

Perturbations of black holes in Einstein-Maxwell-dilaton-axion theories

C. N. Pope^{1,2,*}, D. O. Rohrer^{1,†} and B. F. Whiting^{3,‡}

¹*George P. and Cynthia Woods Mitchell Institute for Fundamental Physics and Astronomy,
Texas A&M University, College Station, Texas 77843, USA*

²*DAMTP, Centre for Mathematical Sciences, Cambridge University,
Wilberforce Road, Cambridge CB3 0WA, United Kingdom*

³*Department of Physics, P.O. Box 118440, University of Florida, Gainesville, Florida 32611-8440, USA*



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We extend our earlier work on the linearized perturbations of static black holes in Einstein-Maxwell-dilaton (EMD) theories to the case where the black holes are solutions in an enlarged theory including also an axion. We study the perturbations in a three-parameter family of such (Einstein-Maxwell-dilaton-axion) EMDA theories. The systems of equations describing the linearized perturbations can always be separated, but they can only be decoupled when the three parameters are restricted to a one-parameter family of EMDA theories, characterized by a parameter b that determines the coupling of the axion to the $\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ term. In the specific case when $b = 1$, the theory is related to an $\mathcal{N} = 2$ supergravity. In this one case we find that the perturbations in the axial and the polar sectors are related by a remarkable transformation, which generalizes one found by Chandrasekhar for the perturbations of Reissner-Nordström in Einstein-Maxwell theory. This transformation is of a form found in supersymmetric quantum mechanical models. The existence of such mappings between the axial and polar perturbations appears to correlate with those cases where there is an underlying supergravity supporting the solution, even though the black hole backgrounds are nonextremal and therefore not supersymmetric. We prove the mode stability of the static black hole solutions in the supersymmetric EMDA theory. For other values of the parameter b in the EMDA theories that allow decoupling of the modes, we find that one of the radial potentials can be negative outside the horizon if b is sufficiently large, raising the possibility of there being perturbative mode instabilities in such a case.

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I. INTRODUCTION

The study of linearized perturbations of black hole solutions in Einstein and Einstein-Maxwell theories has a long history, dating back to the investigations of the perturbations of the Schwarzschild solution of Einstein gravity or Einstein-Maxwell theory by Regge and Wheeler in the 1950s [1,2]. Subsequent work in the 1970s and 1980s extended these results by considering black holes carrying electric charge and/or angular momentum. An important application of these early perturbative analyses was that they allowed one to initiate an investigation of stability. Stability analysis itself requires a multilevel approach, the

first step of which is always the addressing of the question of mode stability. This is a question that can sometimes (but only sometimes) be easily settled by an energy integral analysis, such as that provided in [3] for Schwarzschild, in [4] for Kerr, and by us in [5] for static Einstein-Maxwell-dilaton (EMD) black holes. We will find that an energy integral analysis suffices to establish stability for some, but not all, of the perturbations we discuss.

Our own work closely parallels that of Chandrasekhar and Xanthopoulos, in their study of the linearized perturbations of Reissner-Nordström black holes in Einstein-Maxwell theory [6–10]. This involved making gauge choices so as to isolate the perturbative degrees of freedom of the metric and Maxwell fields into quantities obeying a tractable system of coupled differential equations that could eventually be separated and diagonalized. One of the striking features that was found in this analysis of Reissner-Nordström was that the axial modes and the polar modes, which decouple from each other at the outset of the analysis, obey ostensibly very different systems of second-order differential equations which, nonetheless, turn out to be closely related to one another. In particular, the

*Contact author: pope@physics.tamu.edu

†Contact author: rohrer@physics.tamu.edu

‡Contact author: bernard@phys.ufl.edu

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spectra of modes in the axial and polar sectors are related by a one-to-one mapping, of the kind encountered in supersymmetric quantum mechanical models.

A rather different approach to the study of linearized black-hole perturbations was developed by Teukolsky, who worked in the Newman-Penrose tetrad formulation of the Einstein or Einstein-Maxwell equations, with the perturbative degrees of freedom being described by gauge-invariant quantities involving the Weyl curvature and Maxwell field strength. In the case of the perturbations of the Reissner-Norström black hole, Teukolsky's approach provides a unified description of the axial and polar perturbations of the Chandrasekhar and Xanthopoulos analysis in terms of complex fields obeying a single master equation.

The Teukolsky approach can also be applied to the case of rotating black holes, proving again a complete separation and decoupling of the second-order perturbation equations. However, no analogous complete diagonalization of the equations has been achieved in the case that both charge and rotation are turned on.

In our previous work on the perturbations of static black holes in the EMD theories, we were able to carry out a complete separation and diagonalization of the system of equations, both in the axial and polar sectors [5]. It was evident, however, that the close relation between the modes of the axial and polar sectors that was previously seen in the perturbations of the Reissner-Nordström solution in Einstein-Maxwell theory could no longer occur in the EMD case. Principally, in the EMD case the axial perturbations are governed by two decoupled second-order equations, while the polar perturbations are governed by three decoupled second-order equations, and there can no longer exist a one-to-one mapping between the modes in the two sectors. A natural expectation might therefore be that a Teukolsky type of analysis of the perturbations could be problematical.

In the present paper we extend beyond the EMD system, by including now in addition an axionic scalar field in the theory. Specifically, the class of Einstein-Maxwell-dilaton-axion (EMDA) theories that we shall consider is described by the Lagrangian,

$$\mathcal{L} = \sqrt{-g} \left[R - \frac{1}{2}(\partial\phi)^2 - e^{a\phi}F^2 - \frac{1}{2}e^{c\phi}(\partial\chi)^2 + b\chi\tilde{F}^{\mu\nu}F_{\mu\nu} \right], \quad (1.1)$$

where $\tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ is the Hodge dual of the Maxwell tensor $F_{\mu\nu}$. The constants a , b and c can *a priori* be taken to be arbitrary.

The background static solutions that we shall consider in these EMDA theories will be exactly the same charged static Gibbons-Maeda black holes [11] that served as the background solutions in the EMD theories [5]. In particular, the axion field vanishes in the background solution.

However, the axion does enter at the level of the linearized perturbations around the background solution, and in fact it gives rise to a third degree of freedom obeying a second-order differential equation in the axial sector. This means that there will now be three perturbative modes in each of the axial and polar sectors, raising the possibility that a matching of the axial and the polar modes might now occur. Indeed, as we shall show, for appropriate choices of the coupling constants a , b , and c , we do find that the axial and polar modes are related in an extension of the relations seen for the perturbations of Reissner-Nordström in Einstein-Maxwell theory.

In the EMD truncation of the EMDA system (in which the axion field χ is set to zero), the Lagrangian (1.1) is invariant under a global symmetry,

$$\phi \longrightarrow \phi + \lambda, \quad A_\mu \longrightarrow e^{-\frac{1}{2}a\lambda}A_\mu. \quad (1.2)$$

The kinetic term for the axion in the EMDA theory will share this symmetry too if the axion is scaled also, according to

$$\chi \longrightarrow e^{-\frac{1}{2}c\lambda}\chi. \quad (1.3)$$

However, the Chern-Simons term $b\chi\tilde{F}^{\mu\nu}F_{\mu\nu}$ will break the global symmetry unless the constants a and c are chosen such that

$$c = -2a. \quad (1.4)$$

For any choice of the parameters there is, of course, another global symmetry under which the axion is shifted by an additive constant.

In what follows we shall not, for now, impose the relation (1.4) but, as will emerge later, the diagonalization of the axial perturbation equations seems only to be possible if Eq. (1.4) holds, together with the additional restriction that $a = 1$.

In Sec. II we set up the perturbative analysis, giving the background solution and then obtaining the equations describing linearized perturbations around the background. We focus exclusively on the axial sector in this section, because in the polar sector the linearized analysis is identical to that in the EMD theories that we studied extensively in [5]. As in our previous paper, we consider the full system of scalar, electromagnetic and gravitational perturbations. All the decoupled equations we eventually obtain contain contributions from the metric perturbations, and hence are only applicable when the angular mode number ℓ , which arises in the separation of variables, is greater than or equal to 2.

In Sec. III we separate variables in the axial sector, obtaining three coupled second-order radial equations. For general values of the parameters a , b , and c in the EMDA Lagrangian it does not appear to be possible to diagonalize

the three second-order radial equations. In Sec. IV we follow a slightly different approach in which the linearized axial perturbations are described in terms of a system of six coupled first-order radial equations, and then in Sec. V we show how the system can be diagonalized in the case where the parameters a and c in the EMDA Lagrangian are specialized to the values $a = 1$, $c = -2$. The axial perturbations can then be described in terms of three decoupled second-order radial equations of the form $(\partial_{r_*}^2 + \omega^2 - V_{(p)}^-)Z_{(p)}^- = 0$, where p is a constant that can be any of the three roots of a certain cubic equation.

In Sec. VI we show that if the remaining parameter b in the specialized EMDA Lagrangian is taken to be $b = \pm 1$, then the three potentials $V_{(p)}^-(r)$ in the axial sector can be related, one-to-one, with the three potentials $V_{(p)}^+(r)$ that we found previously in the polar sector in [5]. (In particular, in the $b^2 = 1$ case the constant p in the axial sector then satisfies exactly the same cubic equation as the one arising in the polar sector.) The relation that we find between $V_{(p)}^+(r)$ and $V_{(p)}^-(r)$ for each of the three p values is of the same kind that was previously found by Chandrasekhar in the simpler cases of Schwarzschild or Reissner-Nordström black holes in Einstein or Einstein-Maxwell gravity, in which a pair of axial and polar potentials $V^\pm$ can be written in terms of a superpotential $W = f + \beta$, with $V^\pm = \pm \partial_{r_*} f + f^2 + 2\beta f = \pm \partial_{r_*} W + W^2 - \beta^2$, where β is a constant. The solutions for the radial modes in the axial and polar sectors can then also be mapped into one another. Such systems are known in the literature as “supersymmetric quantum mechanical” models (see, for example, [12–15]).

We also show explicitly in Sec. VI that the axial potentials $V_{(p)}^-(r)$ are all non-negative outside the outer horizon in the $b^2 = 1$ case. An analogous result in the polar sector was obtained previously in [5]. These results are sufficient to establish mode stability for the charged static Gibbons-Maeda black holes, viewed now as solutions in the $a = 1$, $c = -2$, $b^2 = 1$ EMDA theory.

A striking feature of supersymmetric quantum mechanical relations between the axial and polar potentials in the Schwarzschild and Reissner-Nordström cases studied by Chandrasekhar is that the theories in which the black hole backgrounds are embedded can themselves be extended to supergravities. That is to say, the theory of Einstein gravity in which the Schwarzschild perturbations were studied can be supersymmetrized by adding a fermionic sector, giving $\mathcal{N} = 1$ supergravity. Likewise, the Einstein-Maxwell theory in which the Reissner-Nordström perturbations were analyzed can be supersymmetrized to $\mathcal{N} = 2$ supergravity. It is tempting to think that there may be a deeper underlying relation between the “supersymmetry” of the supersymmetric quantum mechanical models that arise in these perturbation analyses and the fact that the underlying theories in which the black holes reside are supergravities.

Our new results in this paper may provide further insight into this question. The supersymmetric quantum mechanical relation between the axial and polar modes that we find here works *only* when the parameter b in the $a = 1$, $c = -2$ family of EMDA Lagrangians takes the value $b = 1$ or $b = -1$. Remarkably, it is precisely in these cases that the EMDA theory is related to a supergravity theory. In this case the $a = 1$, $c = -2$, $b = \pm 1$ Lagrangian is not itself the bosonic sector of a supergravity theory, but we can easily augment the Lagrangian by adding an additional Maxwell field $H_{\mu\nu}$, to give the Lagrangian (6.25), and this theory is then the bosonic sector of a supergravity (in fact, a consistent supersymmetric truncation of STU supergravity, which is $\mathcal{N} = 2$ supergravity coupled to three vector multiplets [19]). If we wish to extend the notion of linking the supersymmetric quantum mechanical relation between axial and polar modes to a possible underlying explanation rooted in a supergravity, then in our current case we should also include the perturbations of the $H_{\mu\nu}$ field in the analysis. As can be seen from the Lagrangian (6.25), the charged static Gibbons-Maeda black holes will still be solutions in this extended theory (with $H_{\mu\nu} = 0$ in the background), and the analysis of the perturbations of $H_{\mu\nu}$ will be rather simple, because they will be totally decoupled from all the other perturbations. We carry out this analysis also in Sec. VI, and show that indeed the axial and polar perturbations in the $H_{\mu\nu}$ sector have potentials that are again related via a supersymmetric quantum mechanical model; in this case, a very simple one. Thus the notion of a linkage between the occurrence of the supersymmetric quantum mechanical relations on the one hand, and the fact that the underlying theories where this arises are supergravities, is somewhat strengthened by our results.

In Sec. VII we consider the perturbations in the more general one-parameter family of EMDA theories where $a = 1$ and $c = -2$ but with b now arbitrary. The polar sector was already fully analyzed in [5] (now specialized to $a = 1$), and in particular the polar potentials were seen to be always non-negative outside the horizon. The situation with the axial potentials obtained in this paper turns out to be more interesting. We find that of the three axial potentials $V_{(p)}^-(r)$, where p is any of the three roots of the cubic equation (5.16), only the case when p is positive (it is always the case that just one root is positive) has the possibility of having the potential be negative outside the outer horizon. We find the conditions under which this can in fact occur, and establish lower bounds on the value of b^2 , beyond which there can be a region outside the horizon for which the potential is negative. The lowest positive value for which this can occur is for the minimum angular mode number $\ell = 2$, and the bound is $b > \sqrt{\frac{365}{112}} = 1.80525\dots$. Thus, if the coupling of the axion to the topological term $\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$ is too large, then the possibility of mode instabilities can arise.

The paper concludes with a summary and further discussion in Sec. VIII. Included in this discussion is an elaboration of the relations between the axial and polar sectors in the $a = 1$, $c = -2$, $b^2 = 1$ theories and the general notation of supersymmetric quantum mechanical models. There are also two Appendixes. In Appendix A, we give a more detailed discussion of the system of three coupled second-order radial equations in the $a = 1$, $c = -2$ theories, and their relation to the diagonalized equations we obtained in Sec. V. In Appendix B we exhibit some plots of the various axial and polar potentials for different values of b , showing in particular how one of the axial potentials can become negative outside the outer horizon when b is sufficiently large.

