

Imprint of the black hole singularity on thermal two-point functions

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(Received 17 November 2025; accepted 11 June 2026; published 6 July 2026)

We consider two-point functions of light fields at finite temperature and large real frequencies in holographic theories. The thermal system is dual to a single-sided anti-de Sitter black hole. We show that the high-frequency expansion obtained from the operator product expansion receives exponentially small nonperturbative corrections, which are controlled by null geodesics bouncing off the black hole singularity in the two-sided eternal black hole geometry. We develop a bulk WKB description of these bouncing geodesics and explain how to calculate reflection coefficients at the singularity.

DOI: [10.1103/1j3s-fzjl](https://doi.org/10.1103/1j3s-fzjl)

I. INTRODUCTION

Within holography [1–3], specific field theories at finite temperature are dual to black holes in asymptotically AdS₅ spacetime at the corresponding Hawking temperature. This framework relates the physics of ordinary, albeit strongly interacting, quantum systems to various calculable quantities in the black hole background [4–6].

Perhaps surprisingly, geodesics approaching the singularity in the interior of the black hole lead to features in exterior correlators [7–10]. This connects the still-mysterious dynamics of probes near the singularity to well-defined exterior quantities. The features can be revealed by considering heavy bulk fields and analytically continuing correlations between the two sides of the thermofield double [11]. The proper time to the singularity can be similarly observed [12].

In this paper we discuss the effect of geodesics that bounce off the singularity in a comparatively simpler setup: single-sided two-point functions with large real frequency and small operator dimension. An example is the spectral density of R-charge currents at zero spatial momentum in $\mathcal{N} = 4$ super Yang-Mills, which was observed numerically in [13], and later analytically [14], to approach its zero-temperature value exponentially fast at high frequencies:

$$\begin{aligned} \text{Im} G_{\text{ret}}^{xx}(\omega, q=0) &= \frac{\pi\omega^2(1 - e^{-\beta\omega})}{(1 - e^{-\frac{\beta\omega}{2}(1-i)})(1 - e^{-\frac{\beta\omega}{2}(1+i)})} \\ &= \pi\omega^2 \left(1 + \sum_{n=1}^{\infty} \left(e^{-\frac{n\beta\omega}{2}(1-i)} + e^{-\frac{n\beta\omega}{2}(1+i)} \right) \right). \end{aligned} \quad (1)$$

The second line shows the high-frequency expansion, with the leading term $\pi\omega^2$ being the zero-temperature result. We will show how the two terms with $n = 1$ can be attributed to a bouncing geodesic and its time reflection, with higher n corresponding to multiple reflections.

More generally, we will study the high-frequency behavior of the retarded function dual to a scalar field in the AdS₅ black hole geometry, where exact solutions are not available; the result will involve a transseries. We will develop a bulk description using the WKB approximation near null geodesics and a novel reflection coefficient capturing the effect of the singularity.

By definition, the retarded response function and its associated spectral density can be measured and/or calculated using only the geometry in the exterior of the black hole. The relative simplicity of this observable is one of its key attractive features: in essence one has to measure thermal conductivity as a function of frequency and subtract off the vacuum contribution.

The fact that this observable receives a contribution from geodesics that go through the *interior* of the black hole may seem surprising. In essence, this happens because the highly oscillatory Fourier transform at high frequency will be dominated by complex times, thus revealing complex solutions to the bulk geodesic equations. The fact that the geodesics appear to explore the interior does not contradict causality, since, as will be clear from the calculation,

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the bouncing geodesics really only explore an analytic continuation of the exterior geometry.

We can thus already anticipate that our description of the bouncing geodesics will not have any obvious relevance to the experience of an infalling observer, nor will it shed light on fine-grained aspects of the information carried by black holes. Nonetheless, the singularity in the (analytically continued) geometry leaves well-defined imprints in exterior observables, which we will specifically discuss for $\mathcal{N} = 4$ super Yang-Mills in four dimensions.

II. BULK SETUP AND WKB PHASE

We consider the momentum space retarded thermal Green function in the CFT_d:

$$G_{\text{ret}}(\omega, q) = i \int d^{d-1}x \int_0^\infty dt e^{i\omega t - iq \cdot x} \langle [\mathcal{O}(t, x), \mathcal{O}(0)] \rangle_\beta. \quad (2)$$

Let us briefly review its calculation in a holographic theory. The bulk dual of the thermal state is the AdS_{d+1} planar black hole metric

$$ds^2 = -r^2 f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 dx^2, \quad (3)$$

where we set the AdS radius $\ell = 1$; $f(r) = 1 - (r_h/r)^d$ with $r_h = \frac{4\pi}{\beta d}$ the horizon radius and dx^2 denotes the flat metric over the $(d-1)$ spatial directions. The bulk field dual to the scalar operator $\mathcal{O}$ of dimension Δ is a scalar of mass $m^2 = \Delta(\Delta - d)$ satisfying the equation of motion

$$\phi'' + \left(\frac{f'}{f} + \frac{d+1}{r} \right) \phi' + \frac{\omega^2/f - q^2 - m^2 r^2}{r^4 f} \phi = 0, \quad (4)$$

where primes denote derivatives with respect to r . According to [4–6], the retarded function can be obtained by finding the solution $\tilde{\phi}_{\omega, q}(r)$ that is purely infalling at the horizon, and dividing the coefficients of the two independent powers of r near the AdS boundary ($r \rightarrow \infty$):

$$G_{\text{ret}}(\omega, q) = C(\tilde{\phi}_{\omega, q}|_{r^\Delta})/(\tilde{\phi}_{\omega, q}|_{r^{\Delta-d}}). \quad (5)$$

The normalization C cancels out when dividing by zero-temperature results and will thus be ignored below.

The large- ω limit motivates a WKB approximation to the radial solution in the limit $\omega \gg m, q$:

$$\phi_{\omega, q} \approx r^{\frac{1-d}{2}} \exp \left(i\omega \int dt + \mathcal{O}(1/\omega) \right), \quad dt = \frac{dr}{r^2 f(r)}. \quad (6)$$

This was discussed recently for null geodesics in AdS in [15–18]. A first hint that null geodesics bouncing off the

singularity are relevant to this limit comes from calculating the exponent for the path shown in Fig. 2, which goes to the singularity and back out to the other side [19]:

$$\Delta t = 2 \int_0^\infty \frac{dr}{r^2 f(r)} = \frac{\beta}{2} \left(\cot \frac{\pi}{d} + i \right) \xrightarrow{d=4} \frac{\beta}{2} (1 + i). \quad (7)$$

We see that the factor $e^{i\omega \Delta t}$ for this path precisely matches the exponentials in (1). The imaginary part of the travel time originates from the pole of dt at the black hole horizon, which we avoided with a $-i0$ shift (to be explained below). It produces the half-Boltzmann suppression $e^{-\beta\omega/2}$ visible in (1) and represents the fact that the geodesic ends on the left boundary despite the original retarded correlator being defined entirely on the right boundary. The real part of Δt is the vertical offset in Fig. 2. It is nonvanishing for $d > 2$, signifying that the Penrose diagram of the AdS black hole is “not a square” [7,20].

The calculation in (7) is suggestive but incomplete: the WKB approximation breaks down near the singularity and we still need to explain why the trajectory reflects there. This will be done in Sec. IV. Before that, we explain on general grounds why the retarded correlator at large *real* frequencies and small Δ probes the singularity; these kinematics differ from the large mass or large imaginary frequencies previously discussed in the literature.

III. OPERATOR PRODUCT EXPANSION

Key properties of high-frequency two-point functions can be understood using the field theory operator product expansion (OPE), which is most readily stated in terms of the definite-order (Wightman) functions:

$$G^>(t, x) \equiv \langle \mathcal{O}(t, x) \mathcal{O}(0) \rangle_\beta, \quad G^<(t, x) \equiv \langle \mathcal{O}(0) \mathcal{O}(t, x) \rangle_\beta. \quad (8)$$

The function $G^>$ coincides with the Euclidean correlator when $t = -i\tau$ with $0 < \tau < \beta$, where it admits the OPE [21]

$$G^>(-i\tau, x) = \sum_{\Delta', \ell} \frac{C_\ell^{(d/2-1)} \left(\frac{\tau}{\sqrt{x^2 + \tau^2}} \right)}{(\sqrt{x^2 + \tau^2})^{2\Delta_0 - \Delta'}} c_{\Delta', \ell} \langle \mathcal{O}_{\Delta', \ell} \rangle_\beta, \quad (9)$$

which runs over the scaling dimension and spin of operators; c are their OPE coefficients, and C_ℓ is a Gegenbauer polynomial. While the general structure of this expansion follows from the translation and rotational invariance of the thermal state, an important fact is that this sum has a finite radius of convergence. For example, convergence for τ real and $\sqrt{x^2 + \tau^2} < \beta$ was established in [21].

The expansion (9) can thus be analytically continued to real times, and the commutator $G_{\text{ret}} = i(G^> - G^<)$ can be calculated as the *discontinuity* of (9) across the real positive

time axis, at least for sufficiently small times. Which operators contribute to the OPE of G_{ret} ?

