Dynamical effects of gluon mass and small longitudinal momentum in heavy-quark bound states

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Precise derivation of the logarithmically scale-dependent Hamiltonian eigenstate picture for hadrons in the space of virtual quark and gluon states of the canonical front form of QCD requires addressing first the problem of divergences stronger than logarithmic and especially the small-p^+ divergences in the dynamics of gluons. We propose to facilitate the regularization and cancellation of these divergences using a gluon mass parameter and an auxiliary color-octet scalar field corresponding to the longitudinally polarized gluons. The auxiliary field decouples from the hadronic constituent dynamics when the mass parameter tends to zero, as required in the gauge theory. The same method applies in the cancellation of the quadratic ultraviolet transverse divergences in the self-interactions. After explaining how the method works in the old-fashioned perturbative computations of the scattering amplitudes to all orders, we describe its application to computing the renormalized, scale-dependent Hamiltonian operator acting in the entire Fock space and to deriving its bound-state eigenvalue problems expressed in terms of the correspondingly scale-dependent creation and annihilation operators for effective quarks and gluons. Our detailed discussion focuses on the weak-coupling expansion of second order for the QCD Hamiltonian of effective heavy quarks and the corresponding eigenvalue problems for quarkonia. The results include the concept of confinement in an effective theory with the gluon mass parameter sent to zero and a heuristic scenario concerning extension of the Hamiltonian approach to the dynamics of light quarks.

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I. INTRODUCTION

For precise application to the physics of hadron constituents, the Hamiltonian of QCD needs to be constructed as an operator acting in the space of states of virtual quarks and gluons in a definite form of dynamics. Among the forms identified in [1], the commonly known way of describing the evolution of physical systems using time t of some observer is called the instant form (IF). The name originates from the space-time hyperplane corresponding to one value of t, an instant. The less familiar form, in which an observer describes the system evolution from one value of $x^+ = t + z$ to another, is called the front form (FF). The name comes from the fact that a space-time hyperplane corresponding to one value of x^+ is swept by the front of a plane wave of light moving against the z axis. The choice of the FF of dynamics for describing how quarks and gluons form hadrons stems from the desire to describe the hadrons in any state of motion, such as being at rest in a laboratory or moving with a speed nearly equal to the speed of light in an accelerator. In the IF, boosting hadrons is associated with complex dynamical effects because the quantum generators of boosts involve interactions. In the FF, there are three independent Lorentz transformations that are free from interaction effects, and the canonical dynamics is invariant with respect to them. This feature suggests a simplification of the task of relating the observed parton structure of the fast moving hadrons and their spectroscopic classification at rest in terms of states of three quarks or a quark-antiquark pair [2,3]. Besides offering an interesting way to study the quantum boost problem for bound states, the FF Hamiltonian formulation of QCD provides the eigenvalue equations determining how the physical hadrons, represented by the eigenstates, are built from the quanta of quark and gluon fields.

However, the canonical FF Hamiltonian of QCD is a singular operator. One needs to identify a computational path that leads from the canonical theory to the finite solutions of the renormalized bound-state eigenvalue problems. Asymptotic freedom suggests that the binding mechanism for hadronic constituents involves logarithms of their relative momenta. But to reliably compute the logarithmic effects in the renormalized bound-state dynamics, one first needs to find a way to control the functions more singular than the logarithms. We propose a method for doing this, including a simpler derivation of previously obtained results

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[4–6]. The difficulty to overcome is that the FF of the theory involves singular functions of two distinct kinds of variables.

The bound-state constituent momenta measured along the z axis greatly differ from the momenta in directions perpendicular to that axis. Consequently, the FF Hamiltonians do not possess any explicit three-dimensional rotation symmetry that the IF Hamiltonians have. Instead of the three-dimensional momenta, say $\vec{p}$, one works with the transverse momenta of quanta, denoted by $p^\perp = (p^1, p^2)$, and the longitudinal momenta, denoted by $p^+ = p^0 + p^3$. The corresponding boost-invariant variables of two kinds are the quanta relative transverse momenta $k^\perp = p^\perp - xP^\perp$ and the ratios $x = p^+/P^+$, where P denotes the total momentum of the quanta involved in a Hamiltonian interaction term. Divergences occur due to $p^\perp$ going to ∞ or 0 and due to p^+ going to 0. For example, there are singular functions that behave as $1/x^2$ or $1/x$. The Hamiltonian requires a renormalization procedure for the divergences in x and $k^\perp$ [4]. The first step of any such procedure is to regulate the diverging terms.

Regularization of the Hamiltonian terms can be arranged by limiting the range of variables $k^\perp$ from above and x from below. Suppose one does it for the transverse and longitudinal coordinates separately, e.g., see [7]. In that case, the ultraviolet divergences associated with the longitudinal and transverse directions pile up on each other, which leads to a complex renormalization procedure. One can correlate the regularization cutoffs on $k^\perp$ and x by limiting the total invariant-mass squared, denoted by $\mathcal{M}^2$, of the quanta involved in an interaction term. Each of the quanta contributes to $\mathcal{M}^2$ via P^+ times,

$$k^- = k^0 - k^3 = \frac{k^{\perp 2} + m^2}{xP^+}, \tag{1}$$

with m being a mass assigned to a quantum in the free part of the Hamiltonian. Limiting $\mathcal{M}$ by a cutoff from above, say Λ, one simultaneously limits $|k^\perp|$ from above and prevents x from approaching 0 because m^2/x is limited. The usage of a mass parameter in regularization is of interest because mass does not distinguish any direction in the space of four-momenta. The issue with gluons, however, is that they are massless, as befits them as gauge bosons. For them,