II. LINEARIZED PERTURBATIONS

A. The background solutions

The equations of motion following from the Lagrangian (1.1) are

$$\begin{aligned} R_{\mu\nu} &= \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}e^{c\phi}\partial_\mu\chi\partial_\nu\chi \\ &\quad + 2e^{a\phi}\left(F_{\mu\rho}F_{\nu}{}^\rho - \frac{1}{4}F^{\rho\sigma}F_{\rho\sigma}g_{\mu\nu}\right), \\ \nabla_\nu(e^{a\phi}F^{\mu\nu} - b\chi\tilde{F}^{\mu\nu}) &= 0, \\ \square\phi &= ae^{a\phi}F^2 + \frac{1}{2}ce^{c\phi}(\partial\chi)^2, \\ \nabla_\mu(e^{c\phi}\nabla^\mu\chi) + b\tilde{F}^{\mu\nu}F_{\mu\nu} &= 0. \end{aligned} \quad (2.1)$$

The static electrically-charged black hole solutions of the EMD theory were constructed by Gibbons and Maeda [11]. In the notation we shall be using here, which is the same as in [5], as solutions of the EMDA theory (1.1) they are given by

$$\begin{aligned} ds^2 &= -\Delta dt^2 + \frac{dr^2}{\Delta} + R^2(d\theta^2 + \sin^2\theta d\varphi^2), \\ e^{-a\phi} &= f_{\pm}^{\frac{2a^2}{1+a^2}}, \quad \chi = 0, \quad A = \frac{Q}{r}dt, \\ \Delta &= f_{\pm}f_{\pm}^{\frac{1-a^2}{1+a^2}}, \quad R = rf_{\pm}^{\frac{a^2}{1+a^2}}, \quad f_{\pm} = 1 - \frac{r_{\pm}}{r}. \end{aligned} \quad (2.2)$$

(Note that we are now including the axion χ , which vanishes in these background solutions.) The electric charge Q is related to the radii $r_{\pm}$ of the outer and inner horizons by

$$Q^2 = \frac{r_+r_-}{1+a^2}. \quad (2.3)$$

Note that in the limit when the dilaton coupling a goes to zero, the metric and Maxwell field reduce to those

of the Reissner-Nordström solution, and the dilaton field vanishes.

B. Parametrization of the perturbations

Since the background is static and spherically symmetric, the perturbations of the metric can, without loss of generality, be taken to have the form,

$$ds^2 = -e^{2\nu}dt^2 + e^{2\mu_1}dr^2 + e^{2\mu_2}d\theta^2 + e^{2\psi}(d\varphi - \mathcal{B}_\alpha dx^\alpha)^2, \quad (2.4)$$

where we index the coordinates (t, r, θ, φ) as $(0, 1, 2, 3)$, the index α ranges over $\alpha = (0, 1, 2)$, and all quantities are taken to depend on t, r and θ , but to be independent of the azimuthal angle φ . Note that the metric has the form of a Kaluza-Klein reduction from 4 to 3 dimensions, where the reduction is performed on the azimuthal coordinate φ .

We shall choose the natural tetrad frame $e^{\underline{a}} = e^{\underline{a}}_\mu dx^\mu$ with

$$\begin{aligned} e^{\underline{0}} &= e^\nu dt, & e^{\underline{1}} &= e^{\mu_1} dr, & e^{\underline{2}} &= e^{\mu_2} d\theta, \\ e^{\underline{3}} &= e^\psi(d\varphi - \mathcal{B}_\alpha dx^\alpha), \end{aligned} \quad (2.5)$$

in order to describe the metric (2.4). Note that we are using the index $\underline{a}$ to run over the tetrad-frame values $(0, \underline{1}, \underline{2}, \underline{3})$ and the index μ to run over the coordinate-frame values $(0, 1, 2, 3)$. The inverse tetrad $E_{\underline{a}}{}^\mu$ is given by

$$\begin{aligned} E_{\underline{0}}{}^\mu \partial_\mu &= e^{-\nu} \left(\frac{\partial}{\partial t} + \mathcal{B}_0 \frac{\partial}{\partial \varphi} \right), & E_{\underline{1}}{}^\mu \partial_\mu &= e^{-\mu_1} \left(\frac{\partial}{\partial r} + \mathcal{B}_1 \frac{\partial}{\partial \varphi} \right), \\ E_{\underline{2}}{}^\mu \partial_\mu &= e^{-\mu_2} \left(\frac{\partial}{\partial \theta} + \mathcal{B}_2 \frac{\partial}{\partial \varphi} \right), & E_{\underline{3}}{}^\mu \partial_\mu &= e^{-\psi} \frac{\partial}{\partial \varphi}. \end{aligned} \quad (2.6)$$

The complete set of fields parametrizing configurations of the perturbed system comprise the metric fluctuations $(\delta\nu, \delta\mu_1, \delta\mu_2, \delta\psi, \delta\mathcal{B}_\alpha)$, the Maxwell field fluctuations $\delta F_{\underline{a}\underline{b}}$, and the dilaton and axion fluctuations $\delta\phi$ and $\delta\chi$. As has been mentioned, these can be divided into the two disjoint sets,

$$\text{Axial: } (\delta\mathcal{B}_0, \delta\mathcal{B}_1, \delta\mathcal{B}_2, \delta F_{\underline{0}\underline{3}}, \delta F_{\underline{1}\underline{3}}, \delta F_{\underline{2}\underline{3}}, \delta\chi), \quad (2.7)$$

$$\text{Polar: } (\delta\nu, \delta\mu_1, \delta\mu_2, \delta\psi, \delta F_{\underline{0}\underline{1}}, \delta F_{\underline{0}\underline{2}}, \delta F_{\underline{1}\underline{2}}, \delta\phi), \quad (2.8)$$

with the axial fluctuations changing sign under a reversal of the azimuthal angle φ , while the polar fluctuations remain invariant. Because the background solution is azimuthally invariant, the equations governing the linearized axial fluctuations will then necessarily be decoupled from those governing the polar fluctuations.

Note that since the axion does not enter into the linearized polar equations, and because the axion vanishes in the background solutions, the analysis of the polar sector will be completely unchanged from that given in our previous paper [5], and so we can just refer to that paper for all the polar results that we shall need.

We now expand all the fields around the values that they take in the Gibbons-Maeda black hole background, writing

$$\begin{aligned}\nu &= \bar{\nu} + \epsilon \delta \nu, & \mu_1 &= \bar{\mu}_1 + \epsilon \delta \mu_1, \\ \mu_2 &= \bar{\mu}_2 + \epsilon \delta \mu_2, & \psi &= \bar{\psi} + \epsilon \delta \psi, \\ \mathcal{B}_\alpha &= \epsilon \delta \mathcal{B}_\alpha, & \phi &= \bar{\phi} + \epsilon \delta \phi, & \chi &= \epsilon \delta \chi.\end{aligned}\quad (2.9)$$

together with the field strength

$$F_{\underline{a}\underline{b}} = \bar{F}_{\underline{a}\underline{b}} + \epsilon \delta F_{\underline{a}\underline{b}}. \quad (2.10)$$

By comparing these expressions, together with Eq. (2.4) with the black hole solution in Eq. (2.2), it follows that the expressions for the background fields (denoted by the overbars) are

$$\begin{aligned}\bar{\nu} &= \frac{1}{2} \log \Delta, & \bar{\mu}_1 &= -\bar{\nu}, \\ \bar{\mu}_2 &= \log R, & \bar{\psi} &= \log R + \log \sin \theta, \\ \bar{\mathcal{B}}_\alpha &= 0, & a\bar{\phi} &= 2 \log r - 2 \log R, & \bar{F}_{01} &= \frac{Q}{r^2},\end{aligned}\quad (2.11)$$

where Δ and R are given in Eq. (2.2), and all components of the background field strength $\bar{F}_{\underline{a}\underline{b}}$ other than $\bar{F}_{01} = -\bar{F}_{10}$ vanish.

C. The axial perturbation equations

In the tetrad basis that we are employing, the equations of motion (2.1) take the form

$$\begin{aligned}(\text{Ein})_{\underline{a}\underline{b}} &\equiv R_{\underline{a}\underline{b}} - \frac{1}{2} R \eta_{\underline{a}\underline{b}} - \frac{1}{2} \phi_{\underline{a}} \phi_{\underline{b}} + \frac{1}{4} \phi^c \phi_c \eta_{\underline{a}\underline{b}} \\ &\quad - \frac{1}{2} e^{c\phi} \chi_{\underline{a}} \chi_{\underline{b}} + \frac{1}{4} e^{c\phi} \chi^c \chi_c \eta_{\underline{a}\underline{b}} \\ &\quad - 2e^{a\phi} \left(F_{\underline{a}\underline{c}} F_{\underline{b}}{}^{\underline{c}} - \frac{1}{4} \eta_{\underline{a}\underline{b}} F^{\underline{c}\underline{d}} F_{\underline{c}\underline{d}} \right) = 0,\end{aligned}\quad (2.12)$$

$$(\text{Max})^{\underline{a}} \equiv e^{\underline{a}}{}_\mu \partial_\nu (\sqrt{g} E^{\underline{b}\mu} E^{\underline{c}\nu} (e^{a\phi} F_{\underline{b}\underline{c}} - b\chi \tilde{F}_{\underline{b}\underline{c}})) = 0, \quad (2.13)$$

$$E_\phi \equiv \square \phi - a e^{a\phi} F^{\underline{a}\underline{b}} F_{\underline{a}\underline{b}} - \frac{1}{2} c e^{c\phi} \chi^{\underline{a}} \chi_{\underline{a}} = 0, \quad (2.14)$$

$$E_\chi \equiv \partial_\mu (\sqrt{g} e^{c\phi} E^{a\mu} \chi_{\underline{a}}) + b \sqrt{g} \tilde{F}^{\underline{a}\underline{b}} F_{\underline{a}\underline{b}} = 0, \quad (2.15)$$

where $\phi_{\underline{a}} = E_{\underline{a}}^\mu \partial_\mu \phi$, etc., and $\square \phi = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} E^{a\mu} \phi_{\underline{a}})$ in terms of tetrad quantities. The Bianchi identities for the Maxwell field are given by

$$(\text{Bianchi})^{\underline{a}} \equiv e^{\underline{a}}{}_\mu \epsilon^{\mu\nu\rho\sigma} \partial_\nu (e^{\underline{b}}{}_\rho e^{\underline{c}}{}_\sigma F_{\underline{b}\underline{c}}) = 0. \quad (2.16)$$

The linearized axial equations of motion, involving the fields listed in Eq. (2.7), come from the components $(\text{Ein})_{1\bar{3}}$ and $(\text{Ein})_{2\bar{3}}$ of the Einstein equations (2.12), the component $(\text{Max})^{\bar{3}}$ of the Maxwell equations (2.13), the axion equation E_χ (2.15), and the components $(\text{Bianchi})^1$ and $(\text{Bianchi})^2$ of the Bianchi identities (2.16). Since the only nonvanishing component of the background Maxwell field $\bar{F}_{\underline{a}\underline{b}}$ is given by $\bar{F}_{01} = \frac{Q}{r^2}$, it follows that the equations for the linearized axial perturbations are as follows:

From the Einstein equations (2.12) we have

$$\begin{aligned}(\text{Ein})_{1\bar{3}}: & e^{\bar{\nu}-\bar{\mu}_2} \frac{1}{\sin^2 \theta} \partial_\theta (\sin^3 \theta \delta \mathcal{G}_{12}) + e^{-\bar{\nu}+\bar{\mu}_2} \sin \theta \partial_t \delta \mathcal{G}_{01} \\ & - \frac{4Q}{r^2} e^{a\bar{\phi}} \delta F_{0\bar{3}} = 0, \\ (\text{Ein})_{2\bar{3}}: & e^{-2\bar{\mu}_2} \partial_r (e^{2\bar{\nu}+2\bar{\mu}_2} \delta \mathcal{G}_{12}) - e^{-2\bar{\nu}} \partial_t \delta \mathcal{G}_{02} = 0.\end{aligned}\quad (2.17)$$

[Note that $\mathcal{G} = d\mathcal{B}$ is the field strength of the Kaluza-Klein 1-form $\mathcal{B}$ appearing in the metric (2.4), with coordinate components $\mathcal{G}_{\alpha\beta}$, with α and β ranging over 0, 1 and 2. Following our conventions the nonunderlined numerical indices on $\mathcal{G}$ in Eq. (2.17) indicate coordinate index values.]

The Maxwell equation (2.13) gives,

$$\begin{aligned}(\text{Max})^{\bar{3}}: & e^{-\bar{\nu}-\bar{\mu}_2} \partial_t \delta F_{0\bar{3}} - e^{-2\bar{\mu}_2-a\bar{\phi}} \partial_r (e^{\bar{\nu}+\bar{\mu}_2+a\bar{\phi}} \delta F_{1\bar{3}}) \\ & - e^{-2\bar{\mu}_2} \partial_\theta \left(\delta F_{2\bar{3}} + e^{-a\bar{\phi}} \frac{bQ}{r^2} \delta \chi \right) \\ & + \frac{Q \sin \theta}{r^2} \delta \mathcal{G}_{01} = 0.\end{aligned}\quad (2.18)$$

The axion equation (2.15) gives

$$\begin{aligned}E_\chi: & -e^{-2\bar{\nu}+c\bar{\phi}} \partial_t^2 \delta \chi + e^{-2\bar{\mu}_2} \partial_r (e^{2\bar{\nu}+2\bar{\mu}_2+c\bar{\phi}} \partial_r \delta \chi) \\ & + e^{-2\bar{\mu}_2+c\bar{\phi}} \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta \delta \chi) - \frac{4bQ}{r^2} \delta F_{2\bar{3}} = 0.\end{aligned}\quad (2.19)$$

Finally, the Bianchi identities (2.16) give

$$\begin{aligned}(\text{Bianchi})^1: & \partial_t \delta F_{2\bar{3}} = e^{\bar{\nu}-\bar{\mu}_2} \frac{1}{\sin \theta} \partial_\theta (\sin \theta \delta F_{0\bar{3}}), \\ (\text{Bianchi})^2: & \partial_t \delta F_{1\bar{3}} = e^{\bar{\nu}-\bar{\mu}_2} \partial_r (e^{\bar{\nu}+\bar{\mu}_2} \delta F_{0\bar{3}}),\end{aligned}\quad (2.20)$$

Note that aside from the axion equation (2.19), all the other equations (absent their axion contributions) are exactly the same as those obtained in Ref. [5], where some more details of their derivation can be found.

III. SECOND-ORDER EQUATIONS FOR AXIAL PERTURBATIONS

One approach to extracting the complete system of equations for the axial perturbations is to construct directly a set of second-order differential equations. This follows essentially the same procedure that was employed in [7] for Reissner-Nordström, and that we employed in [5] for the static charged black holes in the EMD theories; now, of course, with the added complication of the axion field.