At large N , the OPE of thermal correlators is saturated by products of stress tensors and by $\mathcal{O}\mathcal{O}$ double traces, whose scaling dimensions satisfy, respectively, $\Delta = nd$ and $\Delta - \ell = 2n + \mathcal{O}(N^{-2})$ for integer $n \geq 0$ (see [22,23] and also [24,25]). Since the contribution of the latter to (9) is *polynomial*, they cancel out to leading order in the commutator (2). Hence, in a holographic theory, the OPE of the retarded function is saturated at large N by products of stress tensors. Specializing to zero spatial momentum for notational simplicity (this discussion is easily generalized), the retarded function thus admits an expansion:

$$G_{\text{ret}}(t, q = 0) = t^{d-1-2\Delta_{\mathcal{O}}} \sum_{n=0}^{\infty} a_n \left(\frac{t}{\beta}\right)^{nd} \quad (t > 0). \quad (10)$$

This property is explicitly seen in holographic calculations; cf. Appendix A.¹

It would be interesting to characterize the convergence and potential singularities of (10). A simple fact is that the commutator only has support for $|x| < t$. Hence, at small t , the spatial Fourier transform runs over a finite x range, which suggests a finite radius of convergence in a general theory.

Fourier-transforming (10) term by term gives

$$G_{\text{ret}}(\omega, q = 0) \sim (-i\omega)^{2\Delta_{\mathcal{O}}-d} \sum_{n=0}^{\infty} \frac{a_n}{(-i\beta\omega)^{nd}} \times \Gamma(d - 2\Delta_{\mathcal{O}} + nd). \quad (11)$$

We see that the convergent small-time expansion has turned into an asymptotic series in $1/\omega$, due to the factorial growth of the Γ -function.

The spectral density $2 \text{Im} G_{\text{ret}}$ is of particular interest since it determines frequency space Wightman functions by the Kubo-Martin-Schwinger (KMS) conditions:

$$G^>(\omega, q) = \frac{2 \text{Im} G_{\text{ret}}(\omega, q)}{1 - e^{-\beta\omega}}. \quad (12)$$

Taking the imaginary part of (11) reveals an interesting feature: when d and $2\Delta_{\mathcal{O}}$ are both even integers, the series terminate. As noted previously in [26], this explains why

¹When $2\Delta_{\mathcal{O}} + 2k = nd$ for some non-negative k, d , or when $\mathcal{O}$ is the stress tensor, the distinction between double-traces and stress tensors becomes singular and correlators feature logarithms. Nonetheless, these logarithms cancel when taking the discontinuity and the form (10) remains valid.

Independently, when $2\Delta_{\mathcal{O}} - (n+1)d$ is a non-negative integer, the Γ -function in (11) develops poles due to the standard logarithmic ultraviolet divergence of the Fourier transform; $\text{Im} G_{\text{ret}}(\omega)$ discussed below remains finite and unambiguous.

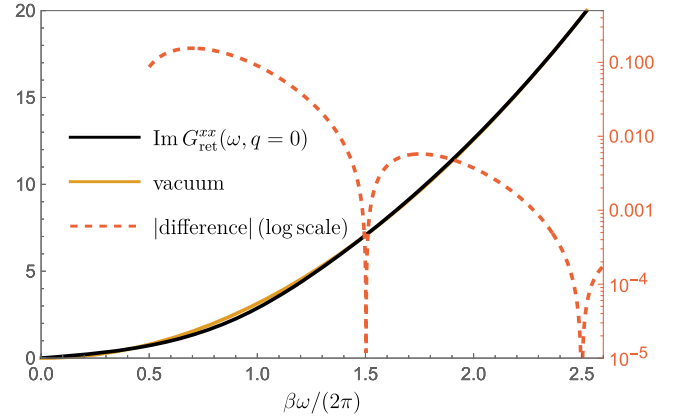


FIG. 1. Current spectral density (1) at finite and zero temperature. We will focus on the exponentially decaying difference at large frequencies; the dips are caused by destructive interference between the two paths in Fig. 2.

thermal contributions to the spectral density of currents and stress tensors in $d = 4$ holographic theories decay exponentially, as was already illustrated in Fig. 1.

For generic values of $\Delta_{\mathcal{O}}$ or d , both the retarded and Wightman functions admit nonterminating asymptotic expansions in $1/\omega$. Borel resumming these expansions is equivalent to analyzing the Fourier integral:

$$G_{\text{ret}}(\omega, q) = \int_0^{\infty} dt e^{i\omega t} G_{\text{ret}}(t, q). \quad (13)$$

A simple steepest-descent analysis of this integral explains why the behavior at large *real* frequencies is controlled by complex times: for large positive ω , it is advantageous to deform the time contour into the upper-half-plane where the Fourier factor decays. The deformed contour picks up the “Borel” singularities of G_{ret} at complex times, as depicted in Fig. 3. A singularity at $t = t_*$ thus produces a nonperturbative effect $\sim e^{it_*\omega}$ in the $1/\omega$ expansion.

Below we will focus on the bouncing geodesic singularities which appear at $\text{Im} t_* = \beta/2$. This complex time value implies geometrically that the geodesic ends on the left boundary, as shown in Fig. 2, even though we emphasize that the original correlator involves only the right boundary. Since $G_{\text{ret}} = i(G^> - G^<)$ where $G^<$ is analytic in the strip $0 < \text{Im} t < \beta$, these singularities appear on the second sheet of $G^>$, thus making contact with the continuation argument of [7].

In summary, the high-frequency expansion of two-point functions is asymptotic in $1/\omega$ and receives nonperturbative corrections associated with complex-time singularities. These are particularly easy to observe when d and $2\Delta_{\mathcal{O}}$ are even integers (which includes the important example of stress tensor correlators in $d = 4$), but they are always present in a holographic theory.

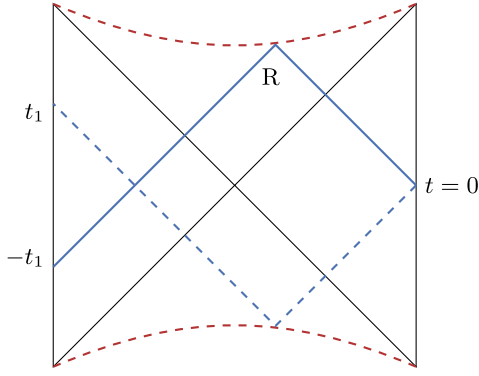


FIG. 2. Nonperturbative contributions $\sim e^{-\frac{1}{2}\beta\omega + i\omega t_1}$ to the retarded function at high frequencies will be explained from a null geodesic that reflect once off future singularity of the eternal black hole. Wightman functions and $\text{Im } G_{\text{ret}}$ also receive contributions from the time-reversed geodesic.

IV. WKB PROPOSAL FOR BOUNCING GEODESICS

We are now ready to state our proposal. We will focus on $d=4$, where we can numerically test it, and set $r_h = \frac{4\pi T}{d} = 1$ for convenience.

A. Leading reflection coefficient

Working in frequency space, we propose to identify the contribution from the vertical steepest descent contour at the boundary in Fig. 3, which represents a canonical Borel resummation of the leading large- ω series, with the solution to bulk equations of motion which decays along a steepest-descent contour in r . Steepest-descent lines in the WKB approximation are shown in Fig. 4. Starting from large real r , we observe that the lines follow a large circle which ends at the attractor point $r = -i$, with infalling solution $\overleftarrow{\phi}_\omega(r)$ initially decaying along this line. Hence, the solution

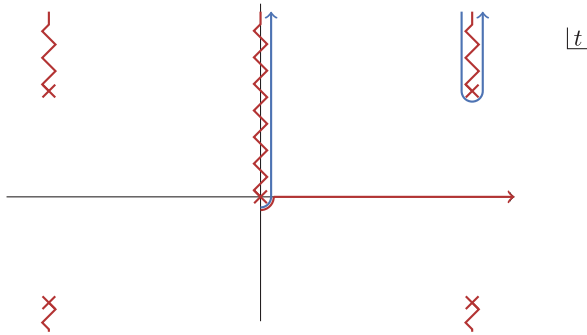


FIG. 3. The original real-time contour (in red) for the Fourier transform of $G_{\text{ret}}(t)$ can be deformed into a steepest-descent contour along the imaginary axis plus a branch cut starting at $t = \frac{\beta}{2}(1+i)$. The same cut appears in the Wightman functions $G^>(t)$ continued to the second sheet.

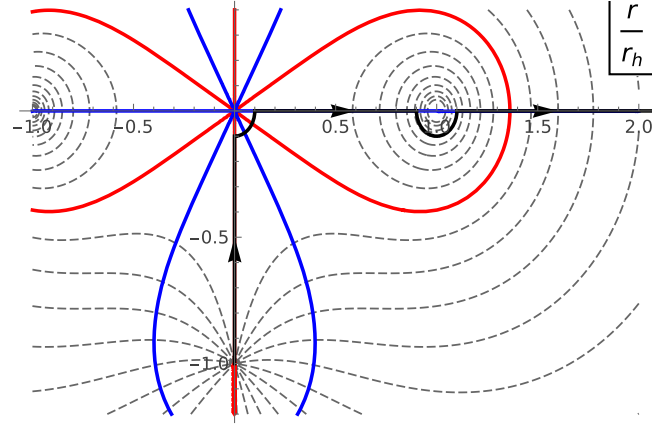


FIG. 4. Stoke's lines (in blue, with phases $n\pi/3$ near the origin), along which WKB solutions are oscillatory, and lines of constant WKB phase (solid red and gray dashed). We track $\phi_\omega^{\text{steepest}}(r)$ from $r = -ir_h$ to large r following the solid path with arrows.