$$P^+k^- = \frac{k^{\perp 2}}{x}. \tag{2}$$

Small x is not limited from below by the upper limit on the invariant mass $\mathcal{M}$ when $k^\perp$ is correspondingly small. Therefore, the ultraviolet renormalization procedure is interfered with by the infrared issues of the theory. Namely, the small-x region belongs to the ultraviolet regime when $k^\perp$ is sizable and it also contributes to the infrared regime when $k^\perp$ is small. This is where the gluon mass parameter, m_g, can be introduced as a regulator disentangling these regimes. Replacement of $k^{\perp 2}$ in Eq. (2) by $k^{\perp 2} + m_g^2$ makes small x correspond to the ultraviolet regime even if $k^\perp$ vanishes. But when one inserts a gluon mass term in the canonical FF Hamiltonian of QCD for the regularization purposes, the resulting virtual quark and gluon transition amplitudes involve the integrands that qualitatively behave as

$$\frac{m_g^2}{k^{\perp 2} + m_g^2}\frac{1}{x^2}. \tag{3}$$

The small-x singularity is associated with a function of the transverse momentum. The singularity strengthens when $k^\perp$ decreases. We propose to counter such gluon mass effects by adding quanta of an auxiliary scalar octet field $\phi = \phi^a T^a$ to the dynamics. These quanta correspond to the longitudinally polarized gluons. In the limit $m_g \to 0$, the longitudinal quanta decouple from the theory when $m_g \to 0$. Such decoupling is known in the case of the FF of QED [8,9] but as far as the author knows has not been taken advantage of in the FF Hamiltonian of QCD as it is done in this paper, cf. [6,10]. In the theory modified by m_g and ϕ, the virtual scattering amplitudes computed using the Hamiltonian approach [11] are shown in Sec. IV to be free from the singularities $\sim 1/x^2$ to all orders in the perturbative expansion. One thus avoids a proliferation of the small-x divergences associated with $m_g > 0$ in the absence of ϕ [4,6]. Since m_g and ϕ can be introduced at the level of the Lagrangian density, these modifications are *a priori* open for studies in other forms of dynamics than the FF considered in this work.

Although the Hamiltonian including m_g and ϕ is less singular than the canonical one, it still generates divergent expressions. For example, the quark and gluon self-interactions are infinite, see Sec. IV C. Therefore, the modified Hamiltonian is regularized by inserting a smooth cutoff function in each of its interaction terms. Such regularization differs from a perturbative loop-by-loop regularization because the cutoff functions introduced in the Hamiltonian terms uniquely regularize every loop. They limit from above the free invariant mass of the bare quanta involved in a term, see Sec. III C. Limiting the invariant mass works in the modified theory because gluons are given the mass m_g. The upper bound on the invariant mass in the interaction terms is set by the inverse of the regularization parameter r. Our regularization respects all kinematic Poincaré symmetries in the FF of dynamics. As a consequence, the self-interaction divergences in second order have only the form of mass squared. The regularization is lifted by sending r to 0 so that the cutoff functions tend to 1, for any value of $m_g > 0$.

However, the limit $r \to 0$ produces diverging quantities, e.g., proportional to $1/r$. The Hamiltonian requires renormalization. For that purpose, we use the renormalization group procedure for effective particles (RGPEP). Review of the RGPEP, including derivation of asymptotic freedom in the QCD Hamiltonian interaction terms, is available in [12]. Its elements that are used here are described in Sec. V.

The main idea of the RGPEP is to rewrite the Hamiltonian, initially expressed using the creation and annihilation operators for bare, pointlike quanta of the canonical quark and gluon fields, in terms of the creation and annihilation operators for effective particles of "size" s. The rewriting is accomplished by solving Eq. (99) of Sec. V. In the process, one computes the counterterms that render the effective dynamics independent of the regularization parameter r in the limit $r \to 0$. Using m_g and ϕ, the only counterterms needed in second-order computations of the finite Hamiltonian are the mass squared counterterms. Subsequently, the solutions to Eq. (99) provide the effective quark and gluon interaction terms that are smoothed out by the form factors of width $1/s$ in p^-. Combination of the boost-invariant regularization with solving Eq. (99) for the effective Hamiltonian including m_g and ϕ, appears to the author's best knowledge most simple computationally in comparison with the earlier variants of the RGPEP, which require more counterterms of second order than only masses squared, cf. [12,13]. Simplifying computations of lowest orders in the weak-coupling expansion, such as by eliminating the need for some counterterms thanks to a suitable choice of regularization, is important because of the complexity of higher-order calculations. For example, the inclusion of the running coupling constant in the dynamics of quarkonia requires at least a fourth-order expansion.

