

Probing continuous spin QED with rare atomic transitions

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An intriguing and elementary possibility is that familiar massless particles like the photon could be “continuous spin” particles (CSPs) with a small but nonzero spin Casimir ρ . In this case, the familiar two polarization states of the photon are accompanied by an infinite tower of integer spaced helicity modes, with couplings dictated entirely by Lorentz symmetry and the parameter ρ . We present a formalism for computing bound state atomic transitions for scalar QED when $\rho \neq 0$, employing path integral methods not often used for bound state computations, but that readily generalize to the CSP case. We compute several illustrative amplitudes and show that $\rho \neq 0$ opens new decay channels for atomic transitions with rates controlled by $\rho\alpha/\omega$ for transition frequency ω . These new channels can appreciably modify the rates of “forbidden” transitions. For example, the lifetime of the hydrogen $2s$ state would be affected at $O(1)$ for $\rho \sim 0.1$ eV, suggesting new directions for fundamental tests of QED in laboratory experiments.

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I. INTRODUCTION

Wigner famously classified all massless particles in $3+1$ dimensions, showing that they can be described by irreducible representations of the ISO(2) Little Group [1]. Additional restrictions on their interactions come from the covariance of soft amplitudes [2–4] and consistency of gauge and general relativity couplings [5–8], which imply that the only long-range forces possible are those mediated by massless particles with spin 0, 1, or 2 (i.e., scalar particles, vector gauge theories, or general relativity). However, these arguments rely on the assumption that massless particles have helicity eigenstates that transform trivially under boosts. This ignores the most general representations of the ISO(2) Little Group, which are defined, for any four momentum k^μ , by their spin Casimir $W^2 = -\rho^2$, where

$$W^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} k_\nu J_{\rho\sigma} \quad (1.1)$$

is the Pauli-Lubanski vector and $J_{\rho\sigma}$ is the covariant generator of rotations in three dimensions. We call W^2 the spin Casimir because for massive particles,

$W^2 = -m^2 S(S+1)$, where m is the mass of the particle and S is its spin. For massless particles, ρ , which we call the spin scale (with units of momentum), parameterizes the degree of helicity mixing under Lorentz boosts, just as spin does in the massive case. Familiar theories enforce $\rho = 0$, and hence assume that helicity is a Lorentz invariant quantity. The lack of theoretical development of the $\rho \neq 0$ alternative has limited our ability to even pose meaningful questions about its compatibility with nature.

A particle with nonzero ρ , called a continuous spin particle (CSP), necessarily requires a description with an infinite tower of helicity modes. In particular, a particle in the state $|k, h\rangle$, with four-momentum k and integer helicity h , will mix with every other integer helicity eigenstate under an infinitesimal Lorentz boost Λ in a transverse direction:

$$U(v_\perp)|k, h\rangle = \sum_{h'} c_{hh'} |\Lambda k, h'\rangle, \quad (1.2)$$

with $h' \in \mathbb{Z}$. The coefficients $c_{hh'}$ are determined by the ISO(2) Little Group representation [9]. For $\rho \neq 0$, all $c_{hh'}$ are nonzero but in the high-energy limit $k^0 \gg \rho$ they approach $\delta_{hh'}$, with helicity-mixing c ’s suppressed by powers of $\rho v_\perp / k^0$.

Because a CSP involves states with arbitrarily high helicity eigenvalues, a description of even a free CSP resisted traditional Lagrangian field theory treatments up until the work of [10]. Moreover, the classic results constraining high spin particles derived from soft limits of scattering amplitudes were generalized to the CSP case only recently in [9–11]. What was uncovered was surprising—nontrivial soft limits of CSP-matter scattering

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amplitudes are compatible with unitarity and Lorentz invariance, and in the $\rho \rightarrow 0$ limit, CSPs faithfully recover the known soft scattering amplitude results with *only* the $h = 0, 1, 2$ modes interacting in this limit. Thus, it would appear that (some or all) known theories of massless particles could be limits of CSP theories, and perhaps known forces could be mediated by CSPs with small ρ .¹

Building on the field theory description of CSPs in [10,13] studied CSP dynamics and derived CSP interactions with spinless matter particles described in the worldline formalism. Like the soft emission limits in [11], the CSP interactions studied in [13] were found to be either scalar-, vector-, or tensor-like, meaning that in the $\rho \rightarrow 0$ limit and at energies $\gg \rho$ they reduce to familiar massless scalar, vector, or tensor field interactions, respectively. Furthermore, deviations from known formulas (e.g., the Larmor frequency) were derived for a photon with nonzero spin scale (CSP photon). These deviations were shown to be controlled by powers of $\rho v/\omega$, with ω the photon's energy and v the matter particle's speed, again echoing the scaling of the soft-limit theorems. These results suggest that a small but nonzero ρ leads to familiar UV physics, with new IR deviations. Taking this analysis one step further, [14] calculated free particle scalar QED amplitudes mediated by a CSP photon by extending the classical matter coupling in [13] to a quantum worldline formalism. The result of that analysis showed how the probability of measuring a final state helicity other than 1 in elementary processes such as Compton scattering can be expanded order by order in ρ , with larger helicity deviations requiring large powers of ρ . Interestingly, while all known photon observations are consistent with $\rho = 0$, they are also consistent with a nonzero $\rho \lesssim \text{neV}$, with the strongest constraints on the size of ρ coming from estimated (though not robust) effects on stellar cooling. It is therefore worthwhile to explore precision laboratory tests of ρ .

With the requisite tools now in place, the observation that a CSP photon can carry away large values of angular momentum, with a probability proportional to powers of ρ , leads us to the present work. From the perspective of atomic transitions, this feature has the direct consequence that single photon emissions traditionally forbidden by angular momentum conservation are now possible. One would expect that transitions of this type are also suppressed for energy splittings $\gg \rho$ by powers of ρ over kinematic invariants, with higher powers being required for transitions more highly forbidden. We will see that this is precisely the case. One can therefore set limits on the size of ρ , or potentially even observe a nonzero ρ , by searching

for photons emitted via such forbidden transitions. We must be careful when carrying out this analysis, however, as the state-of-the-art only includes spin-0 matter particles coupled to a bosonic CSP, not a fermionic spin-1/2 matter particle. We therefore must contend with forbidden transitions that are relativistically suppressed but in principle allowed, e.g., via spin-flip interactions (so-called magnetic dipole transitions). We will not specifically address this issue here, as the purpose of this paper is to set up the techniques required to perform bound state transition amplitudes mediated by CSP photons. Nonetheless, conservative bounds can always be set even ignoring magnetic dipole interactions (and other higher-order QED terms).

The rest of the paper will be set up as follows: in Sec. II we develop a formalism to calculate bound state transition amplitudes via path integral techniques, first in regular QED and then generalizing to nonzero ρ . The main result of this section, and in some sense the main result of this paper, is that in the nonrelativistic limit, the matrix element for a bound state atomic transition mediated by a CSP photon with helicity h takes the simple form:

$$M_h = \left(-q_e \sqrt{2} \frac{\omega}{\rho} \right) \int_0^{2\pi} d\phi e^{ih\phi} \times \int d^3x \psi_{\text{in}}(\vec{x}) \psi_{\text{out}}^* \left(\vec{x} - \frac{\rho}{\omega m} \vec{e}_\phi \right), \quad (1.3)$$

where q_e is the bound particle's charge, and $\vec{e}_\phi = \frac{1}{\sqrt{2}}(e^{i\phi}\vec{e}_- + e^{-i\phi}\vec{e}_+)$ is built from two linearly independent (unit norm) polarization vectors orthogonal to the photon propagation direction $\vec{k}$. As we will see explicitly in Sec. II C, by Taylor-expanding the argument of the final-state wavefunction about $\vec{x}$, one recovers the familiar QED transition amplitude in the limit $\rho \rightarrow 0$. In Sec. III, we apply these techniques to calculate transition amplitudes of a particle in a bound quantum harmonic oscillator (QHO) potential. The QHO serves as a simple toy model for which we derive amplitudes for a variety of forbidden transitions to highlight important features of CSP emissions. In Sec. IV, we repeat this analysis in the more physically relevant setting of a particle bound in a Coulomb potential (i.e., a hydrogen model), focusing on specific transitions of potential interest. The general analysis motivates a focus on the $2s \rightarrow 1s$ transition, from which we derive rough constraints on the spin-scale ρ of the photon. Finally, we conclude and discuss possible directions for future work in Sec. V.

II. BOUND STATE TRANSITION AMPLITUDES FROM THE PATH INTEGRAL

Nonrelativistic transitions of the sort considered in this paper are typically derived from the matter Hamiltonian, treating the radiation field as a background. However,

¹CSPs were previously thought to have infinite heat capacity due to the infinite number of polarization modes. However, [12] recently showed that despite the infinite number of degrees of freedom, CSPs have effectively finite heat capacity because the tower of helicity modes takes infinite time to thermalize, and in particular only one $\pm h$ mode thermalizes efficiently for $\rho \lesssim T$.

recent studies of CSP interactions with matter [13,14] have worked exclusively in the Lagrangian formalism. Because the Lagrangian involves arbitrarily high powers of the matter particle's velocity, a closed-form Legendre transform to an equivalent *interacting* matter Hamiltonian is not currently known in general. Therefore, our first step in analyzing transitions involving CSP photon emission will be to develop a method for computing bound state transition amplitudes starting from the matter Lagrangian, in the path integral formalism; that is the purpose of this section.²

Constructing the path integral in a binding potential is straightforward, but the techniques to solve it exactly [16] are cumbersome. It will be easier to take advantage of the simple evolution of energy eigenstates in the binding potential, essentially paralleling Hamiltonian perturbation theory. To do so, we first expand the path integral to linear order in the interaction Lagrangian describing the electron's coupling to the external CSP field. The resulting path integral involves one insertion of the interaction Lagrangian at a time t^* that must be integrated over, with paths weighted by the static Lagrangian (kinetic + binding potential). We then break up the path-integral into three time-steps: from initial time t to $t^* - \epsilon$, from $t^* - \epsilon$ to $t^* + \epsilon$, and from $t^* + \epsilon$ to the final observation time t' . The initial and final path integrals do not involve the interaction, and are easily mapped to Hamiltonian evolution of the initial and final energy eigenstates. By taking the intermediate interval, which includes the insertion of the interaction Lagrangian, to be infinitesimally small we can neglect the binding potential and use standard techniques for the free path integral.

Note that we are focusing here on CSP corrections to the physics of transition radiation, while neglecting the corrections to the atomic binding potential and wavefunctions. Because the transition radiation and binding potential in an atom arise from the same dynamics, a complete and consistent calculation must consider both. However, this simplification is justified as an *approximation* in precisely the transitions that are most physically interesting. Naively, both effects give rise to effects of $O(\rho^2 v^2 / \omega^2)$ (where ω is the binding energy in the case of potential effects and the transition splitting in the case of radiation effects, and $v \sim \alpha$ is the characteristic speed of the bound electron). However, our focus here is on forbidden transitions—those whose width in QED is parametrically lower than naive dimensional analysis would suggest because of angular momentum selection rules. Angular momentum remains a good quantum number in the presence of nonzero ρ , and so CSP potential effects induce at most fractional corrections of order $O(\rho^2 v^2 / \omega^2)$ to photon-mediated transitions.

²A complementary approach, which is equivalent for problems linear in the interaction, is to transform the interaction Lagrangian to *free* Hamiltonian variables. That approach is illustrated in [15].

However, as we will see, changes to the radiation interaction open entirely new channels and therefore can alter a bound state's lifetime by $O(1)$ even for $\rho v / \omega \ll 1$, where the wavefunction modifications are still parametrically small. A similar hierarchy also arises in the case of fine-structure or hyperfine transitions, where $\omega_{\text{radiation}} \ll \omega_{\text{binding}}$ enhances the effects of nonzero ρ in radiation. Therefore, from here forward we take the binding potential to be purely a function of position and drop any CSP corrections to the potential (considering both a harmonic oscillator potential as a warm-up and a Coulombic potential for a more realistic model hydrogen atom).