$\phi_\omega^{\text{steepest}}(r)$, defined by being regular at $-i$, must *almost* coincide with the infalling solution:

$$\overleftarrow{\phi}_\omega(r) = \phi_\omega^{\text{steepest}}(r) + (\text{exponentially decaying in } \omega). \quad (14)$$

The question is how to calculate the decaying piece, which, based on Fig. 3, we identify with nonperturbative effects.

We must connect $r = -i$ to the horizon $r = 1$ where the boundary condition on $\overleftarrow{\phi}_\omega(r)$ is imposed. Taking inspiration from [27], we follow in turns the imaginary and real axes, as shown in Fig. 4. There the WKB solution is, respectively, purely increasing and purely oscillatory. The axes cannot be connected directly: they are separated by a Stokes line along which the WKB solutions exchange dominance. Rather, we proceed through the small- r region. The WKB approximation breaks down when $r^3\omega \sim 1$ and the ϕ'' and ω^2 terms in (4) compete. Introducing $x = r^3\omega/3$, the wave equation in this scaling limit (large ω with fixed x) becomes

$$0 = \partial_x^2 \phi + \frac{1}{x} \partial_x \phi + \phi + \mathcal{O}(q^2 \omega^{-2/3}, m^2 \omega^{-4/3}). \quad (15)$$

Recall that we are taking ω large with q and m fixed. Luckily, this can be solved exactly. The desired solution, which decays along negative imaginary r , can be written on the real axis as a sum of oscillatory Bessel/Hankel functions:

$$\phi_\omega^{\text{steepest}}(r) \propto K_0(-ix) = \frac{\pi}{2i} (H_0^{(1)}(x) + 2H_0^{(2)}(x)), \quad (16)$$

where the equality can be verified by series expansion in x and continuing termwise through the r fourth quadrant.

Parametrizing the ratio of infalling and outgoing components near the singularity by a *reflection coefficient* R :

$$\phi_\omega^{\text{steepest}}(r) \propto H_0^{(1)}(x) - R H_0^{(2)}(x) \quad (\text{near } r = 0), \quad (17)$$

we conclude from (16) that $R(\omega) = -2 + \mathcal{O}(\omega^{-2/3})$. The number “-2” will be numerically confirmed below.

The final step is to evolve the solution (17) along the real axis toward the AdS boundary. We start from the region $\omega^{-1/3} \ll r \ll 1$ where x is large and each Hankel function can be matched to a WKB solution. We then evolve to large r by integrating the WKB phase, passing below the horizon at $r = 1$, until the region $1 \ll r \ll \omega$ where we can match with boundary Hankel functions discussed in the Appendix A. The relevant limits of the WKB phase are

$$\int_0^r \frac{dr}{r^2 f(r)} \rightarrow \begin{cases} -r^3/3, & r \rightarrow 0, \\ \frac{\beta}{4}(1+i) - 1/r, & r \rightarrow \infty. \end{cases} \quad (18)$$

These limits match with the near-singularity and near-boundary Hankel functions, which are thereby connected:

$$\begin{aligned} H_0^{(1)}(r^3 \omega/3) &\xleftrightarrow{\text{small } r} \frac{\sqrt{6} r^{-3/2}}{\sqrt{i\pi\omega}} e^{-i\omega \int_0^r \frac{dr}{r^2 f(r)}} \\ &\xleftrightarrow{\text{large } r} \sqrt{3} e^{\frac{\beta\omega}{4}(1-i) + \frac{i}{2}\pi\nu} r^{-2} H_\nu^{(1)}(\omega/r). \end{aligned}$$

The real part of the exponent coincides with the change in the infalling mode $\phi \propto (r-1)^{-\frac{i\beta\omega}{4}}$ around $r = 1$ (related to the Schwinger-Keldysh prescription of [28–31]). The continuation of the outgoing solution $H_0^{(2)}$ is similar but with the opposite exponent. Thus (17) evolves as $r \rightarrow \infty$ to

$$\phi_\omega^{\text{steepest}}(r) \rightarrow r^{-2} [H_\nu^{(1)}(\omega/r) - R e^{-\frac{\beta\omega}{2}(1-i) - i\pi\nu} H_\nu^{(2)}(\omega/r)], \quad (19)$$

where we fixed a convenient overall normalization. To the accuracy of our calculation, the first term coincides with the true infalling solution. Hence, as anticipated in (14), the steepest-descent and infalling solutions differ by an exponentially small correction:

$$\overleftarrow{\phi}_\omega(r) \rightarrow \phi_\omega^{\text{steepest}}(r) + R e^{-\frac{\beta\omega}{2}(1-i) - i\pi\nu} r^{-2} H_\nu^{(2)}(\omega/r). \quad (20)$$

According to our proposal, the first contribution represents the canonical Borel resummation of the $1/\omega$ asymptotic expansion. Hence, applying the holographic recipe (5), we have obtained the following transseries solution:

$$G_{\text{ret}}(\omega) = \omega^{2\nu} \left[\begin{aligned} &\frac{e^{-i\pi\nu}}{-2 \sin(\pi\nu)} + \mathcal{O}(\omega^{-4}) \\ &+ i e^{-\frac{\beta\omega}{2}(1-i) - i\pi\nu} R(\omega) \times (1 + \mathcal{O}(\omega^{-4})) \\ &+ \dots \end{aligned} \right], \quad (21)$$

where $\dots$ represent subleading paths which we expect to be more suppressed, $\propto e^{-\beta\omega}$. This is the main result of this paper. Here we chose the normalization C in (5) so that the zero-temperature spectral density is $2 \text{Im } G_{\text{ret}}^{\text{vac}} = \omega^{2\nu}$, and we recall that $\nu = \Delta_\mathcal{O} - \frac{d}{2}$.

Using Fourier transforms from Appendix B, an equivalent statement of (21) is that, in the time domain, the retarded function contains a specific singularity at $t_* = \frac{\beta}{2}(1+i)$ [see (B5)]:

$$G_{\text{ret}}(t, q = 0) \rightarrow -2i \times \frac{\Gamma(2\nu + 1)}{2\pi(i(t - t_*))^{2\nu+1}} + (\text{less singular}), \quad (22)$$

in the same normalization. This controls, for example, the large-order behavior of the OPE coefficients a_n [see (B9)].

B. Subleading reflection coefficient

The series (21) with $R(\omega) \approx -2$ is our main result. Here we explain how perturbation theory near the singularity can be used to obtain the first subleading $\sim \omega^{-4/3}$ correction to R [see (28)] and also explain the size of the error terms in (21) in terms of contributions from other regions.

The correction will come entirely from near the singularity. We will focus here on the case of zero spatial momentum, $q = 0$. In terms of the scaling variable $x = \omega r^3/3$ (we set $r_h = 1$ here) the radial equation reads

$$\phi'' + \frac{1}{x} \phi' + \phi = \frac{12x\phi' - (18x^2 + m^2)\phi}{(3x)^{2/3} \omega^{4/3}} + \mathcal{O}(q, \omega^{-8/3}). \quad (23)$$

The left-hand side is the equation studied in the text. Its zeroth-order solution of interest decays for negative imaginary r : $\phi^{(0)} = K_0(y)$ from (16), where one initially takes $y = i^3 x$ real. The perturbed solution $\phi = \phi^{(0)} + \phi^{(1)} + \dots$ can be found using Green’s function method

$$\begin{aligned} \phi^{(1)}(y') &= - \int_0^\infty y dy G(y', y) \frac{12y \partial_y \phi^{(0)} + (18y^2 - m^2)\phi^{(0)}}{(3y)^{2/3} \omega^{4/3}}, \\ G(y', y) &= \theta(y - y') K_0(y) I_0(y') + (y \leftrightarrow y'), \end{aligned} \quad (24)$$

where we have selected the Green’s function with the same $y \rightarrow \infty$ boundary condition as $\phi^{(0)}$. Although we were not able to evaluate this integral for generic y , its $y' \rightarrow 0$ limit can be evaluated as a complete Bessel integral:

$$\lim_{y \rightarrow 0} \phi^{(1)}(y) = \log \frac{2}{y} - \gamma_E + \frac{C}{\omega^{4/3}} \left(m^2 + \frac{24}{7} \right) + \mathcal{O}(\omega^{-8/3}),$$

$$C \equiv \frac{1}{3^{2/3}} \int_0^\infty y^{1/3} dy K_0(y)^2 = \frac{2^{2/3} \pi^2 \Gamma(\frac{2}{3})}{3^{1/6} \Gamma(\frac{1}{6})^2}. \quad (25)$$

As done above, we now continue r through the fourth quadrant, which replaces the logarithm by $\log \frac{2}{x} - \frac{3i\pi}{2}$, which we can then use as an initial condition for the differential equation (23) at real x . To evolve these data to large x , we follow the same procedure and use Green's functions which have, respectively, the same asymptotics as $H^{(1)}(x)$ and $H^{(2)}(x)$. The large- x asymptotics are then expressed in terms of complete integrals over oscillatory Bessel functions, but we find that all integrals can again be expressed in terms of the same constant C . Omitting details, we find

$$\lim_{x \rightarrow \infty} \phi \propto (1 + i\omega^{-4/3} \delta\varphi) H_0^{(1)}(x) + (1 - i\omega^{-4/3} \delta\varphi) \times H_0^{(2)}(x) [2 + \omega^{-4/3} R^{(1)} e^{i\pi/3}], \quad (26)$$

up to corrections suppressed at large x or by $1/\omega^{8/3}$. Here $R^{(1)}$ is a constant given below and $\delta\varphi = \frac{1}{7}(3x)^{7/3} + \frac{1}{8}(15 + 4m^2)(3x)^{1/3}$ is a phase which does *not* decay at large x . However, the large- x behavior has to be matched with the WKB ansatz expanded to the equivalent order, which at generic r contains the terms

$$\phi(r) \propto r^{-\frac{3}{2}} \exp \left[i\omega \int \frac{dr}{r^2 f(r)} + i \frac{(15 + 4m^2)r^4 - 3}{8\omega r^3} + \mathcal{O}(\omega^{-2}) \right]. \quad (27)$$

