

Yukawa potential of a nonhomogeneous sphere with new limits on an ultralight boson

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Extremely weak long-range forces may lead to apparent violations of the equivalence principle. The final *MICROSCOPE* result, leading at 95% CL to $|\delta| < 4.5 \times 10^{-15}$ or 6.5×10^{-15} for a positive or negative Eötvös parameter δ , requires taking into account the spin of the mediator, and the sign of $\Delta(Q/A_r)_{\text{Ti-Pt}}$ (Q denoting the new charge involved). A coupling to $B-L$ or B should verify $|g_{B-L}| < 1.1 \times 10^{-25}$ or $|g_B| < 8 \times 10^{-25}$, for a spin-1 mediator of mass $m < 10^{-14}$ eV/ c^2 , with slightly different limits of 1.3×10^{-25} or 6.6×10^{-25} in the spin-0 case. The limits increase with m , in a way which depends on the density distribution within the Earth. This involves an hyperbolic form factor, expressed through a bilateral Laplace transform as $\Phi(x = mR) = \langle \sinh mr/mr \rangle$, related by analytic continuation to the Earth form factor $\Phi(ix) = \langle \sin mr/mr \rangle$. It may be expressed as $\Phi(x) = \frac{3}{x^2} (\cosh x - \frac{\sinh x}{x}) \times \bar{\rho}(x)/\rho_0$, where $\bar{\rho}(x)$ is an effective density, decreasing from the average ρ_0 at $m = 0$ down to the density at the periphery. We give general integral or multishell expressions of $\Phi(x)$, evaluating it, and $\bar{\rho}(x)$, in a simplified 5-shell model. $\Phi(x)$ may be expanded as $\sum \frac{x^{2n}}{(2n+1)!} \frac{\langle r^{2n} \rangle}{R^{2n}} \simeq 1 + .0827x^2 + .00271x^4 + 4.78 \times 10^{-5}x^6 + 5.26 \times 10^{-7}x^8 + \dots$, absolutely convergent for all x and potentially useful up to $x \approx 5$. The coupling limits increase at large x like $mR e^{mz/2}/\sqrt{1+mr}$ ($z = r - R$ being the satellite altitude), getting multiplied by $\simeq 1.9, 34$, or 1.2×10^9 , for $m = 10^{-13}, 10^{-12}$ or 10^{-11} eV/ c^2 , respectively.

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I. INTRODUCTION

The identity between the inertial and gravitational masses of a particle, well tested in the past by the Eötvös experiment [1], is one of the most fundamental principles of physics, and a building block of general relativity. The expected limitations of the latter, especially when trying to include gravitational interactions into a quantum framework, as well as the search for new feeble interactions weaker than gravity, have motivated further tests of this equivalence principle [2–6]. These experiments provide strong constraints on very weak new long range forces, usually taken as acting effectively, with strength ϵe , on a charge Q such as $B, B-L$, or L [7]. They could lead to apparent violations of the equivalence principle through the elementary interaction potential

$$V_{ij}(r_{ij}) = \pm \epsilon^2 \alpha Q_i Q_j \frac{e^{-r_{ij}/\lambda}}{r_{ij}} \\ = \pm \epsilon^2 \frac{\alpha}{G_N u^2} \left(\frac{Q}{A_r} \right)_i \left(\frac{Q}{A_r} \right)_j G_N m_i m_j \frac{e^{-r_{ij}/\lambda}}{r_{ij}}. \quad (1)$$

Q and A_r denote the new charge and relative atomic mass (in terms of the atomic mass unit u) of the two interacting bodies considered, the $+$ and $-$ signs are associated with spin-1 and spin-0 mediators, respectively, and the prefactor $\pm \epsilon^2 \alpha / G_N u^2 \simeq \pm 1.25 \times 10^{36} \epsilon^2$, often denoted as α_5 , has to be much smaller than 1 in magnitude if the new force, when supposed to be long-ranged, is to be much smaller than gravity.

The first result of the *MICROSCOPE* experiment [5] led to the constraints [8]

$$|\epsilon_B| < 5 \times 10^{-24}, \quad |\epsilon_{B-L}| < 8.4 \times 10^{-25}, \\ |\epsilon_L| < 8.6 \times 10^{-25}, \quad (2)$$

at the 2σ level, for a long range force coupled to $B, B-L$, or L , with strength ϵe . Its final result, with a gain in

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sensitivity by a factor 4.6, leads to an improvement of these limits by about 2, roughly down to

$$|\epsilon_B| < 2.5 \times 10^{-24}, \quad |\epsilon_{B-L}| \quad \text{or} \quad |\epsilon_L| < 4 \times 10^{-25}, \quad (3)$$

or, for couplings $g = \epsilon e$ to B , $B - L$, or L ,

$$|g_B| < 8 \times 10^{-25}, \quad |g_{B-L}| \quad \text{or} \quad |g_L| < 1.2 \times 10^{-25}. \quad (4)$$

To better estimate these limits, we need to take into account the dissymmetry of the final *MICROSCOPE* result on the Eötvös parameter, $\delta_{\text{Ti-Pt}} = (-1.5 \pm 2.3_{\text{stat}} \pm 1.5_{\text{syst}}) \times 10^{-15}$ [6]. This implies slightly more restrictive limits on the magnitude of δ , and thus on the $|\epsilon|$'s, if δ is supposed to be positive, rather than negative. The expected sign of $\delta_{\text{Ti-Pt}}$ depends on the spin of the new force mediator, and on the sign of the difference $\Delta(Q/A_r)_{\text{Ti-Pt}}$.

Extending the standard model gauge group to an extra $U(1)$, generated by a combination of the weak hypercharge Y with baryon and lepton numbers B and L_i , leads to a new spin-1 boson U , possibly very light and very weakly coupled [7]. It appears as a generalized dark photon, coupled to a combination of electromagnetic with B and L_i currents. Axial couplings, that may lead to new forces acting on particle spins [9] and possibly CP -violating monopole-dipole interactions [10], may also be present but play no role here. The vector coupling of the new U boson to Q_{el} , B and L_i may be parametrized as

$$(\epsilon_{Q_{\text{el}}} Q_{\text{el}} + \epsilon_B B + \epsilon_{L_i} L_i), \quad \text{times } e = \sqrt{4\pi\alpha} \simeq .3028. \quad (5)$$

Acting on neutral matter proportionally to $\epsilon_B B + \epsilon_{L_i} L_i$, or $\epsilon_{B-L}(B - L)$ in a grand-unified theory, the new force (of range $\lambda \simeq 197.327 \text{ km} \times [m/(10^{-12} \text{ eV}/c^2)]^{-1}$ where m is the new boson mass) may lead to very small apparent violations of the equivalence principle. Such an ultralight boson is now frequently considered, also, as a possible dark matter candidate, and many experiments are trying to detect its effects [11–23]. Ultralight spin-0 bosons may also be considered [24–26], although there is no reason to expect a coupling to a combination of Q_{el} with B and L_i in this case.

We shall discuss couplings to B , L , or $B - L$ (or $L_e - L_\mu$ or $L_e - L_\tau$, or to a combination of them with the electric charge Q_{el}), for both spin-1 and spin-0 mediators. As

$$\begin{aligned} \Delta(B/A_r)_{\text{Ti-Pt}} &\simeq 0.00079, \\ \Delta(L/A_r)_{\text{Ti-Pt}} &\simeq 0.05704, \\ \Delta((B - L)/A_r)_{\text{Ti-Pt}} &\simeq -0.05625, \end{aligned} \quad (6)$$

$\delta_{\text{Ti-Pt}}$ is expected negative for a spin-1 induced force coupled to B or L , and positive if coupled to $B - L$; and conversely for a spin-0 induced force. A positive δ being slightly more constrained than a negative one in view of the

dissymmetry of the final result, *a force coupled to $B - L$ gets slightly more strongly constrained if induced by a spin-1 particle, rather than by a spin-0 one*. The situation is opposite for couplings to B , or L .

The Eötvös parameter δ , expressed in terms of the ϵ parameters in (1), (5), is given in the ultralight or massless case by [8]

$$\begin{aligned} \delta_{\text{Ti-Pt}} &= \mp \frac{\alpha}{\underbrace{G_N u^2}_{1.2536 \times 10^{36}}} \epsilon^2 \left(\frac{Q}{A_r} \right)_{\oplus} \Delta \left(\frac{Q}{A_r} \right)_{\text{Ti-Pt}} \\ &\simeq \begin{cases} \mp 1.00 \times 10^{33} & \epsilon_B^2, \\ \pm 3.623 \times 10^{34} & \epsilon_{B-L}^2, \\ \mp 3.482 \times 10^{34} & \epsilon_L^2. \end{cases} \end{aligned} \quad (7)$$

ϵ_L is relative to the electronic number L_e , in case of a nonuniversal coupling to lepton numbers, as for $L_e - L_\mu$ or $L_e - L_\tau$. The upper signs correspond to a spin-1 U boson (or generalized dark photon) mediator, and the lower ones to a spin-0 mediator. The first *MICROSCOPE* result [5], $\delta_{\text{Ti-Pt}} = (-.1 \pm .9_{\text{stat}} \pm .9_{\text{syst}}) \times 10^{-14}$, implies at the 2σ level

$$|\delta_{\text{Ti-Pt}}|_{\text{first}} < 2.55 \times 10^{-14} (2\sigma), \quad (8)$$

providing from (7) the limits (2) on the ϵ 's. With $|\Delta(B/A_r)_{\text{Ti-Pt}}|$ about 72 times smaller than $|\Delta(L/A_r)|$ and $|\Delta((B - L)/A_r)|$, and roughly similar numbers of protons, electrons and neutrons within the Earth, the limits are similar for couplings to L or $B - L$, and about 6 times stronger than for a coupling to B , as shown by (2, 3, 7).

II. A DISSYMMETRIC EXPERIMENTAL RESULT

The final *MICROSCOPE* result has improved the precision down to $\delta_{\text{Ti-Pt final}} = (-1.5 \pm 2.3_{\text{stat}} \pm 1.5_{\text{syst}}) \times 10^{-15}$ [6]. The corresponding uncertainty, still conservatively taken at the 2σ level, decreases from the initial 25.5×10^{-15} in (8) down to

$$2\sigma \simeq 5.5 \times 10^{-15}, \quad (9)$$

with a gain in precision by about 4.6. The global constraint on δ gets reinforced from (8) down to $-7 \times 10^{-15} < \delta_{\text{Ti-Pt}} < 4 \times 10^{-15}$ at 2σ . Or, assuming a Gaussian distribution of the result, at a global 95% confidence level corresponding to about $1.96\sigma \simeq 5.4 \times 10^{-15}$,

$$-6.9 \times 10^{-15} < \delta_{\text{Ti-Pt}} < 3.9 \times 10^{-15} \quad (\text{global 95\% CL}). \quad (10)$$

In the absence of the central value $\delta_o = -1.5 \times 10^{-15}$ one would have $|\delta| < 5.4 \times 10^{-15}$ at the 95% CL. The negative δ_o , however, leads for $\delta > 0$ to a *decrease* of the 95% CL limit on $|\delta|$ below 5.4×10^{-15} , and for $\delta < 0$ to a corresponding *increase* above 5.4×10^{-15} .