A. Hamiltonians and Lagrangians

We begin with a brief review of transition matrix elements in both the Hamiltonian and Lagrangian path-integral formalisms, introducing along the way some helpful notation.

We will assume that the Hamiltonian for the bound particle is given by

$$H = H_0 + H_I, \\ H_0 = \frac{p^2}{2m} + V(\vec{x}), \quad (2.1)$$

where $V(\vec{x})$ is the potential binding the particle, H_I is the interaction term between the particle and the background photon field, and p is the particle's momentum. For a standard ($\rho = 0$) photon, the interaction term is given by

$$H_I = -\frac{q_e}{m} \vec{A} \cdot \vec{p}, \quad (2.2)$$

where $\vec{A}(\vec{x}, t)$ is the photon field and q_e is the particle's charge. The amplitude of moving from one bound eigenstate, $\psi_{\text{in}}(t)$, to another, $\psi_{\text{out}}(t')$, is thus given by

$$A = \langle \psi_{\text{out}} | e^{-i(H_0 + H_I)(t' - t)} | \psi_{\text{in}} \rangle. \quad (2.3)$$

In traditional perturbation theory, we can expand to first order in the interaction Hamiltonian to arrive at

$$A \propto \langle \psi_{\text{out}} | H_I | \psi_{\text{in}} \rangle. \quad (2.4)$$

Rather than take this route, however, we will transform Eq. (2.3) into a path integral. We start by inserting two complete sets of position space basis vectors to yield

$$A = \int d^3x' \psi_{\text{out}}^*(\vec{x}', t') \int d^3x \psi_{\text{in}}(\vec{x}, t) \langle x' | e^{-i(H_0 + H_I)(t' - t)} | x \rangle. \quad (2.5)$$

We will call the last piece of this equation the propagator K :

$$K(x', t', x, t) = \langle x' | e^{-i(H_0 + H_I)(t' - t)} | x \rangle. \quad (2.6)$$

We can then define the static propagator:

$$K_0(x', t', x, t) = \langle x' | e^{-iH_0(t' - t)} | x \rangle, \quad (2.7)$$

which gives the position space transition amplitude in the absence of coupling to external photons.

The transition to the path integral formalism (and vice versa) is straightforward whenever the Hamiltonian is quadratic in momenta [17]. In this case, repeatedly inserting the identity in terms of both position and momentum eigenstates at finely spaced times between t and t' generates a phase-space path integral over paths in (p, x) space. The integrals over intermediate p are Gaussian, and give rise to the position-space path integral form of the amplitude Eq. (2.5):

$$A = \int d^3x' \psi_{\text{out}}^*(\vec{x}', t') \int d^3x \psi_{\text{in}}(\vec{x}, t) \times \int_{\vec{z}(t)=\vec{x}}^{\vec{z}(t')=\vec{x}'} \mathcal{D}\vec{z} e^{i \int_t^{t'} d\tau L(\vec{z}, \tau, \dot{\vec{z}}, \vec{A})}, \quad (2.8)$$

where L is given by the Legendre transform $L = \vec{p} \cdot \dot{\vec{x}} - H$ or equivalently, using (2.1),

$$L = L_0 + L_I, \quad L_0 = \frac{1}{2} m \dot{\vec{z}}^2 - V(\vec{z}). \quad (2.9)$$

When the Lagrangian involves higher powers of $\dot{x}$, or the Hamiltonian higher powers of p , subtleties arise in the equivalency above. The Legendre transform is single-valued for the CSP potential, but it is not analytically tractable. However, following [14], we take Eq. (2.8) using the interaction Lagrangian [13] as the starting point for computing CSP photon emission rates. This is sufficient for practical calculations. We leave further study of equivalent Hamiltonians to future work.

B. Splitting up the path integral

In the next two subsections we will derive transition amplitudes for ordinary QED, starting from the path integral above. The approach generalizes straightforwardly to nonzero ρ , as we will discuss in Sec. II D. We start by demonstrating how the path integral describing a bound state transition induced by coupling to a background field can be broken into three steps.

For the special case of a background photon field, $L_I = q_e \vec{z} \cdot \vec{A}(\vec{z}, \tau)$. We can break this field up into a sum of plane waves:

$$\vec{A}(\vec{z}, t) = \sum_j^N \vec{\epsilon}_j e^{i\omega_j t - i\vec{k}_j \cdot \vec{z}}, \quad (2.10)$$

where $\vec{\epsilon}$ is a polarization vector, and for single photon emission $N = 1$. We then expand the part of the exponential involving the interaction with $\vec{A}$ as a power series, and keep only the leading-order term. In this way we can pull down a vertex insertion:

$$\int_t^{t'} d\tau \hat{V}_k^e(\vec{z}, \tau) = (iq_e) \int_t^{t'} d\tau \vec{\epsilon} \cdot \dot{\vec{z}} e^{i\omega\tau - i\vec{k} \cdot \vec{z}}. \quad (2.11)$$

Note that while we have plugged in the Lagrangian interaction term L_I for QED, this procedure works for generic L_I that couples to a weak background field, and will generalize readily to the CSP case as we will see later. Plugging Eq. (2.11) into Eq. (2.8) gives us

$$A = \int d^3x' \psi_{\text{out}}^*(\vec{x}', t') \int d^3x \psi_{\text{in}}(\vec{x}, t) \times \int_{\vec{z}(t)=\vec{x}}^{\vec{z}(t')=\vec{x}'} \mathcal{D}\vec{z} \left(\int_t^{t'} d\tau^* \hat{V}_k^e(\vec{z}, t^*) \right) e^{i \int_t^{t'} d\tau L_0(\vec{z}, \tau)}, \quad (2.12)$$

where we have written the vertex insertion time variable as t^* for clarity. By changing the order of the path integral and the integral over $d\tau^*$, we can notice three distinct time intervals of importance. At all times except for an infinitesimally small time period surrounding t^* , the noninteracting, bound state path integral is being evaluated. Therefore, in the time periods before and after t^* , we can replace the path integral with the static propagators $K_0(x_1, t^* - \epsilon, x, t)$ and $K_0(x', t', x_2, t^* + \epsilon)$. By utilizing the integral over initial and final state wave functions, one can see that this amounts to a simple phase evolution of each energy eigenstate. Thus, we are left with the only remaining path integral over the time $t^* \mp \epsilon$. The last observation we make here regards the fact that for the infinitesimal integral over $d\tau$, any terms not proportional to $\dot{z}$ (and hence $1/\epsilon$ in discretized form), will be negligible. This allows us to effectively drop the potential term from L_0 and write

$$\lim_{\epsilon \rightarrow 0} \int_{t^* - \epsilon}^{t^* + \epsilon} d\tau L_0 = \lim_{\epsilon \rightarrow 0} \int_{t^* - \epsilon}^{t^* + \epsilon} d\tau \frac{1}{2} m \dot{\vec{z}}^2. \quad (2.13)$$

One has to be careful about this step depending on the form of $V(\vec{z})$, especially for those potentials which appear singular as $\vec{z} \rightarrow 0$, as is the case for a Coulomb potential. Nonetheless, we verify that this step is valid in Appendix A for common potentials that we will use later, including the Coulomb potential. Putting all of these observations together, we arrive at an amplitude that takes the form:

$$A = e^{i\tilde{\epsilon}} \int_t^{t'} d\tau^* e^{-i(E_{\text{in}} - E_{\text{out}})\tau^*} \int d^3x_1 d^3x_2 \psi_{\text{out}}^*(\vec{x}_2) \psi_{\text{in}}(\vec{x}_1) \times \int_{\vec{z}(t^* - \epsilon)=\vec{x}_1}^{\vec{z}(t^* + \epsilon)=\vec{x}_2} \mathcal{D}\vec{z} \hat{V}_k^e(\vec{z}, t^*) e^{i \int_{t^* - \epsilon}^{t^* + \epsilon} d\tau \frac{1}{2} m \dot{\vec{z}}^2}, \quad (2.14)$$

where $\xi = i(-E_{\text{out}}(t' - \epsilon) + E_{\text{in}}(t + \epsilon))$ is a simple phase factor. We can see from this final expression that the only remaining path integral is the free one with a single vertex insertion. Path integrals of this type are well studied in the literature [14,18,19], and we employ many of the same techniques in the next section.

C. Free path integral evaluation

In order to evaluate the remaining path integral in Eq. (2.14), we can make use of some string-inspired techniques. The trick is to realize that for sufficiently simple vertex operators $\hat{V}$, the path integral can be made into a Gaussian and become exactly solvable. If we can write $\hat{V}$ in terms of an exponential in phases linear (or quadratic) in $\vec{z}$ and $\dot{\vec{z}}$, then the path integral will be Gaussian. For the QED vertex operator used in Sec. II B, we can achieve this by writing

$$\hat{V}_{\text{seed}}(\dot{\vec{z}}, t) = q_e e^{i\omega t - i\vec{k} \cdot \vec{z} + i\vec{E} \cdot \dot{\vec{z}}}, \quad (2.15)$$

and defining $\hat{V}_k^\epsilon(\dot{\vec{z}}, t)$ as the piece of $\hat{V}_{\text{seed}}$ linear in $\vec{E}$. The main benefit here is that we can perform the full path integral calculation with $\hat{V}_{\text{seed}}$, and then take only the piece linear in $\vec{E}$ at the end of the calculation. We will see later that we can use the same $\hat{V}_{\text{seed}}$ for a CSP photon insertion. In order to solve the path integral in closed form, it will be helpful to further put it into an unconstrained Gaussian form. Therefore, we want to integrate over all possible end points $\vec{x}_1$ and $\vec{x}_2$, independent from the wave functions ψ_{in} and ψ_{out} . We can achieve such an integration naturally by Fourier transforming the wave functions so that the integrals over $\vec{x}_1$ and $\vec{x}_2$ apply only to end points of the path integral. We are now set up to solve the remaining path integral in Eq. (2.14) and find an amplitude that depends only on the specific wave functions in question. After making the change of variables $\tau \rightarrow \tau - t^*$ and putting $\hat{V}$ into exponential form, we find:

$$\begin{aligned} A_{\text{seed}} &= q_e e^{i\xi} \int dt^* e^{-i(E_{\text{in}} - E_{\text{out}} - \omega)t^*} \int \frac{d^3 p_1}{(2\pi)^{3/2}} \frac{d^3 p_2}{(2\pi)^{3/2}} \tilde{\psi}_{\text{in}}(\vec{p}_1) \tilde{\psi}_{\text{out}}^*(\vec{p}_2) \\ &\quad \times \int d^3 x_1 d^3 x_2 \int_{\vec{z}(-\epsilon)=\vec{x}_1}^{\vec{z}(\epsilon)=\vec{x}_2} \mathcal{D}z e \left[i \int_{-\epsilon}^{\epsilon} d\tau \frac{d\vec{z}}{d\tau}(\tau)^2 \right] - i\vec{k} \cdot \vec{z}(0) + i\vec{E} \cdot \dot{\vec{z}}(0) + i\vec{p}_1 \cdot \vec{z}(-\epsilon) - i\vec{p}_2 \cdot \vec{z}(\epsilon). \end{aligned} \quad (2.16)$$