By expanding this in the scaling region $\omega^{-1/3} \ll r \ll 1$ and expanding the Hankel functions to subleading order we find a precise agreement with $\delta\phi$, enabling a r -independent matching to the WKB ansatz, as it should. This enables an unambiguous definition of the reflection coefficient [generalizing (17)] at this order:

$$R = -2 - \frac{e^{i\pi/3} R^{(1)}}{(\beta\omega/\pi)^{4/3}} + \mathcal{O}\left(\frac{1}{\omega^{8/3}}\right),$$

$$R^{(1)} = \frac{3\sqrt{3}C}{\pi} \frac{24 + 7m^2}{7} = \frac{\pi(24 + 7m^2)\Gamma(\frac{2}{3})}{2 \times 18^{1/3} \Gamma(\frac{1}{6}) \Gamma(\frac{13}{6})}, \quad (28)$$

where we restored $r_h = \pi/\beta$ and $m^2 = \Delta(\Delta - 4)$. This nontrivial prediction is verified numerically below.

In the above we can see that the $1/\omega$ terms present in the WKB exponent for generic r neatly canceled when matching the regions, and we observe similar cancellations near the AdS boundary. Hence we expect that the corrections to the boundary correlator consist of a double series combining powers of $1/\omega^{4/3}$ from the singularity and

powers of $1/\omega^4$ from the boundary, with no other powers of ω .

At nonzero but small q , we expect the calculation of $q^2/\omega^{2/3}$ terms to be qualitatively similar. On the other hand, further orders in $1/\omega$ might be significantly more difficult to calculate since perturbation theory will produce iterated integrals over products of Bessel functions.

V. NUMERICAL TESTS

We can test (21) in two ways. First, for the special values of Δ for which the leading series for $\text{Im } G_{\text{ret}}$ terminates, we can simply measure R by plotting $G^> - G_{\text{vac}}^>$. Second, for general Δ , we can compare the t -plane singularities predicted by (22) with the large-order behavior of the OPE coefficients a_n in (10). We discuss these in turns.

Before going into the details, we describe the results of the first test where we set $\Delta = 4$ in (21) (still in $d = 4$). We observe from (11) that the leading transseries for the imaginary part terminates and consists of a single term, leading to the large- ω prediction

$$2 \text{Im } G_{\text{ret}}^{\Delta=4}(\omega, q=0) \rightarrow \omega^4 + \omega^4 e^{-\frac{\beta\omega}{2}} \times \text{Re } e^{i\frac{\beta\omega}{2}} \left[c_0 + \frac{c_1 e^{i\phi_1}}{(\beta\omega/\pi)^{4/3}} + \frac{c_2 e^{i\phi_2}}{(\beta\omega/\pi)^{8/3}} + \dots \right] \quad (29)$$

with $c_0 = 2R|_{\omega=\infty} = -4$ and known values for c_1 and ϕ_1 (see Table I). Note that the equation of motion for a scalar with $\Delta = 4$ is the same as that of the shear mode T^{xy} of the stress tensor studied in [13], who observed exponential decay in ω after subtracting the vacuum. Figure 5 confirms our description of this decaying term since the remainders after subtracting the c_0 and c_1 terms become up to 5 orders of magnitude smaller. Numerical fits to the coefficients c_0 and c_1 , shown in Table I, also demonstrate precise quantitative agreement.

The second check is to compare the t -plane singularity in (22) with the large-order behavior of the coefficients a_n

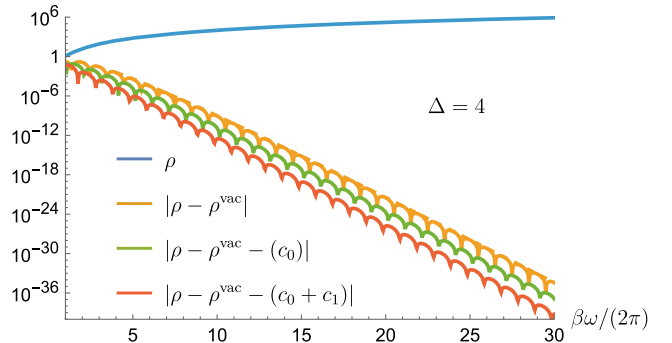


FIG. 5. The spectral density $\rho = 2 \text{Im } G_{\text{ret}}(\omega, q=0)$ for a scalar with $\Delta = 4$, showing the decreasing residuals after subtracting various numbers of terms in (29). A dashed line atop the second (yellow) curve displays the trend $4\omega^4 e^{-\frac{\beta\omega}{2}}$.

TABLE I. Coefficients in the large- ω expansion of the $\Delta = 4$ spectral density in (29). The fit uncertainties are estimated to be on the last digits, based on comparing the fit using $\frac{\beta\omega}{2\pi} \in [20, 30]$ versus $[20, 25]$.

Coefficient	Prediction	Fit
c_0	-4	-3.99999999999994
c_1	$2R^{(1)} _{m=0} \approx 6.46639430\dots$	6.46639431
ϕ_1/π	-2/3	-0.666666668
c_2	Unknown	6.93972
ϕ_2/π	-1/3	-0.3333331

in (10). These coefficients can be calculated efficiently (at least for even d) for any Δ using the algorithm in the Appendix A, which is a momentum space version of the algorithm of [32]. By Fourier transforming (21) one obtains the detailed prediction in (B9), which again agrees exquisitely with numerics as displayed in Table II.

All this confirms the quantitative connections between nonperturbative effects in retarded functions at large real frequencies, complex-time singularities in $G_{\text{ret}}(t)$, and the reflection coefficient [see (17) and (28)] controlling the contribution of null geodesics bouncing off the singularity.

A. Example data for $\Delta = 4$

Here we detail numerical checks performed on the scalar correlator with $\Delta = 4$ in $d = 4$, which as just mentioned is equivalent to the so-called shear channel stress tensor correlator T^{xy} .

It is relatively straightforward to evaluate the frequency space correlator by numerically integrating the radial equation. Working in a variable $u = 1/r^2$, we compute series expansions around $u = 0$ and $u = 1$ and match them at the midpoint where each series converges exponentially fast.

In Fig. 5 we display this numerical result, along with the residual after subtracting the vacuum term (ω^4) and either the c_0 or both c_0 and c_1 terms in (29). Each subtraction is

TABLE II. Coefficients in the large- n behavior (30) of the single-trace coefficients for $\Delta = 4$ using a_n with $n \in [200, 500]$. The analytic prediction for $d_{4/3}$ is $16(4\pi)^{4/3}R^{(1)}$. Numerical results were truncated to the first figure which changed when modifying the range to $[200, 400]$ or changing the ansatz degree.

Coefficient	Prediction	Fit
d_0	2048	2047.9999999999999998
d_1	-5120	-5119.99999999999995
$d_{4/3}$	1511.3509453005...	1511.350945300
d_2	4480	4480.000000
$d_{7/3}/d_{4/3}$	-19/9	-2.111111112
$d_{8/3}$	Unknown	740.420
d_3	-1600	-1600.001

seen to further decrease the residual, as it should, with a net reduction of up to 5 orders of magnitude at $\frac{\beta\omega}{2\pi} = 30$, precisely confirming their values.

We also fitted the correlator to the ansatz (29), observing extremely precise agreement with the predictions for c_0 and c_1 from (21) as shown in Table I. To perform this fit we kept additional subleading coefficients up to c_4 in the ansatz, not shown here. Note that agreement for the phases is somewhat trivial since they are fully determined by the ω exponents together with the reality of $G_{\text{ret}}(t, q = 0)$ at real times; the fitted value of c_2 may be interesting for future work.

The coefficient $c_0 = -4$ in Table I has a simple physical interpretation: the leading nonperturbative correction to $2\text{Im} G_{\text{ret}}$ at large frequencies is the sum of two reflected geodesics (bouncing off, respectively, from the future or past singularity), each weighted by a reflection coefficient $R = -2$. This value helped us single out and identify the derivation presented in Sec. IV.

Another test is to compare the large-order behavior of the single-trace OPE coefficients determined from Appendix A to the prediction in (B9). Again we focus on spatial momentum $q = 0$ ($\gamma = 1$). The advantage of this test is that it can be carried out for any Δ . Inspired by the structure of (B9), we fit the a_n 's obtained from the differential equation to a double expansion in $1/n$ and $1/n^{4/3}$:

$$\frac{2\pi a_n}{(-4)^n n^4} \rightarrow d_0 + \frac{d_1}{n} + \frac{d_{4/3}}{n^{4/3}} + \frac{d_2}{n^2} + \frac{d_{7/3}}{n^{7/3}} + \frac{d_{8/3}}{n^{8/3}} + \dots \quad (30)$$

According to (B9), we expect the coefficients of integer powers of $1/n^{4/3}$ to determine all other coefficients through $r_{p,\nu}$ factors (whose purpose is to cancel $1/\omega$ correction in frequency space), but in our fit we treat them as independent.