Furthermore the asymmetry of the interval (10) allowed at the global 95% level, when the sign of δ is kept free, leads for $\delta > 0$ to a 95% CL limit *somewhat larger than the too strict* 3.9×10^{-15} in (10); and conversely, for $\delta < 0$, to an upper limit on $|\delta|$ *somewhat smaller than* 6.9×10^{-15} . The upper limits on $|\delta|$ should then satisfy, at the 95% CL,

$$\begin{cases} \text{if } \delta > 0: & |\epsilon_B| < [2 \text{ to } 2.3] \times 10^{-24}, & |\epsilon_{B-L} \text{ or } \epsilon_L| < [3.3 \text{ to } 3.9] \times 10^{-25}, \\ \text{if } \delta < 0: & |\epsilon_B| < [2.3 \text{ to } 2.6] \times 10^{-24}, & |\epsilon_{B-L} \text{ or } \epsilon_L| < [3.9 \text{ to } 4.4] \times 10^{-25}, \end{cases} \quad (12)$$

at a 95% CL. In particular for a spin-1 induced force coupled to $B - L$ or B , one has

$$\begin{aligned} |g_B| &= e|\epsilon_B| < [7 \text{ to } 8] \times 10^{-25}, \\ |g_{B-L}| &= e|\epsilon_{B-L}| < [1 \text{ to } 1.2] \times 10^{-25}. \end{aligned} \quad (13)$$

These naive estimates will be confirmed by a more detailed analysis, leading to Eqs. (24), (25).

III. PRECISE DETERMINATION OF THE LIMITS

A. MICROSCOPE limits on the Eötvös parameter δ

The allowed range of values for δ is given, at the global 95% confidence level, by the dissymmetric interval (10). However, the new forces that we are looking for lead, in each case considered, to a δ of a specific sign. The best way of taking this into account when estimating confidence levels may be debatable, and we need additional assumptions, e.g., by considering the possible distribution associated with the experimental result to be Gaussian, as represented in Fig. 1. This hypothesis has, fortunately, little effects on the results, as seen from Eqs. (11)–(13).

Let us consider a Gaussian probability distribution expressed as

$$p(\delta/\sigma) = \frac{e^{-(\delta-\delta_o)^2/2\sigma^2}}{\sqrt{2\pi}}. \quad (14)$$

We take $\text{erf}(x/\sqrt{2}) = \int_{-x}^x \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$ as the probability for $t = (\delta - \delta_o)/\sigma$ to lie between $-x\sigma$ and $x\sigma$, 95% or 90% confidence levels corresponding to $x\sigma \simeq 1.960\sigma$ or 1.645σ . With $\delta_o = -1.5 \times 10^{-15}$, $\sigma \simeq 2.7459 \times 10^{-15}$, $|\delta_o|/(\sigma\sqrt{2}) \simeq .38627$, and $\text{erf}(|\delta_o|/(\sigma\sqrt{2})) \simeq .41512$, about 41.51% of the probability distribution corresponds to δ between $-2|\delta_o|$ and 0. This implies a 29.24% probability for $\delta > 0$, i.e.,

$$\begin{cases} \text{for } \delta > 0, & 3.9 \times 10^{-15} < \lim \delta < 5.4 \times 10^{-15}, \\ \text{for } \delta < 0, & 5.4 \times 10^{-15} < \lim |\delta| < 6.9 \times 10^{-15}. \end{cases} \quad (11)$$

We thus obtain for $\delta > 0$ a gain in precision, compared with the first result, between $25/5.4 \simeq 4.6$ and $25/3.9 \simeq 6.4$, leading to an improvement of the limits by 2.15 to 2.5; and for $\delta < 0$ to a gain in precision between $25/6.9 \simeq 3.6$ and $25/5.4 \simeq 4.6$, for an improvement of the limits by 1.9 to 2.15. We then get from (7), (11), for a sufficiently light mediator of mass $< 10^{-14}$ eV/ c^2 ,

$$\frac{1 - \text{erf}(|\delta_o|/(\sigma\sqrt{2}))}{2} = \int_{|\delta_o|/\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt \simeq 29.24\%, \quad (15)$$

and the remaining 70.76% for $\delta < 0$.

Integrating $p(\delta/\sigma)$ in Eq. (14) we find the upper limits on $|\delta|$ by arranging, for $\delta > 0$, for having 5% of the 29.24% area under the blue curve in Fig. 1 in the positive δ region (i.e., 1.462% of the total area) above the upper limit of $\simeq 4.5 \times 10^{-15}$. And similarly, for $\delta < 0$, for having 5% of the 70.76% area in the negative δ region (i.e., 3.538% of the total) under the limit of -6.5×10^{-15} . Altogether we get

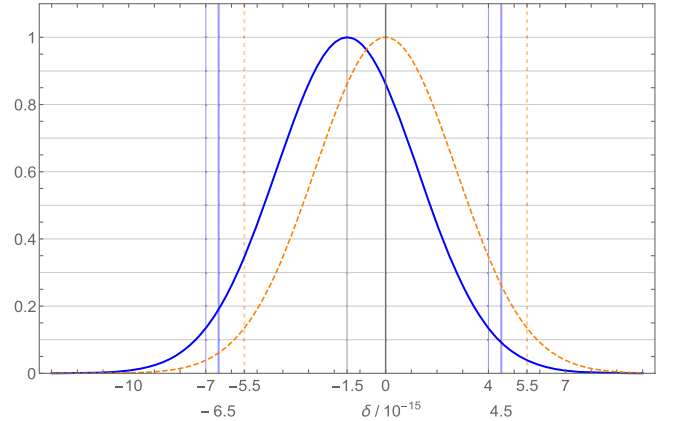


FIG. 1. The MICROSCOPE result $\delta_{Ti-Pt} = (-1.5 \pm 2.3_{\text{stat}} \pm 1.5_{\text{syst}}) \times 10^{-15}$, represented proportionally to $e^{-(\delta-\delta_o)^2/2\sigma^2}$ (in blue), with $\delta_o \simeq -1.5 \times 10^{-15}$ and $\sigma \simeq 2.75 \times 10^{-15}$. It leads for an unconstrained δ to $-7 \times 10^{-15} < \delta < 4 \times 10^{-15}$ at the 2σ CL (versus $-5.5 \times 10^{-15} < \delta < 5.5 \times 10^{-15}$ for $\delta_o = 0$, cf. dashed curve in orange). The negative δ_o leads to limits on $|\delta|$ increased to about 4.5×10^{-15} for $\delta > 0$, and decreased to 6.5×10^{-15} for $\delta < 0$, at the 95% CL. 29.24% of the area under the blue curve corresponds to $\delta > 0$ (including 1.462% for $\delta > 4.5 \times 10^{-15}$), and 70.76% for $\delta < 0$ (including 3.538% for $\delta < -6.5 \times 10^{-15}$).

TABLE I. Upper limits on the Eötvös parameter δ , when positive. Expressed as $\delta_0 + x\sigma$ with $\delta_0 = -1.5 \times 10^{-15}$ and $\sigma \simeq 2.75 \times 10^{-15}$, they are given at the 95%, 91.45% and 90% confidence levels. The resulting 90% CL limits on the $|\epsilon|$'s are smaller than at the 95% level by a factor $\simeq \sqrt{4.5/3.7} \simeq 1.10$.

Confidence level:	95%	91.45%	90%
x	$\sqrt{2} \operatorname{erf}^{-1}(0.97076)$ $\simeq 2.18$	$\sqrt{2} \operatorname{erf}^{-1}(0.95)$ $\simeq 1.96$	$\sqrt{2} \operatorname{erf}^{-1}(0.94152)$ $\simeq 1.89$
$x\sigma$	6×10^{-15}	5.4×10^{-15}	5.2×10^{-15}
$\lim \delta = x\sigma + \delta_0$	4.5×10^{-15}	3.9×10^{-15}	3.7×10^{-15}

$$\begin{cases} \text{for } \delta > 0, & \delta < 4.5 \times 10^{-15} \\ \text{for } \delta < 0, & |\delta| < 6.5 \times 10^{-15} \end{cases} \text{ at the 95\% CL,} \quad (16)$$

as illustrated in Fig. 1, in place of the $[-6.9 \times 10^{-15}, 3.9 \times 10^{-15}]$ interval in (10) and in agreement with (11). These limits come in place of the earlier $|\delta| < 25 \times 10^{-15}$ at the 95% CL from the first *MICROSCOPE* result.

B. Influence of the confidence level on the limits

If δ is positive it has about 2.5% / 29.24% $\simeq 8.55\%$ chances to be above the 3.9×10^{-15} in (10), which provides an upper limit at the $\simeq 91.45\%$ CL. To get an upper limit, expressed as $\delta_0 + x\sigma$, on a positive δ at a given confidence level cl , we express that the probability for a positive δ to be larger than $\delta_0 + x\sigma$ is $1 - cl$ (e.g., 5%, for a 95% CL limit). This one is taken as the probability for an unconstrained δ to be larger than $\delta_0 + x\sigma$, divided by the probability for δ to be found positive, here 29.24%. This reads

$$\mathcal{P}(\delta_{\text{positive}} > \delta_0 + x\sigma) = \frac{\int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt}{\int_{-\delta_0/\sigma}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt} = 1 - cl \quad (17)$$

(this ratio $\mathcal{P}$ is of course 1, for $\delta_0 + x\sigma = 0$), so that

$$\begin{aligned} \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt &= \frac{1 - \operatorname{erf}(x/\sqrt{2})}{2} \\ &= \underbrace{\int_{-\delta_0/\sigma}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt}_{\simeq 29.24\%} \times (1 - cl), \end{aligned} \quad (18)$$

which leads to [27]

$$x \simeq \sqrt{2} \operatorname{erf}^{-1}[1 - 2 \times 29.24\%(1 - cl)]. \quad (19)$$

The resulting limits at the 95%, 91.45%, and 90% confidence levels are shown in Table I.

If δ is negative, it has about 2.5% / 70.76% $\simeq 3.53\%$ chances to be below the -6.9×10^{-15} in (10), $|\delta| < 6.9 \times 10^{-15}$ then providing a limit at the $\simeq 96.47\%$ level. For a limit at a confidence level cl we solve for

$$\int_{-\infty}^{-x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = \frac{1 - \operatorname{erf}(x/\sqrt{2})}{2} \simeq 70.76\% \times (1 - cl), \quad (20)$$

which leads to $x \simeq \sqrt{2} \operatorname{erf}^{-1}[1 - 2 \times 70.76\%(1 - cl)]$. The resulting limits at the 96.47%, 95%, and 90% confidence levels are shown in Table II. We get in particular

$$\begin{cases} \text{for } \delta > 0, & \delta < 3.7 \times 10^{-15} \\ \text{for } \delta < 0, & |\delta| < 5.6 \times 10^{-15} \end{cases} \text{ at the 90\% CL.} \quad (21)$$

C. New limits on the couplings, depending on the mediator spin

The limits on the ϵ 's, found from (7), (16), are

$$\begin{cases} \text{if } \delta > 0: & |\epsilon_B| < 2.1 \times 10^{-24}, \quad |\epsilon_{B-L}| \text{ or } |\epsilon_L| < 3.6 \times 10^{-25}, \\ \text{if } \delta < 0: & |\epsilon_B| < 2.6 \times 10^{-24}, \quad |\epsilon_{B-L}| \text{ or } |\epsilon_L| < 4.3 \times 10^{-25}, \end{cases} \quad (22)$$

TABLE II. Upper limits on $|\delta|$, when δ negative, expressed as $|\delta_0| + x\sigma$, and given at the 96.47%, 95%, and 90% confidence levels. The 90% CL limits on the $|\epsilon|$'s are smaller than at the 95% level by a factor $\simeq \sqrt{6.5/5.54} \simeq 1.08$.