In order to make this integral convergent, we will make the mass redefinition $m = i\tilde{m}$. This is effectively the same as analytically continuing to Euclidean time by replacing $\tau \rightarrow -i\tau$ and $k_0 \rightarrow -ik_0$, but lets us keep track of fewer transformations. In particular, none of the terms in the exponential outside of the brackets will be affected by this transformation. An equivalent transformation is to send $m \rightarrow m + i\tilde{m}$, and take $\tilde{m} \rightarrow 0$ at the end of the calculation. This path integral takes the same form as the unconstrained path integral in Eq. (3.6) of [14], up to a factor of $2\tilde{m}$ and the exponentiated vectors being three- rather than four-dimensional. Thus, we can use standard Green's function methods (see Appendix B for details) to find

$$\begin{aligned} A_{\text{seed}} &= q_e e^{i\xi} \int dt^* e^{-i(E_{\text{in}} - E_{\text{out}} - \omega)t^*} \\ &\quad \times \int d^3 p_1 d^3 p_2 \tilde{\psi}_{\text{in}}(\vec{p}_1) \tilde{\psi}_{\text{out}}^*(\vec{p}_2) e^{\frac{\vec{p}_2}{2\tilde{m}} \cdot (\vec{p}_2 + \vec{p}_1)} \delta^{(3)}(P), \end{aligned} \quad (2.17)$$

where $P = \vec{p}_1 - \vec{p}_2 - \vec{k}$. It will now be interesting to Fourier transform back to position space, as many wave functions are easier to work with in position space. Taking the inverse Fourier transform and performing the integrals over $\vec{p}_1$, $\vec{p}_2$, and $\vec{x}_2$ in Eq. (2.17), we arrive at

$$\begin{aligned} A_{\text{seed}} &= q_e e^{i\xi} \int dt^* e^{-i(E_{\text{in}} - E_{\text{out}} - \omega)t^*} e^{\frac{i\vec{k} \cdot \vec{x}_1}{2m}} \\ &\quad \times \int d^3 x_1 \psi_{\text{in}}(\vec{x}_1) \psi_{\text{out}}^* \left(\vec{x}_1 - \frac{\vec{E}}{m} \right) e^{-i\vec{k} \cdot \vec{x}_1}, \end{aligned} \quad (2.18)$$

where we have reintroduced the real mass m . It is worth stopping here to consider how this seed amplitude relates to the full QED solution. For a massless, transverse photon, $\vec{E} \cdot \vec{k} = 0$, and in the nonrelativistic expansion $\vec{k} \cdot \vec{x} \ll 1$, simplifying Eq. (2.18). When we then take the term linear in $\vec{E}$ to extract the QED amplitude from A_{seed} , we find

$$\begin{aligned} A &= (iq_e) e^{i\xi} \vec{E} \cdot \int dt^* e^{-i(E_{\text{in}} - E_{\text{out}} - \omega)t^*} \\ &\quad \times \int d^3 x_1 \psi_{\text{in}}(\vec{x}_1) \left(\frac{i}{m} \partial_{\vec{x}_1} \right) \psi_{\text{out}}^*(\vec{x}_1), \end{aligned} \quad (2.19)$$

which is just

$$e^{i\xi} \int dt^* e^{-i(E_{\text{in}} - E_{\text{out}} - \omega)t^*} (-iq_e) \vec{E} \cdot \langle \psi_{\text{out}} | \frac{\vec{p}}{m} | \psi_{\text{in}} \rangle, \quad (2.20)$$

where we interpret $-i\partial_{\vec{x}_1} = \vec{p}$. We have thus recovered familiar perturbation theory via a direct path integral calculation. It is also worth noting that when we finally

square the amplitude to extract a physical observable, the remaining time-dependent term will yield an energy-conserving delta function. That is, in the limit of long observation time ($t' - t$),

$$\left| \int_t^{t'} dt^* e^{-i(E_{\text{in}} - E_{\text{out}} - \omega)t^*} \right|^2 = 2\pi\delta(E_{\text{in}} - E_{\text{out}} - \omega)(t' - t), \quad (2.21)$$

but plays no other role in the calculation [20]. We will therefore define the general seed amplitude, in the nonrelativistic limit of a transverse photon, as

$$A_{\text{seed}} = M_{\text{seed}} \times e^{i\xi} \int dt^* e^{-i(E_{\text{in}} - E_{\text{out}} - \omega)t^*},$$

$$M_{\text{seed}} = \mathcal{C} \int d^3x \psi_{\text{in}}(\vec{x}) \psi_{\text{out}}^* \left(\vec{x} - \frac{\vec{b}}{m} \right), \quad (2.22)$$

so that we can focus on the matrix element M . We have written $q_e \rightarrow \mathcal{C}$ and $\vec{\varepsilon} \rightarrow \vec{b}$ in order to keep the solution more general. We will see in the next section that Eq. (2.22) applies also to a CSP photon-mediated transition with the appropriate substitutions for $\mathcal{C}$ and $\vec{b}$.

D. Generalization to nonzero ρ

We now come back to the aforementioned observation that the derivation in the previous sections will hold true even in the case of a CSP photon emission, so long as we can write down a suitable Lagrangian interaction term L_I . An appropriate way to couple matter to a CSP photon that reduces to the regular photon coupling in the $\rho \rightarrow 0$ limit was derived in [13], and used in a similar context to this paper for free particle Compton scattering in [14]. We refer the reader to Secs. II.b and II.c of [14] for a detailed overview of the operator and will simply quote that, in analogy with (2.11), a vertex operator valid on and off-shell for a CSP photon takes the form

$$V_{\gamma, \text{CSP}}^{k, \eta}(t) \equiv i q_e e^{ik \cdot z(t)} (e^{-i\rho \frac{\eta \cdot z(t)}{k \cdot z(t)}} (\sqrt{2} i k \cdot \dot{z}(t) / \rho) + \mathcal{D} X(\eta, z(t))), \quad (2.23)$$

where η is an auxiliary space 4-vector used to parameterize different helicity modes of the CSP photon, and $\mathcal{D} = [k^2 - (-ik \cdot \eta - \frac{1}{2}(\eta^2 + 1)(-ik \cdot \partial_\eta + \rho))]$ is the differential operator in the free CSP equation of motion for the Fourier modes, $\Psi(\eta, k)$, of the CSP gauge field. The function X is not fixed by the arguments of [14], but does not contribute to on-shell CSP amplitudes, so we can safely set $X = 0$ without loss of generality. We therefore take the CSP vertex operator to be

$$\hat{V}_{\text{CSP}}^{k, \eta}(t) \equiv i q_e e^{ik \cdot z(t)} e^{-i\rho \frac{\eta \cdot z(t)}{k \cdot z(t)}} (\sqrt{2} i k \cdot \dot{z}(t) / \rho). \quad (2.24)$$

In the fully nonrelativistic limit, we can set $k \cdot z \approx \omega t$ and $k \cdot \dot{z} \approx \omega$. Using the additional fact that $\eta_0 = 0$ (as we will see below), this approximation yields the nonrelativistic CSP vertex

$$\hat{V}_{\text{CSP}}^{\text{non-rel}}(t) = (i q_e) e^{i\omega t} e^{i\vec{\rho} \cdot \vec{\eta} \cdot \dot{z}(t)} (\sqrt{2} i \omega / \rho). \quad (2.25)$$

Immediately, we can notice that this operator is already in the form required for $\hat{V}_{\text{seed}}$ of Eq. (2.15), after setting $\vec{k} \cdot \vec{z} \approx 0$. Therefore, to recover the nonrelativistic CSP result from $\hat{M}_{\text{seed}}$, we simply make the replacements $\vec{b} = (\rho/\omega)\vec{\eta}$ and $\mathcal{C} = (i q_e)(\sqrt{2} i \omega / \rho)$, without needing to take only the linear part in $\vec{b}$.

While we have therefore arrived at the conclusion that A_{seed} in Eq. (2.22) applies also a CSP photon emission, there is one more step we must take to get a physical result after making the appropriate $\mathcal{C}$ and $\vec{b}$ substitutions. In order to get an observable final state solution out of this operator, defined in the auxiliary η space, we need to integrate it against the free CSP field Fourier amplitude ψ_h , over the appropriately regulated η space measure defined in [13]. This process can be carried out for outgoing CSP photons using the following identity:

$$V_{\text{CSP}}^{\text{out}, k, h} \equiv \int [\bar{d}^A \eta] \psi_h^*(\eta, k) \hat{V}_{\text{CSP}}^{k, \eta} = \int_0^{2\pi} \frac{d\phi_\eta}{2\pi} e^{-i h \phi_\eta} V_{\text{CSP}}^{k, \eta(\phi_\eta)}, \quad (2.26)$$

where ψ_h is the wavefunction defined in [13], and the η -integral expression can be simplified using identities in [13] to the form on the righthand side with $\eta(\phi) \equiv (0, \vec{\eta}(\phi))$, and $\vec{\eta}(\phi)$ a reference polarization vector orthogonal to $\vec{k}$ with unit norm. This is not the only choice for $\eta(\phi)$ that one can take, but it is a convenient one for nonrelativistic computations. In particular, we can take $\vec{\eta}(\phi) = \frac{1}{\sqrt{2}}(e^{i\phi}\vec{\varepsilon}_- + e^{-i\phi}\vec{\varepsilon}_+)$, built from two linearly independent (unit norm) polarization vectors orthogonal to the photon propagation direction $\vec{k}$. Different choices of vectors $\vec{\varepsilon}_\pm$ will all give the same result. Each helicity vertex operator has the interpretation of a Fourier mode of $\hat{V}$, when η is restricted to lie on a unit circle parameterized by ϕ .³ The last step of every CSP photon emission calculation will therefore be to integrate over linear polarizations perpendicular to k with angular weight corresponding to helicity via the integral $\int_0^{2\pi} \frac{d\phi}{2\pi} e^{-i h \phi}$.

We will end this section with a comment on the justification that we can drop all terms proportional to $\vec{z}$ in the CSP photon vertex. One immediate objection may

³In some of the literature, it is this angular basis that is sometimes conflated with the term “continuous spin,” and has originated an unfortunate degree of confusion over the years. Helicity eigenvalues take on only discrete numbers, even for CSPs.

come from dropping terms proportional to $\vec{z}/\rho$, which seem infinite in the $\rho \rightarrow 0$ limit. However, such terms are proportional to a total time derivative, and cancel as proven in Appendix C (by an argument closely related to the QED Ward identity), as does the remaining $O(1/\rho)$ term in Eq. (2.25). It is still possible that the lowest order in ρ vertex term will come with a nonzero power of $\vec{z}$ that we may have erroneously dropped. However, we also detail in Appendix D that for any spherically symmetric potential, the first potentially nonzero term occurs at $O((\frac{\rho v}{\omega})^n)$, for some value of n , determined by which transition is being analyzed. If this specific term were to be zero, then higher-order velocity terms might be needed, but we do not find this to be the case in any of the scenarios analyzed. This justification does, however, rely on being able to apply the form of A_{seed} in Eq. (2.18) to the CSP case, before taking any approximations. Thankfully, it was shown in [14] that the full CSP vertex of Eq. (2.24) can be put into exponential form via a clever integration (see Appendix D), so we are safe to use Eq. (2.25).