An essential step, when analyzing the system of axial equations, is that we decompose all fluctuations into modes with time dependence $e^{-i\omega t}$, with ω assumed to be nonzero. Thus time derivatives in the equations of motion become multiplication by $-i\omega$, allowing certain fields to be solved for algebraically. The general strategy is then, as set out in [5], as follows:

Taking the time derivative of $(\text{Max})^3$ and then using $(\text{Bianchi})^1$ and $(\text{Bianchi})^2$ allows $\delta F_{1\bar{3}}$ and $\delta F_{2\bar{3}}$ to be eliminated. $\delta \mathcal{G}_{01}$ can then be eliminated using $(\text{Ein})_{1\bar{3}}$. Another disentanglement of equations can be achieved by taking the ∂_θ derivative of $(\text{Ein})_{1\bar{3}}$, and the ∂_r derivative of $(\text{Ein})_{2\bar{3}}$, and using the Bianchi identity $\partial_r \delta \mathcal{G}_{02} - \partial_\theta \delta \mathcal{G}_{01} = \delta_t \delta \mathcal{G}_{12}$. The upshot is that the equations can now be separated, yielding ODEs in r , by writing

$$\begin{aligned} \delta F_{0\bar{3}}(t, r, \theta) &= \frac{i\omega \ell(\ell+1)e^{-\bar{v}}}{r \sin \theta} e^{-i\omega t} B_{0\bar{3}}(r) C_{\ell+1}^{-\frac{1}{2}}(\cos \theta) \\ &= \frac{-i\omega e^{-\bar{v}}}{r} e^{-i\omega t} B_{0\bar{3}}(r) \partial_\theta P_\ell(\cos \theta), \quad (3.1) \\ \delta \mathcal{G}_{12}(t, r, \theta) &= \frac{2i\omega \mu \ell(\ell+1)e^{-2\bar{v}-\bar{\mu}_2}}{3\sin^3 \theta} e^{-i\omega t} \mathcal{G}_{12}(r) C_{\ell+2}^{-\frac{3}{2}}(\cos \theta) \\ &= \frac{-2i\omega e^{-2\bar{v}-\bar{\mu}_2}}{\mu} e^{-i\omega t} \mathcal{G}_{12}(r) \partial_\theta (\csc \theta \partial_\theta P_\ell(\cos \theta)), \end{aligned} \quad (3.2)$$

$$\delta \chi(t, r, \theta) = -2\sqrt{\ell(\ell+1)} e^{-\bar{\mu}_2 - \frac{1}{2}c\bar{\phi}} e^{-i\omega t} \chi(r) P_\ell(\cos \theta), \quad (3.3)$$

where we have defined

$$\mu = \sqrt{\ell(\ell+1)} - 2, \quad (3.4)$$

(recall that we are restricting the mode number ℓ to $\ell \geq 2$, since we are considering the coupled perturbations that include the gravitational modes). The quantities $C_n^{(\lambda)}(x)$ are the Gegenbauer polynomials. [See Eq. (139) of Chapter 5 of [6] for some useful recurrence relations for the Gegenbauer polynomials. Note that $C_\ell^{\frac{1}{2}}(x) = P_\ell(x)$, where $P_\ell(x)$ are the Legendre polynomials.] Note that the Bianchi identity $(\text{Bianchi})^1$ can be used to obtain the algebraic relation,

$$B_{2\bar{3}} = -\ell(\ell+1)B_{0\bar{3}}. \quad (3.5)$$

Implementing the above procedures, we arrive at the following three coupled ordinary differential equations for the three radial functions $B_{0\bar{3}}(r)$, $\mathcal{G}_{12}(r)$ and $\chi(r)$,

$$\begin{aligned} (\partial_{r_*}^2 + \omega^2)B_{0\bar{3}} + T_1 B_{0\bar{3}} + S_1 \mathcal{G}_{12} + P_1 \chi &= 0, \\ (\partial_{r_*}^2 + \omega^2)\mathcal{G}_{12} + T_2 \mathcal{G}_{12} + S_1 B_{0\bar{3}} &= 0, \\ (\partial_{r_*}^2 + \omega^2)\chi + T_3 \chi + P_1 B_{0\bar{3}} &= 0, \end{aligned} \quad (3.6)$$

where

$$\begin{aligned} T_1 &= -e^{4\bar{v}} \left(a\bar{v}'\bar{\phi}' + \frac{1}{4}a^2\bar{\phi}'^2 + \frac{1}{2}a\bar{\phi}'' \right) - \frac{1}{r^2} e^{2\bar{v}+a\bar{\phi}} \left(\frac{4Q^2}{r^2} + \ell(\ell+1) \right), \\ T_2 &= -e^{4\bar{v}} \left(a\bar{v}'\bar{\phi}' + \frac{1}{4}a^2\bar{\phi}'^2 + \frac{1}{2}a\bar{\phi}'' + \frac{2}{r^2} - \frac{2}{r}\bar{v}' - \frac{a}{r}\bar{\phi}' \right) - \frac{1}{r^2} e^{2\bar{v}+a\bar{\phi}} (\ell(\ell+1) - 2), \\ T_3 &= -e^{4\bar{v}} \left(-(a-c)\bar{v}'\bar{\phi}' + \frac{1}{4}(a-c)^2\bar{\phi}'^2 - \frac{1}{2}(a-c)\bar{\phi}'' + \frac{2}{r}\bar{v}' - \frac{(a-c)}{r}\bar{\phi}' \right) - \frac{1}{r^2} e^{2\bar{v}+a\bar{\phi}} \ell(\ell+1), \\ S_1 &= -\frac{2\mu Q}{r^3} e^{2\bar{v}+a\bar{\phi}}, \\ P_1 &= -\frac{2b\sqrt{\ell(\ell+1)}Q}{r^3} e^{2\bar{v}-\frac{c}{2}\bar{\phi}}. \end{aligned} \quad (3.7)$$

Note that the derivatives ∂_{r_*} in Eq. (3.6) are with respect to the tortoise coordinate r_* defined by

$$\partial_{r_*} = e^{2\bar{v}} \partial_r. \quad (3.8)$$

We have also made use of the background relations given in Eq. (2.11) to write $\bar{\mu}_2$ in terms of r and $\bar{\phi}$ using

$$e^{\bar{\mu}_2} = r e^{-\frac{1}{2}a\bar{\phi}}. \quad (3.9)$$

For general values of the coupling constants a , b , and c in the EMDA theory (1.1) it does not appear to be possible to diagonalize the three second-order radial equations (3.6). We have, however, succeeded in diagonalizing the equations in the special case when

$$a = 1, \quad c = -2. \quad (3.10)$$

Notice that this is a particular instance of the $c = -2a$ class of EMDA theories that we mentioned in the introduction, for which the Lagrangian has the global $\mathbb{R}$ symmetry given in Eqs. (1.2) and (1.3).

IV. FIRST-ORDER EQUATIONS FOR AXIAL PERTURBATIONS

It will be useful at this stage to adopt an approach that rather closely parallels the discussion of the polar sector that we gave previously in [5], as this will be helpful later on when we find circumstances under which the axial and polar sectors can be related. This approach is based on directly dealing with a system of coupled first-order radial equations. We can obtain our system of separated first-order equations by making the following substitutions in the equations (2.17), (2.18), (2.19), and (2.20):

$$\begin{aligned} \delta\mathcal{G}_{01} &= 2e^{-\bar{\mu}_2}e^{-i\omega t}\mathcal{G}_{01}(r) \csc\theta\partial_\theta P_\ell, \\ \delta\mathcal{G}_{02} &= \frac{2e^{-\bar{\mu}_2}}{\mu}e^{-i\omega t}\mathcal{G}_{02}(r)\partial_\theta(\csc\theta\partial_\theta P_\ell), \\ \delta\mathcal{G}_{12} &= -2i\omega\frac{e^{-2\bar{\nu}-\bar{\mu}_2}}{\mu}e^{-i\omega t}\mathcal{G}_{12}(r)\partial_\theta(\csc\theta\partial_\theta P_\ell), \\ \delta F_{0\bar{3}} &= -i\omega\frac{e^{-\bar{\nu}}}{r}e^{-i\omega t}B_{0\bar{3}}(r)\partial_\theta P_\ell, \\ \delta F_{1\bar{3}} &= \frac{e^{-\bar{\nu}}}{r}e^{-i\omega t}B_{1\bar{3}}(r)\partial_\theta P_\ell, \\ \delta F_{2\bar{3}} &= \frac{e^{-\bar{\mu}_2}}{r}e^{-i\omega t}B_{2\bar{3}}(r)P_\ell, \\ \delta\chi &= -2\sqrt{\ell(\ell+1)}e^{-\bar{\mu}_2-\frac{1}{2}c\bar{\phi}}e^{-i\omega t}\chi(r)P_\ell. \end{aligned} \quad (4.1)$$

Note that the fields $\mathcal{G}_{12}$, $B_{0\bar{3}}$, and χ are the ones that remained in our previous construction of the second-order equations (3.6), after making use of the various Bianchi identities, Maxwell and Einstein equations. Now, by no longer using the Bianchi, Maxwell and Einstein equations to eliminate variables, we can recast the system as an extended set of variables obeying first-order equations, as detailed below.

Substituting Eq. (4.1) into the Bianchi, Maxwell, Einstein, and axion equations, the angular dependence can be seen to factor out upon using standard properties of the Legendre polynomials, as does the time dependence, and we are left with the purely radial equations

$$(\text{Ein})_{1\bar{3}}: \frac{2Q}{r}B_{0\bar{3}} - e^{2\bar{\mu}_2}\mathcal{G}_{01} + \sqrt{(\ell+2)(\ell-1)}\mathcal{G}_{12} = 0, \quad (4.2)$$

$$(\text{Ein})_{2\bar{3}}: e^{-\bar{\mu}_2}\partial_r(e^{\bar{\mu}_2}\mathcal{G}_{12}) - e^{-2\bar{\nu}}\mathcal{G}_{02} = 0, \quad (4.3)$$

$$(\text{Bianchi})^1: \ell(\ell+1)B_{0\bar{3}} + B_{2\bar{3}} = 0, \quad (4.4)$$

$$(\text{Bianchi})^2: r e^{-\bar{\mu}_2}\partial_r\left(\frac{1}{r}e^{\bar{\mu}_2}B_{0\bar{3}}\right) - e^{-2\bar{\nu}}B_{1\bar{3}} = 0, \quad (4.5)$$

$$(\text{Max})^3: \frac{1}{r}e^{\bar{\mu}_2}\partial_r(re^{-\bar{\mu}_2}B_{1\bar{3}}) + \omega^2e^{-2\bar{\nu}}B_{0\bar{3}} + e^{-2\bar{\mu}_2}B_{2\bar{3}} - \frac{2Q}{r}\mathcal{G}_{01} - \frac{2b\sqrt{\ell(\ell+1)}Q}{r^3}e^{-\frac{1}{2}c\bar{\phi}}\chi = 0, \quad (4.6)$$

$$\begin{aligned} E_\chi: \chi'' + 2\bar{\nu}'\chi' + \left[\omega^2e^{-4\bar{\nu}} - \bar{\mu}_2'(\bar{\mu}_2' + 2\bar{\nu}') - c\bar{\phi}'\left(\bar{\mu}_2' + \bar{\nu}' + \frac{1}{4}c\bar{\phi}'\right) - \bar{\mu}_2'' \right. \\ \left. - \frac{1}{2}c\bar{\phi}'' - \ell(\ell+1)e^{-2\bar{\mu}_2-2\bar{\nu}} \right] \chi + \frac{2bQ}{\sqrt{\ell(\ell+1)}}\frac{e^{-2\bar{\nu}-\frac{1}{2}c\bar{\phi}}}{r^3}B_{2\bar{3}} = 0, \end{aligned} \quad (4.7)$$

$$d\mathcal{G}: e^{\bar{\mu}_2}\partial_r(e^{-\bar{\mu}_2}\mathcal{G}_{02}) + \omega^2e^{-2\bar{\nu}}\mathcal{G}_{12} - \sqrt{(\ell+2)(\ell-1)}\mathcal{G}_{01} = 0. \quad (4.8)$$

(We have used the background relations in reducing the equations to this form.)

We now introduce the auxiliary variable $\xi = \chi'$, and we also make use of the purely algebraic equations (4.2) and (4.4) to solve for $\mathcal{G}_{01}$ and $B_{\underline{2}3}$, thus eliminating them from the remaining system of differential equations. We are left with six independent variables,

$$\{\mathcal{G}_{02}, \mathcal{G}_{12}, B_{\underline{0}3}, B_{\underline{1}3}, \chi, \xi\}, \quad (4.9)$$

Thus we arrive at the system of six first-order equations,

$$\begin{aligned} \mathcal{G}'_{02} &= \bar{\mu}'_2 \mathcal{G}_{02} + (2n e^{-2\bar{\mu}_2} - \omega^2 e^{-2\bar{\nu}}) \mathcal{G}_{12} + \frac{2\sqrt{2nQ}}{r} e^{-2\bar{\mu}_2} B_{\underline{0}3}, \\ \mathcal{G}'_{12} &= e^{-2\bar{\nu}} \mathcal{G}_{02} - \bar{\mu}'_2 \mathcal{G}_{12}, \\ B'_{\underline{0}3} &= \left(\frac{1}{r} - \bar{\mu}'_2\right) B_{\underline{0}3} + e^{-2\bar{\nu}} B_{\underline{1}3}, \\ B'_{\underline{1}3} &= \left(2(n+1)e^{-2\bar{\mu}_2} - \omega^2 e^{-2\bar{\nu}} + \frac{4Q^2}{r^2} e^{-2\bar{\mu}_2}\right) B_{\underline{0}3} - \left(\frac{1}{r} - \bar{\mu}'_2\right) B_{\underline{1}3} + \frac{2\sqrt{2nQ}}{r} e^{-2\bar{\mu}_2} \mathcal{G}_{12} + \frac{2b\sqrt{2(n+1)Q}}{r^3} e^{-\frac{1}{2}c\bar{\phi}} \chi, \\ \chi' &= \xi, \\ \xi' &= \left[2(n+1)e^{-2\bar{\mu}_2-2\bar{\nu}} + \bar{\mu}'_2(\bar{\mu}'_2 + 2\bar{\nu}') + c\bar{\phi}'\left(\bar{\mu}'_2 + \bar{\nu}' + \frac{1}{4}c\bar{\phi}'\right) + \bar{\mu}_2'' + \frac{1}{2}c\bar{\phi}'' - \omega^2 e^{-4\bar{\nu}}\right] \chi \\ &\quad + \frac{2b\sqrt{2(n+1)Q}}{r^3} e^{-2\bar{\nu}-\frac{1}{2}c\bar{\phi}} B_{\underline{0}3} - 2\bar{\nu}'\xi. \end{aligned} \quad (4.11)$$

V. DIAGONALIZATION OF THE AXIAL MODES

Having obtained the system of six coupled first-order equations (4.11) describing the axial perturbations of the static black holes in the EMDA theories (1.1), we now proceed by using these to derive diagonalized second-order equations of the form,

$$(\partial_{r_*}^2 + \omega^2)Z^- - V^- Z^- = 0, \quad (5.1)$$

where, for each Z^- , the function $V^-(r)$ will be the potential. The fact that the perturbations in the axial sector can be completely described by three coupled second-order equations for $B_{\underline{0}3}$, $\mathcal{G}_{12}$, and χ , as we saw in Sec. III, suggests that we should try an ansatz of the form,

$$Z^- = f_1 \mathcal{G}_{12} + f_2 B_{\underline{0}3} + f_3 \chi, \quad (5.2)$$

where the functions $f_i(r)$ are to be determined.