In rewriting the Hamiltonian, the transition from the bare, canonical creation and annihilation operators to the scale-dependent ones for the effective particles of size s is of key importance for representing hadrons as its eigenstates. Namely, in the canonical theory, the bare quanta of finite invariant mass can be changed by the local interactions into the bare quanta of arbitrarily large invariant mass. Hamiltonian eigenstates representing a hadron have components made of the bare quanta of unlimited p^-. Such components are degenerate with components made of many bare quanta whose large invariant mass results from the large number of them rather than from their rapid relative motion. Hence, to compute the eigenstates using the canonical basis states, which are made by acting with bare creation operators on the vacuum state, one needs a prohibitively large numbers of the basis states to investigate convergence of the approximate computations using limited momenta and numbers of canonical quanta. In contrast, the Hamiltonian expressed in terms of the creation and annihilation operators for the effective particles cannot make jumps to large p^- because of the presence of the form factors of width $1/s$. A limited number of the effective-particle Fock components, with a limited range of the effective constituents' relative motion, may approximate an eigenstate. Therefore, the expansion of the eigenstates in the effective-particle basis states instead of the bare-particle basis states is expected to facilitate the checks of convergence for the solutions to the bound-state eigenvalue problems obtained numerically using limited sets of basis states and relaxing the limitations. The RGPEP is a mathematical tool enabling such computational program [14–16]. This work describes the main effective interaction terms that one obtains already in the second-order weak-coupling expansion in the case of approaching QCD of heavy quarks through the inclusion of the field ϕ and the limit $m_g \to 0$ while the size s is on the order of the Compton wavelength for the quarks.

The scale-dependent interaction terms in the renormalized Hamiltonian of the modified heavy-quark QCD do not depend on the regularization parameter $r \to 0$, but they do depend on the gluon mass m_g. In our second-order computation of the renormalized interactions corresponding to finite values of the scale parameter s, the mass m_g is treated as a finite parameter of the theory. However, in the limit of $m_g \to 0$ taken after renormalization, the gluon small-p^+ effects generate logarithms of m_g. We show that these logarithms cancel out in the globally colorless effective quark-antiquark and quark-antiquark-gluon components in the quarkonium eigenvalue problem, see Secs. VIII and IX. Hence the colorless states may form eigenstates with finite eigenvalues when $m_g \to 0$. In the globally colored states, the self-interaction strength rises to infinity as $m_g \to 0$, without being canceled by other interactions. Therefore, the colored states cannot form eigenstates with finite eigenvalues. Besides this confining effect, our approach provides analytic second-order expressions, see Eqs. (195) and (196), for the potentials that bind effective quarks in quarkonia in addition to the Coulomb potential [6,17]. There is no need to invent them, e.g., see Sec. I E in [4]. The second-order potentials are quadratic and rotationally symmetric for the quark-antiquark distances smaller than the quarkonium. They rise logarithmically at larger distances, with different strengths in the transverse and longitudinal directions. The logarithmic potentials of such type have been obtained before [5,18–20] using the similarity renormalization group procedure [21,22] and coupling coherence [23,24]. However, the large distance interactions of quarks are foreseen to change due to the involvement of the effective gluons in the eigenstate dynamics. Our method enables the required computations in a systematic and particularly simple form.

The well-defined RGPEP framework, which replaces the canonical Fock-space representation of heavy-quark QCD with the massless-gluon limit of the scale-dependent effective-particle representation of the theory in the same space of states but in a different basis, is expected to produce phenomenologically reasonable results. The Coulomb and

harmonic oscillator potentials obtained in the dominant Fock space component in Sec. VIII have the same form as the ones obtained in an earlier version of the RGPEP in [17]. There, fitting the quark masses and using the coupling constant evolved from the electro-weak scale to an optimized s yields an approximate description of the ground and lowest excited states of quarkonia made of quarks c and b. The calculated oscillator frequencies are universally of order of $\Lambda_{\rm QCD}$. There are then no free parameters remaining in the effective Hamiltonian when it is applied to triply charmed baryons. Their masses predicted using the RGPEP formulas are closely matched by the results of recent lattice calculations [25].

Since the paper is focused on the use of gluon mass for the purpose of regularization and renormalization of the FF Hamiltonian of QCD, it does not report on or directly refer to most of the rich literature on the FF formulation of quantum field theory and its applications. The literature can be traced following [4,26–34].

Section II describes the bare FF Hamiltonian of QCD including our addition of the mass m_g and field ϕ. Regularization is introduced in Sec. III. Cancellation of the small-x singularities in virtual scattering amplitudes is discussed in Sec. IV. The renormalized Hamiltonian eigenvalue problem, defined using the renormalization group procedure for effective particles (RGPEP) and computed using its weak-coupling expansion, is described in Sec. V. The Hamiltonian dynamics of the quark-antiquark component of a heavy quarkonium is derived in Sec. VII, demonstrating the cancellation of small-x singularities. The limit of $m_g \to 0$ is obtained in Sec. VIII, identifying the nature of dominant terms with the help of a nonrelativistic approximation for the relative motion of the constituents and commenting on the issue of confinement. Small-x singularities in components with effective gluons are addressed in Sec. IX, including interactions that change the number of effective gluons. Discussion in Sec. X concludes the paper with an outlook concerning further computations of the effective quark and gluon dynamics in hadrons.

II. BARE HAMILTONIAN

Canonical FF Hamiltonian of QCD is derived [27,28] from the Lagrangian density,

$$\mathcal{L}_{\rm QCD} = \bar\psi(i\partial\!\!\!/ - gA\!\!\!/ - m)\psi - \frac{1}{2}{\rm Tr}G_{\mu\nu}G^{\mu\nu}, \tag{4}$$

where the sum over quark flavors is assumed but not indicated. The derivation is recalled below for it is used later to explain our procedure for dealing with the small-x divergences caused by gluons.