III. 3D QUANTUM HARMONIC OSCILLATOR

In this section, we will apply the techniques developed in Sec. II to an easy-to-solve toy problem, namely a particle bound in a 3D harmonic oscillator potential. Such potentials are ubiquitous in physics and will serve as a good warm up that will highlight many important features of bound state CSP photon emissions before moving on to a hydrogen atom model in Sec. IV. The 3D isotropic harmonic oscillator has a central potential that depends only on distance:

$$V(x, y, z) = \frac{1}{2} m \Omega^2 (x^2 + y^2 + z^2) = \frac{1}{2} m \Omega^2 r^2. \quad (3.1)$$

We use Ω for the quantum harmonic oscillator (QHO) frequency to distinguish it from the photon energy ω . The QHO potential yields wavefunction eigenstates in spherical coordinates:

$$\psi_{q\ell m}(r, \theta, \phi) = N_{q\ell} r^\ell e^{-\nu r^2/2} L_q^{(\ell+1/2)}(\nu r^2) Y_{\ell m}(\theta, \phi), \quad (3.2)$$

where

$$N_{q\ell} = \sqrt{\frac{\nu^3}{4\pi} \frac{2^{q+\ell+3} q! \nu^\ell}{(2q+2\ell+1)!}} \text{ is a normalization constant,} \\ \nu = m\Omega, \\ L_q^{(\ell+1/2)}(\nu r^2) \text{ is the generalized Laguerre polynomial,} \\ Y_\ell^m(\theta, \phi) \text{ is a spherical harmonic,} \quad (3.3)$$

with ℓ and m labeling the total angular momentum and its $\hat{z}$ component as usual, and q counts the number of nodes in

the radial wavefunction. The energy levels depend only on the principal quantum number $n = 2q + \ell$ as:

$$E_n = \Omega \left(n + \frac{3}{2} \right). \quad (3.4)$$

A. $(q, \ell, m) : (0, \ell, \ell) \rightarrow (0, 0, 0)$ transitions

The simplest types of transitions we can calculate for this potential will be those for which the radial part of the wave function does not change between the initial and final state. We will call these types of transitions “angular momenta only” transitions. If we furthermore restrict ourselves to $m = \ell$ in the initial state, then the only difference in wave function dependencies between different ℓ states will come from $\mathcal{R}^\ell \sin^\ell \theta e^{i\ell\phi}$ in spherical coordinates, which becomes $r^\ell e^{i\ell\phi}$ in cylindrical coordinates, with r now the cylindrical radius. Cylindrical coordinates are ultimately the most natural for this problem as the spherical symmetry of the system is broken by the photon emission, but cylindrical symmetry remains. Finally, we also restrict our final state to always be the ground state, for simplicity. It will be useful to keep in mind that we can define a direction of propagation for the emitted photon (CSP or otherwise), which we will set to be along $\hat{z}$. This is not a completely general choice, as we have already chosen the angular momentum of the initial state to be quantized along the same direction. Nonetheless, a photon emitted in the $\hat{z}$ direction will dominate the phase space of the emission, and can be used to get a parametric understanding of the allowed processes. Let us now recall that for the generic bound state amplitude given in Eq. (2.22), in the non-relativistic limit, the matrix element M_{seed} takes the form:

$$M_{\text{seed}} = C \int d^3x \psi_{\text{in}}(\vec{x}) \psi_{\text{out}}^* \left(\vec{x} - \frac{\vec{b}}{m} \right). \quad (3.5)$$

We can begin in Cartesian coordinates in order to address the dot product $\vec{b} \cdot \vec{x}$ that will come out of $\psi_{\text{out}}^* (\vec{x} - \frac{\vec{b}}{m})$. By only considering transitions to the ground state, the angular dependence on $\vec{b}$ will be the same for all values of ℓ . Taking this first step we have:

$$M_{\text{seed}} = C \tilde{N}_{00} \int d^3x \psi_{\text{in}}(\vec{x}) e^{-\frac{\nu}{2} |\vec{x}|^2 - \frac{\nu}{2} |\frac{\vec{b}}{m}|^2 + \frac{\nu}{m} \vec{b} \cdot \vec{x}}, \quad (3.6)$$

where $\tilde{N}_{q\ell}$ refers to the wave function normalization $N_{q\ell}$ multiplied by the normalization of the spherical harmonic $Y_{\ell, \ell}$, for example $\tilde{N}_{00} = \frac{1}{\sqrt{4\pi}} N_{00}$. Recalling that for a QED photon, $\vec{b}$ will eventually be replaced with the polarization vector $\vec{\epsilon}$, while for a CSP photon, $\vec{b}$ is also proportional to the CSP polarization vector $\vec{b} = \frac{\rho}{\omega} \vec{\eta}$, we will evaluate Eq. (3.6) for real vectors $\vec{b}$ that lie in the $\hat{x} - \hat{y}$ plane.

Such an evaluation will apply to linearly polarized QED photons, or any CSP photon. For the circularly polarized QED photon that we will be interested in, we simply combine the results from x and y polarizations appropriately. We therefore write the dot product $\vec{b} \cdot \vec{x}$ in cylindrical coordinates as

$$\vec{b} \cdot \vec{x} = |\vec{b}|r \cos(\phi - \phi_b), \quad (3.7)$$

where ϕ_b is the position in the $\hat{x} - \hat{y}$ plane of $\vec{b}$. We can rewrite $\phi_b = \phi - \tilde{\phi}$, sending the integration over angles ϕ to

$$\int_0^{2\pi} d\phi \rightarrow \int_{-\phi_b}^{2\pi-\phi_b} d\tilde{\phi}. \quad (3.8)$$

Because the function being integrated is periodic in $\tilde{\phi}$, we can then simply shift the integration to $0 \rightarrow 2\pi$. Ultimately this trick will pull out an overall $e^{i\ell\phi_b}$ and simplify the integration over angles. Using Eq. (3.7) and plugging in $\psi_{\text{in}}(x) = \psi_{0,\ell,\ell}$, we find

$$M_{\text{seed}} = C\tilde{N}_{00}\tilde{N}_{0\ell}e^{i\ell\phi_b}e^{-\frac{\nu|\vec{b}|^2}{2m^2}} \times \int dr dz d\tilde{\phi} r r^\ell e^{-\nu(r^2+z^2)-iBr \cos \tilde{\phi} + i\ell\tilde{\phi}}, \quad (3.9)$$

where $B = \frac{i\nu}{m}|\vec{b}|$. For the QED case, we set $\phi_b = 0$ for a photon linearly polarized in the $\hat{x}$ direction, and $\phi_b = \pi/2$ for the $\hat{y}$ direction, while for a CSP photon ϕ_b becomes ϕ_η . The integral over dz is simply a Gaussian, and the integral over $d\tilde{\phi}$ yields a function involving an ℓ 'th order Bessel function, which we can then integrate against the remaining r dependencies to yield

$$M_{\text{seed}} = C\tilde{N}_{00}\tilde{N}_{0\ell}\pi^{3/2}e^{i\ell\phi_b}e^{-\frac{\nu|\vec{b}|^2}{4m^2}}\frac{|\vec{b}|^\ell}{2^\ell\nu^{3/2}m^\ell}. \quad (3.10)$$

To go further and get a physically understandable amplitude, we now need to decide whether to extract the QED or CSP photon solution, which we do in the following sections.

1. Extracting the QED result

We can now notice from the form of Eq. (3.10) that the only transition with a nonzero piece linear in $\vec{b}$ comes from $\ell = 1$. Each piece of the polarization vector will therefore yield a component (when setting $C = q_e$, and linearizing in $\vec{b} = \vec{\epsilon}_{x,y}$):

$$M_{x,y}(\phi_b) = e^{i\phi_b}q_e\left[\frac{1}{2\nu^{3/2}m}\right]\tilde{N}_{00}\tilde{N}_{01}\pi^{3/2}. \quad (3.11)$$

Finally, by setting $\phi_b = 0$ for M_x , $\phi_b = \pi/2$ for M_y , and combining the two to get a circularly polarized photon with

angular momentum quantized along the $\hat{z}$ axis, we get a total amplitude:

$$M = \frac{1}{\sqrt{2}}(M_x - iM_y) = q_e\sqrt{\frac{\Omega}{2m}}, \quad (3.12)$$

where we have also plugged in the appropriate normalizations. One can easily verify that this is the exact matrix element that a perturbative Hamiltonian calculation yields, up to a normalization factor in the photon field $\vec{A}$.

2. Extracting the CSP result

To get the CSP result is actually somewhat simpler, as we can simply insert the proper definitions of $\vec{b} = (\rho/\omega)\vec{\eta}$ and $C = (iq_e)(\sqrt{2}i\omega/\rho)$ to get:

$$M(\phi_\eta) = (iq_e)(\sqrt{2}i\omega/\rho) \times \left[\tilde{N}_{00}\tilde{N}_{0\ell}\pi^{3/2}e^{i\ell\phi_\eta}e^{-\frac{\nu\rho^2}{4\omega^2m^2}}\frac{(\rho/\omega)^\ell}{2^\ell\nu^{3/2}m^\ell} \right], \quad (3.13)$$

where we have used the fact that $|\vec{\eta}| = 1$. From this expression we can immediately see both the helicity support in $e^{i\ell\phi_\eta}$ and the power of ρ dependence in ρ^ℓ . To formally extract the helicity support, we need to integrate over linear CSP polarizations orthogonal to $\vec{k}$ as per Eq. (2.26). We therefore have that the matrix element for a given helicity mode is

$$M_h = \int_0^{2\pi} \frac{d\phi_\eta}{2\pi} e^{-ih\phi_\eta} M(\phi_\eta). \quad (3.14)$$

The integral

$$\int_0^{2\pi} \frac{d\phi_\eta}{2\pi} e^{-i\phi_\eta(h-\ell)} \quad (3.15)$$

will yield 1 for $h = \ell$ and 0 otherwise. Therefore, we have that the only nonzero helicity mode allowed by this transition is $h = \ell$, giving us

$$M_\ell = (iq_e)|P_\ell^\ell|\sqrt{\frac{2(2\ell+1)}{2^\ell(2\ell)!(2\ell+1)!!}} \times \left[\frac{\rho^{\ell-1}}{\omega^{\ell-1}} \left(\sqrt{\frac{\Omega}{m}} \right)^\ell e^{-\frac{\nu\rho^2}{4\omega^2m^2}} \right], \quad (3.16)$$

where $|P_\ell^\ell|$ is the constant multiplying $\sin^\ell\theta$ in the associated Legendre polynomial $P_\ell^\ell(\sin\theta)$. Furthermore, in the limit that $\rho \rightarrow 0$, the only nonzero transition occurs for $\ell = 1$, yielding a matrix element

$$M_{\ell=1} = iq_e\sqrt{\frac{\Omega}{2m}}. \quad (3.17)$$

This expression exactly matches up with the QED result in Eq. (3.12), up to an overall phase. For larger values of initial angular momentum ℓ , Eq. (3.16) tells us that higher powers of $\frac{\rho v}{\omega}$ are required, where we have used the fact that $v \sim \sqrt{\Omega/m}$ for the QHO. This fact encapsulates the general result of this paper, that forbidden transitions can be mediated by a CSP photon at the price of a $\frac{\rho v}{\omega}$ suppression. While we have only considered transitions between the ground state and a state with no radial mode excitation here, conservation of angular momentum tells us that this pattern will persist for nonzero n as well. In the following section, we will consider a different type of transition for the QHO, and see that it follows the same trend.

B. A $\Delta\ell = 0$ transition

The previous section discussed generic transitions of the type $\Delta\ell = \Delta m = \Delta n$. While those transitions highlight important features of a CSP photon-mediated emission and can be computed rather generally, they do not encompass the only type of forbidden transition that a CSP photon can mediate. We will perform here another example of a transition forbidden by QED that differs from the angular momenta only type, namely a $\Delta\ell = 0$ transition. Unlike the case of angular momenta only transitions, a $\Delta\ell = 0$ transition necessarily requires starting in an excited state of the radial part of the wave function. The simplest such transition, and the one that we will consider here, is the transition from $\psi_{1,0,0} \rightarrow \psi_{0,0,0}$, i.e., the first excited radial state down to the ground state. It is worth noting that due to the specific symmetry of the QHO, there is no $n = 1, \ell = 0$ state, so $\psi_{1,0,0}$ is actually an $n = 2$ state. By angular momentum conservation, this transition requires emission of an $h = 0$ CSP photon. Because $h = 0$ is one integer away from the QED photon's $h = \pm 1$, we can expect that the lowest-order contribution to this transition will come at first order in ρ . We now show that this is the case.