Substituting (5.2) into (5.1) and making repeated use of the six first-order equations (4.11) yields an equation of the form,

$$e_1 \mathcal{G}_{02} + e_2 \mathcal{G}_{12} + e_3 B_{\underline{0}3} + e_4 B_{\underline{1}3} + e_5 \chi + e_6 \xi = 0. \quad (5.3)$$

which will satisfy first-order equations. As in the Reissner-Nordström and EMD cases we define

$$n = \frac{1}{2}\ell(\ell+1) - 1 = \frac{1}{2}\mu^2. \quad (4.10)$$

Since the six fields $(\mathcal{G}_{02}, \mathcal{G}_{12}, B_{\underline{0}3}, B_{\underline{1}3}, \chi, \xi)$ are treated as being independent here, this means that the coefficients e_j must all vanish,

$$e_j = 0, \quad 1 \leq j \leq 6. \quad (5.4)$$

Each of the coefficients e_j will depend upon the functions f_i , the background metric functions, and derivatives thereof. The coefficients e_2 , e_3 and e_5 will also depend on the unknown potential function V^- .

The coefficients e_1 , e_4 , and e_6 are independent of V^- , and each consists of a single term that can immediately be solved for f_1 , f_2 , and f_3 . In fact, we simply have

$$\begin{aligned} e_1 &= 2f_+ f_+^{\frac{1-a^2}{1+a^2}} f'_1 = 0 \Rightarrow f_1 = c_1, \\ e_4 &= 2f_+ f_+^{\frac{1-a^2}{1+a^2}} f'_2 = 0 \Rightarrow f_2 = c_2, \\ e_6 &= 2(f_+ f_-)^2 f_+^{\frac{-4a^2}{1+a^2}} f'_3 = 0 \Rightarrow f_3 = c_3, \end{aligned} \quad (5.5)$$

for constants c_i to be determined. Substituting these constant expressions for the f_i back into Eq. (5.3), the remaining equations for $j = 2, 3$ and 5 lead to the following conclusions.

First we solve $e_2 = 0$ for the potential V^- , finding

$$V^- = \frac{1}{(1+a^2)^2 c_1 r^5} f_+ f_+^{\frac{-4a^2}{1+a^2}} [2(n+1)(1+a^2)^2 c_1 r^3 - (1+a^2)[(2n(1+a^2) + (5-a^2))r_- + 3(1+a^2)r_+]c_1 - 2\sqrt{2n}Q(1+a^2)c_2]r^2 + [(3r_- + 7(1+a^2)r_+)r_- c_1 - 2\sqrt{2n}Q(1+a^2)^2 c_2 r_-]r - (4+a^2)c_1 r_-^2 r_+]. \quad (5.6)$$

Up until this point, the calculations have proceeded rather analogously to those that we performed when analyzing the polar perturbation sector in the EMD theory (and therefore also in the polar sector in our present EMDA case). However, we now encounter obstacles to proceeding further, for generic values of the coupling constants a , b , and c in the theory. Substituting the solutions for f_i in Eq. (5.5) and the solution (5.6) for V^- into the remaining equations $e_3 = 0$ and $e_5 = 0$, gives the conditions,

$$\left(\frac{3-a^2}{1+a^2}r_- + 3r_+\right)c_1 c_2 + 2\sqrt{2n}Q(c_1^2 - c_2^2) + 2b\sqrt{2(n+1)}Qf_-^{\frac{a(2a+c)}{1+a^2}}c_1 c_3 = 0, \quad (5.7)$$

$$\sqrt{2n}Qr(r-r_-)c_2 c_3 - b\sqrt{2(n+1)}Qr(r-r_-)f_-^{\frac{a(2a+c)}{1+a^2}}c_1 c_2 - B(r)c_1 c_3 = 0, \quad (5.8)$$

where $B(r)$ is quadratic in r and independent of the constants c_i ,

$$B(r) = \frac{1}{2(1+a^2)^2} [4(1+a^2)[(1+a^2)r_+ + (1-a^2)r_-]r^2 + r_-[(a^2 c^2 - 4)r_- - (1+a^2)(ac + 10)r_+]r + (ac + 2)(3 + a^2 - ac)r_-^2 r_+]. \quad (5.9)$$

Because of the occurrence generically of terms with incommensurate fractional powers of the metric function f_- , Eq. (5.7) requires that we must impose the condition $c = -2a$. This then implies that

$$B(r) = \frac{(r-r_-)}{1+a^2} [2[(1+a^2)r_+ + (1-a^2)r_-]r - 3(1-a^2)r_- r_+], \quad (5.10)$$

and therefore to be able to solve Eq. (5.8) for all r we must have $a = 1$ (since, without loss of generality, we may assume a is non-negative). Thus we can only solve the full system of equations $e_j = 0$ if the constants a and c in the Lagrangian (1.1) are restricted to take the values

$$a = 1, \quad c = -2. \quad (5.11)$$

Accordingly, we shall assume from now on that Eq. (5.11) are imposed.

With these assumptions, the equations $e_3 = 0$ and $e_5 = 0$ now imply the algebraic conditions,

$$(r_- + 3r_+)c_1 c_2 + 2\sqrt{2n}Q(c_1^2 - c_2^2) + 2b\sqrt{2(n+1)}Qc_1 c_3 = 0, \quad (5.12)$$

$$(\sqrt{2n}c_3 - b\sqrt{2(n+1)}c_1)Qc_2 - 2r_+ c_1 c_3 = 0. \quad (5.13)$$

Introducing a new dimensionless parameter p in place of c_1 by defining

$$c_1 = -\frac{p c_2 r_-}{4\sqrt{2n}Q}, \quad (5.14)$$

Eq. (5.13) then implies

$$c_3 = -\frac{b\sqrt{2(n+1)}Qc_2 p}{2(p+2n)r_+}, \quad (5.15)$$

and so Eq. (5.12) implies that p must satisfy the cubic equation

$$\tilde{\Lambda}(p) \equiv p^3 r_- + p^2 [2n(1+b^2)r_- - 2(1-b^2)r_- - 6r_+] - 4np(r_- + 7r_+) - 32n^2 r_+ = 0. \quad (5.16)$$

The final conclusion, then, is that we have three linearly independent diagonalized axial eigenfunctions,

$$Z_{(p)}^- = c_1 \mathcal{G}_{12} + c_2 B_{03} + c_3 \mathcal{X}, \quad (5.17)$$

obeying the second-order wave equations,

$$(\partial_{r_*}^2 + \omega^2)Z_{(p)}^- - V_{(p)}^- Z_{(p)}^- = 0, \quad (5.18)$$

where c_1 and c_3 are given in terms of c_2 by Eqs. (5.14) and (5.15) and p is any of the three roots of Eq. (5.16). With the requirement that $r_+ \geq r_-$ in Eq. (5.16), p is then restricted to lie in three specific, real ranges, as established below, and in Sec. VII A. The corresponding potentials $V_{(p)}^-$ are given by substituting Eqs. (5.14) and (5.15) into Eq. (5.6) with $a = 1$; that is,

$$V_{(p)}^- = \frac{(r - r_+)[8(n+1)r^3 - 4(2(n+1)r_- + 3r_+)r^2 + (3r_- + 14r_+)r - 5r_-^2r_+]}{4(r - r_-)^2r^4} - \frac{8n(r - r_+)r_+}{p(r - r_-)r^3}. \quad (5.19)$$

It is straightforward to see, as we shall now show, that for all valid values of the parameters, the three roots p_i of the cubic polynomial (5.16) are all real, and furthermore that two are negative and the third is positive. The discriminant of cubic (5.16) is given by

$$\begin{aligned} D_p = & 64n^2[16(n+1)^3r_-^3r_+b^6 + (n+1)^2(r_-^2 + (48n-34)r_-r_+ - 95r_+^2)r_-^2b^4 \\ & + 2(n+1)[(n-1)r_-^3 + (24n^2+2n+7)r_-^2r_+ + (157n+53)r_-r_+^2 + 69r_+^3]r_-b^2 \\ & + [(n+1)r_- - r_+]^2[r_-^2 + 2(8n+3)r_-r_+ + 9r_+^2]]. \end{aligned} \quad (5.20)$$

This can be written as

$$\begin{aligned} D_p = & 64n^2[16(n+1)b^2r_-r_+[3r_+ - (n+1)b^2r_-]^2 + r_+^2[3r_+ - (n+1)(3+b^2)r_-]^2 \\ & + 4(n+1)(77n+25)b^2r_-^2r_+^2 + r_-[r_- + 2(8n+3)r_+][r_+ - (n+1)r_-]^2 \\ & + 2(n+1)b^2r_-^3[(n-1)r_- + (24n^2+2n+7)r_+] + (n+1)^2b^4r_-^3[r_- + (48n-34)r_+]], \end{aligned} \quad (5.21)$$

which is a sum of manifestly positive terms for all b^2 , since we are concerned only with modes with $\ell \geq 2$ and therefore $n \geq 2$. This establishes that the three roots p_i are all real.

From the cubic (5.16) it follows that the roots p_i obey

$$\begin{aligned} p_1p_2p_3 &= \frac{32n^2r_+}{r_-} > 0 \quad \text{and} \\ p_1p_2 + p_2p_3 + p_1p_3 &= -\frac{4n(r_- + 7r_+)}{r_-} < 0. \end{aligned} \quad (5.22)$$

Ordering the roots as $p_1 \leq p_2 \leq p_3$, the first equation here implies that either $p_i > 0$ for all i , or else $p_1 < 0$ and $p_2 < 0$ with $p_3 > 0$. The second equation in (5.22) implies that we cannot have all $p_i > 0$, and therefore it must be that

$$p_1 \leq p_2 < 0, \quad p_3 > 0. \quad (5.23)$$

Having successfully separated and diagonalized the equations describing the linearized axial fluctuations around the static black hole solutions in the $a=1$, $c=-2$ class of EMDA theories, we are in a position to study the question of mode stability. This would be established, by the energy integral method we used previously [5], if the potentials $V_{(p)}^-$ could be shown to be non-negative everywhere outside the horizon at $r = r_+$. Before addressing this question for general values of the remaining parameter b in the EMDA theories [see Eq. (1.1)], we shall first consider

the special case where $b^2 = 1$. This is of particular interest because then, the theory is actually closely related to the bosonic sector of an $\mathcal{N} = 2$ supergravity. As we shall now see, not only are the axial potentials manifestly non-negative outside the horizon in this case but also, we can establish a remarkable one-to-one mapping between the potentials, and also the modes, in the axial and the polar sectors. This generalizes earlier results obtained by Chandrasekhar for the axial and polar modes in the perturbations of Schwarzschild [16] and Reissner-Nordström [10].

VI. THE SUPERGRAVITY CASE:

$$a = 1, c = -2, b^2 = 1$$

A. Relation between V^- and V^+

If we assume not only $a = 1$ and $c = -2$, as in Eq. (5.11), but in addition

$$b^2 = 1, \quad (6.1)$$

then the cubic equation (5.16) for p becomes

$$\begin{aligned} \tilde{\Lambda}_1(p) \equiv & p^3r_- + p^2[4nr_- - 6r_+] \\ & - 4np(r_- + 7r_+) - 32n^2r_+ = 0. \end{aligned} \quad (6.2)$$

This is identical, with p replaced by q , to the cubic equation that we found in [5] characterizing the diagonalized modes and potentials in the polar sector, after specializing that result to $a = 1$. [See Eq. (5.18) in [5].]

This raises the possibility that we might be able to establish a one-to-one mapping between the potentials, and between the modes, of the axial sector and the polar sector in the EMDA theory. This would be a striking extension of the analogous relation between the axial and polar sectors of the perturbations of the Reissner-Nordström black holes, which was established in [10] and elaborated upon in [6].

The relation that was found in [10] between the potentials in the axial and polar sectors for the Reissner-Nordström perturbations took the form,

$$V^\pm = \pm \partial_{r_*} f + \alpha f^2 + 2\beta f, \quad (6.3)$$

where f is related to a superpotential $W = f + \beta$, and α and β are certain constants. In our present case, we shall look for precisely such a relation (6.3) also. Note that from Eq. (6.3) it follows that:

$$V^+ - V^- = 2\partial_{r_*} f. \quad (6.4)$$

The expression in [5] for V^+ in the EMD case is quite complicated, even under the specialization $a = 1$ that we now require, and we shall not reproduce it here. [It is given by the function g in Eq. (5.14) of [5], and associated equations].

Rather than viewing the potentials $V^\pm$ as being dependent on the black hole parameters r_- and r_+ , with p as being the roots of the cubic equation (6.2), it is easier to view r_- and p as being the two adjustable parameters, with r_+ then being obtained by solving Eq. (6.2) to give

$$r_+ = \frac{p[p^2 + 4np - 4n]r_-}{2(2n + p)(8n + 3p)}. \quad (6.5)$$

Using this, and inserting the expressions for $V_{(p)}^\pm$ into Eq. (6.4), we find that

$$f_{(p)} = \frac{r_-(r - r_+)[(4n + 2p)r - pr_-]}{2r^2(r - r_-)(4nr + pr_-)}, \quad (6.6)$$

It can now be verified that there do indeed exist constants α and β such that Eq. (6.3) both hold, with

$$\alpha_{(p)} = 1, \quad \beta_{(p)} = \frac{4n(n + 1)}{(2n + p)r_-}, \quad (6.7)$$

and hence

$$V_{(p)}^\pm = \pm \partial_{r_*} f_{(p)} + f_{(p)}^2 + \frac{8n(n + 1)}{(2n + p)r_-} f_{(p)}. \quad (6.8)$$

B. Relation between the Z^+ and Z^- eigenfunctions

The radial equations in the axial and polar sectors obey

$$[\partial_{r_*}^2 + \omega^2 - V^\pm]Z^\pm = 0, \quad (6.9)$$

for each of the three roots p , with $-$ corresponding to the axial sector and $+$ to the polar sector. In view of Eq. (6.8) it is evident that Eq. (6.9) can be written as

$$D_- D_+ Z^- + (\omega^2 + \beta^2)Z^- = 0, \quad (6.10)$$

$$D_+ D_- Z^+ + (\omega^2 + \beta^2)Z^+ = 0, \quad (6.11)$$

where we have defined

$$D_\pm = \partial_{r_*} \pm (f + \beta), \quad (6.12)$$

and the constant β is given in Eq. (6.7).