Confidence level:	96.47%	95%	90%
x	$\sqrt{2} \operatorname{erf}^{-1}(0.95)$ $\simeq 1.96$	$\sqrt{2} \operatorname{erf}^{-1}(0.92924)$ $\simeq 1.81$	$\sqrt{2} \operatorname{erf}^{-1}(0.85848)$ $\simeq 1.47$
$x\sigma$	5.4×10^{-15}	5×10^{-15}	4.04×10^{-15}
$\lim \delta = x\sigma + \delta_0 $	6.9×10^{-15}	6.5×10^{-15}	5.54×10^{-15}

with an improvement by $\simeq(25/4.5)^{1/2} \simeq 2.36$ for $\delta > 0$, and $\simeq(25/6.5)^{1/2} \simeq 1.96$ for $\delta < 0$. This leads to the final result

$$\begin{cases} \text{spin-1 mediator: } |\epsilon_B| < 2.6 \times 10^{-24}, & |\epsilon_{B-L}| < 3.6 \times 10^{-25}, & |\epsilon_L| < 4.3 \times 10^{-25}, \\ \text{spin-0 mediator: } |\epsilon_B| < 2.1 \times 10^{-24}, & |\epsilon_{B-L}| < 4.3 \times 10^{-25}, & |\epsilon_L| < 3.6 \times 10^{-25}, \end{cases} \quad (23)$$

at the 95% confidence level [28].

For a given sign of δ the limits on the ϵ 's, and couplings $g = \epsilon e$, are larger for B and L than for $B - L$, by factors $\simeq \sqrt{36.23} \simeq 6$ and $\sqrt{36.23/34.82} \simeq 1.02$, respectively, as seen from (7). For a given type of coupling they are larger for $\delta < 0$ than for $\delta > 0$ by a factor $\simeq \sqrt{6.5/4.5} \simeq 1.2$. The 90 % CL limits are obtained for $\delta > 0$ (resp. < 0) from $\delta < 3.7 \times 10^{-15}$ instead of 4.5×10^{-15} (resp. $|\delta| < 5.6 \times 10^{-15}$ instead of 6.5×10^{-15}),

cf. Tables I and II. They are only very slightly smaller, by factors $\simeq 1.1$ or 1.08 (close to the $(1.960/1.645)^{1/2} \simeq 1.09$ that would be obtained for a symmetric distribution), illustrating the modest effect of the chosen confidence level on the coupling limits.

We finally remember, for a spin-1 U boson somewhat lighter than $\approx 10^{-14}$ eV/ c^2 effectively coupled to $B, B - L$, or L ,

$$\text{spin-1 mediator: } \begin{cases} |\epsilon_B| < 2.6 \times 10^{-24}, & \text{or} & |g_B| < 7.7 \times 10^{-25}, \\ |\epsilon_{B-L}| < 3.6 \times 10^{-25}, & \text{or} & |g_{B-L}| < 1.1 \times 10^{-25}, \\ |\epsilon_L| < 4.3 \times 10^{-25}, & \text{or} & |g_L| < 1.3 \times 10^{-25}, \end{cases} \quad (95\% \text{ CL}) \quad (24)$$

as expected from (12), (13) [29]. This is almost always more constraining than the limit from a torsion-balance search for ultralow-mass dark matter [12] constraining, at its maximum sensitivity near $m = 8 \times 10^{-18}$ eV/ c^2 , $|g_{B-L}|$ to be less than 10^{-25} (at 95 % CL), assuming that dark matter predominantly consists of ultralow-mass vector bosons coupled to $B - L$. For a spin-0 mediator we get

$$\text{spin-0 mediator: } \begin{cases} |\epsilon_B| < 2.1 \times 10^{-24}, & \text{or} & |g_B| < 6.4 \times 10^{-25}, \\ |\epsilon_{B-L}| < 4.3 \times 10^{-25}, & \text{or} & |g_{B-L}| < 1.3 \times 10^{-25}, \\ |\epsilon_L| < 3.6 \times 10^{-25}, & \text{or} & |g_L| < 1.1 \times 10^{-25}. \end{cases} \quad (95\% \text{ CL}) \quad (25)$$

The 90% CL limits are about 8% or 9% lower as seen from Tables I and II, down to 1×10^{-25} for a $B - L$ coupling in the spin-1 case.

IV. GENERAL EXPRESSION OF THE FORM

$$\text{FACTOR } \Phi(x) \text{ AS } \left\langle \frac{\sinh mr}{mr} \right\rangle$$

These limits are valid for a mediator mass m smaller than about 10^{-14} eV/ c^2 , corresponding to a range $\lambda = \hbar/mc \gtrsim 20\,000$ km, larger than the diameter of the Earth. They still remain approximately valid somewhat above this value, before increasing significantly above $\approx 10^{-13}$ eV/ c^2 , corresponding to $\lambda \approx 2\,000$ km, for which only a smaller part of the Earth can act efficiently as a source of the new force. This may be taken into account using a spherical or ellipsoidal-layered Earth model, as done in [3] for $\lambda > 1\,000$ km. Furthermore with a satellite orbiting at a mean altitude $z \simeq 710$ km, the strength of a

new force of range $\lambda = \hbar/mc$, and the resulting Eötvös parameter δ , also include, for the smaller values of λ , an exponentially small factor proportional to e^{-mz} , leading to an increase in the coupling limits by a large factor proportional to $e^{mz/2}$. δ also includes, for the lower values of λ down to ≈ 20 km, a factor $\bar{\rho}/\rho_0$ decreasing down to $\simeq 1/2$, taking into account that the effective density $\bar{\rho}$ of the regions at the origin of the new force decreases, near the surface of the Earth, down to about $1/2$ of its average density $\rho_0 \simeq 5.51$ g/cm³.

The Earth will still be considered as spherically symmetric with a radius $R \simeq 6371$ km. The analysis involves density ratios $\rho(r)/\rho_0$, where $\rho_0 = \int_0^R \rho(r) \frac{3r^2 dr}{R^3}$ is the average density. We shall consider for simplicity that these ratios are approximately the same for the new charge (denoted by Q), and for the mass, viewing the new charge distribution as approximately proportional to the mass distribution.

A. Solution of the Poisson-like equation for the potential

The Yukawa potential of a pointlike new charge Q at the origin, $\mathcal{V}(r) = Qe^{-mr}/4\pi r$, is replaced, outside the sphere of radius R , by

$$\mathcal{V}_{\text{out}}(r) = Q \frac{e^{-mr}}{4\pi r} \Phi(x), \quad (26)$$

with $x = mR = R/\lambda$. $\Phi(x)$ takes into account the extension of the sphere and its nonhomogeneity, as compared to a pointlike source. The potential within the sphere, $\mathcal{V}_{\text{in}}(r)$, may be obtained by solving the Poisson-like equation with the new charge density $\rho(r)$,

$$(-\Delta + m^2)\mathcal{V}_{\text{in}}(r) = \rho(r). \quad (27)$$

A thin shell of radius r' , density $\rho(r')$, and thickness dr' generates, at $r \geq r'$, an outside potential proportional to e^{-mr}/r ; and at $r \leq r'$, an inside potential proportional to $\sinh mr/mr$. Both are equal at $r = r'$, for which the induced potential is proportional to the product $(\sinh mr'/mr')(e^{-mr'}/r')$. It is then equal to $\rho(r')r'^2 dr'$ $(\sinh mr'/mr')(e^{-mr'}/r')$, up to a possible proportionality

factor found to be 1 as for $m = 0$, corresponding to a Coulomb potential $[\rho(r')4\pi r'^2 dr']/4\pi r'$.

This shell of radius r' thus generates the elementary potential

$$d\mathcal{V}(r) = \begin{cases} \rho(r') \frac{\sinh mr'}{mr'} \frac{e^{-mr}}{r} r'^2 dr', & \text{for } r \geq r', \\ \rho(r') \frac{\sinh mr}{mr} \frac{e^{-mr'}}{r'} r'^2 dr', & \text{for } r \leq r'. \end{cases} \quad (28)$$

For a sphere of radius R the inside potential $\mathcal{V}_{\text{in}}(r)$ is the sum of the contributions (28) from the inner and outer parts of the sphere, i.e.,

$$\mathcal{V}_{\text{in}}(r) = \frac{e^{-mr}}{r} \int_0^r \rho(r') \frac{\sinh mr'}{mr'} r'^2 dr' + \frac{\sinh mr}{r} \int_r^R \rho(r') \frac{e^{-mr'}}{mr'} r'^2 dr'. \quad (29)$$

Equation (27) is indeed satisfied, as

$$\begin{aligned} \frac{1}{r} \frac{d}{dr} r \mathcal{V}_{\text{in}}(r) &= -m \frac{e^{-mr}}{r} \int_0^r \rho(r') \frac{\sinh mr'}{mr'} r'^2 dr' + m \frac{\cosh mr}{r} \int_r^R \rho(r') \frac{e^{-mr'}}{mr'} r'^2 dr', \\ \frac{1}{r} \frac{d^2}{dr^2} r \mathcal{V}_{\text{in}}(r) &= m^2 \mathcal{V}_{\text{in}}(r) - \rho(r) e^{-mr} (\sinh mr + \cosh mr) = m^2 \mathcal{V}_{\text{in}}(r) - \rho(r), \end{aligned} \quad (30)$$

so that $(-\Delta + m^2)\mathcal{V}_{\text{in}}(r) = \rho(r)$.

B. Expression of $\Phi(x)$

In a similar way, the outside potential reads

$$\mathcal{V}_{\text{out}}(r) = \frac{e^{-mr}}{r} \int_0^R \rho(r') \frac{\sinh mr'}{mr'} r'^2 dr' = \frac{Q}{4\pi} \frac{e^{-mr}}{r} \Phi(x), \quad (31)$$

with $Q = (4\pi R^3/3)\rho_0$, so that

$$\Phi(x) = \int_0^R \frac{\rho(r)}{\rho_0} \frac{\sinh mr}{mr} \frac{3r^2 dr}{R^3} = \left\langle \frac{\sinh mr}{mr} \right\rangle \geq 1. \quad (32)$$

This can also be expressed, independently of R and ρ_0 , as

$$\Phi(x) = \frac{1}{m} \int_0^\infty 4\pi \rho(r) \sinh mr r dr, \quad (33)$$

with ρ normalized to unity through $\int_0^\infty 4\pi \rho(r) r^2 dr = 1$ (but still considering a sphere of finite radius). The outside potential at distance r is indeed the above Yukawa potential, reobtained as

$$\mathcal{V}_{\text{out}}(r) = \int \rho(r) d^3 \vec{r}' \frac{e^{-m|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|} = \left\langle \frac{\sinh mr}{mr} \right\rangle \times Q \frac{e^{-mr}}{4\pi r}, \quad (34)$$

as verified using $l^2 = (\vec{r} - \vec{r}')^2 = r^2 + r'^2 - 2rr' \cos \theta$ so that $l dl = rr' \sin \theta d\theta$, with l varying from $r - r' > 0$ to $r + r'$, and

$$\begin{aligned} \mathcal{V}_{\text{out}}(r) &= \int_0^R \rho(r') r'^2 dr' \int_0^\pi \frac{1}{2} \frac{e^{-ml}}{l} \sin \theta d\theta \\ &= \int_0^R \rho(r') r'^2 dr' \frac{e^{-m(r-r')} - e^{-m(r+r')}}{2mr r'}. \end{aligned} \quad (35)$$

It is interesting to note, in view of future developments, that $\rho(r)$ may be extended to an even function of r , so that

$$\Phi(x = mR) = g(m) = \frac{2\pi}{m} \int_{-\infty}^\infty r \rho(r) e^{mr} dr, \quad (36)$$

is equal to $2\pi/m$ times the bilateral Laplace transform of the odd function $r\rho(r)$, with $g(0) = \Phi(0) = 1$.