Starting again with our generic matrix element:

$$M_{\text{seed}} = C \int d^3x \psi_{\text{in}}(\vec{x}) \psi_{\text{out}}^*\left(\vec{x} - \frac{\vec{b}}{m}\right), \quad (3.18)$$

we can make the same initial steps as with the angular-momenta-only transitions because once again ψ_{out} is the ground state. We can therefore write

$$M_{\text{seed}} = C \tilde{N}_{00} \tilde{N}_{10} e^{-\frac{\nu|\vec{b}|^2}{2m^2}} \int dr dz d\tilde{\phi} r \left(\frac{3}{2} - \nu(r^2 + z^2) \right) \times e^{-\nu(r^2+z^2) - i\text{Br} \cos \tilde{\phi}}, \quad (3.19)$$

where $B = \frac{i\nu}{m} |\vec{b}|$. Taking the integrals over dz and $d\tilde{\phi}$, we get

$$M_{\text{seed}} = C \tilde{N}_{00} \tilde{N}_{10} e^{-\frac{\nu|\vec{b}|^2}{2m^2}} \times \int dr 2\pi J_0(\text{Br}) e^{-\nu r^2} \left[r \sqrt{\frac{\pi}{\nu}} - r^3 \sqrt{\pi\nu} \right], \quad (3.20)$$

with $J_0(\text{Br})$ the zeroth-order Bessel function. While $J_0(\text{Br})$ naively has an $O(1)$ term independent of ρ , one can see that such a term is proportional to a total r derivative of $r^2 e^{-\nu r^2}$, and so reduces to boundary terms which do not contribute. In fact, this term was guaranteed to cancel by the Ward identity argument in Appendix C, since it is proportional to $\frac{1}{\rho}$ when we replace C with $(iq_e)(\sqrt{2}i\frac{\omega}{\rho})$. Carrying out the final r integral and plugging this expression back in to our full amplitude, we find:

$$M_{\text{seed}} = -C \tilde{N}_{00} \tilde{N}_{10} e^{-\frac{\nu|\vec{b}|^2}{2m^2}} e^{\frac{\nu|\vec{b}|^2}{4\omega^2 m^2}} \left[\frac{\pi^{3/2} |\vec{b}|^2}{4m^2 \sqrt{\nu}} \right]. \quad (3.21)$$

From here, we can see immediately that there is no term linear in $\vec{b}$, so the QED solution is zero, as expected. Performing the steps to recover the nonrelativistic CSP photon solution, we insert the proper definitions of $\vec{b} = (\rho/\omega)\vec{\eta}$ and $C = (iq_e)(\sqrt{2}i\omega/\rho)$ and normalizations to get

$$M(\phi_\eta) = \frac{1}{\sqrt{12}} e^{-\frac{\nu\rho^2}{4\omega^2 m^2}} \left[\frac{\rho}{\omega m} \right]. \quad (3.22)$$

When we take the final integral over $d\phi_\eta$, we find helicity support only for $h = 0$, with a suppression of $\frac{\rho v}{\omega}$ relative to allowed QED transitions, as expected for a helicity mode one integer value away from $h = 1$. Other $\Delta\ell = 0$ transitions are possible for higher initial energy states, and one can verify the same helicity support and ρ dependence.

IV. SCALAR HYDROGEN ATOM

In this section, we move beyond the simple QHO oscillator model and highlight the application of our techniques to a more realistic problem of interest, namely a hydrogen atom. In particular, we will be concerned only with a scalar hydrogen atom, i.e., ignoring any spin-dependent corrections. Understanding how to couple CSP photons to fermions is an important subject that we leave to future work. Nonetheless, we can still learn something about how forbidden transition amplitudes will change in the presence of a CSP photon, even ignoring spin. The 3D coulomb potential is given simply by

$$V(x, y, z) = -\frac{q_e^2}{\sqrt{x^2 + y^2 + z^2}} = -\frac{q_e^2}{r}. \quad (4.1)$$

Such a potential yields wavefunctions in spherical coordinates:

$$\psi_{n\ell m}(r, \theta, \phi) = N_{n\ell} e^{-\nu/2} \nu^\ell L_{n-\ell-1}^{2\ell+1}(\nu) Y_\ell^m(\theta, \phi), \quad (4.2)$$

where

$$N_{n\ell} = \sqrt{\left(\frac{2}{na_0}\right)^3 \frac{(n-\ell-1)!}{2n(n+\ell)!}} \text{ is a normalization constant;}$$

$$\nu = \frac{2r}{na_0};$$

$$a_0 = \frac{4\pi}{mq_e^2} \text{ is the Bohr radius;}$$

$$L_{n-\ell-1}^{2\ell+1} \text{ is the generalized Laguerre polynomial;}$$

$$Y_\ell^m(\theta, \phi) \text{ is a spherical harmonic.} \quad (4.3)$$

The energy levels for the hydrogen atom also depend only on the quantum number n :

$$E_n = \frac{a_0 q_e^2}{2n^2}. \quad (4.4)$$

A. $(n, \ell, m) : (\ell+1, \ell, \ell) \rightarrow (1, 0, 0)$ transitions, order by order

Just as we did for the harmonic oscillator, we will begin by computing angular-momenta-only transitions that end up in the ground state, where the initial state satisfies $\ell = m$. In this section, we will forego notation to keep the solution general to a QED photon, and use the values for C and $\vec{b}$ appropriate for a CSP photon from the start. As we

saw, one can always recover the QED solution by taking the $\rho \rightarrow 0$ limit anyway. With these substitutions, bound state transition elements take the form

$$M_{\text{seed}} = (iq_e)(\sqrt{2}i\omega/\rho) \int d^3x \psi_{\text{in}}(\vec{x}) \psi_{\text{out}}^*\left(\vec{x} - \frac{\rho}{\omega m} \vec{\eta}\right). \quad (4.5)$$

We could take the same initial steps for dealing with dot products that we took in the case of a harmonic oscillator, starting with:

$$M_{\text{seed}} = (iq_e)(\sqrt{2}i\omega/\rho) \tilde{N}_{10} \int d^3x \psi_{\text{in}}(\vec{x}) e^{-\frac{|\vec{x} - \frac{\rho}{\omega m} \vec{\eta}|}{a_0}}, \quad (4.6)$$

where

$$\left|\vec{x} - \frac{\rho}{\omega m} \vec{\eta}\right| = \sqrt{|\vec{x}|^2 + \frac{\rho^2}{\omega^2 m^2} |\vec{\eta}|^2 - 2 \frac{\rho}{\omega m} \vec{\eta} \cdot \vec{x}}. \quad (4.7)$$

However, we will not get a Gaussian integral over z due to the square root in the exponential. This square root will prove generally problematic, so for the hydrogen atom it becomes beneficial to work order by order in ρ/ω from the start. We replace $\psi_{\text{out}}^*(\vec{x} - \frac{\rho}{\omega m} \vec{\eta})$ in Eq. (4.5) with its Taylor expansion, getting an expression for M_{seed} :

$$M_{\text{seed}} = (iq_e)(\sqrt{2}i\omega/\rho) \int d^3x \psi_{\text{in}}(\vec{x}) \left[1 - i \frac{\rho}{\omega m} \vec{\eta} \cdot \vec{p} - \frac{1}{2} \frac{\rho^2}{\omega^2 m^2} \vec{\eta} \cdot P_{ij} \cdot \vec{\eta} + \mathcal{O}\left(\left(\frac{\rho}{\omega}\right)^3\right) \right] \psi_{\text{out}}^*(\vec{x}), \quad (4.8)$$

where $\vec{p} = -i \frac{\partial}{\partial \vec{x}}$ and $P_{ij} = -\frac{\partial^2}{\partial x_i \partial x_j}$ are momentum operators. Immediately we can notice that the zeroth-order term in the expansion will give 0, as it is simply proportional to the overlap of two orthogonal wavefunctions. This is expected from the Ward Identity as discussed in Sec. IID.

The QED amplitude corresponds to the first-order term of the expansion, which is independent of ρ . Having already extracted the piece of the exponential dependent on the photon polarization through perturbation theory, the problem is easier to solve in spherical rather than cylindrical coordinates. The $\mathcal{O}(\rho^0)$ piece of the matrix element, in spherical coordinates, is given by:

$$M_{\text{QED}} = (-q_e \sqrt{2}) \left(\frac{2}{(\ell+1)a_0} \right)^\ell \frac{\tilde{N}_{(\ell+1)\ell} \tilde{N}_{10}}{ma_0} e^{i\ell\phi_\eta} \\ \times \int dr d\theta d\tilde{\phi} r^{\ell+2} \sin^{\ell+2}(\theta) e^{i\ell\tilde{\phi}} \cos(\tilde{\phi}) e^{-r \frac{(\ell+2)}{(\ell+1)a_0}}, \quad (4.9)$$

where we have made the substitution $\tilde{\phi} = \phi - \phi_\eta$, with ϕ_η the angle that $\vec{\eta}$ takes on in the $x-y$ plane. The integral over $d\tilde{\phi}$ will yield π if $\ell = 1$ and 0 otherwise. As mentioned, this term corresponds to the QED piece, so the only allowed transition should be one which changes angular momentum by 1. After setting $\ell = 1$ and taking the remaining integrals, we get:

$$M_{\text{QED}} = (-q_e \sqrt{2}) \alpha \frac{2^4}{3^4}, \quad (4.10)$$

where α is the fine structure constant. This result is once again readily verified in a perturbative, Hamiltonian QED calculation.

Finally, let us look at the second-order term in the expansion, which corresponds to the first ρ -dependent transition amplitude. This time the only nonzero transition amplitude occurs when $\ell = 2$, for which the $\tilde{\phi}$ integral yields $\pi/2$. For an initial state with $n \neq 0$, there also exists an $\ell = 0 \rightarrow \ell = 0$ transition at this same order in $\frac{\rho}{\omega}$, but we

do not consider that case here. Plugging in $\ell = 2$ and taking the integrals in spherical coordinates, we find:

$$M_\rho = (-q_e)\alpha \frac{3}{20} \left(\frac{\rho}{\omega} \alpha \right), \quad (4.11)$$

which is parametrically the same as the allowed QED transition, suppressed by a factor of $\frac{\rho v}{\omega}$ (with $v \sim \alpha$). So we have seen that even for a hydrogen atom, forbidden transitions can be mediated by a single CSP photon, suppressed by the spin scale ρ . From symmetry of the angular momentum structure we can note that this pattern persists, such that higher angular momentum transitions occur at higher orders in $\frac{\rho v}{\omega}$, and new corrections to the $\Delta\ell = 1$ transition occur at $\mathcal{O}((\frac{\rho v}{\omega})^2)$. Of course, we could also calculate the forbidden transitions where $\Delta n \neq \Delta\ell$, and the total angular momentum change $\Delta\ell$ (for $\Delta\ell = \Delta m$) will still set the scale of the ρ dependence. For example, one can verify that a $\Delta\ell = \Delta m = 0$ transition will come at $\mathcal{O}(\frac{\rho v}{\omega})$, as was shown for the QHO potential in the Sec. III B transition. We set about calculating one such transition of particular importance in the next section.