Acting with D_+ on the axial equation (6.10) it can be seen that $D_+ Z^-$ obeys the polar equation (6.11). Similarly, acting with D_- on the polar equation (6.11) it can be seen that $D_- Z^+$ obeys the axial equation (6.10). In fact, we can choose normalizations of the solutions so that the relations between Z^- and Z^+ take the form,

$$D_- Z^+ = (i\omega - \beta)Z^-, \quad D_+ Z^- = (i\omega + \beta)Z^+. \quad (6.13)$$

Thus we have an explicit one-to-one mapping between each mode of the radial wave equation for the axial modes Z^- and the polar modes Z^+ . This generalizes the analogous relation between the axial and polar modes in the case of the perturbations of the Reissner-Nordström black hole, which was established by Chandrasekhar in [10].

C. Algebraically special modes

It was observed previously, for example in [17,18], that there can exist certain exceptional solutions of the equations for linearized perturbations of black hole metrics that have the feature of being purely ingoing or purely outgoing. These solutions, which are not relevant when considering putative unstable modes obeying physically-motivated boundary conditions on the horizon and at infinity, have nonetheless been the subject of some investigations. In the context of perturbations of Schwarzschild or Kerr, these modes were referred to as ‘‘Algebraically Special’’ perturbations, since they preserved the Algebraically Special form of the Weyl tensor. In [18], Chandrasekhar extended the notion of algebraically special to Reissner-Nordström, even though gravitational and electromagnetic perturbations are inextricably coupled there. Here, for convenience, we do the same, and use the term ‘‘Algebraically Special’’ to describe the similar, special generalizations that we find here of the Schwarzschild and Reissner-Nordström modes.

We observe from Eqs. (6.10) and (6.11) that one can obtain such algebraically special solutions for Z^- and Z^+

having a very simple form, in the case where the frequency ω takes the purely imaginary values,

$$\omega = \pm i\beta, \quad (6.14)$$

and where Z^+ or Z^- obey the first-order equations

$$D_+ Z^- = 0, \quad D_- Z^+ = 0, \quad (6.15)$$

respectively. Thus, the algebraically special solutions are given by

$$Z^\pm = \exp\left[\pm \int^r (f(r') + \beta) e^{-2\bar{v}(r')} dr'\right]. \quad (6.16)$$

Substituting in the expressions for $f_{(p)}$ and $\beta_{(p)}$ from Eqs. (6.6) and (6.7), we find

$$\begin{aligned} Z_{(p)}^+ &= e^{\beta_{(p)} r_*} \left(1 - \frac{r_-}{r}\right)^{\frac{1}{2}} \left(4n + \frac{pr_-}{r}\right)^{-1}, \\ Z_{(p)}^- &= e^{-\beta_{(p)} r_*} \left(1 - \frac{r_-}{r}\right)^{-\frac{1}{2}} \left(4n + \frac{pr_-}{r}\right), \end{aligned} \quad (6.17)$$

where $r_* = \int^r e^{-2\bar{v}(r')} dr' = r + r_+ \log(\frac{r}{r_+} - 1)$ is the tortoise coordinate. [Note that these algebraically special polar and axial solutions are inversely related, with $Z_{(p)}^+ Z_{(p)}^- = 1$.] Including the time dependence $e^{-i\omega t}$, with $\omega = \pm i\beta$, these special explicit solutions have the forms,

$$\begin{aligned} e^{-i\omega t} Z_{(p)}^+ &= e^{\beta_{(p)}(r_* \pm t)} \left(1 - \frac{r_-}{r}\right)^{\frac{1}{2}} \left(4n + \frac{pr_-}{r}\right)^{-1}, \\ e^{-i\omega t} Z_{(p)}^- &= e^{-\beta_{(p)}(r_* \mp t)} \left(1 - \frac{r_-}{r}\right)^{-\frac{1}{2}} \left(4n + \frac{pr_-}{r}\right), \end{aligned} \quad (6.18)$$

Note that although these are of interest as explicit exact solutions of the linearized equations, they do not obey physically interesting boundary conditions. This is because if they are ingoing on the future horizon they are also ingoing at past null infinity $\mathcal{I}^-$, and conversely if outgoing on future null infinity $\mathcal{I}^+$ they are also outgoing on the past horizon.

As well as the pair of algebraically special solutions obtained above, one can also, of course, construct in each case the corresponding “second solution” by the standard Wronskian method. Thus one also has the explicit (up to quadrature) second solutions of the form,

$$\begin{aligned} \tilde{Z}_{(p)}^+(r) &= Z_{(p)}^+(r) \int^r \frac{e^{-2\bar{v}(r')}}{(Z_{(p)}^+(r'))^2} dr', \\ \tilde{Z}_{(p)}^-(r) &= Z_{(p)}^-(r) \int^r \frac{e^{-2\bar{v}(r')}}{(Z_{(p)}^-(r'))^2} dr', \end{aligned} \quad (6.19)$$

where $Z_{(p)}^+$ and $Z_{(p)}^-$ are given in Eq. (6.17). The integrals in Eq. (6.19) are rather involved to evaluate explicitly. We just remark here that these second solutions $\tilde{Z}_{(p)}^+$ and $\tilde{Z}_{(p)}^-$ do not have the feature of being purely ingoing or purely outgoing, unlike the solutions (6.17).

D. Positivity of the V^- potentials

The positivity of the axial potentials V^- outside the horizon in the supergravity case $a = 1$, $c = -2$, $b^2 = 1$ may be established using the same techniques that we employed for the polar potentials V^+ in the general EMD case in [5].

We already showed in Sec. V that for valid values of the parameters the three roots p of the cubic equation (6.2) are real, and furthermore two are negative and one positive. Thus, when we instead choose to solve (6.2) for r_+ , as in Eq. (6.5), there will be three disjoint real ranges for p , two negative and one positive, that correspond to the situations of interest to us where $r_+ > r_-$. Writing n in terms of ℓ , as in Eq. (4.10), the solution for r_+ in Eq. (6.5) implies that $(r_+ - r_-)$ factorizes over the rationals,

$$\begin{aligned} r_+ - r_- &= \frac{[p + 2(\ell - 1)][p - 2(\ell + 2)][p + 2(\ell + 2)(\ell - 1)]r_-}{2[p + (\ell + 2)(\ell - 1)][3p + 4(\ell + 2)(\ell - 1)]}. \end{aligned} \quad (6.20)$$

Thus, we can easily identify the three segments in the ranges of p for which $(r_+ - r_-) > 0$, namely,

$$\begin{aligned} (1): & p > 2(\ell + 2), \\ (2): & -(\ell + 2)(\ell - 1) < p < -2(\ell - 1), \\ (3): & -2(\ell + 2)(\ell - 1) < p < -\frac{4}{3}(\ell + 2)(\ell - 1). \end{aligned} \quad (6.21)$$

Generically, we need to consider modes with $\ell \geq 2$. We can then parametrize ℓ and r (outside the horizon) by

$$\ell = 2 + y, \quad r = r_+(1 + z), \quad (6.22)$$

where $y \geq 0$ and $z \geq 0$. For the three cases in Eq. (6.21) we can parametrize the valid ranges of p in terms of $x \geq 0$, with

$$\begin{aligned} \text{Case (1): } & p = 2(\ell + 2)(1 + x), \\ \text{Case (2): } & p = -(\ell + 2)(\ell - 1) + \frac{\ell(\ell - 1)}{1 + x}, \\ \text{Case (3): } & p = -2(\ell + 2)(\ell - 1) + \frac{2(\ell + 2)(\ell - 1)}{3(1 + x)}. \end{aligned} \quad (6.23)$$

For each of the three cases, we can plug these parametrizations into the expression (5.19) for the potential $V_{(p)}^-$. We find that in each case the potential $V_{(p)}^-$ then takes the factorized form,

$$V^- = RP, \quad (6.24)$$

where for each of the three cases R is a simple manifestly positive rational function of x , y , and z , and P is a somewhat complicated multinomial in x , y , and z . In fact in case (1) P has 177 terms, in case (2) P has 207 terms, and in case (3) P has 169 terms. Remarkably, the coefficient of each term in each of the three multinomials P is positive, thus proving that V^- is positive outside the horizon for each of the three cases in (6.21).

It was already established in [5] that the polar potential V^+ was positive outside the horizon for the case of static black holes in the whole class of EMD theories with arbitrary values of a . Thus, in particular, V^+ is positive outside the horizon for the case $a = 1$, which corresponds to the potential in the polar sector in our present analysis of the perturbations around static black holes in the $a = 1$, $c = -2$, $b^2 = 1$ EMDA theory. Thus, combined with our new analysis here of the axial sector in this theory, we have obtained a complete demonstration of the linearized mode stability of the static black holes in this EMDA theory.

Some illustrative plots showing examples of the three axial and three polar potentials, for the case of the $\ell = 2$ modes, can be found in Fig. 1 in Appendix B. In particular, it can be seen that for $b = 1$, all the potentials are indeed positive outside the horizon, consistent with the positivity proof we gave above.

E. Relation to $\mathcal{N} = 2$ supergravity

STU supergravity is $\mathcal{N} = 2$ supergravity coupled to three vector multiplets [19]. Thus its bosonic sector comprises the metric, four $U(1)$ Maxwell gauge fields, three dilatons and three axions. There exists a supersymmetric consistent truncation in which two dilatons and two axions are set to zero and the four Maxwell fields are set pairwise equal, together with corresponding truncations in the fermionic sector. (The truncation is discussed in a number of papers, including, for example, [20,21].) The bosonic sector of this truncated theory is described by the Lagrangian,

$$\mathcal{L} = \sqrt{-g} \left[R - \frac{1}{2} (\partial\phi)^2 - e^\phi F^2 - \frac{1}{2} e^{-2\phi} (\partial\chi)^2 + \chi \tilde{F}^{\mu\nu} F_{\mu\nu} - \frac{e^{-2\phi}}{1 + \chi^2 e^{2\phi}} (e^\phi H^2 + \chi \tilde{H}^{\mu\nu} H_{\mu\nu}) \right], \quad (6.25)$$

where $H_{\mu\nu}$ is the field strength of the second Maxwell field. Note that the first line in (6.25) is exactly the same as our

EMDA Lagrangian (1.1) when $a = 1$, $c = -2$, and $b = 1$. It can be seen that the Gibbons-Maeda static black hole solutions that we are taking to form the backgrounds in the EMDA theories will also be solutions of the theory (6.25), in which we take the additional Maxwell field to vanish, $H_{\mu\nu} = 0$. Furthermore, the linearized perturbations around the static Gibbons-Maeda black holes within this supergravity theory will comprise precisely the modes we have already obtained in the EMDA theory with $a = 1$, $c = -2$, $b^2 = 1$, together with a rather simple and totally decoupled system of modes for the $H_{\mu\nu}$ field, obeying the equation $\bar{\nabla}^\mu (e^{-\tilde{\phi}} \delta H_{\mu\nu}) = 0$.

F. Inclusion of the $H_{\mu\nu}$ field of the supergravity theory

As discussed in Sec. VI E, we can also view the Gibbons-Maeda black hole for $a = 1$ as being a solution of the $\mathcal{N} = 2$ supergravity whose bosonic sector is described by the Lagrangian (6.25). As we mentioned, the complete set of linearized bosonic perturbations within this theory will comprise exactly the perturbations within the EMDA theory as we have discussed already earlier in this section, together with a decoupled system describing the perturbations of the $H_{\mu\nu}$ field; these obey $\bar{\nabla}^\mu (e^{-\tilde{\phi}} \delta H_{\mu\nu}) = 0$. (As usual, the overbars denote background quantities.) These perturbations will themselves divide into disjoint axial and polar sectors.

Because of the fact that the $\delta H_{\mu\nu}$ perturbations are decoupled from the other perturbations in the EMDA sector, it is in fact rather straightforward to implement the separation of variables, and the construction of the second-order radial equations for each of the axial and polar perturbations, by making appropriate truncations and modifications to results we have already presented here and in [5]. For example, in the axial sector we just take the Bianchi identities (4.4) and (4.5), with the components $B_{\underline{a}\underline{b}}$ replaced by the corresponding components $C_{\underline{a}\underline{b}}$ associated with the $\delta H_{\underline{a}\underline{b}}$ perturbations, and then we take the Maxwell equation (4.6), with $B_{\underline{a}\underline{b}}$ replaced by $e^{-2\tilde{\phi}} C_{\underline{a}\underline{b}}$ (to take account of the opposite sign of the dilaton coupling in the exponential prefactor in the kinetic term for $H_{\underline{a}\underline{b}}$). At the same time the $\mathcal{G}_{01}$ and χ terms should be omitted (since $\delta H_{\underline{a}\underline{b}}$ does not couple to any other perturbations). In the polar sector, we can analogously read off the equations from Eqs. (4.16)–(4.18) of [5], sending $B_{\underline{a}\underline{b}} \rightarrow C_{\underline{a}\underline{b}}$ in the Bianchi identity and $B_{\underline{a}\underline{b}} \rightarrow e^{-2\tilde{\phi}} C_{\underline{a}\underline{b}}$ in the Maxwell equations, and also with the perturbations of the metric and the dilaton field omitted. In each of the axial and polar sectors, appropriate additional r -dependent scaling redefinitions are also then needed, in order to arrive at radial functions $Z_H^\pm$ that obey standard $(\partial_{r_*}^2 + \omega^2 - V_H^\pm) Z_H^\pm = 0$ wave equations.