$\Phi(x)$ appears, for a finite-range Yukawa interaction, as an isotropic form factor taking into account the extension of

the spherical body considered, and its inhomogeneous character. $\Phi(x)$ is close to 1 for a long range interaction with $x \simeq 0$ i.e. $\lambda = 1/m \gg R$, for which the internal structure of the sphere plays essentially no role. For an homogeneous sphere $\Phi(x)$ in Eq. (32) reduces to [30–34]

$$\phi(x) = \frac{3}{x^3} \int_0^x \sinh u u du = \frac{3(x \cosh x - \sinh x)}{x^3}, \quad (37)$$

with the potential obtained from (29) as

$$\begin{cases} \mathcal{V}_{\text{in}}(r) = \frac{\rho_0}{m^2} \left[1 - (x+1)e^{-x} \frac{\sinh mr}{mr} \right], \\ \mathcal{V}_{\text{out}}(r) = \frac{\rho_0}{m^3} (x \cosh x - \sinh x) \frac{e^{-mr}}{r}. \end{cases} \quad (38)$$

This may also be found directly, the inside and outside potentials being expressed as $\rho_0/m^2 + A \sinh mr/mr$ and Be^{-mr}/r , respectively, which also leads to Eqs. (38) [35,36].

V. ANALYTIC EXPANSION AND CONTINUATION OF $\Phi(x)$

$\Phi(x = mR)$ may be viewed as an hyperbolic form factor, expressed as in (32) as the weighted average of $\sinh mr/mr = \sum (mr)^{2n}/(2n+1)!$. It may be expanded as a power series in x , absolutely convergent for all x , involving the even moments of the density $\rho(r)$,

$$\langle r^{2n} \rangle = \frac{1}{\rho_0 4\pi R^3/3} \int_0^R \rho(r) r^{2n} 4\pi r^2 dr = \int_0^R \frac{\rho(r)}{\rho_0} \frac{3r^{2n+2} dr}{R^3}, \quad (39)$$

so that

$$\begin{aligned} \Phi(x) &= \left\langle \frac{\sinh mr}{mr} \right\rangle = \sum_0^\infty \frac{x^{2n}}{(2n+1)!} \frac{\langle r^{2n} \rangle}{R^{2n}} = \sum_0^\infty \frac{1}{(2n+1)!} \frac{\langle r^{2n} \rangle}{\lambda^{2n}} \\ &= 1 + x^2 \frac{\langle r^2 \rangle}{6R^2} + x^4 \frac{\langle r^4 \rangle}{120R^4} + x^6 \frac{\langle r^6 \rangle}{5040R^6} + \dots \end{aligned} \quad (40)$$

This may also be reexpressed, with no reference to R , in terms of a density distribution (vanishing at large r) normalized to 1, as

$$\Phi(x) = \frac{1}{m} \int_0^\infty 4\pi \rho(r) \sinh mr r dr = \sum_0^\infty \frac{1}{(2n+1)!} \frac{\langle r^{2n} \rangle}{\lambda^{2n}}, \quad (41)$$

with

$$\langle r^{2n} \rangle = \int_0^\infty 4\pi \rho(r) r^{2n+2} dr. \quad (42)$$

For a pointlike distribution at the origin $\Phi(x) = 1$, in agreement with Eq. (40), all momenta $\langle r^{2n} \rangle$ being 0 for

$n \geq 1$. For an homogeneous distribution $\rho = \rho_0$ for $r < R$, for which

$$\langle r^{2n} \rangle = \frac{3}{2n+3} R^{2n}, \quad (43)$$

Eq. (40) reads

$$\begin{aligned} \phi(x) &= \sum_0^\infty \frac{x^{2n}}{(2n+1)!} \frac{3}{2n+3} \\ &= 1 + \frac{x^2}{10} + \frac{x^4}{280} + \frac{x^6}{15120} + \frac{x^8}{1330560} + \dots, \end{aligned} \quad (44)$$

which is, not surprisingly, the expansion of

$$\begin{aligned} \phi(x) &= \frac{3}{x^2} \left(\cosh x - \frac{\sinh x}{x} \right) \\ &= 3 \sum_0^\infty x^{2n} \left[\frac{1}{(2n+2)!} - \frac{1}{(2n+3)!} \right]. \end{aligned} \quad (45)$$

The expansion (40) of $\Phi(x) = \langle \frac{\sinh mr}{mr} \rangle$ allows for its analytic continuation as an holomorphic function of x . The change $m \rightarrow \mp ik$, i.e., $x^2 \rightarrow -x^2$ with now $x = kR$, leads to define its dual as

$$\begin{aligned} \tilde{\Phi}(x) &= \Phi(ix) = \left\langle \frac{\sin kr}{kr} \right\rangle = \sum_0^\infty (-1)^n \frac{x^{2n}}{(2n+1)!} \frac{\langle r^{2n} \rangle}{R^{2n}} \\ &= 1 - x^2 \frac{\langle r^2 \rangle}{6R^2} + x^4 \frac{\langle r^4 \rangle}{120R^4} - x^6 \frac{\langle r^6 \rangle}{5040R^6} + \dots \end{aligned} \quad (46)$$

The expression obtained,

$$\begin{aligned} f(k) &= \tilde{\Phi}(kR) = \Phi(ikR) = \left\langle \frac{\sin kr}{kr} \right\rangle \\ &= \int_0^R \frac{\rho(r)}{\rho_0} \frac{\sin kr}{kr} \frac{3r^2 dr}{R^3}, \end{aligned} \quad (47)$$

also written with a density distribution normalized to unity as

$$\begin{aligned} f(k) &= \langle e^{i\vec{k} \cdot \vec{r}} \rangle = \int_0^\infty \rho(r) 2\pi r^2 dr \int_{-1}^1 e^{ikr \cos \theta} d(\cos \theta) \\ &= \int_0^\infty 4\pi \rho(r) \frac{\sin kr}{kr} r^2 dr, \end{aligned} \quad (48)$$

is identified as the form factor of the spherically symmetric body considered [37]. For an homogeneous body it is simply $\tilde{\phi}(x) = \phi(ix) = \frac{3}{x^2} (\frac{\sin x}{x} - \cos x)$, with $x = kR$.

VI. EXPRESSING $\Phi(x)$ FROM AN EFFECTIVE DENSITY $\bar{\rho}(x)$

An homogeneous sphere has a moment of inertia $\frac{2}{5}MR^2$ corresponding to $\langle r^2 \rangle = \frac{3}{5}R^2$. For the Earth it has a smaller value, usually taken as $I \simeq .3308 MR^2$ [38,39], even if other values, from .3307 to about .332 MR^2 , have been considered [40,41]. This value, close to $\frac{1}{3}MR^2$, corresponds to

$$\langle r^2 \rangle = \frac{3}{2} \frac{I}{MR^2} \simeq .4962 R^2. \quad (49)$$

The Earth density $\rho(r)$ is larger within the core, which results in somewhat smaller values of $\langle r^{2n} \rangle$, and globally of $\Phi(x)$, as compared to $\phi(x)$.

We can compare, more generally, the form factor $\Phi(x)$ with the corresponding factor $\phi(x)$ for an homogeneous sphere. This leads us to introduce, for each value of the range λ , an effective average density $\bar{\rho}(x = R/\lambda)$, such that the inhomogeneous sphere of radius R generates the same outside potential $\mathcal{V}_{\text{out}}(r)$ in (26) as an homogeneous sphere with the constant density $\bar{\rho}(x)$. This one is thus defined by

$$\frac{\bar{\rho}(x)}{\rho_0} = \frac{\Phi(x)}{\phi(x)}, \quad (50)$$

or, more explicitly, as

$$\bar{\rho}(x) = \rho_0 \frac{\Phi(x)}{\phi(x)} = \frac{\int_0^R \rho(r) \sinh mr \, dr}{\int_0^R \sinh mr \, dr}. \quad (51)$$

This may also be expressed in terms of the bilateral Laplace transform of $r\rho(r)$ (now considered as an odd function of r), as

$$\bar{\rho}(x) = \frac{\int_{-R}^R r\rho(r)e^{mr} \, dr}{\int_{-R}^R re^{mr} \, dr} = \frac{\int_{-R}^R r\rho(r)e^{-m(R-r)} \, dr}{\int_{-R}^R re^{-m(R-r)} \, dr}. \quad (52)$$

$\bar{\rho}(x)$ may be viewed as an average of the density $\rho(r)$, weighted proportionally to $(\sinh mr/mr)$ to take into account the finite range of the interaction considered. This favors the larger values of r , ultimately approaching R , when m increases so that the range $\lambda = 1/m$ gets smaller. The ratio $\bar{\rho}(x)/\rho_0$ can be shown to be ≤ 1 , and decreasing with increasing x i.e. decreasing λ , for a density $\rho(r)$ decreasing with r (resp. ≥ 1 and increasing with r , for a density increasing with r). This effective $\bar{\rho}(x)$ decreases regularly with decreasing λ 's, from the average density ρ_0 at large $\lambda \gg R$, down to the density at the periphery of the sphere, or some average of it, for the smaller values of λ .

For x large enough (typically $\gtrsim 5$), $\bar{\rho}(x)$ may in general be approximated by

$$\bar{\rho}(x) \simeq \frac{\int_0^R \rho(r) \frac{e^{-d/\lambda}}{r} r^2 \, dr}{\int_0^R \frac{e^{-d/\lambda}}{r} r^2 \, dr}, \quad (53)$$

$\rho(r)$ being simply weighted proportionally to $e^{-d/\lambda}/r$, exponentially decreasing with the depth $d = R - r$ below surface. $\bar{\rho}(x)$ involves mainly the external shells of the sphere, within a depth $\lesssim 2\lambda$ below surface, ultimately decreasing, for small λ 's, down to the relevant external density ρ_e ,

$$\frac{\Phi(x)}{\phi(x)} = \frac{\bar{\rho}(x)}{\rho_0} \rightarrow \frac{\rho_e}{\rho_0}. \quad (54)$$

This one may be taken here as the crust density, close to $\rho_0/2 \simeq 2.755 \text{ g/cm}^3$, for small λ 's of a few tens down to $\approx 20 \text{ km}$, below which the experiment loses its sensitivity given the mean satellite altitude of $\approx 710 \text{ km}$. (We can thus ignore the oceanic contribution to $\rho(r)$, that would lead to a lower ρ_e .) This provides a reduction factor $\rho_e/\rho_0 \approx \frac{1}{2}$ in the short-range effects of a new force as compared to the homogeneous case, responsible for *an increase by $\approx \sqrt{2}$ of the upper limits on its couplings*.

The new force is proportional to $\Phi(x)(1 + mr)e^{-mr}/r^2$, i.e., to $\Phi(x)(1 + mr)e^{-mr}$ as compared to the massless case. For small λ 's this reads

$$\begin{aligned} \Phi(x)(1 + mr)e^{-mr} &\simeq \frac{3(x-1)}{2x^3} (1 + mr)e^{-m(r-R)} \frac{\bar{\rho}(x)}{\rho_0} \\ &\simeq \frac{3\lambda}{2R} \frac{\rho_e}{\rho_0} \frac{r}{R} e^{-mz}. \end{aligned} \quad (55)$$

Near the surface, i.e., at low z compared to λ , this reduces approximately to $\frac{3\lambda}{2R} \frac{\rho_e}{\rho_0}$ [42].