B. 2s $\rightarrow$ 1s transition

In this section, we compute the $(n, \ell, m): (2, 0, 0) \rightarrow (1, 0, 0)$ hydrogen transition amplitude. This is not only an example of a forbidden $\Delta\ell = 0$ transition, similar to what was done for the quantum harmonic oscillator in Sec. III B, but it is also of particular experimental importance as it is forbidden to all orders in standard, nonrelativistic QED. In fact, the leading-order contribution to the decay lifetime of the 2s hydrogen state comes from two-photon emission. On the other hand, as we will see here, a single CSP photon can mediate this transition. We once more start with the matrix element seed for a CSP photon:

$$M_{\text{seed}} = (iq_e)(\sqrt{2}i\omega/\rho) \int d^3x \psi_{\text{in}}(\vec{x}) \psi_{\text{out}}^* \left(\vec{x} - \frac{\rho}{\omega m} \vec{\eta} \right), \quad (4.12)$$

and take the Taylor expansion of ψ_{out}^* . We can use the knowledge that the first term in the expansion cancels for all transitions, and that the second gives the forbidden QED piece, to jump straight to the third term which is proportional to ρ/ω . Plugging in ψ_{in} and ψ_{out}^* we have, in spherical coordinates,

$$M_{\text{seed}} = \left(\frac{-q_e \rho}{\sqrt{2} \omega} \right) \tilde{N}_{20} \tilde{N}_{10} \int dr d\theta d\phi r^2 \sin(\theta) \left(2 - \frac{r}{a_0} \right) e^{-r/2a_0} \left(-\frac{\vec{\eta}}{m} \cdot \vec{P}_{ij} \cdot \frac{\vec{\eta}}{m} \right) e^{-r/a_0}, \quad (4.13)$$

where

$$\left(-\frac{\vec{\eta}}{m} \cdot \vec{P}_{ij} \cdot \frac{\vec{\eta}}{m} \right) = \left[\left(\frac{\sin^2(\theta) \cos^2(\tilde{\phi})}{m^2 a_0^2} \right) - \left(\frac{1}{m^2 a_0 r} \right) + \left(\frac{\sin^2(\theta) \cos^2(\tilde{\phi})}{m^2 a_0 r} \right) \right], \quad (4.14)$$

with $\tilde{\phi} = \phi - \phi_\eta$. Plugging this in and taking the required integrals yields:

$$M_{\text{seed}} = q_e \alpha \frac{2^3}{3^4} \left(\frac{\rho}{\omega} \alpha \right). \quad (4.15)$$

We have thus confirmed that a single CSP photon can mediate a 2s $\rightarrow$ 1s transition. The remaining step of integrating over $d\phi_\eta$ simply tells us that the CSP photon emitted will have helicity $h = 0$. Relative to the allowed 2p $\rightarrow$ 1s transition rate (4.10), this transition amplitude is suppressed by a factor of $\frac{1}{2\sqrt{2}} \frac{\rho\alpha}{\omega}$. Since the characteristic speed of the bound electron in hydrogen is $\sim \alpha$, this is precisely the $\rho v/\omega$ suppression characteristic of CSP partner-mode couplings. Since the helicity of the emitted CSP photon matches the $\Delta\ell$ of the transition, its emission experiences no further suppression.

C. Interpreting constraints on ρ

It is worth commenting briefly on what these results mean for our experiment. Given that we have considered only a scalar hydrogen atom, our results are only valid for electric transitions, where spin effects are not important. Moreover, for CSP effects to be observable, their rates must be at least comparable to the leading QED process mediating a similar transition. In the first row of Table I, we show the lowest order in ρ at which CSPs can mediate a given $\Delta\ell$ transition, without requiring an electron spin-flip. In contrast, the second row shows the largest nonzero contribution to the same transition, mediated by a QED photon. As we can see, $\Delta\ell \neq 1$ “forbidden” transitions are in fact allowed in QED, but they are mediated by higher magnetic or electric moment interactions and therefore suppressed by powers of α relative to their electric dipole counterparts.

TABLE I. Largest nonzero contribution for a given hydrogen atom transition amplitude. Each column corresponds to a given orbital angular momentum transition. Each row corresponds to either a CSP photon with scalar coupling (as used throughout the paper), or a full QED photon. The entry then gives the lowest-order nonzero contribution to the transitions. For the CSP photon, we parameterize the contribution by orders in ρ , while for the QED photon we parameterize the contribution by the type of transition required (E_1 = Electric Dipole, M_1 = Magnetic Dipole, E_2 = Electric Quadrupole, E_3 = Electric Octopole). It should be noted that $M1$ transitions can only mediate $\Delta\ell = 0, \Delta n \neq 0$ transitions when relativistic corrections to the wave function are included.

	$\Delta\ell = 0$	$\Delta\ell = 1$	$\Delta\ell = 2$	$\Delta\ell = 3$
CSP	$\mathcal{O}(\rho)$	$\mathcal{O}(1)$	$\mathcal{O}(\rho)$	$\mathcal{O}(\rho^2)$
QED	M_1	E_1	E_2	E_3

In most of these cases, we find that the $\rho\alpha/\omega$ suppression of the CSP-emission amplitude can only dominate over the α suppression of the higher-multipole QED transitions if $\rho \gtrsim \omega$, leading to a rather weak bound. Moreover, although for example direct $3d \rightarrow 1s$ transitions are forbidden in the electric dipole, the $3d$ state is short-lived because it can decay through other dipole transitions, e.g., to $2p$. Thus, these high- ℓ states are not particularly efficient probes of CSP effects.

Of particular interest, however, is the $2s \rightarrow 1s$ transition calculated in the previous section. Not only does the $2s$ state have a particularly long lifetime, any single-photon emission is fully forbidden to all orders in QED, unless relativistic corrections to the wavefunction are included, which gives a large suppression $\propto \alpha^{11}$ [21]. In fact, the primary decay channel for the $2s$ hydrogen atom in regular QED is via two-photon emission. To convert the matrix element computed in Sec. IV B into a decay rate, we need to use Fermi's golden rule with the appropriate normalizations [20]. Because none of the states involved carry angular dependence, the integral over angular phase space is trivial. Using $\Gamma = \frac{\omega}{2\pi\hbar} |M|^2$ and the matrix element (4.15), we find

$$\tau_{2s}^{\text{CSP}} \approx \left(\frac{\omega^2}{\rho^2} \right) 1.6 \times 10^{-4} \text{ s}. \quad (4.16)$$

Comparing this value to the known lifetime of 0.12 s [21,22], we can say with certainty that $\rho \lesssim \omega\sqrt{10^{-3}}$, or else the dominant decay channel for the $2s$ state of the hydrogen atom would be through a single CSP photon emission and the observed lifetime would be shorter. Using the fact that $\omega \simeq 10.2$ eV, this sets the spin scale at $\rho \lesssim \mathcal{O}(0.1 \text{ eV})$. This bound can be strengthened with precision measurement and calculation of the $2s$ lifetime, improving as the square root of the relative precision.

This relatively weak bound leads one to consider more highly forbidden and lower-energy transitions. The $2s$ state of a helium atom with both electrons aligned, for example, cannot decay via a two-photon electric dipole transition, and thus has a much longer lifetime than that of hydrogen. However, any decay at all requires a spin flip of the excited electron. Similarly, the ground state of hydrogen can decay due to a hyperfine energy shift, giving the famous 21 cm line, but requires a spin flip to do so. These potentially sensitive decay channels strongly motivated understanding how to couple CSP photons to particles with spin. The required formalism and phenomenology will be developed in forthcoming papers [23,15], respectively. Other possible detection schemes include the use of nuclear decay and subsequent gamma cascades with missing energy, such as the experiment proposed in [24]. The energy of emitted photons in these decays is typically much larger than those of low-lying atomic transitions, but the sensitivity due to coincidence with other cascade photons allows for high precision measurements.

V. DISCUSSION AND OUTLOOK

In this work, we have derived a method for calculating atomic transition amplitudes in the case in which the photon is a continuous-spin particle. We first showed how to calculate bound state transitions directly in velocity space via path integral techniques for regular QED. We then extended this method to a CSP photon using the matter coupling of a CSP field derived in [13]. We confirmed, in a new setting, that the physics of a CSP photon differs from the familiar QED photon in the deep IR, parameterized by the ratio of the CSP spin scale ρ to its energy ω , times the matter particle's velocity v . In particular, we have shown that forbidden atomic transitions can occur when mediated by a CSP photon, with rates suppressed by powers of $\frac{\rho\alpha}{\omega}$, proportional to how forbidden the transition is. After computing a few example transitions for a simple harmonic oscillator model, we calculated the matrix elements for various hydrogen atom transitions. Finally, we put these transition rates in context by comparing the spin scale suppression to the suppression coming from higher electric and magnetic moments that can mediate the same processes for regular QED. While moderate constraints on ρ are set by analyzing the $2s \rightarrow 1s$ hydrogen atom transition, further study is merited to understand the extent to which atomic transitions can be used to probe ρ . In addition to the study of spin flip transitions initiated in [15], studies of potential interest include the examination of more exotic atomic and nuclear transitions and of systems engineered to have longer-lived excited states.

The power of the atomic transition constraints we have considered here comes from the presence of (approximate) angular momentum selection rules in atomic systems.

It would be interesting to extend this study to other systems with selection rules. A particularly interesting case is self-interaction of photons through matter loops, which are constrained, e.g., by Furry's theorem. It seems likely that $O(\rho)$ effects would lead to nonzero amplitudes that violate Furry's theorem, e.g., by involving an even number of helicity ± 1 photons and an odd number of helicity 0 partner photons. On one hand, these effects are incredibly suppressed because they involve fermion loops. On the other, the QED selection rule is exact and so even small effects may be detectable in a suitably designed experiment.

A second natural question raised by this work is the impact of nonzero ρ on QED radiative corrections. These are measured with incredible precision through observables like the Lamb shift and electron $g - 2$, and it is unreasonable to expect that they would *not* be modified by nonzero ρ at some level. However, the theoretical formalism for CSP interactions is not yet sufficiently mature to enable robust calculations of effects involving off-shell CSPs. This is why we have confined our attention to questions of CSP radiation, but it would be fascinating to return to the question of radiative corrections when the structure of CSP interactions is better developed. A formalism for computing off-shell CSP effects would also enable exploration of precision effects in other domains, including strong-field QED and loaded resonant cavities.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: DROPPING THE POTENTIAL TERM IN THE PATH INTEGRAL

In this section we elaborate on the point made in Sec. II B, that we may drop the potential term of the Lagrangian when computing the path integral over the infinitesimally short emission time. Let us start by recalling the argument made, that any contribution from the potential in the action becomes negligible:

$$\lim_{\epsilon \rightarrow 0} \int_{t^*-\epsilon}^{t^*+\epsilon} d\tau \frac{1}{2} m \dot{\vec{z}}^2 + V(\vec{z}) = \int_{t^*-\epsilon}^{t^*+\epsilon} d\tau \frac{1}{2} m \dot{\vec{z}}^2. \quad (\text{A1})$$

In order to understand why the velocity term remains while the potential term vanishes, we can look at the discretized path integral:

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \int d^3x_1 d^3x_2 \psi_{\text{in}}(\vec{x}_1) \psi_{\text{out}}^*(\vec{x}_2) \int_{\vec{z}(t^*-\epsilon)=\vec{x}_1}^{\vec{z}(t^*+\epsilon)=\vec{x}_2} \mathcal{D}\vec{z} \hat{V}(\dot{\vec{z}}, t^*) e^{i \int_{t^*-\epsilon}^{t^*+\epsilon} d\tau L_0(\vec{z}, \tau)} \\ &= \lim_{\epsilon \rightarrow 0} \int d^3x_1 d^3x_2 \psi_{\text{in}}(\vec{x}_1) \psi_{\text{out}}^*(\vec{x}_2) \hat{V}\left(\frac{\vec{x}_2 - \vec{x}_1}{\epsilon}, t^*\right) e^{i\epsilon L_0(\vec{x}_1, \vec{x}_2, \epsilon)} \\ &= \lim_{\epsilon \rightarrow 0} \int d^3x_1 d^3x_2 \psi_{\text{in}}(\vec{x}_1) \psi_{\text{out}}^*(\vec{x}_2) \hat{V}\left(\frac{\vec{x}_2 - \vec{x}_1}{\epsilon}, t^*\right) e^{i\epsilon [\frac{1}{2}m(\frac{\vec{x}_2 - \vec{x}_1}{\epsilon})^2 - V(\vec{x}_2)]} \end{aligned} \quad (\text{A2})$$

where we have made a couple of jumps. First, we have entirely gotten rid of the path integral measure $\mathcal{D}\vec{z}$ by assuming $\vec{x}_1$ and $\vec{x}_2$ are a ϵ step away in the path integral. Due to this same description, instead of a factor of 2ϵ that would show up naively, we have written only ϵ , absorbing the factor of 2 into this small parameter. Furthermore, we have rewritten velocity terms as $\lim_{\epsilon \rightarrow 0} (\frac{\vec{x}_2 - \vec{x}_1}{\epsilon})$. Writing velocity in this way highlights why it does not drop out of the exponentiated action term in the $\epsilon \rightarrow 0$ limit. The potential term on the other hand is well behaved in the $\epsilon \rightarrow 0$ limit so long as there is no regime in which $V(\vec{x}_2) \rightarrow \infty$. Strictly speaking this is not always true, but for the two potentials analyzed in this paper it is effectively true. For the Harmonic oscillator potential, the wave functions are Gaussian suppressed outside of a finite

region, beyond which the amplitude has no support. We can therefore focus on a finite volume for $\vec{x}_2$, within which ϵ can be taken to be small enough that the potential term vanishes. The Coulomb potential, on the other hand, blows up as $\vec{x}_2 \rightarrow 0$, where the wavefunction does not vanish. However, there is a physical cutoff to the potential, given by the radius of the atom, a_0 . We can therefore write the Coulomb potential as

$$V(x) = -\frac{q_1 q_2}{|x| + a_0}, \quad (\text{A3})$$

so that $V(x)$ remains finite everywhere. If desired, a_0 can even be taken to 0 at the end of the calculation with no issues.

