}, \frac{-i\lambda}{\sqrt{2}}, \epsilon_z^* \right)^T \left(\frac{1}{\sqrt{2}}, \frac{i\lambda}{\sqrt{2}}, \epsilon_z \right) = \frac{1}{2} \begin{pmatrix} 1 & i\lambda & \frac{-k_x - i\lambda k_y}{k_z} \\ -i\lambda & 1 & \frac{-k_y + i\lambda k_x}{k_z} \\ \frac{-k_x + i\lambda k_y}{k_z} & \frac{-k_y - i\lambda k_x}{k_z} & 0 \end{pmatrix} + \mathcal{O}\left(\frac{k_{\perp}^2}{k_z^2}\right) \quad (\text{B3})$$

which agrees with (B2) in the same limit.

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