VII. TWO WAYS OF VIEWING A MULTISHELL MODEL

A. Expressions of $\Phi(x)$

Let us approximate the sphere as a series of n homogeneous hollow spheres of radii R_i from R_1 to $R_n = R$, each one with a density ρ_i varying from $\rho_1 = \rho_c$ at the center to $\rho_n = \rho_e$ at the periphery. The contribution to $\Phi(x)$ of a full homogeneous sphere of radius R_i is $[\rho_i R_i^3 / (\rho_0 R^3)] \phi(x_i)$, with $x_i = R_i/\lambda = x R_i/R$. Summing on the contributions of the n hollow spheres one gets

$$\Phi(x) = \sum_{i=1}^n \frac{\rho_i}{\rho_0} \left[\frac{R_i^3}{R^3} \phi(x_i) - \frac{R_{i-1}^3}{R^3} \phi(x_{i-1}) \right], \quad (56)$$

with an inside radius $R_0 = 0$ for the first sphere of radius R_1 .

One may view the same situation as a superposition of n full homogeneous spheres of charges q_i (or masses m_i , when $\rho(r)$ is viewed as a mass density) and radius R_i from R_1 to $R_n = R$, with densities equal to the excess density $\Delta\rho_i = \rho_i - \rho_{i+1}$ (or $\Delta\rho_n = \rho_e$ for the last sphere). With $\Phi(x)$ providing the external potential as

$$\mathcal{V}_{\text{out}}(r) = \mathcal{Q} \Phi(x) \frac{e^{-mr}}{4\pi r} = \sum_i q_i \phi(x_i) \frac{e^{-mr}}{4\pi r}, \quad (57)$$

we get the equivalent expression

$$\Phi(x) = \sum_1^n \frac{q_i}{\mathcal{Q}} \phi(x_i) = \sum_1^{n-1} \frac{\rho_i - \rho_{i+1}}{\rho_0} \frac{R_i^3}{R^3} \phi(x_i) + \frac{\rho_e}{\rho_0} \phi(x), \quad (58)$$

with the normalization condition

$$\sum_1^n \frac{q_i}{\mathcal{Q}} = \sum_1^{n-1} \frac{\rho_i - \rho_{i+1}}{\rho_0} \frac{R_i^3}{R^3} + \frac{\rho_e}{\rho_0} = 1. \quad (59)$$

This one expresses that for $\lambda = \infty$ so that $x_i = x = 0$, all ϕ and Φ in (58) are equal to 1. Expression (58) of $\Phi(x)$ is a simple rewriting of (56) by reordering its terms.

B. $\Phi(x)$ in the continuum limit

In the continuum limit, the discrete sums (56), (58) provide the two integral expressions

$$\begin{aligned} \Phi(x) &= \int_0^R \frac{\rho(r)}{\rho_0} d \left[\frac{r^3}{R^3} \phi(r/\lambda) \right] \\ &= \int_0^R \frac{-d\rho(r)}{\rho_0} \frac{r^3}{R^3} \phi(r/\lambda) + \frac{\rho_e}{\rho_0} \phi(x), \end{aligned} \quad (60)$$

which are equal, thanks to an integration by parts. The normalization condition (59), which reads

$$\begin{aligned} \int \frac{dq}{\mathcal{Q}} &= \int \frac{dm}{M} = \int_0^R \frac{\rho(r)}{\rho_0} \frac{3r^2 dr}{R^3} \\ &= \int_0^R \frac{-d\rho(r)}{\rho_0} \frac{r^3}{R^3} + \frac{\rho_e}{\rho_0} = 1, \end{aligned} \quad (61)$$

expresses that the total new charge (or mass) of the sphere may be decomposed as the one of an homogeneous sphere of radius R and density $\rho(R) = \rho_e$, plus the contributions of all the homogeneous full spheres of radii r , with the infinitesimal densities $-\frac{d\rho(r)}{dr} dr$. The second expression in (60) explicates how $\Phi(x)$ depends both of the external density ρ_e and on the variations of the relative density inside the sphere, described by $\frac{-1}{\rho_0} \frac{d\rho(r)}{dr}$.

With

$$\begin{aligned} d \left[\frac{r^3}{R^3} \phi(r/\lambda) \right] &= \frac{3}{m^3 R^3} d(mr \cosh mr - \sinh mr) \\ &= \frac{\sinh mr}{mr} \frac{3r^2 dr}{R^3}, \end{aligned} \quad (62)$$

the first expression of $\Phi(x)$ in (60) reads

$$\Phi(x) = \int_0^R \frac{\rho(r)}{\rho_0} \frac{\sinh mr}{mr} \frac{3r^2 dr}{R^3} = \left\langle \frac{\sinh mr}{mr} \right\rangle. \quad (63)$$

This provides another way to Eq. (32), again obtained as an integral over the contributions of the infinitesimally small thin shells of radius r from 0 to R .

The second expression in (60) may be explicated as

$$\begin{aligned} \Phi(x) &= \int_0^R \frac{1}{\rho_0} \frac{-d\rho(r)}{dr} dr \frac{3(mr \cosh mr - \sinh mr)}{(mR)^3} \\ &\quad + \frac{\rho_e}{\rho_0} \frac{3(x \cosh x - \sinh x)}{x^3}. \end{aligned} \quad (64)$$

One can also reinclude the last term within the integral, to write

$$\Phi(x) = \int_0^\infty \frac{1}{\rho_0} \frac{-d\rho(r)}{dr} dr \frac{3(mr \cosh mr - \sinh mr)}{(mR)^3}, \quad (65)$$

again leading back to Eq. (63). The last term proportional to ρ_e in (64) is recovered from $\frac{-d\rho(r)}{dr} = \rho_e \delta(r - R) + \dots$. Equation (65) also provides back $\phi(x)$ for an homogeneous sphere from the discontinuity of $\rho(r)$ at its surface, as

$$\begin{aligned} \phi(x) &= \int_0^\infty \delta(r - R) dr \frac{3(mr \cosh mr - \sinh mr)}{(mR)^3} \\ &= \frac{3(x \cosh x - \sinh x)}{x^3}. \end{aligned} \quad (66)$$

C. $\Phi(x)$ from the evaluation of $\langle r^{2n} \rangle$ in a multishell model

Let us consider again the expression (58) of $\Phi(x)$ for a p -shell model of the sphere, viewed as a superposition of p homogeneous full spheres of radii R_i and charges q_i (or masses m_i), or from expression (60) obtained in the continuum limit. It reads, replacing the charge ratios $q_i/\mathcal{Q}$ by the corresponding mass ratios m_i/M ,

$$\Phi(x) = \sum_{i=1}^p \frac{m_i}{M} \phi(R_i/\lambda) \rightarrow \int \frac{dm}{M} \phi(r/\lambda), \quad (67)$$

where $\phi(R_i/\lambda)$ may be expanded as in (44), according to

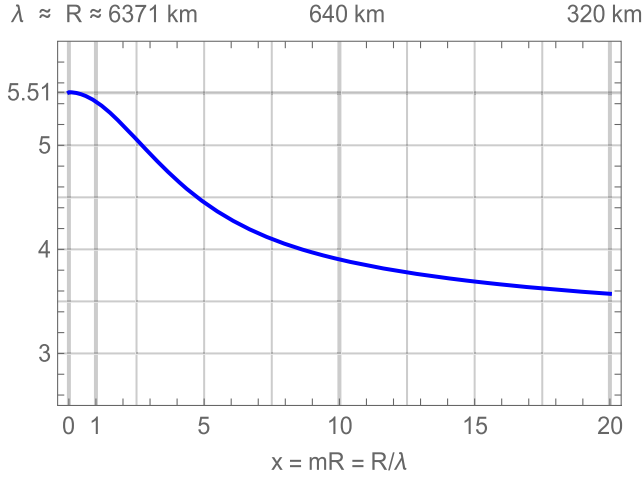


FIG. 2. For an interaction of range λ the Earth generates the same outside Yukawa potential as an homogeneous sphere of density $\bar{\rho}(x) = \rho_0 \Phi(x)/\phi(x)$, function of $x = mR = R/\lambda$. It decreases regularly from about $\rho_0 \simeq 5.51$ g/cm³ for λ larger than the Earth radius, down to $\lesssim 3.5$ g/cm³ for $\lambda \lesssim 300$ km. $\bar{\rho}(x)$, given by Eqs. (50)–(54), is then representative of an average density around a depth $d = R - r \approx \lambda$, and is evaluated in Sec. VIII in a simple 5-shell model. For $x \simeq 3$, i.e., $\lambda \simeq 2100$ km, $\bar{\rho} \simeq 4.92$ g/cm³ is close to the inner mantle density, while for $x \simeq 20$ i.e. $\lambda \simeq 320$ km, $\bar{\rho} \simeq 3.57$ g/cm³ is close to the outer mantle density (cf. Table IV).

$$\phi(R_i/\lambda) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)!} \frac{R_i^{2n}}{R^{2n}} \frac{3}{2n+3}. \quad (68)$$

With the moments of $\rho(r)$ given by

$$\langle r^{2n} \rangle = \frac{3}{2n+3} \sum_{i=1}^p \frac{m_i}{M} R_i^{2n} \rightarrow \frac{3}{2n+3} \int \frac{dm}{M} r^{2n}, \quad (69)$$

this reads

$$\begin{aligned} \Phi(x) &= \int \frac{dm}{M} \phi(r/\lambda) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)!} \int \frac{dm}{M} \frac{r^{2n}}{R^{2n}} \frac{3}{2n+3} \\ &= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)!} \frac{\langle r^{2n} \rangle}{R^{2n}}, \end{aligned} \quad (70)$$

which reconstitutes the expansion (40), absolutely convergent for all x , and identified as $\langle \frac{\sinh mr}{mr} \rangle$.

This provides a fourth way to obtain the general expression of $\Phi(x)$, in addition to summing the contributions of thin shells and solving the Poisson-like equation for the potential as in (27)–(32), decomposing the sphere into small hollow spheres of radius r and thickness dr as in (56)–(63), or viewing it as a superposition of an homogeneous sphere of radius R and density ρ_e with a set of smaller full spheres of radius r from 0 to R as in (58)–(64).

To go further we need to specify the density distribution $\rho(r)$. We have, from the Earth moment of inertia $I = \frac{2}{3} M \langle r^2 \rangle \simeq .3308 MR^2$, $\langle r^2 \rangle \simeq .4962 R^2$, so that

$$\Phi(x) \simeq 1 + .0827 x^2 + \dots \quad (71)$$

This is slightly below $\phi(x) = 1 + \frac{x^2}{10} + \dots$ for an homogeneous sphere, in agreement with the fact that

$$\frac{\bar{\rho}(x)}{\rho_0} = \frac{\Phi(x)}{\phi(x)} = 1 - .0173 x^2 + \dots \quad (72)$$

is a decreasing function of x , as illustrated in Fig. 2. For $\lambda = R$ i.e., $x = 1$ this leads to $\bar{\rho}(1)/\rho_0 \simeq .983$, close to the .9835 shown later in Table IV (in agreement with the small x expansion $\bar{\rho}(x)/\rho_0 \simeq 1 - .0173 x^2 + 8.65 \times 10^{-4} x^4 + \dots$).

VIII. THE EARTH AS A SUPERPOSITION OF 5 HOMOGENEOUS FULL SPHERES

We now consider a simplified model of the Earth, taken as composed of five sectors of external radii R_i and densities ρ_i . We then view it as a superposition of five homogeneous full spheres of radii R_i increasing from $R_1 = R_{ic}$ to $R_5 = R$, each one having for density the excess density

$$\Delta\rho_i = \rho_i - \rho_{i+1}, \quad (73)$$

the last one with density

$$\Delta\rho_5 = \rho_5 = \rho_{\text{crust}} = \rho_0/2 = 2.755 \text{ g/cm}^3. \quad (74)$$

We take $R_5 = R = 6371$ km, an outer mantle radius $R_4 = R_{om} = 6341$ km (for a crust of thickness $\simeq 30$ km), an inner mantle radius $R_3 = R_{im} = 5701$ km, and outer and inner core radii $R_2 = R_{oc} = 3480$ km and $R_1 = R_{ic} = 1221$ km [38].