A. FF Hamiltonian of QCD

By minimizing action described by $\mathcal{L}_{\rm QCD}$ of Eq. (4), one obtains equations

$$(i\partial\!\!\!/ - gA\!\!\!/ - m)\psi = 0, \tag{5}$$

$$\partial_\alpha G^{a\alpha\beta} + ig[A_\alpha, G^{\alpha\beta}]^a + J^{a\beta}_\psi = 0, \tag{6}$$

where the quark current is $J^{a\beta}_\psi = -g\bar\psi T^a\gamma^\beta\psi$. In the gauge $A^+ = 0$, Eq. (5) relates the field $\psi_- = \Lambda_-\psi$ to the fields $\psi_+ = \Lambda_+\psi$ and $A^\perp$, where $\Lambda_\pm = \frac{1}{2}\gamma^0\gamma^\pm$. Namely,

$$\psi_- = \frac{1}{i\partial^+}[\alpha^\perp(i\partial^\perp - gA^\perp) + \beta m]\psi_+. \tag{7}$$

Similarly, Eq. (6) with $\beta = +$ provides the constraint

$$A^- = \frac{2\partial^\perp}{\partial^+}A^\perp + \frac{2}{\partial^{+2}}(J^+_A + J^+_\psi), \tag{8}$$

with $J^{a\mu}_A = -ig[\partial^\mu A^\perp, A^\perp]^a$. Inversion of ∂^{+2} is a remote analog of inverting Laplacian in the Poisson equation to obtain the Coulomb potential in the IF of Hamiltonian dynamics. Generally, inversion of the derivative ∂^+ is understood here in terms of the inverse of momentum in the Fourier transform. Field modes constant in x^- will not appear in the regulated theory, see below. When the coupling constant g is set to 0, the constraints simplify to the constraint equations for free particles,

$$\psi_{f-} = \frac{1}{i\partial^+}(\alpha^\perp i\partial^\perp + \beta m)\psi_+, \tag{9}$$

$$A^-_f = \frac{2\partial^\perp}{\partial^+}A^\perp. \tag{10}$$

One uses these equations to define the fields

$$\psi_f = \psi_+ + \psi_{f-}, \tag{11}$$

$$A_f = (A^-_f, A^+ = 0, A^\perp). \tag{12}$$

The canonical formula for the FF Hamiltonian of QCD expressed in terms of the fields ψ_f and A_f is provided by the component P^- of the total four-momentum carried by the quark and gluon fields. Thus, P^- is an integral of Noether's energy-momentum tensor-density component $+-$ over the front,

$$H_{\rm QCD} = \int_F T^{+-}_{\rm QCD}, \tag{13}$$

where $\int_F = \int d^4x\delta(x^+)$. The Noether tensor is

$$T^{\mu\nu}_{\rm QCD} = \sum_{\chi} \frac{\partial \mathcal{L}_{\rm QCD}}{\partial \partial_\mu \chi} \partial^\nu \chi - g^{\mu\nu} \mathcal{L}_{\rm QCD}. \tag{14}$$

The sum extends over all fields χ in $\mathcal{L}_{\rm QCD}$ with $A^+ = 0$. The Hamiltonian reads

$$H_{\rm QCD} = \int d^2x^\perp dx^- \mathcal{H}_{\rm QCD}, \tag{15}$$

where the integrated density is, cf. [35],

$$\begin{aligned}\mathcal{H}_{\rm QCD} = {} & \bar\psi_f \frac{\gamma^+(-\partial^{\perp 2} + m^2)}{2i\partial^+} \psi_f + \frac{1}{2} A^a_{f\mu} \partial^{\perp 2} A^{a\mu}_f \\ & - (J^{a\mu}_{\psi f} + J^{a\mu}_{Af}) A^a_{f\mu} + \frac{1}{2} g^2 \bar\psi_f \mathcal{A}_f \frac{\gamma^+}{i\partial^+} \mathcal{A}_f \psi_f \\ & - \frac{1}{4} g^2 [A_{f\mu}, A_{f_\nu}]^a [A^\mu_f, A^\nu_f]^a \\ & - \frac{1}{2} (J^{a+}_{\psi f} + J^{a+}_{Af}) \frac{1}{\partial^{+2}} (J^{a+}_{Af} + J^{a+}_{\psi f}),\end{aligned} \tag{16}$$

with currents

$$J^{a\mu}_{\psi f} = -g \bar\psi_f T^a \gamma^\mu \psi_f, \tag{17}$$

$$J^{a\mu}_{Af} = ig[\partial^\mu A_{f\nu}, A^\nu_f]^a. \tag{18}$$

These currents depend on the fields $A^\perp$ and ψ_+ only, since they include explicit solutions for ψ_{f-} and A^-_f. They differ from the currents

$$J^{a\mu}_{\psi} = -g \bar\psi T^a \gamma^\mu \psi, \tag{19}$$

$$J^{a\mu}_{A} = ig[A_\alpha, G^{\alpha\mu}]^a. \tag{20}$$

B. Gluon mass and auxiliary field

The gluon mass term and the associated term for the auxiliary octet scalar field $\phi = \phi^a T^a$ are added to the density $\mathcal{H}_{\rm QCD}$ of Eq. (16) to form the amended density,

$$\begin{aligned}\mathcal{H}^{m_g}_{\rm QCD} = {} & \mathcal{H}_{\rm QCD} - \frac{1}{2} m_g^2 A^a_{f\mu} A^{a\mu}_f + \frac{1}{2} \phi^a (-\partial^{\perp 2} + m_g^2) \phi^a \\ & + m_g \phi^a \frac{1}{\partial^+} (J^{a+}_{Af} + J^{a+}_{\psi f}).\end{aligned} \tag{21}$$

The gluon mass parameter m_g and auxiliary field ϕ introduced in Eq. (21) suffice for the purpose of regulating the gluon dynamics, see Sec. III. When m_g is set to 0, the amended density becomes equal to the canonical $\mathcal{H}_{\rm QCD}$ of Eq. (16) plus a density for a decoupled, free massless field ϕ. The latter can be ignored. Thus, the limit $m_g \to 0$ recovers the canonical FF Hamiltonian of QCD.