APPENDIX B: SOLVING THE FREE PATH INTEGRAL

In this section, we solve the path integral from Sec. II C, starting with Eq. (2.16):

$$A = \mathcal{C} e^{i\xi} \int dt^* e^{-i(E_{\text{in}} - E_{\text{out}} - \omega)t^*} \int \frac{d^3 p_1}{(2\pi)^{d/2}} \frac{d^3 p_2}{(2\pi)^{d/2}} \tilde{\psi}_{\text{in}}(\vec{p}_1) \tilde{\psi}_{\text{out}}^*(\vec{p}_2) \\ \times \int d^3 x_1 d^3 x_2 \int_{\vec{z}(-\epsilon) = \vec{x}_1}^{\vec{z}(\epsilon) = \vec{x}_2} \mathcal{D}\vec{z} e^{[i \int_{-\epsilon}^{\epsilon} d\tau \frac{m}{2} \dot{\vec{z}}(\tau)^2] - i\vec{k} \cdot \vec{z}(0) + i\vec{\epsilon} \cdot \dot{\vec{z}}(0) + i\vec{p}_1 \cdot \vec{z}(-\epsilon) - i\vec{p}_2 \cdot \vec{z}(\epsilon)}. \quad (\text{B1})$$

In order to make this integral convergent, we will make the mass redefinition $m = i\tilde{m}$. It is also now useful to define the following variables:

$$j(\tau) = \frac{i}{2\tilde{m}} [\bar{j}(\tau) + \hat{j}(\tau)], \\ \bar{j}(\tau) = -\vec{p}_1 \delta(\tau + \epsilon) + \vec{p}_2 \delta(\tau - \epsilon) + \vec{k} \delta(\tau), \\ \hat{j}(\tau) = \vec{\epsilon} \delta'(\tau). \quad (\text{B2})$$

Plugging in these substitutions, our amplitude becomes:

$$A = \mathcal{C} e^{i\xi} \int dt^* e^{-i(E_{\text{in}} - E_{\text{out}})t^*} \int \frac{d^3 p_1}{(2\pi)^{d/2}} \frac{d^3 p_2}{(2\pi)^{d/2}} \tilde{\psi}_{\text{in}}(\vec{p}_1) \tilde{\psi}_{\text{out}}^*(\vec{p}_2) \int d^3 x_1 d^3 x_2 \int_{\vec{z}(-\epsilon) = \vec{x}_1}^{\vec{z}(\epsilon) = \vec{x}_2} \mathcal{D}\vec{z} e^{-2\tilde{m} \int_{-\epsilon}^{\epsilon} d\tau \dot{\vec{z}}(\tau)^2/4 + j(\tau) \vec{z}(\tau)}. \quad (\text{B3})$$

This path integral takes the same form as the unconstrained path integral in Eq. (3.6) of [14], up to a factor of $2\tilde{m}$ and the exponentiated vectors being three- rather than four-dimensional. Thus, we can use all of the same tricks, first defining

$$\vec{w}(\tau) = \vec{z}(\tau) - \int_{-\epsilon}^{\epsilon} d\tau' G(\tau, \tau') j(\tau'), \quad (\text{B4})$$

where

$$\frac{1}{2} \partial_\tau^2 G(\tau, \tau') = \delta(\tau - \tau'), \quad (\text{B5})$$

(i.e., G is the Green's function). It is also worth defining $\vec{w}_1$ and $\vec{w}_2$, which will simply be:

$$\vec{w}_1 = \vec{w}(-\epsilon) = \vec{x}_1 - \int_{-\epsilon}^{\epsilon} d\tau' G(-\epsilon, \tau') j(\tau'), \\ \vec{w}_2 = \vec{w}(\epsilon) = \vec{x}_2 - \int_{-\epsilon}^{\epsilon} d\tau' G(\epsilon, \tau') j(\tau'). \quad (\text{B6})$$

From here, we can express $\vec{z}(\tau)$ in terms of $\vec{w}(\tau)$, G , and $j(\tau)$, and integrate by parts, yielding an expression that depends on $\vec{w}(\tau)$ only through boundary terms and the free action $\frac{\tilde{m}}{2} \dot{\vec{w}}(\tau)^2$. Thus, we can write our amplitude as

$$A = \mathcal{C} e^{i\xi} \int dt^* e^{-i(E_{\text{in}} - E_{\text{out}})t^*} \int \frac{d^3 p_1}{(2\pi)^{d/2}} \frac{d^3 p_2}{(2\pi)^{d/2}} \tilde{\psi}_{\text{in}}(\vec{p}_1) \tilde{\psi}_{\text{out}}^*(\vec{p}_2) e^{-2\tilde{m} \int_{-\epsilon}^{\epsilon} d\tau d\tau' \frac{1}{2} j(\tau) G(\tau, \tau') j(\tau')} \\ \times \int d^3 w_1 d^3 w_2 \int_{\vec{w}(-\epsilon) = \vec{w}_1}^{\vec{w}(\epsilon) = \vec{w}_2} \mathcal{D}\vec{w} e^{-2\tilde{m} \int_{-\epsilon}^{\epsilon} d\tau \dot{\vec{w}}(\tau)^2/4} e^{-2\tilde{m} [\frac{1}{2} \int d\tau' (\vec{w}(\tau) + \frac{1}{2} \int d\tau'' G(\tau, \tau'') j(\tau'')) \partial_\tau G(\tau, \tau') j(\tau')]_{\tau=-\epsilon}^{\tau=\epsilon}}, \quad (\text{B7})$$

where we have used the fact that $\vec{w}$ is just a finite shift from $\vec{z}$ such that $\mathcal{D}\vec{z} = \mathcal{D}\vec{w}$, and that an integral over all $\vec{x}_{1,2}$ is the same as an integral over all $\vec{w}_{1,2}$. In the last two lines we have the massive, nonrelativistic, free path integral and a boundary term, both of which will be integrated over $\vec{w}_1$ and $\vec{w}_2$. Starting with the remaining path integral, we have

$$\int_{\vec{w}(-\epsilon) = \vec{w}_1}^{\vec{w}(\epsilon) = \vec{w}_2} \mathcal{D}\vec{w} e^{-2\tilde{m} \int_{-\epsilon}^{\epsilon} d\tau \dot{\vec{w}}(\tau)^2/4} = \left(\frac{\tilde{m}}{2\pi(2\epsilon)} \right)^{3/2} e^{-\frac{\tilde{m}(w_2 - w_1)^2}{2(2\epsilon)}}. \quad (\text{B8})$$

Now, using the Green's function definition $G(\tau, \tau') = |\tau - \tau'|$, we can take the integrals over w_1 and w_2 to find:

$$\int d^3 w_1 d^3 w_2 \left(\frac{\tilde{m}}{2\pi(2\epsilon)} \right)^{3/2} e^{-\frac{\tilde{m}(w_2 - w_1)^2}{2(2\epsilon)}} 2\tilde{m} \left[\frac{1}{2} \int d\tau' (\vec{w}(\tau) + \frac{1}{2} \int d\tau'' G(\tau, \tau'') j(\tau'')) \partial_\tau G(\tau, \tau') j(\tau') \right]_{\tau=-\epsilon}^{\tau=\epsilon} = (2\pi)^{(3)} \delta^{(3)}(P), \quad (\text{B9})$$

where $P = \vec{p}_1 - \vec{p}_2 - \vec{k}$. With that said, let us look at the integral

$$-\tilde{m} \int_{-\epsilon}^{\epsilon} d\tau d\tau' j(\tau) G(\tau, \tau') j(\tau'), \quad (\text{B10})$$

which has three pieces. First we can look at the $\bar{j} \circ \bar{j}$ piece:

$$\begin{aligned} \bar{j}(\tau) \circ \bar{j}(\tau') &= \int d\tau d\tau' \bar{j}(\tau) G(\tau, \tau') \bar{j}(\tau') \\ &= \int d\tau d\tau' [-\vec{p}_1 \delta(\tau + \epsilon) + \vec{p}_2 \delta(\tau - \epsilon) + \vec{k} \delta(\tau)] |\tau - \tau'| [-\vec{p}_1 \delta(\tau' + \epsilon) + \vec{p}_2 \delta(\tau' - \epsilon) + \vec{k} \delta(\tau')] \\ &= \int d\tau [-\vec{p}_1 \delta(\tau + \epsilon) + \vec{p}_2 \delta(\tau - \epsilon) + \vec{k} \delta(\tau)] \times [-\vec{p}_1 |\tau + \epsilon| + \vec{p}_2 |\tau - \epsilon| + \vec{k} |\tau|] \\ &= \vec{p}_1^2(0) - 2\vec{p}_1 \cdot \vec{p}_2(2\epsilon) - 2\vec{p}_1 \cdot \vec{k}_1(\epsilon) + \vec{p}_2^2(0) + 2\vec{p}_2 \cdot \vec{k}(\epsilon) + \vec{k}^2(0). \end{aligned} \quad (\text{B11})$$

This piece gives terms only proportional to ϵ , so we can neglect it. On the other hand, by moving the derivative on $\delta'(\tau)$ in $\hat{j}$ to the Green's function, we can find

$$\begin{aligned} \hat{j}(\tau) \circ \bar{j}(\tau') &= \int_{-\epsilon}^{\epsilon} d\tau d\tau' \hat{j}(\tau) G(\tau, \tau') \bar{j}(\tau') \\ &= \int_{-\epsilon}^{\epsilon} d\tau d\tau' [\vec{\epsilon} \delta'(\tau)] |\tau - \tau'| [-\vec{p}_1 \delta(\tau' + \epsilon) + \vec{p}_2 \delta(\tau' - \epsilon) + \vec{k} \delta(\tau')] \\ &= - \int_{-\epsilon}^{\epsilon} d\tau d\tau' \vec{\epsilon} \delta(\tau) \text{sign}(\tau - \tau') [-\vec{p}_1 \delta(\tau' + \epsilon) + \vec{p}_2 \delta(\tau' - \epsilon) + \vec{k} \delta(\tau')] \\ &= -\vec{\epsilon} \cdot \int_{-\epsilon}^{\epsilon} d\tau' \text{sign}(-\tau') [-\vec{p}_1 \delta(\tau' + \epsilon) + \vec{p}_2 \delta(\tau' - \epsilon) + \vec{k} \delta(\tau')] \\ &= -\vec{\epsilon} \cdot (-\vec{p}_1 \text{sign}(\epsilon) + \vec{p}_2 \text{sign}(-\epsilon) + \vec{k} \text{sign}(0)) \\ &= \vec{\epsilon} \cdot (\vec{p}_2 + \vec{p}_1). \end{aligned} \quad (\text{B12})$$