We include for completeness a distinction between inner and outer cores, even if it is inessential in view of the small effect of an increased inner core density, over .7% of the Earth volume, corresponding to $\lesssim 3 \times 10^{-3}$ of its mass (or $\simeq 2.3 \times 10^{-3}$, with the excess density 1.8 g/cm³ chosen here). In fact the structure of the density distribution within the core has only very small or negligible effects for the smaller ranges λ . For larger ones (i.e., small or moderate $x = R/\lambda$), once $\langle r^2 \rangle$ is fixed to $.4962 R^2$ by the moment of inertia $I \simeq .3308 MR^2$, the effects of an increased inner core density mainly occur through the adjustment of the $x^4 \langle r^4 \rangle / (120 R^4)$ term in expression (40) of $\Phi(x)$, and are very small. We shall consider here that the excess of the inner over outer core densities is $\rho_1 = \rho_{ic} - \rho_{oc} \simeq 1.8$ g/cm³, the results being largely insensitive to this choice.

TABLE III. A 5-shell model of the Earth, with five sectors of densities ρ_i between radii R_{i-1} and R_i . It is then viewed as a superposition of five full spheres of radii R_i , masses $m_i = y_i M$ and densities $\Delta\rho_i = \rho_i - \rho_{i+1}$, the fifth one, of radius R , with density $\Delta\rho_5 = \rho_{\text{crust}} = \rho_0/2 = 2.755 \text{ g/cm}^3$. The outer mantle density, and the difference between inner and outer core densities, are taken as 3.5 and 1.8 g/cm^3 . The mass and moment of inertia of the Earth are reproduced with $\sum_i y_i = 1$ and $\sum_i y_i R_i^2/R^2 = I/(\frac{2}{5}MR^2) \simeq .827$.

	$R_5 = R$ = 6 371 km	$R_4 = R_{om}$ = 6 341 km	$R_3 = R_{im}$ = 5 701 km	$R_2 = R_{oc}$ = 3 480 km	$R_1 = R_{ic}$ = 1 221 km
R_i^2/R^2	1	.9906	.8007	.2984	.0367
$\Delta\rho_i = \rho_i - \rho_{i+1} \text{ (g/cm}^3\text{)}$	$\rho_{\text{crust}} = 2.755$	$\rho_{om} - \rho_{\text{crust}} = .745$	$\rho_{im} - \rho_{om} = 1.319$	$\rho_{oc} - \rho_{im} = 6.522$	$\rho_{ic} - \rho_{oc} = 1.8$
$\rho_i \text{ (g/cm}^3\text{)}$	2.755	3.5	4.82	11.34	13.14
Mass fraction: $y_i = \frac{m_i}{M} = \frac{\Delta\rho_i R_i^3}{\rho_0 R^3}$	.5	.1333	.1715	.1929	.0023
$y_i R_i^2/R^2$	.5	.1320	.1373	.0576	.0001

We take the outer mantle density as $\rho_{om} = 3.5 \text{ g/cm}^3$, and adjust the inner mantle and outer core densities to reproduce the Earth mass and moment of inertia $I \simeq .3308 MR^2$. The proportions of the total mass M in each one of the five full-sphere contributions are denoted as

$$y_i = \frac{m_i}{M} = \frac{\Delta\rho_i}{\rho_0} \frac{R_i^3}{R^3}, \quad \text{with} \quad \sum_i y_i = 1, \quad (75)$$

and $I = \sum_i \frac{2}{5} m_i R_i^2 = \sum_i y_i \frac{R_i^2}{R^2} \times \frac{2}{5} MR^2 \simeq .3308 MR^2$, so that

$$\sum_i y_i \frac{R_i^2}{R^2} = .8270. \quad (76)$$

The mass fractions $y_5 = .5$, $y_4 \simeq .1333$ and $y_1 \simeq .0023$ are fixed by the chosen values of the crust and outer mantle densities (for y_5 and y_4), and excess of the inner over outer core densities (for y_1). The two remaining fractions y_3 and y_2 , satisfying $y_3 + y_2 \simeq .3644$, are adjusted to $y_3 \simeq .1715$ and $y_2 \simeq .1929$ so as to reproduce the Earth moment of inertia. Altogether we get, as shown in Table III,

$$\begin{cases} y_5 \simeq .5, & y_4 \simeq .1333, & y_3 \simeq .1715, & y_2 \simeq .1929, & y_1 \simeq .0023, \\ y_5 \simeq .5, & y_4 \frac{R_{om}^2}{R^2} \simeq .1320, & y_3 \frac{R_{im}^2}{R^2} \simeq .1373, & y_2 \frac{R_{oc}^2}{R^2} \simeq .0576, & y_1 \frac{R_{ic}^2}{R^2} \simeq .0001. \end{cases} \quad (77)$$

This fixes the density differences

$$\begin{cases} \Delta\rho_3 = \rho_{im} - \rho_{om} = y_3 \frac{R_{im}^3}{R^3} \rho_0 \simeq 1.319 \text{ g/cm}^3, \\ \Delta\rho_2 = \rho_{oc} - \rho_{im} = y_2 \frac{R_{oc}^3}{R^3} \rho_0 \simeq 6.522 \text{ g/cm}^3, \end{cases} \quad (78)$$

leading to

$$\rho_{im} \simeq 4.819 \text{ g/cm}^3, \quad \rho_{oc} \simeq 11.341 \text{ g/cm}^3, \quad \rho_{ic} \simeq 13.141 \text{ g/cm}^3. \quad (79)$$

Within this simplified 5-shell model one can estimate, from Eq. (58),

$$\begin{aligned} \Phi(x) = \sum_i y_i \phi(R_i/\lambda) &\simeq .5 \phi(x) + .1333 \phi(.9953x) + .1715 \phi(.8948x) \\ &+ .1929 \phi(.5462x) + .0023 \phi(.1916x). \end{aligned} \quad (80)$$

The resulting values of $\Phi(x)$, and $\bar{\rho}(x)/\rho_0$, are given in Table IV. They are represented in Fig. 2 for $\bar{\rho}(x)$, and Fig. 3 for $\phi(x)$ and $\Phi(x)$. As an illustration for $x = 3$ (i.e. $\lambda \simeq 2124 \text{ km}$ or $m \simeq .929 \times 10^{-13} \text{ eV}/c^2$), $\Phi(3) \simeq 2.0021$ as compared to $\phi(3) \simeq 2.2428$ for an homogeneous sphere, corresponding to $\bar{\rho}(3)/\rho_0 \simeq .893$, or to an effective density

$$\bar{\rho}(3) = \frac{\Phi(3)}{\phi(3)} \rho_0 \simeq .893 \rho_0 \simeq 4.92 \text{ g/cm}^3. \quad (81)$$

TABLE IV. $\phi(x = R/\lambda)$ for an homogeneous sphere, and $\Phi(x) = \sum_i \frac{m_i}{M} \phi(R_i/\lambda)$ in Eq. (80) for the 5-shell model of Table III. Their ratio defines the effective density $\bar{\rho}(x) = \rho_0 \Phi(x)/\phi(x)$, decreasing from $\simeq \rho_0$ for large $\lambda \gtrsim R$, down to somewhat above $\rho_{\text{crust}} \simeq \frac{1}{2}\rho_0$, for shorter λ 's down to a few tens of km. Cf. Figs. 2 and 3.

$x = mR = R\lambda$	$\lambda(\text{km})$	$\phi(x)$	$\Phi(x)$	$\bar{\rho}(x)/\rho_0 = \Phi(x)/\phi(x)$
.5	12 742	1.0252	1.0208	.996
1	6 371	1.1036	1.0855	.984
2	3 185	1.4616	1.3773	.942
3	2 124	2.2428	2.0021	.893
5	1 274	7.1243	5.7563	.808
10	637	297.36	210.7	.708
20	319	1.728×10^6	1.121×10^6	.648
30	212	1.722×10^{10}	1.078×10^{10}	.626
50	127	3.049×10^{18}	1.852×10^{18}	.607
100	63.7	3.992×10^{39}	2.332×10^{39}	.584
200	31.9	2.696×10^{82}	1.490×10^{82}	.553

This is not far from the inner mantle density $\rho_{im} \simeq 4.82 \text{ g/cm}^3$ obtained in Eq. (79), and in agreement with expression (53) of $\bar{\rho}(x)$. For $x = 20$ (i.e. $\lambda \simeq 319 \text{ km}$ or $m \simeq 6.2 \times 10^{-13} \text{ eV}/c^2$), $\Phi(20) \simeq 1.121 \times 10^6$, to be compared with $\phi(20) \simeq 1.728 \times 10^6$, leading to

$$\bar{\rho}(20) = \frac{\Phi(20)}{\phi(20)} \rho_0 \simeq .648 \rho_0 \simeq 3.57 \text{ g/cm}^3, \quad (82)$$

not far from the chosen value of the outer mantle density $\rho_{om} = 3.5 \text{ g/cm}^3$ (cf. Fig. 2).

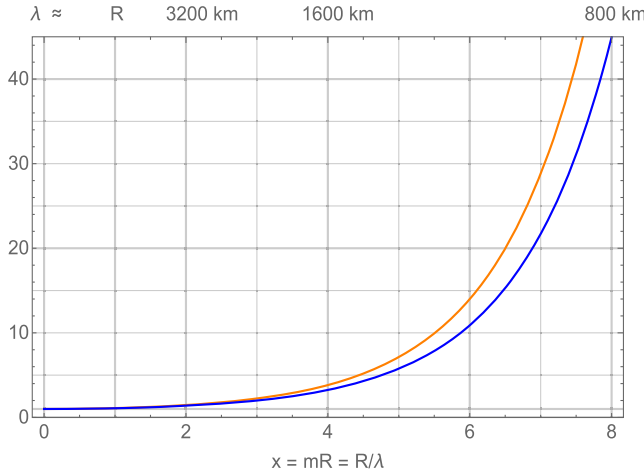


FIG. 3. The hyperbolic form factors $\phi(x) = 3(x \cosh x - \sinh x)/x^3$ for an homogeneous sphere (in orange), and $\Phi(x)$, as approximated in a 5-shell model of the Earth (in blue), as functions of $x = mR = R/\lambda$. Their ratio $\Phi(x)/\phi(x) = \bar{\rho}(x)/\rho_0$ defines the effective density $\bar{\rho}(x)$ represented earlier in Fig. 2.