For $\Delta = 4$ we obtained 500 a_n terms and used $n \in [200, 500]$ to fit with (30) including terms up to $1/n^7$. The results for the first few coefficients are presented in Table II, again revealing precise agreement.

This large-order analysis was important in our study for several reasons. First, the value of d_0 is equivalent to the leading term $c_0 = -4$ discussed above, thus independently confirming that $R \approx -2$ to leading order. The large-order numerics further indicated that this value was independent of Δ , an important hint of its universal origin from near the singularity.

Second, our initial large- n fits included only powers of $1/n$ (and eventually, logarithms) but this led to poor fits and the extracted values were not stable beyond the leading figures. Modeling the corrections by powers of $1/n^{4/3}$ immediately stabilized and improved the fits, which suggested the calculation in Sec. IV B.

Finally, we also repeated the large- n analysis for many $\mathcal{O}(1)$ values of Δ , again finding perfect agreement within

errors, thus confirming the simple functional dependence of $R^{(1)}$ on Δ in (28).

VI. DETAILED ALL-ORDER ANALYSIS FOR R -CURRENT CORRELATORS

To illustrate the ideas in a case where analytic solution is possible, we look at the radial equation corresponding to the transverse current correlators G^{xx} [14]. We find a different near-singularity scaling limit controlled by Airy instead of Bessel functions, and a corresponding reflection coefficient $Re^{-i\pi\nu} = 1$ in agreement with (1). Here we will also analyze sub-subleading saddles, e.g., multiply-bouncing geodesics.

We analyze the two point functions of transverse R -currents J^x with $q = 0$ in the planar AdS_5 black hole. The analysis will be simplified by defining a “periodic coordinate” z by

$$\tanh(z) = \frac{r_h^2}{r^2}. \quad (31)$$

The wave equation for the R -charge currents [13] at $q = 0$ in these coordinates becomes (we set $\beta = 2\pi$ in this section)

$$\chi''(z) + \omega^2 \coth(z)\chi(z) = 0, \quad (32)$$

where $\phi(r) = \frac{\sqrt{r^4 - r_h^4}}{\sqrt{r}}\chi(z)$. The complex r -plane and z -planes are shown in Fig. 6. The AdS boundary at $r = \infty$ is mapped to $z = 0$ and the horizon at $r = r_h$ is mapped to $z = +\infty$; the singularity at $r = 0$ is mapped to $z = i\frac{\pi}{2}$ and its images at $z = i\frac{\pi}{2} + in\pi$ for $n \in \mathbb{Z}$. In these coordinates one finds an infinite series of simple turning points in the z -plane, hence the name. The exact solution to the wave equation with ingoing boundary conditions at the horizon is given by

$$\begin{aligned} \chi(z) = & \frac{\Gamma(1 + \frac{1-i}{2}\omega)\Gamma(1 - \frac{1+i}{2}\omega)}{2^{-\frac{1+i}{2}\omega}\Gamma(1-i\omega)} \times \frac{(\coth(z) - 1)^{-\frac{i\omega}{2}}}{(\coth(z) + 1)^{\frac{\omega}{2}}} \\ & \times {}_2F_1\left[1 - \frac{1+i}{2}\omega, -\frac{1+i}{2}\omega, 1-i\omega, \frac{1-\coth(z)}{2}\right], \end{aligned} \quad (33)$$

which we normalized to 1 at the AdS boundary $z = 0$. Expanding in this limit we find

$$\chi(z) = 1 + z\omega^2(1 - \log(z)) + z\kappa_2 + O(z^2), \quad (34)$$

where κ_2 gives the retarded Green's function as [14]

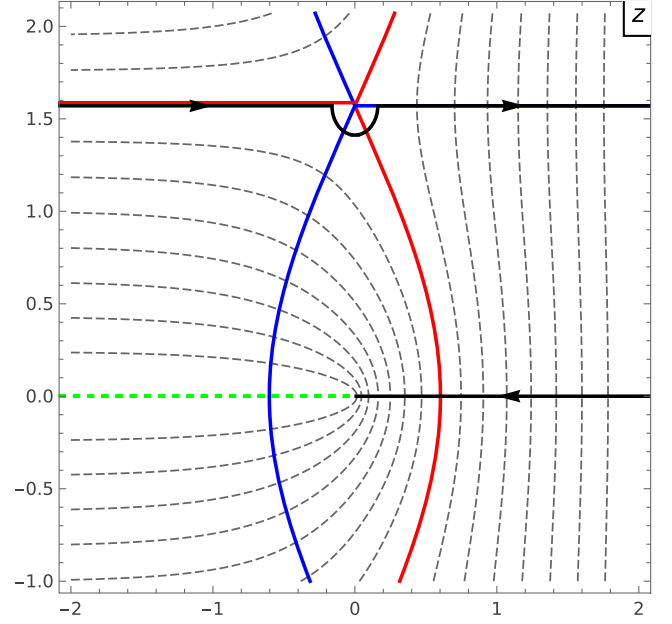


FIG. 6. Stokes lines (blue, at angles 0 and $\pm 2\pi/3$) where saddles exchange dominance, and anti-Stokes lines (red), where $\text{Re}S = 0$, in the complex z -plane. The AdS boundary is at $z = 0$, the black hole singularity at $z = i\pi/2$, and the horizon at $z \rightarrow +\infty$. We placed branch cuts of the solution going to the left (in dashed green). The black line depicts the contour described in the text.

$$\begin{aligned} G_{\text{ret}}^{xx}(\omega, q = 0) = & -\omega^2 \left[\psi^{(0)} \left(1 + \frac{1-i}{2}\omega \right) \right. \\ & + \psi^{(0)} \left(-\frac{1+i}{2}\omega \right) \\ & \left. + 2\gamma_E + \log 2 - \omega^{-1} \right]. \end{aligned} \quad (35)$$

Taking the imaginary part reproduces the spectral density quoted in (1):

$$\text{Im } G_{\text{ret}}^{xx}(\omega, q = 0) = \frac{\pi\omega^2 \sinh(\pi\omega)}{\cosh(\pi\omega) - \cos(\pi\omega)}. \quad (36)$$

In the rest of this section, we will rederive these expressions using WKB methods.

A. Leading WKB approximation

The leading WKB approximation to differential equations of the form

$$\chi''(z) + Q(z)\chi(z) = 0 \quad (37)$$

is given by

$$\chi(z) = \frac{1}{Q(z)^{1/4}} \left(A e^{i \int \sqrt{Q(z)} dz} + B e^{-i \int \sqrt{Q(z)} dz} \right). \quad (38)$$

In our case, we have $Q(z) = \omega^2 \coth(z)$, and to compute the retarded Green's function, we need to impose ingoing boundary conditions at the horizon $z = \infty$, which sets $B = 0$. The WKB solution near the horizon is then given by

$$\chi(z) = \frac{e^{iS_0}}{(\omega^2 \coth[z])^{1/4}}, \quad (39)$$

where $S_0 = \omega z - \int_z^\infty \omega (\sqrt{\coth(z')} - 1) dz'$ and we have expressed the action in this way to reabsorb the divergent piece into the normalization constant. We wish to know the solution near the boundary at $z = 0$. For small z , the general solution (34) is readily obtained by a Frobenius expansion. To connect the WKB solution to the boundary, we need to consider an intermediate matching regime $\frac{1}{\omega^2} \ll z \ll 1$, where the differential equation simplifies to the Bessel form

$$\chi''(z) + \frac{\omega^2}{z} \chi(z) \approx 0. \quad (40)$$

The oscillatory solution in (39) then matches specifically to the Hankel function

$$\chi(z) \rightarrow \sqrt{\pi z} e^{-\frac{i}{4}\omega(\pi + \log(4)) + \frac{3i\pi}{4}} H_1^{(1)}(2\sqrt{z}\omega). \quad (41)$$

Expanding near the AdS boundary $z \rightarrow 0$ as above, we find the leading approximating to the retarded Green's function

$$G_{\text{ret}}^{xx}(\omega, q = 0) \approx \omega^2 (-2 \log(\omega) - 2\gamma_E + i\pi), \quad (42)$$

which agrees, of course, with the leading term of (35) as $\omega \rightarrow \infty$. The corrections are given by a nontrivial asymptotic series in $1/\omega^4$ with real coefficients. We now show how the method from Sec. IV captures the nonperturbative corrections to it.

B. First nonperturbative correction

The red path used in Fig. 4 is shown in the z -plane in Fig. 6. The coordinate map $\tanh z = r_h^2/r^2$ sends the point $r = -ir_h$ (the attractor point of steepest-descent contours in the complex r -plane) to

$$z_* = -\infty. \quad (43)$$

As in the text, we denote as $\chi^{\text{steepest}}(z)$ the solution which is exponentially decaying as $\text{Re} z_* \rightarrow -\infty$; it is approximately equal to the infalling solution at large ω , but is not exactly the same.