Similarly, we will get the exact same contribution from $\bar{j}(\tau) \circ \hat{j}(\tau')$. Finally, the last piece, proportional to $\hat{j} \circ \hat{j}$, will be:

$$\begin{aligned} \hat{j}(\tau) \circ \hat{j}(\tau') &= \int_{-\epsilon}^{\epsilon} d\tau d\tau' \hat{j}(\tau) G(\tau, \tau') \hat{j}(\tau') \\ &= \int_{-\epsilon}^{\epsilon} d\tau d\tau' \vec{\epsilon} \delta'(\tau) |\tau - \tau'| \vec{\epsilon} \delta'(\tau') = 2\vec{\epsilon}^2 \int_{-\epsilon}^{\epsilon} d\tau d\tau' \delta(\tau) \delta(\tau - \tau') \delta(\tau') = 2\vec{\epsilon}^2 \delta(0), \end{aligned} \quad (\text{B13})$$

which we set to 0 by the Veltman prescription, following the conventions used in [14]. This leads to the final amplitude:

$$A = C e^{i\xi} \int dt^* e^{-i(E_{\text{in}} - E_{\text{out}} - \omega)t^*} \int \frac{d^3 p_1}{(2\pi)^{d/2}} \frac{d^3 p_2}{(2\pi)^{d/2}} \tilde{\psi}_{\text{in}}(\vec{p}_1) \tilde{\psi}_{\text{out}}^*(\vec{p}_2) e^{\frac{\vec{p}_2}{2\tilde{m}} \cdot (\vec{p}_2 + \vec{p}_1)} (2\pi)^3 \delta^{(3)}(P). \quad (\text{B14})$$

APPENDIX C: VANISHING $1/\rho$ VERTEX CONTRIBUTION

In this section, we prove that contributions of $O(\rho^{-1})$ from the vertex operator in Eq. (2.24) do not contribute to bound state transition amplitudes. In the full vertex operator, the $O(\rho^{-1})$ terms will come from

$$e^{-ik \cdot z(t)}(k \cdot \dot{z}(t)/\rho). \quad (\text{C1})$$

This term is proportional to a standard QED vertex where the polarization ϵ_μ is replaced by the momentum k_μ . The vanishing of QED amplitudes under this same substitution is guaranteed by the Ward identity, and so we expect that the effects of (C1) should similarly vanish in the computation of full amplitudes. To see this in the worldline formalism and in the present context, we can write Eq. (C1) as

$$e^{ik \cdot z}(k \cdot \dot{z}) = -i\partial_t e^{ik \cdot z}, \quad (\text{C2})$$

showing that this term is a total z_0 derivative, and reduces to a boundary condition evaluated at the start and end times t and t' , which does not contribute to the amplitude. To prove this claim, we can plug Eq. (C2) into our original amplitude

$$\begin{aligned} A = & -i \int d^3x' \psi_{\text{out}}^*(\vec{x}') \int d^3x \psi_{\text{in}}(\vec{x}) \\ & \times \int_{\vec{z}(t)=\vec{x}}^{\vec{z}(t')=\vec{x}'} \mathcal{D}z \left(\int_t^{t'} dt^* \frac{d}{dt^*} (e^{i\omega t^* - i\vec{k} \cdot \vec{z}(t^*)}) \right) e^{i \int_t^{t'} d\tau L_0(\vec{z}, \tau)}. \end{aligned} \quad (\text{C3})$$

This interaction clearly reduces to boundary conditions, yielding a simple path integral that can be computed easily. Carrying out these steps, we find

$$\begin{aligned} A = & i(e^{i\omega t' - iE_{\text{out}}(t'-t)} - e^{i\omega t' - iE_{\text{in}}(t'-t)}) \\ & \times \int d^3x \psi_{\text{in}}(\vec{x}) e^{-i\vec{k} \cdot \vec{x}} \psi_{\text{out}}(\vec{x}). \end{aligned} \quad (\text{C4})$$

Now we can stop and think about what the squared amplitude will look like. Starting with just the first term, which has the only time dependence, we have

$$\begin{aligned} & |(e^{i\omega t' - iE_{\text{out}}(t'-t)} - e^{i\omega t' - iE_{\text{in}}(t'-t)})|^2 \\ & = 2 - 2 \cos((E_{\text{out}} + \omega - E_{\text{in}})(t' - t)), \end{aligned} \quad (\text{C5})$$

which is of course 0 if energy is conserved. Therefore, all $O(\rho^{-1})$ contributions to transition amplitudes in bound states vanish.

We note that although this paper has focused on computations at leading order in perturbation theory, the cancellation of $1/\rho$ vertices should hold to all orders in perturbation theory. One way to see this is again based on the close relation of (C1) to the Ward identity substitution $\epsilon_\mu \rightarrow k_\mu$ on the standard QED vertex—thus, our cancellation should hold everywhere the QED Ward identity applies. More explicitly, the total t derivative (C2) vanishes when integrated over closed matter loops, just as it does when integrated against on-shell external states as was done here. We caution, however, that as discussed in [14], and again just as with the QED Ward identity, the $1/\rho$ terms do not vanish when considering off-shell matter correlation functions. It would be interesting to further explore the implications of these corrections in, e.g., radiative corrections.

APPENDIX D: NONRELATIVISTIC VERTEX OPERATOR

In this section, we will justify the nonrelativistic expansion of the CSP vertex operator in Sec. II D. To do this, we show that for a given transition, the lowest $O(\rho)$ contribution allowed by parity and symmetry corresponds to the result obtained from the nonrelativistic operator in Eq. (2.25). To get a sense for what we mean, consider Table II, showing which matrix elements of the CSP vertex operator are zero, order by order in ρ , for different Δm transitions (where m is the $\hat{z}$ component of the atom's angular momentum). It is clear that for a given order in ρ , the matrix element given by the vertex piece in Eq. (2.25):

$$\hat{V}_{\text{CSP}}^{k,\eta}(t) \approx (iq_e) e^{i\omega t} e^{i\vec{L}_0 \cdot \vec{z}(t)} (\sqrt{2}i\omega/\rho), \quad (\text{D1})$$

contains the leading-order contribution, since all corrections will be higher order in velocity. The task therefore becomes showing that the leading order in the ρ result we

TABLE II. The lowest order in ρ contributions to a given Δm transition are shown. Entries that are zero vanish to all powers in velocity.

	$\Delta m = 0$	$\Delta m = 1$	$\Delta m = 2$	$\Delta m = 3$
ρ^{-1}	0 by Ward	0 by symmetry	0 by symmetry	0 by symmetry
ρ^0	0 by parity	Nonzero $\rightarrow$ QED	0 by symmetry	0 by symmetry
ρ^1	Nonzero	$\vdots$	Nonzero	0 by symmetry
ρ^2	$\vdots$	$\vdots$	$\vdots$	Nonzero

find by using this nonrelativistic operator matches with the first allowed order in ρ contribution to all orders in velocity.

As a starting point, it is necessary to show that any term that shows up as $\vec{k} \cdot \vec{z}$ or $\vec{k} \cdot \vec{z}$ in the vertex operator will also show up in the same form in the final integral against wave functions. It is for this reason that we need the full, relativistic result in Eq. (2.18):

$$A = \mathcal{C} e^{i\xi} \int dt^* e^{-i(E_{\text{in}} - E_{\text{out}} - \omega)t^*} e^{\frac{i\vec{b} \cdot \vec{k}}{2m}} \times \int d^3x \psi_{\text{in}}(\vec{x}) \psi_{\text{out}}^*\left(\vec{x} - \frac{\vec{b}}{m}\right) e^{-i\vec{k} \cdot \vec{x}}. \quad (\text{D2})$$

To prove that this equation holds true for the full CSP vertex operator requires only that we put $\hat{V}_{\text{CSP}}^{k,\eta}$ into the exponentiation form of $\hat{V}_{\text{seed}}$ as in Eq. (2.11). [14] showed that we can do this via the following integration technique:

$$\hat{V}_{\text{CSP}}^{k,\eta}(t) = \int [d\mu] e^{ib^\mu(\eta,k)\dot{z}_\mu(t)} e^{ik \cdot z(t)},$$

$$[d\mu] \equiv (\sqrt{2}i)^w \frac{d\theta}{\theta^w} d\lambda ds e^{-i\lambda\rho} \frac{d}{ds} \left(\frac{2s}{s^2 + e^2} \right)$$

$$b^\mu \equiv \theta \eta^\mu - (s - \lambda\theta) k^\mu. \quad (\text{D3})$$

The integrand of Eq. (D3) matches the form of $\hat{V}_{\text{seed}}$ up to a redefinition of $\vec{e}$ and an overall constant. Eq. (D2) then allows us to recover any interaction term that is polynomial in $\vec{z}$ by extracting the appropriate $O(b)$ term. Setting $\vec{b}$ in this scenario either to $\frac{\rho}{\omega} \vec{\eta}$ or $\vec{k}$ lets us analyze interaction terms $\frac{\rho}{\omega} \vec{\eta} \cdot \vec{z}$ and $\vec{k} \cdot \vec{z}$ order by order. One can see that such a Taylor expansion in b , means that velocity corrections will show up as $(\vec{k} \cdot \vec{x})^n$ corrections to the final integral (and one can make a similar expansion in the term $e^{-i\vec{k} \cdot \vec{x}}$).

With that understood, we can now analyze Table II. Let us begin by considering any transition for which

$\Delta m > [\rho] + 1$ (i.e., the top right of the table). In the same way that any vertex element of the form $(\vec{k} \cdot \vec{z})^n$ results in $(\vec{k} \cdot \vec{x})^n$ corrections to the final integral, so too will vertex elements of the form $(\frac{\rho}{\omega} \vec{\eta} \cdot \vec{z})^n$ result in $(\frac{\rho}{\omega} \vec{\eta} \cdot \vec{x})^n$ corrections to the final integral. In cylindrical coordinates, each power of $\vec{\eta} \cdot \vec{x}$ will result in one power of $\cos \tilde{\phi}$ (where $\tilde{\phi}$ is the angle between $\vec{\eta}$ and $\vec{x}$ in the x-y plane), while $\vec{k} \cdot \vec{x}$ will have no $\tilde{\phi}$ dependence for a CSP photon propagating in the $\hat{z}$ direction. The other $\tilde{\phi}$ dependent piece comes from the starting state wavefunction in the form of $e^{im\tilde{\phi}}$. Therefore, the integral over $d\tilde{\phi}$ becomes

$$\int_0^{2\pi} d\tilde{\phi} \cos^n(\tilde{\phi}) e^{im\tilde{\phi}} = \int_0^{2\pi} d\tilde{\phi} \left(\frac{e^{i\tilde{\phi}} + e^{-i\tilde{\phi}}}{2} \right)^n e^{im\tilde{\phi}}. \quad (\text{D4})$$

We can now notice that for any integer m and n , the only nonzero term will be those for which the exponents cancel. Therefore, for any $n < m$, this integral is zero. This argument will set the entire top right of the table above the diagonal to zero. We therefore only have two cells of the table left to analyze, which are the $O(\rho^{-1})$ and $O(\rho^0)$ rows of the $\Delta m = 0$ column. The $O(\rho^{-1})$ term is set to zero by the argument in Appendix C. In fact, the entire top row of Table II is zero independent of the symmetry argument presented here for the same reason. Finally, we can invoke parity argument to prove that the $O(\rho^0)$, $\Delta m = 0$ cell is zero. With no contribution from the wave functions to the $d\tilde{\phi}$ integral, the one power of $\cos \tilde{\phi}$ at order $O(\rho^0)$ leads immediately to a vanishing matrix element. We have therefore shown the first potentially nonzero matrix element for each Δm transition order by order in ρ . As long as the lowest order in ρ result we find using Eq. (D1) matches with the first nonzero elements in Table II, we can ignore any velocity corrections.

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