In Eq. (21), the mass term for the gluon field A involves only the constrained field A_f of Eq. (12). The kinetic term for the field ϕ has the standard form for scalar bosons. The coupling of ϕ to the quark current is similar in structure to the couplings introduced in massive QED [8,36,37]. The auxiliary field ϕ couples to the color current of gluons. However, it does not couple directly to itself, despite that it carries color. Our scheme differs from the generalized Stueckelberg formalism for massive Yang–Mills fields, in which additional constraints are imposed to eliminate terms inversely proportional to the gauge-boson mass parameter [38]. Such additional constraints are not needed here.

The Fourier modes of the field ϕ with momentum p couple to the color currents proportionally to m_g/p^+. This ratio diverges for $p^+ \to 0$. Precisely such coupling yields the cancellation of the severe small-x divergences due to the assignment of mass m_g to the transverse gluons, see Sec. IV.

The density $\mathcal{H}^{m_g}_{\rm QCD}$ can also be derived from the Lagrangian density

$$\mathcal{L}^{m_g}_{\rm QCD} = \mathcal{L}_{\rm QCD} + \frac{1}{2} (m_g A^a_\mu + \partial_\mu \phi^a)^2, \tag{22}$$

proceeding in parallel to Sec. II A, keeping $A^+ = 0$. The term added to $\mathcal{L}_{\rm QCD}$ in Eq. (22) is not gauge invariant, and the regularization purpose it serves is not. Since the field ϕ is assumed very weakly coupled to gluons and since color is confined, any physical effects due to that field, if it existed, would be hard to detect. In principle, ϕ might contribute to the gravitational effects.

Quark fields ψ obey the same equations in the case of $\mathcal{L}^{m_g}_{\rm QCD}$ as in the case of $\mathcal{L}_{\rm QCD}$, for the added term does not depend on the fermion fields. Therefore, the constraint on ψ_- is the same as in QCD with $m_g = 0$, Eq. (7). Keeping $A^+ = 0$, variation of A^- yields the constraint

$$A^- = A^-_f + \frac{2}{\partial^{+2}} (J^+_{Af} + J^+_{\psi f}) + \frac{2m_g}{\partial^+} \phi, \tag{23}$$

which differs from the canonical Eq. (8) by the last term only, given the identities $J^+_A = J^+_{Af}$ and $J^+_\psi = J^+_{\psi f}$ that follow from $\gamma^{+2} = 0$. The field ϕ obeys

$$\Box \phi + m_g \partial_\mu A^\mu = 0. \tag{24}$$

The constraint Eq. (23) implies

$$\partial_\mu A^\mu = \frac{1}{\partial^+} (J^+_A + J^+_\psi) + m_g \phi, \tag{25}$$

$$(\Box + m_g^2) \phi = -\frac{m_g}{\partial^+} (J^+_A + J^+_\psi). \tag{26}$$

The latter equation exhibits the so-called "longitudinal" or "good" currents [39] as the only sources of the field ϕ.

Evaluating P^- that follows from the energy-momentum tensor density implied by $\mathcal{L}^{m_g}_{\rm QCD}$, one obtains the amended Hamiltonian density $\mathcal{H}^{m_g}_{\rm QCD}$.

C. Quantum Hamiltonian

The Hamiltonian,

$$H^{m_g}_{\rm QCD} = \int d^2x^\perp dx^- \mathcal{H}^{m_g}_{\rm QCD}, \tag{27}$$

where the density $\mathcal{H}^{m_g}_{\rm QCD}$ is a function of fields ψ_f, A_f, and ϕ, given in Eq. (21), is turned into a quantum operator $\hat{H}^{m_g}_{\rm QCD}$ by replacing the fields at $x^+ = 0$ with operators,

$$\hat{\psi}_f = \sum_{c,\sigma} \int [p] \Big[u_{p\sigma} \chi_c \hat{b}_{p\sigma c} e^{-ipx} + v_{p\sigma} \chi_c \hat{d}^\dagger_{p\sigma c} e^{ipx} \Big], \tag{28}$$

$$\hat{A}^\mu_f = \sum_{c,\sigma} \int [p] \Big[\varepsilon^\mu_{p\sigma} T^c \hat{a}_{p\sigma c} e^{-ipx} + \varepsilon^{\mu *}_{p\sigma} T^c \hat{a}^\dagger_{p\sigma c} e^{ipx} \Big], \tag{29}$$

$$\hat{\phi} = \sum_{c} \int [p] \Big[-iT^c \hat{a}_{p3c} e^{-ipx} + iT^c \hat{a}^\dagger_{p3c} e^{ipx} \Big]. \tag{30}$$