To evolve $\chi^{\text{steepest}}(z)$ to positive z where we can most easily compare with (33), we stay on the line $\text{Im} z = \pi/2$ so

as to avoid crossing a Stokes line. The price to pay is that we hit the turning point of the potential $\omega^2 \coth(z)$ at

$$z_s = i \frac{\pi}{2}, \quad (44)$$

which corresponds physically to the curvature singularity $r = 0$ in the original radial coordinate. Near this point the WKB expansion breaks down because $\coth z$ vanishes linearly. To resolve this region we introduce the scaling variable

$$x = \omega^{2/3}(z - z_s). \quad (45)$$

Keeping x fixed as $\omega \rightarrow \infty$ transforms the wave equation (37) into the universal Airy form

$$\chi''(x) + x\chi(x) \approx 0. \quad (46)$$

The solution that decays toward negative x (and thus negative z) is given by

$$\chi^{\text{steepest}}(x) \propto \text{Ai}(-x). \quad (47)$$

The Airy function is entire, but, as is well known, its asymptotics at positive x are not a simple continuation of the decaying exponential. Rather, it is a combination of two oscillatory exponentials:

$$\text{Ai}(-x) \rightarrow \frac{e^{i\pi/4}}{2\sqrt{\pi}x^{1/4}} \left(e^{i\frac{2}{3}x^{3/2}} - iRe^{-i\frac{2}{3}x^{3/2}} \right), \quad R = -1, \quad (48)$$

where we defined a reflection coefficient R using the same phase convention as in (17). This solution can be matched to the WKB form (38) in an overlap region where both the WKB and Airy approximations are valid,

$$\omega^{-\frac{2}{3}} \ll \left| z - \frac{i\pi}{2} \right| \ll 1, \quad x \gg 1. \quad (49)$$

Continuing each WKB solution and matching to the Bessel functions at $z \rightarrow 0$ (the AdS boundary) as in (41) then gives, up to a proportionality constant,

$$\chi^{\text{steepest}}(z) \rightarrow \sqrt{\pi z} [H_1^{(1)}(2\sqrt{z}\omega) + Re^{(-1+i)\pi\omega} H_1^{(2)}(2\sqrt{z}\omega)], \quad (50)$$

which is equivalent to (19) in with $\nu = 1$. Taking $z \rightarrow 0$ as in (34) we find the analog of the transseries (21) for the retarded function, with $e^{-i\pi\nu} R = +1$:

$$G_{\text{ret}}^{xx}(\omega, q = 0) \propto \omega^2 \left[\frac{i}{2} - \frac{\log \omega}{\pi} + \mathcal{O}(\omega^{-4}) + i e^{-\frac{\beta\omega}{2}(1-i)} \times (1 + \mathcal{O}(\omega^{-4})) + \dots \right]. \quad (51)$$

In particular, the imaginary part is proportional to $\omega^2(1 + 2e^{-\beta\omega/2}\cos(\frac{\beta\omega}{2}) + \dots)$, in precise agreement with (1) and (36). The only difference between the leading nonperturbative contribution to the current and scalar correlators is the value of R [compare (48) with (16)], which is caused by the different behavior near the singularity.

C. All-order nonperturbative corrections

For the retarded function of currents, the strategy just described can be implemented to go beyond the leading correction in (51). The exact solution to (32) that is regular at $z \rightarrow -\infty$ is

$$\chi^{\text{steepest}}(z) = \frac{\Gamma(1 + \frac{1-i}{2}\omega)\Gamma(1 + \frac{1+i}{2}\omega)}{2^{\frac{1-i}{2}\omega}\Gamma(1+\omega)} \frac{(-\coth(z) - 1)^{\frac{\omega}{2}}}{(-\coth(z) + 1)^{\frac{\omega}{2}}} \times {}_2F_1\left[1 + \frac{1-i}{2}\omega, \frac{1-i}{2}\omega, 1 + \omega, \frac{\coth(z) + 1}{2}\right]. \quad (52)$$

Upon analytically continuing to positive z with a small positive imaginary part, it becomes a combination of infalling and outgoing modes at the horizon, with the latter suppressed by a relative $e^{-\beta\omega/2}$ as anticipated in (14). Let us quantify the remainder more precisely.

A good way to impose the infalling condition at the horizon ($z \rightarrow +\infty$) is to impose the correct monodromy $\sim e^{-\beta\omega/2}$ under $z \mapsto z + i\pi$ at positive z . Since the similar shift on the left only rescales χ^{steepest} by a phase $e^{i\beta\omega/2}$, this can be phrased in terms of two analytic continuations of the same function:

$$\chi(z) \propto \chi^{\text{steepest}, \nearrow}(z) + e^{-\frac{1+i}{2}\beta\omega} \chi^{\text{steepest}, \searrow}(z) \quad (53)$$

where $\chi^{\text{steepest}, \searrow}$ is defined by continuing slightly above the image of the AdS boundary at $z = -i\pi$, to ensure the correct periodicity on the right. However, because the solution has no singularities between 0 and $-i\pi$, this is the same as continuing slightly below the origin. The difference between the two continuations is thus regular at the AdS boundary $z = 0$:

$$\begin{aligned} \chi^{\text{reg}}(z) &\equiv \chi^{\text{steepest}, \nearrow}(z) - \chi^{\text{steepest}, \searrow}(z) \\ &= ie^{iz\omega} \pi \omega^2 (1 - e^{-2z}) \\ &\times {}_2F_1\left[1 + \left(\frac{1-i}{2}\right)\omega, 1 - \left(\frac{1+i}{2}\right)\omega, 2, 1 - e^{-2z}\right]. \end{aligned} \quad (54)$$

Combining the previous two equations we can write the infalling solution exactly as

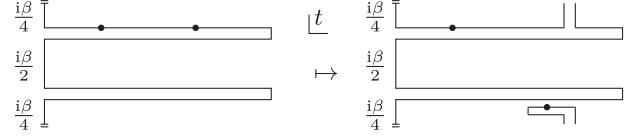


FIG. 7. Analytic continuation of the single-sided correlator in the thermofield double state to the configuration which displays the bouncing geodesics. The two initial timefolds represent the left and right exteriors of the Kruskal geometry.

$$\chi(z) = \chi^{\text{steepest}, \nearrow}(z) + \frac{1}{e^{(1-i)\beta\omega/2} - 1} \chi^{\text{reg}}(z). \quad (55)$$

The second term is exponentially small at large real ω , confirming (14). The above is normalized to $1 + \mathcal{O}(z)$ at the AdS boundary, and following (34) we get the exact retarded function in the form

$$\begin{aligned} G_{\text{ret}}^{xx}(\omega, q=0) &= -\omega^2 \left[\psi^{(0)} \left(1 + \frac{1-i}{2}\omega \right) \right. \\ &\quad \left. + \psi^{(0)} \left(1 + \frac{1+i}{2}\omega \right) + 2\gamma_E + \log 2 \right. \\ &\quad \left. - \omega^{-1} - i\pi \right] + \frac{2\pi i \omega^2}{e^{(1-i)\beta\omega/2} - 1}. \end{aligned} \quad (56)$$

This can be verified to be equivalent to (35) using a polygamma identity. The upshot of this form is that the square bracket, which originates from the first term of (55), can be interpreted as a canonical Borel resummation of the large- ω asymptotic series for real ω ; this interpretation is further confirmed in Appendix C. The second term captures the genuine nonperturbative corrections. We see that they naturally take the form of a geometric series, representing multiple reflections off the singularity and AdS boundary.

For general correlation functions, the step leading to (54) would be significantly more complicated since the solutions will not be analytic near the black hole singularity at $z = \frac{-i\pi}{2} + i\pi n$. The trivial monodromy there is the main simplifying feature of the $G^{xx}(\omega, q=0)$ correlator.

VII. DISCUSSION

As in the Introduction, we emphasize that our analysis describes measurements done in the *exterior* of a black hole: the “interior” gets generated by analytic continuation of the exterior geometry along a steepest-contour path. All effects of the singularity are contained in the reflection coefficient $R(\omega)$ in (21).² As was also stressed in the Introduction, there is no obvious connection between these data and the experience of an infalling observer.

²The reflection coefficient is expected to receive stringy corrections at frequencies higher than those we considered; see, for example, [33].

We elaborate on this point in Fig. 7 by showing the analytic continuation to $\text{Im } t = +i\beta/2$, where the signal is found, of the path integral which calculates the correlation function. The analytic continuation from the right exterior of the Kruskal geometry appears to spawn a copy of the left exterior that is distinct from the “original” left region. For example, modifying the thermofield double state by inserting a unitary to add infalling matter in the original left region would have strictly no effect on its copy. It would also not affect the spectral density measured in the right exterior, as is also clear from the commutativity of left and right operators.

The continued contour in Fig. 7 explicitly violates the Kontsevich-Segal criterion [34,35] for convergence of the path integral: it features backward Euclidean time evolution. This simply means that there is no simple axiomatic reason for the analytic continuation of $G^>(t)$ to exist. It is unclear whether this complex saddle point can be precisely defined at finite N . Finite- N effects, which are beyond the accuracy of our calculations, could thus modify the exponential decay exemplified in Figs. 1 and 5 at sufficiently large frequencies, either suppressing or enhancing the (already exponentially small) signal from the bouncing geodesics.

From the field theory perspective, the *absence* of other nonanalyticity in the strip $|\text{Im } t| < \beta/2$ is notable. By conformally mapping the strip to the unit disk via $z = \tanh(\pi t/(2\beta))$, convergence of the resulting series for $|z| < 1$ implies that the small- t OPE series of the retarded function can be resummed to reach late times. Fourier-transforming the KMS condition (12), this would also explicitly reconstruct all Wightman functions from the retarded function’s OPE; this could be relevant for bootstrap approaches [22–24]. Given the difficulty of producing complex-time singularities at weak coupling, it is tempting to conjecture convergence of the OPE for $|z| < 1$ in any relativistic thermal field theory.