The upshot is that in the axial sector we can write

$$\delta H_{03}(t, r, \theta) = -\frac{i\omega e^{-\bar{\nu}+\bar{\phi}}}{r} e^{-i\omega t} Z_H^-(r) \partial_\theta P_\ell(\cos \theta), \quad (6.26)$$

and in the polar sector,

$$\delta H_{12}(t, r, \theta) = -\frac{i\omega e^{-\bar{\nu}+\bar{\phi}}}{r^2} e^{-i\omega t} Z_H^+(r) \partial_\theta P_\ell(\cos \theta), \quad (6.27)$$

with the other components of δH_{ab} being expressed just in terms of $Z_H^-(r)$ and $Z_H^+(r)$. Specifically, we find

$$\begin{aligned} \delta H_{13} &= e^{\bar{\nu}-\bar{\mu}_2} \partial_r (r e^{-\bar{\mu}_2} Z_H^-(r)) e^{-i\omega t} \partial_\theta P_\ell(\cos \theta), \\ \delta H_{23} &= -\frac{\ell(\ell+1) e^{-\bar{\mu}_2+\bar{\phi}}}{r} Z_H^-(r) e^{-i\omega t} P_\ell(\cos \theta), \end{aligned} \quad (6.28)$$

in the axial sector, and

$$\begin{aligned} \delta H_{12} &= -\frac{i\omega e^{-\bar{\nu}+\bar{\phi}}}{r^2} Z_H^+(r) e^{-i\omega t} \partial_\theta P_\ell(\cos \theta), \\ \delta H_{01} &= \frac{\ell(\ell+1) e^{-\bar{\mu}_2+\bar{\phi}}}{r^2} Z_H^+(r) e^{-i\omega t} P_\ell(\cos \theta), \end{aligned} \quad (6.29)$$

in the polar sector.

The radial equations then turn out to be

$$(\partial_{r_*}^2 + \omega^2 - V_H^\pm) Z_H^\pm = 0, \quad (6.30)$$

where the axial and polar potentials are given by

$$V_H^\pm = \frac{2(n+1) e^{2\bar{\nu}+\bar{\phi}}}{r^2} + e^{4\bar{\nu}} \left[\frac{1}{4} \bar{\phi}'^2 \pm \left(\bar{\nu}' \bar{\phi}' + \frac{1}{2} \bar{\phi}'' \right) \right], \quad (6.31)$$

Thus, we can find a function f_H such that

$$V_H^\pm = \pm \partial_{r_*} f_H + f_H^2 + 2\beta_H f_H, \quad (6.32)$$

with

$$f_H = \frac{1}{2} e^{2\bar{\nu}} \bar{\phi}', \quad \beta_H = \frac{2(n+1) e^{\bar{\phi}}}{r^2 \bar{\phi}'}. \quad (6.33)$$

Explicitly, the two potentials in Eq. (6.31) are given by

$$V_H^- = \frac{(r-r_+)[8(n+1)r^3 - 4(2n+3)r_-r^2 + 3(r_-+2r_+)r_-r - 5r_-^2r_+]}{4r^4(r-r_-)^2}, \quad (6.34)$$

$$V_H^+ = \frac{(r-r_+)[8(n+1)r^3 - 4(2n+1)r_-r^2 - (r_-+6r_+)r_-r + 3r_-^2r_+]}{4r^4(r-r_-)^2}, \quad (6.35)$$

and, as can be seen from Eq. (2.2), the function f_H and the constant β_H in Eq. (6.33) are given by

$$f_H = -\frac{r_-(r-r_+)}{2r^2(r-r_-)}, \quad \beta_H = -\frac{2(n+1)}{r_-}. \quad (6.36)$$

It can easily be seen, by writing $r = (1+z)r_+$, $r_+ = (1+w)r_-$ and $n = 2+y$, that V_H^+ and V_H^- are both non-negative everywhere outside the horizon.

In the case of these axial and polar modes of the $H_{\mu\nu}$ field, the explicit algebraically special solutions that we obtained in Eqs. (6.15) and (6.16) for the axial and polar modes in the EMDA sector are extremely simple, and we find

$$\omega = \pm i\beta_H = \mp \frac{2i(n+1)}{r_-}, \quad (6.37)$$

$$Z_H^+ = e^{\beta_H r_*} \left(1 - \frac{r_-}{r}\right)^{-\frac{1}{2}}, \quad Z_H^- = e^{-\beta_H r_*} \left(1 - \frac{r_-}{r}\right)^{\frac{1}{2}}, \quad (6.38)$$

where $r_* = r + r_+ \log(\frac{r}{r_+} - 1)$ is the tortoise coordinate. For the two sign choices for ω in these special solutions, the t and r dependences for these waves have the form,

$$e^{-i\omega t} Z_H^+ = e^{\beta_H(r_* \pm t)} \left(1 - \frac{r_-}{r}\right)^{-\frac{1}{2}} \quad (6.39)$$

and

$$e^{-i\omega t} Z_H^- = e^{-\beta_H(r_* \mp t)} \left(1 - \frac{r_-}{r}\right)^{\frac{1}{2}}. \quad (6.40)$$

For the same reason as we already discussed for the triplets of explicit modes $Z_{(p)}^\pm$ obtained in Eq. (6.17), these exact solutions do not obey physically interesting boundary conditions. The second solutions of the analogous forms to those given in Eq. (6.19) can also be constructed here. In fact it happens that in the polar sector, the second solution $\tilde{Z}_H^+$ turns out to have a relatively simple explicit form, namely,

$$\tilde{Z}_H^+ = \frac{(2\beta_H r_- e^{\beta_H(r-2r_+)}) \text{Ei}[1 + 2\beta_H r_+, 2\beta_H(r-r_+)] - e^{-\beta_H r} \sqrt{r}}{2\beta_H \sqrt{r-r_-} (r-r_+)^{\beta_H r_+}}, \quad (6.41)$$

where $\text{Ei}[\alpha, z] = \int_1^\infty e^{-zt} t^{-\alpha} dt$ is the exponential integral function.

To summarize, we have shown that if we view the $a = 1$ Gibbons-Maeda charged static black holes as solutions in the $\mathcal{N} = 2$ supergravity theory whose bosonic sector is described by the Lagrangian (6.25), then the complete set of linearized perturbations are described by the radial functions $Z_i^\pm$, for $i = 1, 2$, and 3 , together with the radial functions $Z_H^\pm$. For each of these, the associated potentials $(V_1^+, V_2^+, V_3^+, V_H^+)$ in the polar sector are related to the corresponding potentials $(V_1^-, V_2^-, V_3^-, V_H^-)$ in the axial sector, as given in Eqs. (6.8) and (6.32). All the potentials are non-negative outside the outer horizon, thus demonstrating linearized mode stability for the $a = 1$ Gibbons-Maeda black holes as solutions of $\mathcal{N} = 2$ supergravity.

VII. THE $a = 1, c = -2$ EMDA THEORIES WITH b^2 ARBITRARY

When $b^2 \neq 1$, the axial sector of the $a = 1, c = -2$ EMDA theories is no longer related to the polar sector in the way we saw in the $b^2 = 1$ case in the previous section. It is still of interest, however, to investigate the properties of the potential V^- in the axial radial wave equation.

A. Allowed ranges for p

When b^2 is arbitrary, we can again view the cubic equation $\tilde{\Lambda}(p)$ in Eq. (5.16) as a linear equation for r_+ , with p rather than r_+ being taken to be a parameter of the solutions. As in the $b^2 = 1$ case, there are generally three allowable ranges for p , now dependent on b , which we find by examining the expression for $r_+ - r_-$, namely,

$$r_+ - r_- = \frac{r_-(p^3 + [2n(1+b^2) - 2(1-b^2) - 6]p^2 - 32np - 32n^2)}{2(p+2n)(3p+8n)}, \quad (7.1)$$

and requiring $(r_+ - r_-) > 0$.

There are two special values for b^2 such that the numerator of (7.1) has a factor which is also in the denominator, namely when $b^2 = 0$ and when $b^2 = \frac{4n+3}{12(n+1)}$. These correspond to cases where the cubic (5.16) factorizes, with a linear factor that is independent of r_+ and r_- ,

$$\begin{aligned} b^2 = 0: \quad \tilde{\Lambda}(p) &= (p+2n)[r_- p^2 - 2(r_- + 3r_+)p - 16nr_+], \\ b^2 = \frac{4n+3}{12(n+1)}: \quad \tilde{\Lambda}(p) &= \frac{1}{6}(3p+8n)[2r_- p^2 - 3(r_- + 4r_+)p - 24nr_+]. \end{aligned} \quad (7.2)$$

In each of these cases there is therefore a discrete root for p that is independent of r_+ and r_- , and the two other roots lie at points within ranges that depend upon the ratio $\frac{r_+}{r_-}$ (which must exceed 1). Thus, when $b^2 = 0$ we have

$$\text{Fixed: } p = -2n, \quad (7.3)$$

$$\text{Range (i): } -\frac{8n}{3} < p < 4(1 - \sqrt{n+1}) < 0, \quad (7.4)$$

$$\text{Range (ii): } 0 < 4(1 + \sqrt{n+1}) < p < \infty, \quad (7.5)$$

and when instead $b^2 = \frac{4n+3}{12(n+1)}$ we have

$$\text{Fixed: } p = -\frac{8n}{3}, \quad (7.6)$$

$$\text{Range (i): } -2n < p < \frac{15}{4} \left(1 - \sqrt{1 + \frac{64n}{75}} \right) < 0, \quad (7.7)$$

$$\text{Range (ii): } 0 < \frac{15}{4} \left(1 + \sqrt{1 + \frac{64n}{75}} \right) < p < \infty. \quad (7.8)$$

For all other values of $b^2 > 0$, there are three valid ranges for p , and these divide into two classes, one for $0 < b^2 < \frac{4n+3}{12(n+1)}$, and the other for $\frac{4n+3}{12(n+1)} < b^2 < \infty$. In the first case, the numerator in Eq (7.1) has a zero, $p_z = p_a$, between $-\frac{8n}{3}$ and $-2n$, and an allowable range for p is $-\frac{8n}{3} < p < p_a < -2n$; in the second case, the numerator has a zero, $p_z = p_b$, between $-\infty$ and $-\frac{8n}{3}$, and an allowable range for p is $-\infty < p_b < p < -\frac{8n}{3}$. Using the location of each zero, we can solve $r_+ - r_- = 0$ in Eq. (7.1) for b^2 in terms of that zero ($p_z = p_a$ or $p_z = p_b$),

$$b^2 = \frac{(p_z + 2n)[2(\ell + 2) - p_z][p_z + 2(\ell - 1)]}{2p_z^2(n + 1)} > 0 \quad \forall p_z < -2n, \quad (7.9)$$

and then factor the numerator of eq (7.1) into $(p - p_z)(p - p_-)(p - p_+)$, where $p_{\pm}$ are solutions (see below) to the resulting quadratic equation. The additional allowable ranges for p are then $-2n < p < p_- < 0$ and $0 < p_+ < p < +\infty$.

While it would be most convenient to express our results in terms of b^2 , the task of then solving the cubic for p makes this impractical. What we have found is that, as we change b^2 from 0 to ∞ , one root of the cubic moves from $p = -2n$ to $-\infty$. If we choose a p in this latter range, then we can simultaneously find the value of b^2 for which this is a root, and solve the remaining quadratic for the other two roots for p . This is the approach we use, whereby we can obtain appropriate ranges for all roots p , and for b^2 , that are parametrized in terms of a real variable x lying in the range $0 < x < \infty$.

For b^2 lying in the range $0 < b^2 < \frac{4n+3}{12(n+1)}$, it will be parametrized as

$$b^2 = \frac{4(4n+3)x^3 + 12(2n+1)x^2 + 9nx}{3(n+1)(3+4x)^2(1+x)}, \quad \text{with } 0 < x < \infty, \quad (7.10)$$

and we find that the three valid ranges for p are

$$\text{Range (i): } -\frac{8n}{3} < p < p_a < -2n \quad \text{where } p_a = -\frac{2n(3+4x)}{3(1+x)}, \quad (7.11)$$

$$\text{Range (ii): } -2n < p < p_{-,a} < 0 \quad \text{where } p_{-,a} = \frac{4(3(1+x)(3+5x) - \sqrt{P_a})}{(3+4x)^2}, \quad (7.12)$$

$$\text{Range (iii): } 0 < p_{+,a} < p < \infty \quad \text{where } p_{+,a} = \frac{4(3(1+x)(3+5x) + \sqrt{P_a})}{(3+4x)^2}, \quad (7.13)$$

$$\text{where } P_a = 3n(1+x)(3+4x)^3 + [3(1+x)(3+5x)]^2. \quad (7.14)$$

Similarly, for b^2 lying in the range $\frac{4n+3}{12(n+1)} < b^2 < \infty$ it will be parametrized as

$$b^2 = \frac{(1+4x)(4nx^2 + 8nx + 4n + 12x + 3)}{12(1+x)^2(n+1)} < \infty, \quad \text{with } 0 < x < \infty, \quad (7.15)$$

and we find that the three valid ranges for p are

$$\text{Range (i): } -\infty < p_b < p < -\frac{8n}{3} \quad \text{where } p_b = -\frac{8n}{3}(1+x), \quad (7.16)$$

$$\text{Range (ii): } -2n < p < p_{-,b} < 0 \quad \text{where } p_{-,b} = \frac{3(5+8x) - \sqrt{P_b}}{4(1+x)^2}, \quad (7.17)$$

$$\text{Range (iii): } 0 < p_{+,b} < p < \infty \quad \text{where } p_{+,b} = \frac{3(5+8x) + \sqrt{P_b}}{4(1+x)^2}, \quad (7.18)$$

$$\text{where } P_b = 3n[4(1+x)]^3 + [3(5+8x)]^2. \quad (7.19)$$

Note that the special case $b^2 = 1$ here corresponds to $x = \frac{1}{2}$.

It should also be noted that in all cases, we have that for any given choice of the parameter values, there are two negative roots and one positive root, in accordance with what we found in Sec. V and Sec. VII A.

B. Conditions under which V^- can be negative outside the horizon

The three axial potentials, corresponding to the three roots p_i of the cubic polynomial (5.16), are given by

$$A = \frac{8n(r - r_+)r_+}{(r - r_-)r^3},$$

$$B = \frac{(r - r_+)[8(n + 1)r^3 - 4(2(n + 1)r_- + 3r_+)r^2 + (3r_- + 14r_+)r_-r - 5r_-^2r_+]}{4(r - r_-)^2r^4}. \quad (7.21)$$

The function A is manifestly positive outside the outer horizon, and by writing

$$r = r_+(1 + z), \quad r_+ = r_-(1 + w), \quad (7.22)$$

it follows that B can be written as

$$B = \frac{(r - r_+)r_-^2r_+}{4r^4(r - r_-)^2} [8(n + 1)(1 + w)^2z^3 + 4(1 + w)[4n + 1 + 3(2n + 1)w]z^2 + [8n + 1 + 2(16n - 1)w + 24nw^2]z + 2w[4n - 1 + 2(2n - 1)w]]. \quad (7.23)$$

Since $n \geq 2$ it can be seen that B is also manifestly positive outside the outer horizon. It follows that if p is negative then the potential $V_{(p)}^-$ is definitely positive outside the horizon. In other words, the axial potential $V_{(p)}^-(r)$ only has the possibility of being negative outside the horizon in the case where the root p is positive. It may be recalled that we showed in Sec. V that the three roots are always real, and that two of them are negative while the third is positive.