In this notation, c refers to colors of $SU(3)$, and σ denotes spin projections on z axis. The momentum integration measure reads $[p] = dp^+ d^2p^\perp/[2p^+(2\pi)^3]$ with $p^+ > 0$, $p^\perp = (p^1, p^2)$, $px = p^- x^+/2 + p^+ x^-/2 - p^\perp x^\perp$. The quark spinors, $u_{p\sigma}$, $v_{p\sigma}$, and gluon polarization vectors, $\varepsilon^\mu_{p\sigma}$, are given in [28], Appendix A. Flavors of quarks are kept in mind. The creation and annihilation operators, $\hat{b}, \hat{b}^\dagger, \hat{d}, \hat{d}^\dagger, a, a^\dagger$, are commonly referred to as particle operators. They are assumed to satisfy the anticommutation relations for fermions and commutation relations for bosons,

$$\{\hat{b}_{p\sigma c}, \hat{b}^\dagger_{p'\sigma' c'}\} = \{\hat{d}_{p\sigma c}, \hat{d}^\dagger_{p'\sigma' c'}\} = [\hat{a}_{p\sigma c}, \hat{a}^\dagger_{p'\sigma' c'}] \tag{31}$$

$$= 2p^+(2\pi)^3 \delta^3(p - p') \delta_{\sigma\sigma'} \delta_{cc'}. \tag{32}$$

Other commutators or anticommutators vanish. Fermion operators commute with the boson operators. By definition, the annihilation operators b, d, a produce 0 when they act on the vacuum state, denoted by $|0\rangle$. Products of particle operators in all terms of the Hamiltonian $\hat{H}^{m_g}_{\rm QCD}$ are normal ordered according to the pattern $a^\dagger b^\dagger d^\dagger dba$. The normal ordering is indicated by double dots. Thus,

$$\hat{H}^{m_g}_{\rm QCD} = \int d^2x^\perp dx^- : \mathcal{H}^{m_g}_{\rm QCD}(\hat{\psi}_f, \hat{A}_f, \hat{\phi}) : . \tag{33}$$

This Hamiltonian is singular and requires regularization as a part of the renormalization procedure described in next sections. The normal ordering is always present but usually not indicated. Hats are omitted to farther simplify the notation.

III. REGULARIZATION

Regulating the Hamiltonian of Eq. (33) begins with limiting from below the range of momentum p^+ labeling creation and annihilation operators in the quantum fields in Eqs. (28), (29), and (30). The lower bound on p^+ is denoted by $\epsilon^+ > 0$. This lower bound is meant to be smaller than any p^+ needed to be considered in any description of any physical process.

A. Particle momenta and cutoff ϵ^+

To introduce the cutoff ϵ^+, we define the integrals over the momenta that label creation and annihilation operators in Eqs. (28), (29), and (30) as discrete sums. The continuous notation is meant to correspond to the discrete momentum spacing that is too small to notice in physical applications. Care needs to be exercised because there are two limits to consider in discussing physical processes, a large-volume limit and a long-time limit, cf. [11]. We introduce a large cuboid extending from $-2L$ to $2L$ in x^- and from $-L$ to L in x^1 and x^2. The cuboid is assumed large enough to contain all processes of physical interest without difficulty. The physical processes are meant to develop over the front "time" x^+ much shorter than $4L$.

The discrete plane-wave momenta in the quark and gluon fields are defined using the periodic boundary conditions on the cuboid walls at $x^+ = 0$,

$$p_n = \epsilon n, \qquad n = (n^+, n^1, n^2), \tag{34}$$

where $\epsilon = \pi/L$, n^+ is a natural number while n^1 and n^2 are integers. The number n^+ is positive because a free particle of positive mass and finite energy E has $p^+ = E + p^z > 0$ no matter how fast and in what direction the particle moves. The assumption is that, in the absence of interaction, the theory describes the energy of free particles, which we consider to be the field quanta, cf. [40], p. 297. Since the box can be arbitrarily large, ϵ can be arbitrarily small.

Suppose one sets an ultraviolet cutoff Δ on the particle energy [41–44],

$$E_n = (p^+_n + p^-_n)/2 < \Delta, \tag{35}$$

where $p^-_n = (p^{\perp 2}_n + \mu^2)/p^+_n$. Such cutoff implies that

$$\Delta - \Delta_n < p^+_n < \Delta + \Delta_n, \tag{36}$$

with $\Delta_n = \sqrt{\Delta^2 - p^{\perp 2}_n - \mu^2}$. Thus, $p^+_n > \epsilon^+$, where $\epsilon^+ = (\mu^2 + p^{\perp 2}_n)/(2\Delta) > 0$ for $\Delta \gg \mu^2 \gg \epsilon^2$. Thus, the energy cutoff implies that the momentum p^+_n for particles with positive masses is separated by the positive gap ϵ^+ from zero. In the infrared box limit, $L \to \infty$, and the ultraviolet cutoff limit, $\Delta \to \infty$, the question of which quantity is

greater, ϵ or ϵ^+, has an answer depending on the order of limits,

$$\frac{\epsilon^+}{\epsilon} = \frac{\mu L}{2\pi}\frac{\mu}{\Delta}. \tag{37}$$

This formula relates the ratio ϵ^+/ϵ to the ratio of the number of the particle Compton wavelengths that fit in the cube edge and the number of free particles at rest whose total energy equals the cutoff Δ. We assume the infrared, large L limit is taken first, for a fixed mass μ. In that case, $\epsilon^+ \gg \epsilon$. In the field expansions, the sum over momenta p_n involves p_n^+ separated from 0 by the gap $\mu^2/(2\Delta)$. Therefore, the integrals in Eqs. (28), (29), and (30) do not extend down to $p^+ = 0$. Consequently, the Hamiltonian $H_{\rm QCD}^{m_g}$ does not contain any terms that are made only of creation or only of annihilation operators. Such terms would imply that a natural multiple of ϵ^+ equals 0.