IX. EXPLICIT EXPANSION OF $\Phi(x)$

In the continuum limit, treating apart the contribution of the last sphere of radius R ,

$$\begin{aligned} \sum_i \frac{m_i R_i^{2n}}{M R^{2n}} &\rightarrow \int_0^R \frac{-d\rho(r)}{\rho_0} \frac{r^3}{R^3} \frac{r^{2n}}{R^{2n}} + \frac{\rho(R)}{\rho_0} \\ &= (2n+3) \int_0^R \frac{\rho(r)}{\rho_0} \frac{r^{2n+2} dr}{R^{2n+3}}, \end{aligned} \quad (83)$$

which provides

$$A_{2n} = \frac{\langle r^{2n} \rangle}{R^{2n}} = \sum_i \frac{m_i}{M} \frac{3}{2n+3} \frac{R_i^{2n}}{R^{2n}} \rightarrow \int_0^R \frac{\rho(r)}{\rho_0} \frac{3r^{2n+2} dr}{R^{2n+3}}, \quad (84)$$

and

$$\begin{aligned} \Phi(x) &= \sum_0^\infty \frac{x^{2n}}{(2n+1)!} \frac{\langle r^{2n} \rangle}{R^{2n}} \\ &= \sum_0^\infty \frac{x^{2n}}{(2n+1)!} \int_0^R \frac{\rho(r)}{\rho_0} \frac{3r^{2n+2} dr}{R^{2n+3}}. \end{aligned} \quad (85)$$

In the above 5-shell model of the Earth, for which $\Phi(x)$ is given by Eq. (80),

$$\begin{aligned} \sum_i y_i \frac{R_i^{2n}}{R^{2n}} &\simeq .5 + .1333 \times .9953^{2n} + .1715 \times .8948^{2n} \\ &\quad + .1929 \times .5462^{2n} + .0023 \times .1916^{2n}, \end{aligned} \quad (86)$$

with $y_i = m_i/M$, we get from Eq. (84)

$$\begin{cases} \langle r^2 \rangle = \sum_i y_i \frac{R_i^2}{R^2} \times \frac{3}{5} R^2 = .4962 R^2, \\ \langle r^{2n} \rangle = \sum_i y_i \frac{R_i^{2n}}{R^{2n}} \times \frac{3}{2n+3} R^{2n} = A_{2n} R^{2n}, \end{cases} \quad (87)$$

with

$$A_0, A_2, \dots, A_{14}, \dots$$

$$\simeq 1, .4962, .3248, .2409, .1910, .1579, .1343, .1166, \dots \quad (88)$$

A_{2n} behaves at large n much like $3/(2n+3)$ times .6333, then .5.

This provides

$$\begin{aligned} \Phi(x) &= \sum_0^\infty \frac{x^{2n}}{(2n+1)!} \frac{3}{2n+3} \sum_i y_i \frac{R_i^{2n}}{R^{2n}} \\ &= \sum_0^\infty \frac{A_{2n}}{(2n+1)!} x^{2n}, \end{aligned} \quad (89)$$

or explicitly, in the 5-shell model considered,

$$\begin{aligned} \Phi(x) &\simeq 1 + \frac{.4962}{6} x^2 + \frac{.3248}{120} x^4 + \frac{.2409}{5040} x^6 + \frac{.1910}{9!} x^8 \\ &\quad + \frac{.1579}{11!} x^{10} + \frac{.1343}{13!} x^{12} + \frac{.1166}{15!} x^{14} + \dots, \end{aligned} \quad (90)$$

i.e.

$$\begin{aligned} \Phi(x) &\simeq 1 + .0827 x^2 + (2.707 \times 10^{-3}) x^4 + (4.78 \times 10^{-5}) x^6 + (5.26 \times 10^{-7}) x^8 \\ &\quad + (3.95 \times 10^{-9}) x^{10} + (2.16 \times 10^{-11}) x^{12} + (8.92 \times 10^{-14}) x^{14} + \dots, \end{aligned} \quad (91)$$

as also obtained directly from the expansion of Eq. (80). More specifically we have

$$\begin{aligned} &\Phi(1, 2, 3, 4, 5; 6, 7, 8, 9, 10) \\ &\simeq 1.0855, 1.3773, 2.0021, 3.2510, 5.756; \\ &10.89, 21.70, 44.96, 96.11, 210.7; \end{aligned} \quad (92)$$

which illustrates how $\Phi(x)$ increases rapidly with x . This expansion, absolutely convergent for all x , already provides, from its first five terms up to x^8 (summing up, e.g., to

$\simeq 10.60$ for $x = 6$, not far from $\Phi(6) \simeq 10.89$) a good approximation of $\Phi(x)$ up to $x \simeq 6$.

Equations (87)–(90) show how improvements of the Earth model considered would affect, normally in a modest way, the above expansion (91) of $\Phi(x)$. Any adjustment of the moments as compared to Eqs. (87), (88) would lead to a change in the expression of $\Phi(x)$, by

$$\delta\Phi(x) = \sum_1^\infty \frac{x^{2n}}{(2n+1)!} \frac{\delta\langle r^{2n} \rangle}{R^{2n}}. \quad (93)$$

TABLE V. The strength of a finite range force, as compared to the massless or ultralight case, is fixed by $F(x) = (1 + mr)e^{-mr}\Phi(x)$, with $mr \simeq 1.1114 x$ at $z = 710$ km. $\Phi(x) = 3(x \cosh x - \sinh x)/x^3 \times \bar{\rho}(x)/\rho_0$ is given in Table IV. The coupling limits in Eqs. (24), (25) get multiplied by $G(x) = F(x)^{-1/2} = e^{mr/2}/(\sqrt{1 + mr}\sqrt{\Phi(x)})$, behaving at large x like $mR e^{mz/2}/\sqrt{1 + mr}$. For $m = 10^{-13}, 10^{-12}$ or 10^{-11} eV/ c^2 , $G(x) \simeq 1.9, 34$, or 10^9 . The resulting coupling limits are shown in Table VI and Fig. 4.

$m(\text{eV}/c^2)$	$\lambda(\text{km})$	$x = R/\lambda$	$\bar{\rho}/\rho_0 = \Phi/\phi$	$\Phi(x)$	$F(x) = (1 + mr)e^{-mr}\Phi(x)$	$G(x) = \frac{e^{mr/2}}{\sqrt{1 + mr}\sqrt{\Phi(x)}}$
10^{-14}	19 733	.323	.998	1.0087	.957	1.022
2×10^{-14}	9 866	.646	.993	1.0350	.867	1.074
5×10^{-14}	3 947	1.614	.960	1.2348	.574	1.320
10^{-13}	1 973	3.23	.882	2.217	.281	1.886
2×10^{-13}	987	6.46	.766	14.8	9.27×10^{-2}	3.28
5×10^{-13}	395	16.14	.664	3.68×10^4	1.124×10^{-2}	9.43
10^{-12}	197	32.3	.623	9.13×10^{10}	8.77×10^{-4}	33.8
2×10^{-12}	98.7	64.6	.600	2.35×10^{24}	1.16×10^{-5}	294
5×10^{-12}	39.5	161.4	.563	4.14×10^{65}	8.93×10^{-11}	1.06×10^5
10^{-11}	19.7	323	.530	1.26×10^{135}	6.46×10^{-19}	1.24×10^9

TABLE VI. Upper limits (95% CL) on the couplings of a spin-1 or spin-0 particle effectively coupled to B , $B - L$, or L . They follow from the expressions and limits on δ in (7), (16), leading to the $m = 0$ limits (24), (25), multiplied by $G(x)$ from Table V. For a given sign of δ the limits on $|g_B|$ and $|g_L|$ are larger than for $|g_{B-L}|$ by about 6 and 1.02, respectively; for $\delta < 0$ they are larger than for $\delta > 0$ by $\simeq 1.2$. The limits on $|g_{B-L}|$, $|g_B|$ and $|g_L|$, are approximately proportional to $[1, (6 \times 1.2) \text{ and } (1.02 \times 1.2)]$ and $[1.2, 6 \text{ and } 1.02]$ in the spin-1 and spin-0 cases, respectively. These limits, $\simeq 1.07 \times 10^{-15}$, 7.7×10^{-15} and 1.31×10^{-15} in the ultralight spin-1 case (and 1.28×10^{-15} , 6.4×10^{-15} and 1.09×10^{-15} for spin-0) are represented in Fig. 4 as functions of m .

$m(\text{eV}/c^2)$	$G(x)$	$\lim g_B $	$\lim g_B $	$\lim g_{B-L} [\simeq \lim g_L]$	$\lim g_L [\simeq \lim g_{B-L}]$
		(spin-1)	(spin-0)	(spin-1) (spin-0)	(spin-1) (spin-0)
		$\delta < 0$ $\propto 6 \times 1.2$	$\delta > 0$ $\propto 6$	$\delta > 0$ $\propto 1 \text{ [or 1.02]}$	$\delta < 0$ $\propto 1.02 \times 1.2 \text{ [or 1.2]}$
$\ll 10^{-14}$	$\simeq 1$	7.7×10^{-25}	6.4×10^{-25}	1.07×10^{-25}	1.31×10^{-25}
10^{-14}	1.022	7.9×10^{-25}	6.6×10^{-25}	1.09×10^{-25}	1.34×10^{-25}
2×10^{-14}	1.074	8.3×10^{-25}	6.9×10^{-25}	1.15×10^{-25}	1.40×10^{-25}
5×10^{-14}	1.320	1.02×10^{-24}	8.5×10^{-25}	1.41×10^{-25}	1.73×10^{-25}
10^{-13}	1.886	1.46×10^{-24}	1.21×10^{-24}	2.01×10^{-25}	2.47×10^{-25}
2×10^{-13}	3.28	2.53×10^{-24}	2.11×10^{-24}	3.50×10^{-25}	4.30×10^{-25}
5×10^{-13}	9.43	7.3×10^{-24}	6.1×10^{-24}	1.01×10^{-24}	1.23×10^{-24}
10^{-12}	33.8	2.61×10^{-23}	2.17×10^{-23}	3.60×10^{-24}	4.42×10^{-24}
2×10^{-12}	294	2.27×10^{-22}	1.89×10^{-22}	3.14×10^{-23}	3.84×10^{-23}
5×10^{-12}	1.06×10^5	8.2×10^{-20}	6.8×10^{-20}	1.13×10^{-20}	1.38×10^{-20}
10^{-11}	1.24×10^9	9.6×10^{-16}	8×10^{-16}	1.33×10^{-16}	1.63×10^{-16}

X. LIMITS ON THE COUPLINGS AS FUNCTIONS OF m , or λ

The Yukawa potential $\mathcal{V}(r) = (Qe^{-mr}/4\pi r)\Phi(x)$ leads to a field $\mathcal{E}(r) = (Qe^{-mr}/4\pi r^2)(1 + mr)\Phi(x)$, and to an Eötvös parameter expressed from (7) as

$$\delta = \mp \frac{\alpha}{G_N u^2} \epsilon^2 \left(\frac{Q}{A_r} \right)_{\oplus} \Delta \left(\frac{Q}{A_r} \right) e^{-mr} (1 + mr) \Phi(x). \quad (94)$$

It is multiplied by $F(x) = e^{-mr}(1 + mr)\Phi(x)$, as compared to the ultralight or massless case. For small λ , i.e., large x one has

$$\begin{aligned} F(x) &\simeq \frac{3(1 - \frac{1}{x})}{2x^2} (1 + mr) e^{-m(r-R)} \frac{\bar{\rho}(x)}{\rho_0} \\ &\approx \frac{3\lambda}{2R} \left(1 + \frac{z}{R} \right) e^{-mz} \frac{\bar{\rho}(x)}{\rho_0}. \end{aligned} \quad (95)$$