Numerical examples show that under certain circumstances the potential can in fact be negative in some region outside the horizon. For example, if the parameters are taken to be such that

$$r_+ = 2r_-, \quad \ell = 2, \quad b = 5, \quad (7.24)$$

then for the positive root, $p \approx 1.83155$, the potential is negative in the region $r_+ < r < r_{\max}$ near to the horizon, where $r_{\max} \approx 1.925r_+$. It is of interest, therefore, to investigate more extensively to establish when the potential can be negative outside the outer horizon.

C. Some properties of the potential

The potential $V_{(p)}^-(r)$ takes the form,

$$V_{(p)}^- = \frac{(r - r_+)}{4r^4(r - r_-)^2} U(r), \quad (7.25)$$

Eq. (5.19). Note that the parameter b does not appear explicitly in the expression (5.19); it enters only in the cubic (5.16) whose roots determine the three values p_i for the parameter p . Note also that the potential $V_{(p)}^-$ in Eq. (5.19) is of the form,

$$V_{(p)}^- = -\frac{A}{p} + B, \quad (7.20)$$

where A and B are independent of p , and given by

where

$$U(r) = 8(n + 1)r^3 - 4[(2(n + 1)r_- + 3r_+)r^2 + (3r_- + 14r_+)r_-r - 5r_-^2r_+] - \frac{32n(r - r_-)r_+r}{p}. \quad (7.26)$$

As we have already observed, the potential outside the horizon cannot be negative unless p is positive. The potential will be negative there if and only if $U(r)$ is negative for some $r > r_+$.

The function $U(r)$ is cubic in r , and therefore has three roots r_1 , r_2 , and r_3 . These are either all real, or else one is real with the other two being a complex-conjugate pair. In the limits of large negative and positive r we have

$$U(r) \longrightarrow -\infty \quad \text{when} \quad r \longrightarrow -\infty,$$

$$U(r) \longrightarrow +\infty \quad \text{when} \quad r \longrightarrow +\infty. \quad (7.27)$$

We also have

$$U(0) = -5r_+r_-^2 < 0, \quad U(r_-) = -3(r_+ - r_-)r_-^2 < 0, \quad (7.28)$$

and

$$r_1 r_2 r_3 = \frac{5}{8(n+1)} r_+ r_-^2 > 0. \quad (7.29)$$

If the roots are all real, then with $r_1 \leq r_2 \leq r_3$ it must be that r_3 at least is positive (r_1 and r_2 could be both positive or both negative). If there is only one real root, r_3 , then r_3 must again be positive. We can now establish the following:

- (1) *$U(r)$ can have at most one root outside the horizon at r_+ :*

It follows from Eq. (7.29) that in the case of three real roots, at most one can lie outside the horizon at r_+ . Suppose all three were outside r_+ ; in this case

$$\frac{r_1}{r_+} \frac{r_2}{r_+} \frac{r_3}{r_+} = \frac{5}{8(n+1)} \frac{r_-^2}{r_+^2}, \quad (7.30)$$

implying a contradiction since the left-hand side would be greater than 1 while the right-hand side would be less than 1. Suppose instead two roots lay outside the outer horizon; in this case

$$\frac{r_1}{r_-} = \frac{5}{8(n+1)} \frac{r_+ r_-}{r_3 r_2}. \quad (7.31)$$

The right-hand side would be less than 1, and hence $0 < r_1 < r_-$. But there cannot exist just one root between 0 and r_- , since as seen in Eq. (7.28), $U(r)$ is negative both at $r = 0$ and $r = r_-$. Thus, $U(r)$ can have at most one root outside the outer horizon.

Suppose, then, that there exists just one root, r_3 , outside the outer horizon. Solving $U(r_3) = 0$ for p and substituting this back into $U(r)$ gives

$$U(r) = \frac{(r - r_3)}{r_3(r_3 - r_-)} P_2(r), \quad (7.32)$$

where

$$\begin{aligned} P_2(r) = & 8(n+1)r_3(r_3 - r_-)r^2 \\ & - r_-[8(n+1)r_3^2 - (8n+5)r_3r_- \\ & + 2r_3r_+ - 5r_-r_+]r + 5(r_3 - r_-)r_-^2r_+. \end{aligned} \quad (7.33)$$

Letting $r_+ = r_-(1+w)$, $r = r_+(1+z)$ and $n = 2+y$ it can be seen that for positive y , z , and w the discriminant of the quadratic $P_2(r)$ is positive, and furthermore that the two (real) roots r_1 and r_2 must both be positive, with $0 < r_1 \leq r_2 < r_-$. In other words, if $U(r)$ has one root outside the outer horizon, then the other two roots are real and lie between 0 and r_- .

- (2) *If $U(r_+) > 0$ then $U(r) > 0$ for all $r_+ \leq r < \infty$:*
We have

$$U(r_+) = \frac{2(r_+ - r_-)r_+}{p} [pr_- + (4np - 2p - 16n)r_+], \quad (7.34)$$

and hence if this is to be positive we must have

$$p = \frac{ur_- + 16nr_+}{2(2n-1)r_+ + r_-}, \quad (7.35)$$

with $u > 0$. Writing $r_+ = r_-(1+w)$, it follows that

$$\begin{aligned} U'(r_+) = & \frac{r_-^3}{[2(2n-1)r_+ + r_-]} [48n + (8n+1)u \\ & + 2[64n^2 + 72n + (16n-1)u]w \\ & + 8n(32n + 28 + 3u)w^2 \\ & + 128n(n+1)w^3], \end{aligned} \quad (7.36)$$

which is manifestly positive since $n \geq 2$. Since, as we showed above, $U(r)$ can have at most one zero for $r > r_+$, and since $U(r) \rightarrow +\infty$ as $r \rightarrow +\infty$, it follows that if $U(r_+) > 0$ then $U(r)$ cannot be negative outside the outer horizon.

The conclusion from the properties established above is that if the potential $V^-(r)$ is to be negative in some region outside the outer horizon, then it must be that $U(r)$ would have to be negative on the horizon, i.e., $U(r_+) < 0$.

We have

$$\begin{aligned} U(r_+) = & 2(r_+ - r_-)r_+ W(r_+), \\ W(r_+) = & \frac{pr_- - 2(p + 8n - 2np)r_+}{p}. \end{aligned} \quad (7.37)$$

If $U(r_+) < 0$ it therefore follows that $W(r_+) < 0$.

D. Lower bounds on b^2 for having $V^-(r) < 0$ outside r_+

We know that $V^-(r)$ is always non-negative outside the outer horizon if $b^2 = 1$, and we know from numerical examples that $V^-(r)$ can become negative in some region outside the horizon if b^2 is sufficiently large. It is therefore of interest to determine bounds on b^2 , and, in particular, to determine the lowest value of b^2 for which the potential can be negative outside the horizon. This can be done as follows.

We already showed that if $U(r_+) > 0$ then $U'(r_+) > 0$ and furthermore $U(r)$ is then greater than zero outside the horizon. Thus, in order to be able to have $U(r) < 0$ in some region outside the horizon, $U(r_+)$ would have to be negative. The threshold case is then when $U(r_+) = 0$.

Solving this for p , we have that

$$U(r_+) = 0 \quad \Rightarrow \quad p = \frac{16nr_+}{2(2n-1)r_+ + r_-}. \quad (7.38)$$

Substituting this value for p into the cubic equation $\tilde{\Lambda}(p)$ given in Eq. (5.16), we can solve for b^2 , finding

$$b^2 = \frac{[r_- + 2(2n+3)r_+][3r_-^2 - 2r_-r_+ + (n+1)(2n-1)r_+^2]}{16(n+1)r_-r_+[r_- + 2(2n-1)r_+]}. \quad (7.39)$$

It can easily be seen by writing

$$r_+ = r_-(1+w), \quad n = 2+y, \quad (7.40)$$

that b^2 is positive for all valid values of the parameters (that is, $y \geq 0$ and $w \geq 0$).

It can also easily be seen that viewing b^2 in Eq. (7.39) as a function of y and w , and differentiating with respect to w , gives

$$\frac{\partial b^2}{\partial w} > 0, \quad \forall w \geq 0. \quad (7.41)$$

This implies that the threshold value of b^2 , above which the potential can become negative outside the horizon, is as small as possible when w approaches zero, that is, in the extremal limit

$$r_+ \longrightarrow r_-. \quad (7.42)$$

Setting $r_+ = r_-$ in Eq. (7.38) and Eq. (7.39) then implies that at threshold,

$$p = \frac{16n}{4n-1}, \quad b^2 = \frac{(4n+7)(16n^2+8n-7)}{16(n+1)(4n-1)}. \quad (7.43)$$

This gives the threshold values of b^2 for each $n = \frac{1}{2}\ell(\ell+1) - 1$. Thus, we have for $\ell = 2, 3, 4, 5 \dots$ that

$$b_{\min} = \{1.80525, 2.53171, 3.23273, 3.93299, \dots\}. \quad (7.44)$$

If b exceeds these values, then the potential will be negative in some region outside the outer horizon. Thus the smallest lower bound is for the $\ell = 2$ angular mode, and is $b > \sqrt{\frac{365}{112}}$.

VIII. SUMMARY AND DISCUSSION

The goal of this paper was to study the linearized perturbations of black hole solutions in theories of the kind that one encounters in supergravity. Such theories typically involve scalar fields in addition to the metric and

Maxwell fields, and consequently the analysis of the linearized perturbations is considerably more complicated than in pure Einstein or Einstein-Maxwell gravity. Our focus in this paper has been on the perturbations of static charged black holes in a three-parameter class of EMDA theories, described by the Lagrangian (1.1).

A number of general features emerged in the course of our analysis of the linearized perturbations. The equations describing the dynamics of the perturbed fields can be separated for all choices of the three parameters in the EMDA Lagrangian. That is to say, the angular and time dependence can be factored off, thereby reducing the problem to the analysis of ordinary differential equations satisfied by the radial functions. The equations describing the perturbations in the polar sector, which are disjoint from those for the axial sector, are identical to those arising in the study of linearized perturbations in a family of Einstein-Maxwell-Dilaton theories. We studied these previously in [5]. The polar perturbations were reduced to three functions obeying three decoupled second-order radial equations. In the axial sector, we showed in the first five sections of the current paper that the perturbations are described by a coupled system of three second-order radial equations, which cannot in general be diagonalized. Diagonalization is possible in the special case where two of the three parameters in the EMDA Lagrangian (1.1) are fixed, namely $a = 1$ and $c = -2$.

By this means, we arrived at a one-parameter family of EMDA theories,

$$\mathcal{L} = \sqrt{-g} \left[R - \frac{1}{2}(\partial\phi)^2 - e^\phi F^2 - \frac{1}{2}e^{-2\phi}(\partial\chi)^2 + b\chi\tilde{F}^{\mu\nu}F_{\mu\nu} \right], \quad (8.1)$$

for which the linearized perturbations can be essentially fully analyzed, in the sense that they can be reduced to three axial functions obeying three decoupled second-order equations, and three polar functions obeying three decoupled radial equations. The remaining free parameter in the family of EMDA theories (8.1) is the constant b in the Lagrangian (1.1) (with $a = 1$ and $c = -2$), which characterizes the strength of the axionic coupling to the topological term $\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$.

We showed that the potentials $V_{(p)}^-(r)$ in the diagonalized radial equations describing the linearized perturbations in the axial sector are all non-negative outside the horizon when $b^2 = 1$, and in fact for all values of b^2 that are sufficiently small. If the potentials are all non-negative, then this establishes the linearized mode stability of the black hole solution.

If, however, b^2 is larger than certain threshold values (which themselves depend on the angular mode number ℓ), then one of the three axial potentials will become negative in a region outside, and near to, the outer horizon. Under such circumstances there no longer exists a simple energy argument establishing mode stability, and a more detailed investigation, perhaps numerical, would be needed in order to see whether mode instabilities could then actually arise. The threshold values of b^2 are given in Eq. (7.43). Thus, taking b to be positive without loss of generality, then the smallest of the thresholds, which occurs for the $\ell = 2$ mode, is at $b = \sqrt{\frac{365}{112}} \approx 1.80525$. If b exceeds this value then one of the $\ell = 2$ modes in the axial sector has a potential that is negative in a region outside and near to the outer horizon.

One specific member of the one-parameter family of theories described by the Lagrangian (8.1) is particularly notable, namely when $b = 1$ (or, equivalently, and related by an orientation reversal, $b = -1$). We showed that in this specific case the potentials $V_i^+(r)$ appearing in the three decoupled polar radial equations $(\partial_{r_*}^2 + \omega^2 - V_i^+)Z_i^+ = 0$ (for $i = 1, 2$, and 3) are related to the three potentials $V_i^-(r)$ appearing in the three decoupled axial radial equations $(\partial_{r_*}^2 + \omega^2 - V_i^-)Z_i^- = 0$. In fact, for each value of i the corresponding polar and axial potentials are expressible in terms of a “superpotential,” with

$$V^\pm = \pm \partial_{r_*} W + W^2 - \beta^2, \quad (8.2)$$

where $W = f + \beta$ and the superpotential function f and constant β are given in Eqs. (6.6) and (6.7). The axial and polar fields are also related, with $D_\pm Z^\mp = (i\omega \pm \beta)Z^\pm$, where $D_\pm = \partial_{r_*} \pm W$, as discussed earlier.