The largest value of p^+ allowed in the operator labels in Eqs. (28), (29), and (30), say Q^+, is assumed large enough to consider all physical processes of interest. Since every physical process is characterized by some conserved value of its total plus momentum, say P^+, and since the plus momentum of each participating field quantum is smaller than P^+, it is sufficient to assume that Q^+ is finite but greater than the total P^+ in any process of physical interest.

B. The weak-coupling issue of vacuum

The Hamiltonian of Eq. (33) is normal ordered, and according to Sec. III A, all its terms contain at least one annihilation operator. Such terms annihilate the bare vacuum state, denoted by $|0\rangle$. Therefore, the state $|0\rangle$ is an eigenstate of H with an eigenvalue 0. It is also an eigenstate with eigenvalue 0 of the normal-ordered operators P^+ and $P^\perp$, which makes it a candidate for the theory vacuum state.

The issue is that the interactions can cause formation of Hamiltonian eigenstates with eigenvalues smaller than 0, for any finite eigenvalues of P^+ and $P^\perp$. But when the regulated interaction terms are made arbitrarily small, by making the bare coupling constant g arbitrarily small, all eigenstates with positive P^+ must have eigenvalues P^- greater than 0. It has to be so because all the quanta of the regulated theory are assigned masses squared greater than 0. Therefore, all quanta necessarily contribute only positive amounts to the total value of free P^-. The interaction energy is too small to cancel that free positive P^- when g tends to 0. Hadrons are then considered to be states obtained by acting with the creation operators on $|0\rangle$.

The situation changes when the coupling constant is increased and the Hamiltonian develops negative eigenvalues due to the interactions. Such eigenvalues would invalidate the assumption that the eigenstate $|0\rangle$ of eigenvalue 0 is the theory's ground state. Therefore, we assume that no such strong binding effect occurs in QCD with physically justified values of the coupling constant and masses of the quanta. The issue becomes whether the squares of masses in the renormalized effective-particle FF kinetic energies, including the adjustment of the mass-squared counterterms to be discussed later, are large enough to prevent interactions from generating negative eigenvalues P^- for bound states of quarks and gluons. The author does not know any example of such negative eigenvalues in any realistic theory.

C. Regulating interaction terms

Interaction terms in $H_{\rm QCD}^{m_g}$ do not converge as functions of the particle momenta. Such interactions cause arbitrarily large changes of momenta and produce diverging transition amplitudes. The pertinent measure of momentum is p^-, an eigenvalue of the free part of the Hamiltonian, obtained from $H_{\rm QCD}^{m_g}$ by setting the bare coupling constant g to 0. The free part is denoted by H_f,

$$H_f = \sum_{\sigma=1}^{2}\int [p]\,\frac{p^{\perp 2}+m^2}{p^+}\left(b_{p\sigma}^\dagger b_{p\sigma} + d_{p\sigma}^\dagger d_{p\sigma}\right) + \sum_{\sigma=1}^{3}\int [p]\,\frac{p^{\perp 2}+m_g^2}{p^+}\,a_{p\sigma}^\dagger a_{p\sigma}. \tag{38}$$

It results from the first two density terms on the right-hand side of Eq. (16) and from the terms second and third on the right-hand side of Eq. (21). Quanta of the auxiliary gluon field ϕ are accounted for by summing over three instead of only two polarizations. The interaction part of $H_{\rm QCD}^{m_g}$ is denoted by H_I,

$$H_{\rm QCD}^{m_g} = H_f + H_I. \tag{39}$$

With all annihilation operators commonly denoted by a, the structure of H_I reads

$$H_I = \sum_i H_i, \tag{40}$$

$$H_i = \left[\prod_{m=1}^{c_i}\int [p_m]\,a_{p_m}^\dagger\right]\left[\prod_{n=1}^{a_i}\int [q_n]\,a_{q_n}\right]\tilde{\delta}_{c.a}\,h_i(\bar{p},\bar{q}), \tag{41}$$

where c_i and a_i are the numbers of creation and annihilation operators in the term i, respectively. The coefficient $h_i(\bar{p},\bar{q})$ depends on the set of created particles' quantum numbers, $\bar{p}$, which includes p_m with $m = 1,\ldots,c_i$, and the set of annihilated particles' quantum numbers, $\bar{q}$, which includes q_n with $n = 1,\ldots,a_i$. Canonical coefficients h_i are proportional to the coupling constant g or its square. The symbol $\tilde{\delta}_{c.a}$ stands for the δ function that secures conservation of the total kinematic momentum of quanta involved in a term,

$$\tilde{\delta}_{c.a} = 2(2\pi)^3 \delta(p_c^+ - p_a^-)\delta^2(p_c^\perp - p_a^\perp), \tag{42}$$

where the subscripts c and a refer to quanta created and annihilated by a term, respectively,

$$p_c = \sum_{m=1}^{c_i} p_m, \qquad p_a = \sum_{n=1}^{a_i} q_n. \tag{43}$$

Our regularization scheme sets a limit on the magnitude of change of the particle momenta that the interaction terms can cause. We first describe the regularization of the interaction terms that are linear in g and then of the terms that are quadratic in g.