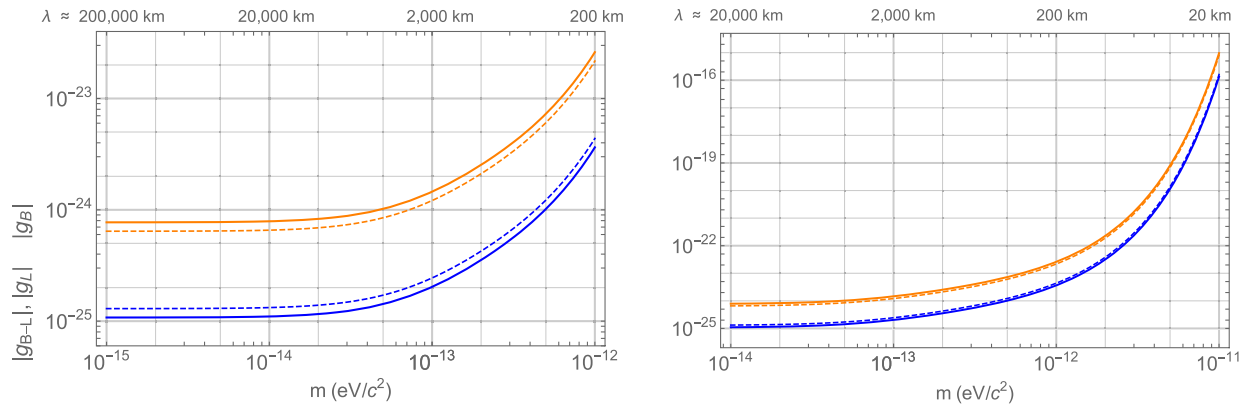


FIG. 4. Upper limits on $|g_{B-L}|$ or $|g_L|$ (in blue), and $|g_B|$ (in orange), depending on the mediator mass m (or range $\lambda = \hbar/mc$), and spin, at the 95% CL. The limits for $\delta < 0$ are larger than for $\delta > 0$ by $\simeq \sqrt{6.5/4.5} \simeq 1.2$. For $\lambda \gg R$ they are about 1.1×10^{-25} for $|g_{B-L}|$ in the spin-1 case or $|g_L|$ in the spin-0 case (solid blue line); and 1.3×10^{-25} for $|g_{B-L}|$ in the spin-0 case or $|g_L|$ in the spin-1 case (dashed blue line). For $|g_B|$ they are about 6 times larger, at 7.7×10^{-25} for spin 1 (solid orange) and 6.4×10^{-25} for spin-0 (dashed orange). The limits increase with m proportionally to $e^{mr/2}[\phi(x)(1 + mr)\bar{\rho}(x)/\rho_0]^{-1/2}$, behaving at large x like $mR e^{mz/2}/\sqrt{1 + mr}$. The limits at the 90% CL are slightly lower, down to 10^{-25} in the spin-1 case for $B - L$.

The resulting limits on the ϵ parameters, and on the couplings $g = \epsilon e$, get increased by the factor

$$G(x) = F(x)^{-1/2} = \frac{e^{mr/2}}{\sqrt{\phi(x)\sqrt{1+mr}}} \sqrt{\frac{\rho_0}{\bar{\rho}(x)}}, \quad (96)$$

with $\phi(x) = \frac{3}{x^2} (\cosh x - \frac{\sinh x}{x})$.

The values of $\Phi(x)$, $F(x)$ and $G(x)$ are given in Table V [43]. At large x , with $mr = x(1 + z/R) \simeq 1.1114x$ and $mz/2 \simeq .0557x$, one gets the simplified expression

$$G(x) \simeq \sqrt{\frac{2}{3(1-\frac{1}{x})}} x \frac{e^{mz/2}}{\sqrt{1+mr}} \sqrt{\frac{\rho_0}{\bar{\rho}(x)}} \approx .77 \sqrt{x} e^{0.0557x} \sqrt{\frac{\rho_0}{\bar{\rho}(x)}}, \quad (97)$$

where $\sqrt{\rho_0/\bar{\rho}(x)}$ increases from $\simeq 1$ at small x up to $\simeq \sqrt{2}$ at larger values of x , which provides a good approximation of $G(x)$. The large factor $\Phi(x)$ is essential for the smaller λ 's, as the effect of the new force does not decrease proportionally to e^{-mr} but only to e^{-mz} ($z \simeq 710$ km being the mean satellite altitude). Ignoring Φ would lead to overestimate the limits by a factor $\sqrt{\Phi}$, which is $\approx 3 \times 10^5$ for $m \simeq 10^{-12}$ eV/ c^2 .

The resulting limits on the couplings of a spin-1 or spin-0 particle effectively coupled to B , $B-L$, or L , are summarized in Table VI and represented in Fig. 4, for a mediator mass m from 10^{-15} or less, up to 10^{-11} eV/ c^2 .

XI. CONCLUSIONS

The dissymmetry of the final *MICROSCOPE* result on the Eötvös parameter δ requires taking into account its sign, leading to

$$\delta < 4.5 \times 10^{-15} (\delta > 0), \quad |\delta| < 6.5 \times 10^{-15} (\delta < 0) \quad (95\% \text{CL}) \quad (98)$$

(or 3.7×10^{-15} and 5.6×10^{-15} , respectively, at the 90% CL). For $\delta > 0$ we get almost the same limits (up to $\simeq 2\%$) for a spin-1 coupling to $B-L$ and a spin-0 coupling to L ; and for $\delta < 0$, for a spin-0 coupling to $B-L$ and a spin 1 coupling to L . The limits for B are about 6 times larger. For a massless or ultralight mediator we have

$$|g_{B-L}| < \begin{cases} 1.1 \times 10^{-25} & (\text{spin } 1), \\ 1.3 \times 10^{-25} & (\text{spin } 0), \end{cases} \quad (95\% \text{CL})$$

$$|g_B| < \begin{cases} 7.7 \times 10^{-25} & (\text{spin } 1), \\ 6.4 \times 10^{-25} & (\text{spin } 0), \end{cases} \quad (99)$$

with slightly lower 90% CL limits, including 10^{-25} for a spin-1 coupling to $B-L$.

The Yukawa potential of a sphere can be expressed in terms of an hyperbolic form factor $\Phi(x = mR)$. We have shown by several methods, including solving the Poisson-like equation for the inside potential, that it is given by

$$\Phi(x) = \left\langle \frac{\sinh mr}{mr} \right\rangle, \quad (100)$$

related by analytic continuation to the form factor

$$f(k) = \Phi(ikR) = \langle e^{i\vec{k}\cdot\vec{r}} \rangle = \left\langle \frac{\sin kr}{kr} \right\rangle. \quad (101)$$

One can also define the hyperbolic form factor

$$g(\vec{k}) = f(-i\vec{k}) = \langle e^{\vec{k}\cdot\vec{r}} \rangle = \frac{\int \rho(\vec{r}) e^{\vec{k}\cdot\vec{r}} d^3\vec{r}}{\int \rho(\vec{r}) d^3\vec{r}}, \quad (102)$$

as the bilateral Laplace transform of the distribution $\rho(\vec{r})$, in general taken with compact support. For a spherically symmetric distribution $\rho(r)$ normalized to unity, $\Phi(x)$ may be expressed from the bilateral Laplace transform of $r\rho(r)$, as

$$\Phi(x = kR) = g(k) = \langle e^{\vec{k}\cdot\vec{r}} \rangle = \frac{2\pi}{k} \int_{-R}^R r\rho(r) e^{kr} dr = \left\langle \frac{\sinh kr}{kr} \right\rangle, \quad (103)$$

reducing to $\phi(x) = 3(x \cosh x - \sinh x)/x^3$ for an homogeneous sphere. An inhomogeneous sphere generates the same outside potential as an homogeneous one with density $\bar{\rho}(x) = \rho_0 \Phi(x)/\phi(x)$, decreasing from the average ρ_0 at small x down to values representative of an average density around a depth $d \approx \lambda$ (cf. Figs. 2 and 3).

Viewing the Earth as a superposition of n homogeneous full spheres of radii R_i , densities $\Delta\rho_i = \rho_i - \rho_{i+1}$ and masses m_i , one has

$$\Phi(x) = \sum_1^n \frac{m_i}{M} \phi(mR_i) = \sum_1^{n-1} \frac{\rho_i - \rho_{i+1}}{\rho_0} \frac{R_i^3}{R^3} \phi\left(x \frac{R_i}{R}\right) + \frac{\rho_e}{\rho_0} \phi(x), \quad (104)$$

or in the continuum limit

$$\Phi(x) = \int_0^R \frac{-d\rho(r)}{\rho_0} \frac{r^3}{R^3} \phi\left(x \frac{r}{R}\right) + \frac{\rho_e}{\rho_0} \phi(x), \quad (105)$$

evaluated here in a simplified 5-shell model. $\Phi(x)$ may be developed in terms of the even moments of the density distribution, as

$$\Phi(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)!} \frac{\langle r^{2n} \rangle}{R^{2n}} = 1 + x^2 \frac{\langle r^2 \rangle}{6R^2} + x^4 \frac{\langle r^4 \rangle}{120R^4} + x^6 \frac{\langle r^6 \rangle}{5040R^6} + \dots$$

$$\simeq 1 + .0827 x^2 + (2.707 \times 10^{-3}) x^4 + (4.78 \times 10^{-5}) x^6 + (5.26 \times 10^{-7}) x^8 + \dots \quad (106)$$

The ratio of a new force induced by a mediator of mass m , as compared to a massless one, is obtained in terms of the effective density $\bar{\rho}(x) = \rho_0 \Phi(x)/\phi(x)$, as

$$F(x) = \phi(x)(1 + mr) e^{-mr} \frac{\bar{\rho}(x)}{\rho_0}, \quad (107)$$

the coupling limits being rescaled by $F(x)^{-1/2}$, $\simeq 1.9$ or 34 for $m = 10^{-13}$ or 10^{-12} eV/ c^2 . This factor increases very rapidly proportionally to $e^{mz/2}$ where z is the satellite altitude, up to $\approx 10^9$ at 10^{-11} eV/ c^2 (cf. Table VI and Fig. 4). In particular, for a spin-1 coupling to $B - L$, or B ,

$$\begin{cases} \text{for } m \simeq 10^{-13} \text{ eV}/c^2: & |g_{B-L}| < 2 \times 10^{-25}, & |g_B| < 1.5 \times 10^{-24}, \\ \text{for } m \simeq 10^{-12} \text{ eV}/c^2: & |g_{B-L}| < 3.6 \times 10^{-24}, & |g_B| < 2.6 \times 10^{-23}. \end{cases} \quad (108)$$

Beyond the specific limits derived here, useful in particular in the context of ultralight dark matter, these methods are general, and may be applied to other situations involving finite-range forces, interactions, or phenomena.

DATA AVAILABILITY

The data supporting this study's findings are available within the article.

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- [28] Strictly speaking for a δ of a given sign the limits on $|\epsilon_L|$ are larger than for $|\epsilon_{B-L}|$ by about 2%, a difference that we shall usually consider as negligible.
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- $$\begin{aligned}\mathcal{V}_{\text{in}}(R) &= \frac{\rho_o}{m^2 x} [x - (x+1)e^{-x} \sinh x] \\ &= \frac{\rho_o}{2m^2 x} [x - 1 + (x+1)e^{-2x}] \\ &= \frac{\rho_o}{m^2 x} (x \cosh x - \sinh x) e^{-x} = \mathcal{V}_{\text{out}}(R),\end{aligned}$$
- $$-\frac{d\mathcal{V}_{\text{in}}}{dr}(R) = \frac{\rho_o}{m} \frac{(x+1)e^{-x}(x \cosh x - \sinh x)}{x^2} = -\frac{d\mathcal{V}_{\text{out}}}{dr}(R).$$
- [36] For large $\lambda \gg R$ i.e. small $x = R/\lambda$, $\phi(x) \rightarrow 1$ and $\mathcal{V}_{\text{out}}(r)$ tends to be the same Yukawa potential as for a pointlike sphere. As $\sinh mr/mr \simeq 1 + x^2 r^2/6R^2$, we also recover in this massless limit the same inside potential $\mathcal{V}_{\text{in}}(r) = \frac{Q}{4\pi R} (\frac{3}{2} - \frac{1}{2} \frac{r^2}{R^2})$ as for a uniformly charged sphere.
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