Our analysis has been limited to zero spatial momentum. Assuming that the Fourier transform to position space is controlled by the q -dependence of the WKB phase, $\omega\Delta t \supset |q|^{3/2}/\omega^{1/2}$, one would expect $|q| \sim \omega^{1/3}$ to dominate. Unfortunately, this is exactly where the corrections to (15) become important, invalidating the use of $R \approx -2$ within the Fourier transform. In general d , there is a nontrivial large- ω scaling regime where $q \sim \omega^{\frac{d-2}{2(d-1)}}$ and $m \sim \omega^{\frac{d}{2(d-1)}}$ and all effects compete near the singularity:

$$r^2\phi'' + r\phi' + (\omega^2 r^d + m^2 r^2 + q^2)r^{d-2}\phi \approx 0. \quad (57)$$

Calculating the reflection coefficient for this equation would enable Fourier transforming our results to position space for general d and mass.

Several open questions remain. Can the reflection coefficient at the singularity be related to a final-state projection in black hole quantum mechanics? Since the

contour in Fig. 4 runs along the imaginary r axis, where the radial coordinate plays the role of a time coordinate, does that mean we can interpret quantities in the dual CFT in terms of the cosmology that lives behind the event horizon? How will our analysis change when extended to black holes in asymptotically flat spacetime (see also [36])? Do higher-point correlators allow us to gain further insight into the black hole interior (see also [37])? For black holes with both inner and outer horizons, does the presence of timelike singularities and multiple copies of the asymptotic boundaries lead to new “bouncing” saddles in addition to those we have considered (see also [38–40])? What happens in dimension $d \neq 4$? Are there connections with the Heun equation and integrability, as in [41]? What are the broader implications of our results for the physics of the black hole interior and ways to probe it, including quasinormal modes [27], near-singularity Kasner-behavior [42,43] and stringy features [33,44]?

ACKNOWLEDGMENTS

We thank Nejc Čeplak, Sean Hartnoll, Yoav Zigdon, and especially Keivan Namjoo for discussions. The work of N. A.-J., S. C.-H. and J.-C. is supported by the National Science and Engineering Council of Canada (NSERC), funding reference SAPIN/00028-2022, and the Canada Research Chair program, reference number CRC-2022-00421. The work of A. M. is supported in part by the Simons Foundation Grant No. 12574 and the Natural Sciences and Engineering Research Council of Canada (NSERC), funding Reference No. SAPIN/00047-2020.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: HIGH-FREQUENCY EXPANSION OF $G_{\text{ret}}(\omega)$

Here we present our algorithm to expand the retarded correlator at large $(\omega^2 - k^2)$ when d is an even integer. The algorithm can be viewed as a momentum space version of that in [32]. The perturbative contributions originate from the region where r is large but the ratio $z = \ell^2 \frac{\sqrt{\omega^2 - k^2}}{r}$ is fixed. In the limit, the wave equation (4) expressed in terms of $\phi(r) = ((r/r_h)^d - 1)^{-1/2} \chi(z)$ becomes the Bessel equation plus perturbative corrections:

$$[z^2 \partial_z^2 + z \partial_z + z^2 - \nu^2] \chi(z) \equiv \mathcal{D}_\nu \chi(z) = U \chi(z), \quad (\text{A1})$$

where $\nu = \Delta - \frac{d}{2}$. The perturbing potential is

$$U = \epsilon z^d \left[\frac{\nu^2 - z^2}{1 - \epsilon z^d} - \frac{\gamma^2 z^2 + (d/2)^2}{(1 - \epsilon z^d)^2} \right] \\ = \sum_{n=1}^{\infty} \epsilon^n z^{nd} \left[\nu^2 - z^2 - n(\gamma^2 z^2 + (d/2)^2) \right], \quad (\text{A2})$$

where $\epsilon = \left(\frac{4\pi T/d}{\sqrt{\omega^2 - q^2}} \right)^d$ is a dimensionless ratio of the energy density and probe energy and $\gamma = \omega / \sqrt{\omega^2 - q^2}$.

At order ϵ^0 , the solution to (A1) are Bessel/Hankel functions. Infalling boundary conditions at the horizon pick the solution with asymptotic $\chi \propto e^{iz}$ (up to nonperturbative effects that do not contribute at any finite order in $1/\omega$):

$$\mathcal{D}_\nu \chi^{(0)}(z) = 0 \Rightarrow \chi^{(0)}(z) \propto H_\nu^{(1)}(z). \quad (\text{A3})$$

We see from (A2) that to calculate the $\mathcal{O}(\epsilon^1)$ correction to χ , we need to solve the inhomogeneous equation $\mathcal{D}_\nu \delta\chi = z^a H_\nu^{(1)}$ with $a \in \{d, d+2\}$. In general this cannot be solved in terms of elementary functions, and therefore the procedure becomes difficult to iterate. However, when d is an even integer, elementary solutions turn out to exist.

Our method consists in making an ansatz for $\phi(z)$ in terms of polynomials multiplying Bessel functions. This was inspired by the position space ansatz in [32], which used terms of the form $(t^2 - \vec{x}^2)^a t^b$; the Fourier transform of each term gives a Bessel function times a polynomial when b is an even integer (this is the only case where the procedure works). Staying in momentum space, we find that it suffices to use the original Bessel function, and one with index shifted by 1, for example,

$$\mathcal{D}_\nu^{-1}(2z^2 H_\nu^{(1)}) = z H_{\nu+1}^{(1)}, \quad (\text{A4a})$$

$$\mathcal{D}_\nu^{-1}(4z^3 H_{\nu+1}^{(1)}) = -z^2 H_\nu^{(1)} + 2z(1 + \nu) H_{\nu+1}^{(1)}, \quad (\text{A4b})$$

$$\mathcal{D}_\nu^{-1}(6z^4 H_\nu^{(1)}) = z^2(1 - \nu) H_\nu^{(1)} + z(z^2 + 2\nu^2 - 2) H_{\nu+1}^{(1)}. \quad (\text{A4c})$$

Here the argument of every Hankel function is z . The second line illustrates that the procedure can be iterated: the set of functions H_ν and $H_{\nu+1}$ times polynomials form a closed set under the action of $\mathcal{D}_\nu^{-1}$. This enables solving (A1) to arbitrarily high order, at least when d is an even integer. The solutions (A4) are unchanged if $H_c^{(1)}$ is replaced by any other Bessel function (e.g., $H_c^{(2)}$ or J_c or Y_c).

The inverse $\mathcal{D}_\nu^{-1}$ is unique up to homogeneous solutions; we impose the (perturbative) infalling condition and vanishing of the $z^{-\nu}$ coefficient, which modifies (A4a) by addition of $H_\nu^{(1)}$ terms. The following recursion then efficiently computes all required primitives:

$$\mathcal{D}_\nu^{-1}(z^k H_\nu^{(1)}) = \frac{z^{k-1} H_{\nu+1}^{(1)}}{2(k-1)} - \frac{n_{k,\nu}}{2(k-1)} \mathcal{D}_\nu^{-1}(z^{k-1} H_{\nu+1}), \\ \mathcal{D}_\nu^{-1}(z^k H_{\nu+1}^{(1)}) = -\frac{z^{k-1} H_\nu^{(1)}}{2(k-1)} + \frac{k-1+2\nu}{2} \mathcal{D}_\nu^{-1}(z^{k-1} H_\nu^{(1)}) \quad (\text{A5})$$

with $n_{k,\nu} = (k-2)(k-2-2\nu)$. We use these for $k \geq 4$ even in the first line and $k \geq 3$ odd in the second line. The recursion then terminates on (the modified) (A4a).

Using this method, for example, for $\Delta = 4$ (equivalent to the shear channel stress-tensor correlator $G_R^{xy,xy}$), we find the following order-by-order solution to the wave equation

$$\chi_{\gamma=1}^{\Delta=4}(z) = \left[1 - \epsilon^2 \left(\frac{2z^6}{15} + \frac{z^8}{45} + \frac{z^{10}}{50} \right) + \mathcal{O}(\epsilon^3) \right] H_2(z) \\ + \left[-\epsilon \frac{z^5}{5} + \epsilon^2 \left(\frac{4z^5}{5} - \frac{2z^7}{15} - \frac{z^9}{90} \right) + \mathcal{O}(\epsilon^3) \right] H_3(z). \quad (\text{A6})$$

It is also straightforward to retain the γ dependence, which is polynomial, and so we record it below. Taking the ratio of the z^ν and $z^{-\nu}$ coefficients as $z \rightarrow 0$ according to the standard recipe (5), we thus obtain the retarded Green's function

$$G_{\text{ret}}^{\Delta=4}(\omega, q) \propto (\omega^2 - q^2)^2 \left[-\log \frac{q^2 - (\omega + i0)^2}{\mu^2} \right. \\ \left. - \frac{64\epsilon}{5}(\gamma^2 + 1) + \frac{512\epsilon^2}{35}(28\gamma^4 - 24\gamma^2 + 3) \right. \\ \left. + \mathcal{O}(\epsilon^3) \right]. \quad (\text{A7})$$

The log term is the $T = 0$ result, which also controls the short-distance limit of the correlator; the overall convention-dependent normalization, left unspecified here, could be determined from this limit. Note that the $T = 0$ term is logarithmically ultraviolet divergent, whence the arbitrary holographic renormalization scale μ ; this generically happens for integer Δ . For real positive ω , the imaginary part of the square bracket, $i\pi$, comes entirely from the first term as explained in the text. We expect the expansion (A7) to be valid for spacelike or timelike momentum ($\gamma < 1$ or $\gamma > 1$), although its validity near the light cone [when $\epsilon\gamma^2 \propto T^4\omega^2/(\omega^2 - q^2)^3 \gtrsim 1$] is less clear (similar scales appeared in [45]).