Systems where there exists such a relation between two potentials are known in the literature as “supersymmetric quantum mechanical” models. (See, for example, [12–15, 22, 23].) Defining a supercharge

$$\mathcal{Q} = (\mathbf{p} + iW)\mathbf{b}, \quad (8.3)$$

where $\mathbf{p} = -i\partial_{r_*}$ is a momentum operator and $\mathbf{b}$ is a fermionic operator obeying

$$\begin{aligned} \{\mathbf{b}^\dagger, \mathbf{b}\} &= 1, & \{\mathbf{b}, \mathbf{b}\} &= 0, & \{\mathbf{b}^\dagger, \mathbf{b}^\dagger\} &= 0, \\ [\mathbf{p}, \mathbf{b}] &= 0, & [\mathbf{p}, \mathbf{b}^\dagger] &= 0, \end{aligned} \quad (8.4)$$

it can be seen that $\{\mathcal{Q}^\dagger, \mathcal{Q}\} = 2H$, with the Hamiltonian H being given by

$$H = \frac{1}{2}(-\partial_{r_*}^2 + V^+ + \beta^2)\mathbf{b}\mathbf{b}^\dagger + \frac{1}{2}(-\partial_{r_*}^2 + V^- + \beta^2)\mathbf{b}^\dagger\mathbf{b}. \quad (8.5)$$

Representing $\mathbf{b}$ by the matrix $\mathbf{b} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, this implies

$$H = \frac{1}{2} \begin{pmatrix} -\partial_{r_*}^2 + V^+ + \beta^2 & 0 \\ 0 & -\partial_{r_*}^2 + V^- + \beta^2 \end{pmatrix}, \quad (8.6)$$

and so by defining

$$\mathbf{Z} = \begin{pmatrix} Z^+ \\ Z^- \end{pmatrix}, \quad (8.7)$$

the polar and axial equations can be written in the unified form

$$H\mathbf{Z} = \frac{1}{2}(\omega^2 + \beta^2)\mathbf{Z}. \quad (8.8)$$

There is another special property of the one-parameter family of Lagrangians (8.1) when $b = 1$, namely that in this case it is a consistent truncation of the bosonic theory given in Eq. (6.25). This latter theory is the bosonic sector of an $\mathcal{N} = 2$ supergravity, which is itself the consistent truncation of a larger $\mathcal{N} = 2$ theory known as STU supergravity.¹ This supersymmetric consistent truncation of STU supergravity is implemented in the bosonic sector by setting its four Maxwell fields equal in pairs, yielding $F_{\mu\nu}$ and $H_{\mu\nu}$ in Eq. (6.25), while at the same time setting two of its three dilatons to zero and two of its three axions to zero. (Corresponding truncations in the fermionic sector must also be implemented.)

As we showed in Sec. VI F, the axial and polar radial functions $Z_H^\pm$ associated with the perturbations of the additional $H_{\mu\nu}$ field in the $\mathcal{N} = 2$ supergravity theory also have potentials $V_H^\pm$ that are related via a superpotential. Thus, the feature of the $a = 1$, $c = -2$, $b = 1$ EMDA theory, of having axial and polar sectors that are related as a supersymmetric quantum mechanical model, persists also for all the bosonic perturbations within the full $\mathcal{N} = 2$ supergravity theory.

¹It should be noted that although the Maxwell field $H_{\mu\nu}$ in the bosonic Lagrangian (6.25) can be consistently truncated by setting $H_{\mu\nu} = 0$, thereby yielding the EMDA theory described by the Lagrangian (8.1), there is no analogous *supersymmetric* truncation of the supergravity theory whose bosonic sector is given by (6.25). That is to say, the bosonic theory described by the Lagrangian (8.1) is not *itself* the bosonic sector of any supersymmetric theory.

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DATA AVAILABILITY

The graphs in Figs. 1 and 2 were plotted from equations appearing in this paper and in reference [5], as detailed in the figure captions.

APPENDIX A: THE COUPLED $a=1, c=-2$ SECOND-ORDER EQUATIONS

In Sec. III we obtained the system of three coupled second-order equations describing the axial perturbations in the three-parameter EMDA theories in Eq. (3.6), but we were unable to diagonalize the equations for general values of the parameters a , b , and c . We then went on to study the system in terms of a set of six first-order equations, and we showed how we could diagonalize the system if the special choice of parameters $a = 1$ and $c = -2$ was made. It is of interest now to return to the direct second-order analysis of Sec. III, but now imposing the specialization,

$$a = 1, \quad c = -2. \quad (\text{A1})$$

With this done, we find that the coupled second-order system given by Eqs. (3.6) and (3.7) takes the form,

$$(\partial_{r_*}^2 + \omega^2 + T_1)\Psi + \frac{e^{2\bar{v}+\bar{\phi}}}{r^3}\mathbf{M}\Psi = 0, \quad \Psi = \begin{pmatrix} \mathcal{G}_{12} \\ B_{03} \\ \chi \end{pmatrix}, \quad (\text{A2})$$

where T_1 is the given in Eq. (3.7), but now with $a = 1$, and $\mathbf{M}$ is the constant matrix,

$$\mathbf{M} = \begin{pmatrix} r_- + 3r_+ & -2\sqrt{2n}Q & 0 \\ -2\sqrt{2n}Q & 0 & -2\sqrt{2(n+1)}bQ \\ 0 & -2\sqrt{2(n+1)}bQ & r_- - r_+ \end{pmatrix}. \quad (\text{A3})$$

Since the nondiagonal term in Eq. (A2) is of the form of an overall function times the symmetric constant matrix $\mathbf{M}$, it is evident that the system can be diagonalized by means of an orthogonal transformation. The final conclusion for the form of the resulting decoupled equations will be equivalent to the decoupled equations that we obtained in Sec. V.

To make this more precise, the orthogonal matrix S that diagonalizes $\mathbf{M}$, that is, $\mathbf{M} \rightarrow S\mathbf{M}S^T = \mathbf{M}_{\text{diag}}$, has components S_{ij} such that

$$\frac{S_{i1}}{S_{i2}} = \frac{2\sqrt{n}\sqrt{r_-r_+}}{(r_- + 3r_+ - s_i)}, \quad \frac{S_{i3}}{S_{i2}} = -\frac{4nr_-r_+ + (r_- + 3r_+ - s_i)s_i}{2b\sqrt{n+1}\sqrt{r_-r_+}(r_- + 3r_+ - s_i)}, \quad (\text{A4})$$

for $i = 1, 2$ and 3 , where s_i denotes the roots of the cubic equation,

$$s^3 - 2(r_- + r_+)s^2 + (r_-^2 - 3r_+^2 + 2r_-r_+[1 - 2n - 2(n+1)b^2])s + 4r_-r_+[n(r_- - r_+) + (n+1)b^2(r_- + 3r_+)] = 0. \quad (\text{A5})$$

Comparing with the discussion in Sec. V, it is evident that we must have

$$\frac{S_{i1}}{S_{i2}} = \frac{c_1}{c_2}, \quad \frac{S_{i3}}{S_{i2}} = \frac{c_3}{c_2}, \quad (\text{A6})$$

where c_1 and c_3 are given in terms of c_2 in Eqs. (5.14) and (5.15), with p taken to be the root p_i of the cubic (5.16). Indeed, it can be verified that these equations hold, provided that p is related to s by

$$s = r_- + \left(3 + \frac{8n}{p}\right)r_+. \quad (\text{A7})$$

In particular one finds that the cubic equation (A5) becomes equivalent to the cubic equation (5.16) if Eq. (A7) holds.

APPENDIX B: SOME EXAMPLE POTENTIALS

Here we present some plots of the three axial potentials V_i^- and the three polar potentials V_i^+ appearing in the radial wave equations describing the linearized perturbations of the static charged Gibbons-Maeda black holes, viewed as solutions in the family of $a = 1$, $c = -2$ EMDA theories described by the Lagrangian (8.1). The six plots here show the potentials for $b = 0, 0.5, 1, 2, 3$ and 4 , as described in the caption to Fig. 1. Note that the potentials all vanish on the outer horizon, being proportional to $(r - r_+)$. At large r the potentials all go to zero, with the asymptotic behaviour $V \sim \frac{\ell(\ell+1)}{r^2}$.

In the case that $b = 1$ we can also include the potentials $V_H^\pm$ for the axial and polar perturbations of the additional Maxwell field $H_{\mu\nu}$ of the $\mathcal{N} = 2$ supergravity theory whose bosonic sector is described by the Lagrangian (6.25). This gives the plots shown in Fig. 2.

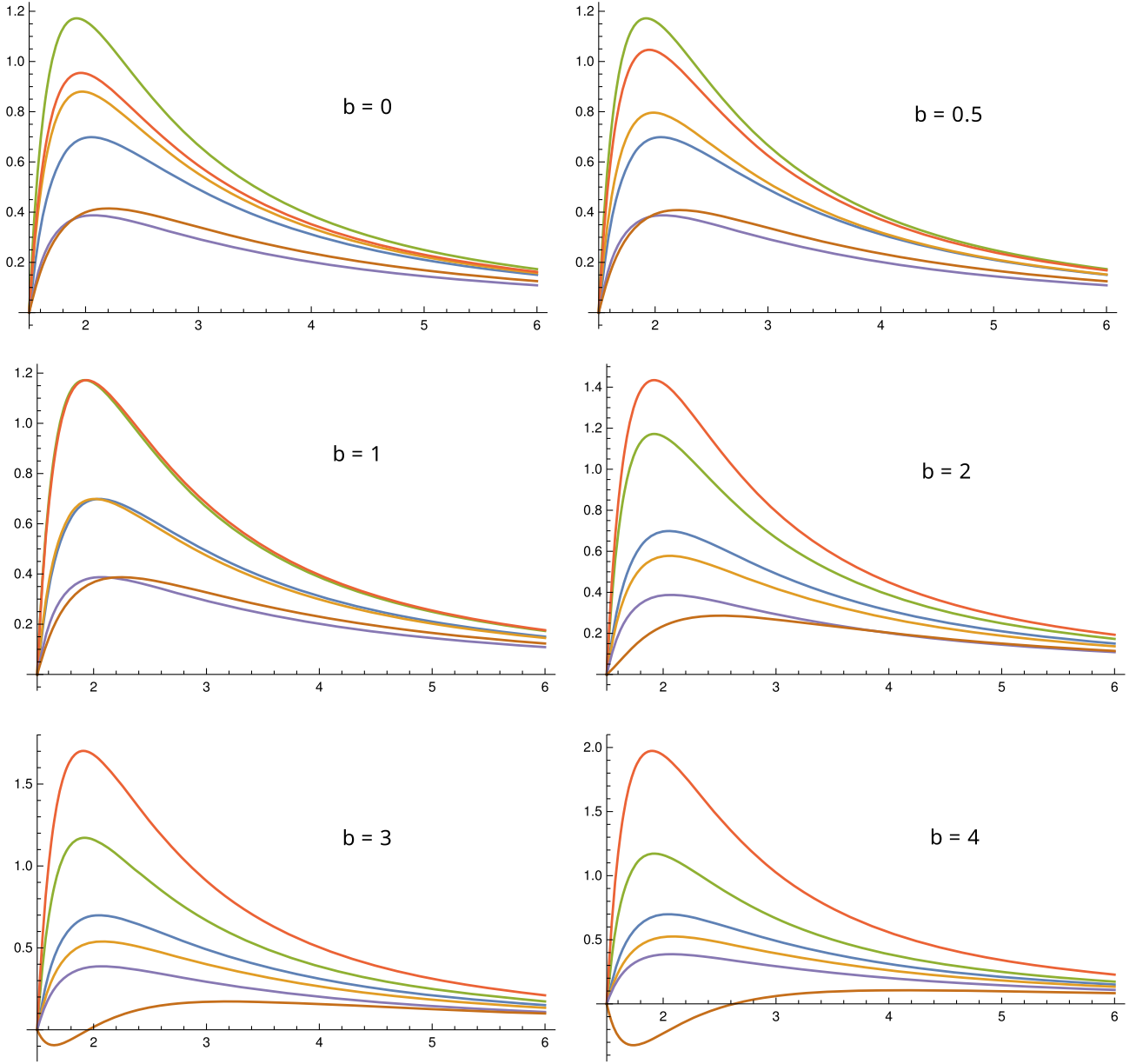


FIG. 1. Some example plots of the axial and polar potentials ($V_1^-, V_1^+, V_2^-, V_2^+, V_3^-, V_3^+$) corresponding to the roots (p_1, p_2, p_3) ordered as $p_1 < p_2 < p_3$, for $r_+ = 1.5r_-$, $\ell = 2$, for various values of b , as indicated. The dimensionless ratio r/r_- is plotted from the outer horizon at $r/r_- = 1.5$ to $r/r_- = 6$ along the horizontal axis. The dimensionless quantity $V(r)r_-^2$ is plotted along the vertical axis, for the various potentials ($V_1^-, V_1^+, V_2^-, V_2^+, V_3^-, V_3^+$). The top pair of almost superimposed potentials in the $b = 1$ plots correspond to the middle root p_2 , with the red line being the axial potential V_2^- . The middle pair of almost superimposed $b = 1$ plots correspond to the smallest root p_1 , with the orange line being the axial potential V_1^- . The bottom pair of almost superimposed $b = 1$ plots correspond to the largest root p_3 (that is, the positive root), with the brown line being the axial potential V_3^- . One can see how the potentials change as b is adjusted, and in particular how the axial potential V_3^- becomes negative in a region outside the outer horizon when b is large enough. The axial potentials V_i^- are given in Eq. (5.19), with p being any of the three roots p_i of the cubic equation (5.16). The polar potentials V_i^+ are given by the function g in Eq. (5.14) of Reference [5], with the constants c_1 and c_3 being given by Eqs. (5.16) and (5.17) of [5], and q being any of the three roots q_i of the cubic equation (5.18) of [5] (in all equations, the constant a is set equal to 1).

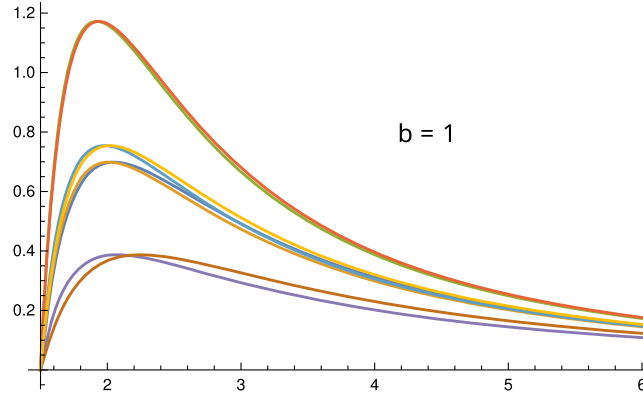


FIG. 2. This is the same as the middle-left panel above, showing the potentials (V_1^- , V_1^+ , V_2^- , V_2^+ , V_3^- , V_3^+) for $b = 1$, but now with the potentials (V_H^- , V_H^+) included as well. These are the pair of lines that are just above the previous middle pair (V_1^- , V_1^+). The light blue line is the axial potential V_H^- . The potentials V_H^- and V_H^+ are given in Eqs. (6.34) and (6.35) of this paper.

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