The method just detailed enables the calculation of $1/\omega$ series to very high order in ϵ . The bottleneck is multiplication of the potential U by the lower-order solutions, and multiplication of the resulting coefficients with the tabulated $\mathcal{D}_\nu^{-1}$ primitives [which are obtained essentially instantaneously from (A5)]. Using exact rational arithmetic

in our *Mathematica* implementation, obtaining the correlator to order ϵ^{400} for $\Delta = 4$ and $q = 0$ takes about an hour on a laptop.

APPENDIX B: ASYMPTOTICS IN DIFFERENCE SPACES

Here we record various formulas relating the small-time and large-frequency expansions of the Wightman functions, focusing on zero spatial momentum ($q = 0$). We reproduce here for convenience the expansion coefficients defined in the text for the retarded function in (10) and (11):

$$G_{\text{ret}}(t, q = 0) = t^{-1-2\nu} \sum_{n=0}^{\infty} a_n \left(\frac{t}{\beta}\right)^{nd} \quad (t > 0), \quad (\text{B1})$$

$$G_{\text{ret}}(\omega, q = 0) \sim (-i\omega)^{2\nu} \sum_{n=0}^{\infty} \frac{a_n}{(-i\beta\omega)^{nd}} \Gamma(nd - 2\nu), \quad (\text{B2})$$

where we recall that $\nu = \Delta_{\mathcal{O}} - \frac{d}{2}$. Taking the imaginary part of the second gives the asymptotic series for $G^>(\omega, q = 0) = \frac{2\text{Im } G_{\text{ret}}(\omega)}{1 - e^{-\beta\omega}}$, which is the same, up to corrections $\sim e^{-\beta\omega}$ that are nonperturbative at large ω , as that for $2\text{Im } G_{\text{ret}}(\omega)$:

$$G^>(\omega) \sim \sum_{n=0}^{\infty} \frac{a_n \omega^{2\nu - nd} \beta^{-nd}}{\Gamma(1 + 2\nu - nd) \cos(\frac{\pi}{2}(2\nu - nd))}. \quad (\text{B3})$$

Fourier-transforming back to the time domain, this high-frequency expansion only determines the *nonanalytic* terms in $G^>(t)$ near $t = 0$:

$$G^>(t, q = 0) = (it)^{-1-2\nu} \sum_{n=0}^{\infty} \frac{a_n}{\cos(\frac{\pi}{2}(nd - 2\nu))} \left(\frac{it}{\beta}\right)^{nd} + (\text{integer powers of } t^2). \quad (\text{B4})$$

The second line can be interpreted as the contribution to the OPE from $\mathcal{OO}$ double traces. Note that we have written $G^>(t)$ in a form which respects the analyticity of this correlator in the strip $-\beta < \text{Im } t < 0$; i.e., a small negative imaginary part is allowed.

Using the complex conjugate expansion for $G^<(t)$, (B1) can be verified directly to be the discontinuity of (B4), that is, $G_{\text{ret}} = i(G^> - G^<)$.

Starting from the large-frequency expansion of $G_{\text{ret}}(\omega)$ in (B2), the coefficients a_n can thus be obtained by dividing each term by a Gamma function and a phase.

Although closely related expansions have been considered in the literature, direct comparison did not seem possible since results typically are for $\vec{x} = 0$ [for example, (3.9) of [10]], whereas we focus on $\vec{q} = 0$. The full Gegenbauer expansion in (9) (for the retarded function)

could be obtained in future work by Fourier-transforming (A7) termwise, keeping all the dependence on γ .

Consider now a nonperturbative contribution $G_{\text{ret}}(\omega, q = 0) \supset b_p e^{i\omega t_*} \omega^p$ appearing in a large- ω transseries such as (21). Evaluating the Fourier transform on the contour in Fig. 3, this is equivalent to a t -plane singularity:

$$G_{\text{ret}}(\omega) \supset b_p \omega^p e^{i\omega t_*} \Leftrightarrow G_{\text{ret}}(t) \supset \frac{b_p \Gamma(p+1)}{2\pi} \frac{1}{(i(t - t_*))^{p+1}}. \quad (\text{B5})$$

Let us translate this into a prediction for the large-order behavior of the coefficients a_n around $t = 0$, assuming that the singularity at t_* is the one closest to the origin. The basic idea is to use the following identity for the singular behavior as $x \rightarrow 1$ of a sum with power-law coefficients:

$$\sum_{n=1}^{\infty} n^p x^n = \frac{\Gamma(p+1)}{(1-x)^{p+1}} \left(1 - \frac{p+1}{2}(1-x) + \mathcal{O}((1-x)^2) \right) + (\text{terms analytic at } x \rightarrow 1). \quad (\text{B6})$$

Applying this to $x = (t/t_*)^d$ and comparing with (B1),

$$\lim_{n \rightarrow \infty} a_n = n^p \frac{b_p (id)^{p+1}}{2\pi t_*^{p-2\nu}} \left(\frac{\beta}{t_*}\right)^{nd} \times r_{p,\nu}(n) \quad (\text{B7})$$

with remainder $r_{p,\nu}(n) = 1 + \mathcal{O}(n^{-1})$.

The subleading corrections to (B7), assuming a pure power law in (B5) the ω or t planes, are interesting because they can be detected numerically. They have two origins. First, the pure power in $(t - t_*)$ in (B5) does not map to a pure power of $(1 - x)$. Second, a pure power of $(1 - x)$ does not translate into a pure power of n , due to the series of corrections in (B6). Both effects can be systematically accounted for order by order in $1/n$. Accounting for these we obtain the more refined large-order prediction

$$r_{p,\nu}(n) = 1 + \frac{p}{n} \frac{p-1-4\nu}{2d} + \frac{p(p-1)}{n^2} \times \frac{3p^2 - p(24\nu + 7) + 48\nu^2 + 24\nu + 2}{24d^2} + \mathcal{O}(n^{-3}). \quad (\text{B8})$$

Applying these formulas to the transseries in (21) including the subleading reflection coefficient in (28), we obtain the following prediction for the large- n behavior of the expansion coefficients [in the same normalization as (21), where $a_0 = \Gamma(2\nu + 1) \cos(\pi\nu)/\pi$]:

$$\lim_{n \rightarrow \infty} \frac{a_n}{(-4)^n} = (4n)^{2\nu} \frac{4}{\pi} r_{2\nu, \nu} + (4n)^{2\nu-4/3} (2\pi)^{1/3} R^{(1)} r_{2\nu-4/3, \nu} + \mathcal{O}(n^{2\nu-8/3}). \quad (\text{B9})$$

APPENDIX C: BOREL RESUMMATION FOR R-CURRENT CORRELATORS

In this Appendix we confirm the identification of the square bracket of (56) with the canonical Borel resummation of the large- ω series. Using the standard series expansion of $\psi^{(0)}$ we find

$$G_{\text{ret}}^{xx}(\omega, q=0) \sim \omega^2 \left[i\pi - 2 \log(\omega) + \sum_{n=1}^{\infty} \left(\frac{-4}{\omega^4} \right)^n \frac{B_{4n}}{2n} \right], \quad (\text{C1})$$

where B_n are Bernoulli numbers. The $\sim$ indicates that this is an asymptotic expansion: the coefficients grow factorially. In the notation of (B2), this gives the coefficients $a_0^{xx} = -4$ and

$$a_n^{xx} = -\frac{(-64\pi^4)^n B_{4n}}{2n\Gamma(4n-2)} \quad (n \geq 1). \quad (\text{C2})$$

(Note that the normalization here is 2π larger than that used for scalar correlators in the rest of this paper.) In the time domain the series (B1) has a finite radius of convergence and, in fact, can be summed analytically:

$$G_{\text{ret}}^{xx}(t, q=0) = -\partial_t^2 \left[\frac{\pi(1-i)/\beta}{\tan \frac{\pi t}{\beta}(1-i)} + \frac{\pi(1+i)/\beta}{\tan \frac{\pi t}{\beta}(1+i)} \right]. \quad (\text{C3})$$

The canonical Borel resummation of the series (C1), for large real ω , is defined by integrating the Fourier transform over imaginary t which is the steepest-descent path for $e^{i\omega t}$. As depicted in Fig. 3, the full Fourier transform over the real axis is equal to this, plus the sum over the poles at $t = k \frac{1+i}{2} \beta$. Accounting for ultraviolet regularization and a small arc near the origin (responsible for the $i\pi$ below), we can write these explicitly as

$$\begin{aligned} G_{\text{ret}}^{xx}(\omega, q=0) &\equiv \int_0^\infty dt e^{i\omega t} G_{\text{ret}}^{xx}(t, q=0) \\ &= \left[\int_0^\infty d\tau e^{-\omega\tau} \left(i G_{\text{ret}}^{xx}(i\tau, q=0) - \frac{4}{\tau^3} \right) \right. \\ &\quad \left. + \omega^2 (i\pi - 2 \log(\omega) - 2\gamma_E) \right] \\ &\quad + 2\pi i \omega^2 \sum_{k=1}^{\infty} e^{i(1+i)k\beta\omega/2}. \end{aligned} \quad (\text{C4})$$

The square bracket and remainder can be verified to be precisely equal to those in (56). This confirms the bulk-boundary identification proposed in Sec. IV: canonically Borel-resumming the boundary correlator using the steepest-descent time contour, coincides with solving the bulk radial equation along a steepest-descent contour in r .

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