1. Terms linear in g

Regularization of terms linear in g amounts to inserting in them regulating factors $r_{\bar{p},\bar{q}}$,

$$h(\bar{p},\bar{q}) \to r_{\bar{p},\bar{q}} h(\bar{p},\bar{q}), \tag{44}$$

where, using notation of Eq. (43),

$$r_{\bar{p},\bar{q}} = e^{-[r^2(\mathcal{M}_c^2 - \mathcal{M}_a^2)]^2}, \tag{45}$$

with $\mathcal{M}_c^2 = p_c^2$ and $\mathcal{M}_a^2 = p_a^2$. The form of $r_{\bar{p},\bar{q}}$ in Eq. (45) is an example of a factor that limits the invariant-mass changes and hence respects the seven kinematic Poincaré symmetries of the FF of dynamics. Since our renormalization procedure aims to remove dependence of the effective dynamics on the regularization, as it is lifted by taking the limit $r \to 0$, the choice of function $r_{\bar{p},\bar{q}}$ is *in principle* irrelevant. The choice in Eq. (45) is one of many that the author tried in various computations. For example, one could choose $\exp[-r^2(\mathcal{M}_c^2 + \mathcal{M}_a^2)]$ and obtain different expressions for the counterterms.

The insertion of regulating factors in terms linear in g is illustrated below using the three-gluon interaction term. The term originates from the density term $-J^{a\mu}_{Af} A^a_{f\mu}$ for transverse gluons in Eq. (16). Before regularization, the canonical term reads

$$H_{A^3} = g \sum_{c,\sigma} \int [123] a_1^\dagger a_2^\dagger a_3 Y_{123} \tilde{\delta}_{12.3} + \text{H.c.}, \tag{46}$$

where

$$Y_{123} = i f^{c_1 c_2 c_3} [\varepsilon_1^* \varepsilon_2^* \varepsilon_3 k_{12} - \varepsilon_1^* \varepsilon_3 \varepsilon_2^* k_{12}/x_2 - \varepsilon_2^* \varepsilon_3 \varepsilon_1^* k_{12}/x_1]. \tag{47}$$

The symbols ε denote the gluon polarization four-vectors. The three-gluon vertex function Y_{123} grows proportionally to $k_{12}^\perp$, the relative transverse momentum of the created gluons 1 and 2, $k_{12}^\perp = x_2 p_1^\perp - x_1 p_2^\perp$, $x_1 = p_1^+/p_3^+$, $x_2 = p_2^+/p_3^+ = 1 - x_1$, $k_{12}^+ = 0$. This growth leads to the ultraviolet transverse divergences in transition amplitudes. The vertex also grows as $1/x_1$ or $1/x_2$ when x_1 or x_2 tend to 0, which leads to the small-x divergences. Additional complexity results from the ratios $k_{12}^\perp/x_1$ and $k_{12}^\perp/x_2$. For example, a term with an infrared $k^\perp$ may become ultraviolet when $x \to 0$. We use the gluon mass parameter m_g to untangle such mixtures of the ultraviolet and infrared divergences. The regulated three-gluon interaction term reads

$$H^r_{A^3} = g \sum_{c,\sigma} \int [123] a_1^\dagger a_2^\dagger a_3 Y_{123} \tilde{\delta}_{12.3} r_{12,3} + \text{H.c.}, \tag{48}$$

$$r_{12,3} = e^{-[r^2(\mathcal{M}_{12}^2 - m_g^2)]^2}. \tag{49}$$

The invariant mass $\mathcal{M}_{12}$ only depends on the relative motion variables of gluons 1 and 2. Denoting $x = x_1$ and $k^\perp = k_{12}^\perp$,

$$\mathcal{M}_{12}^2 = \frac{k^{\perp 2} + m_g^2}{x(1-x)}. \tag{50}$$

When the invariant mass is finite and $r \to 0$, the regulating factor $r_{12,3}$ tends to 1. When the invariant mass diverges and the parameter r is fixed, the factor tends to 0. The regulating suppression of the vertex occurs when $\mathcal{M}_{12}$ exceeds the magnitude of $1/r$ or, for $|k^\perp|$ smaller than m_g, when x decreases below $r^2 m_g^2$. The presence of m_g guarantees that $\mathcal{M}_{12} \to \infty$ when $x \to 0$. Thus, the small-x gluon divergences are turned into ultraviolet divergences. Without m_g, the region of small x would not be regulated by the factor $r_{12,3}$, for one could have $k^\perp \sim x^\alpha$ with $\alpha \geq 1/2$. The nonlocality of vertices regulated using factors $r_{12,3}$ is analyzed in [45,46].

All terms linear in g in the Hamiltonian $\hat{H}^{m_g}_{\text{QCD}}$ are regulated using factors $r_{\bar{p},\bar{q}}$ as in the above example. Such terms contain two creation operators, one for a quantum of some mass m_1 and another one for a quantum of some mass m_2, and an annihilation operator for some quantum of mass m_3, in correspondence to $a_1^\dagger$, $a_2^\dagger$ and a_3 in Eq. (48). One uses the invariant mass of quanta 1 and 2, squared,

$$\mathcal{M}_{12}^2 = \frac{k_{12}^{\perp 2} + m_1^2}{x_1} + \frac{k_{12}^{\perp 2} + m_2^2}{x_2}. \tag{51}$$

The regulating factor is

$$r_{12,3} = e^{-[r^2(\mathcal{M}_{12}^2 - m_3^2)]^2}. \tag{52}$$

2. Terms quadratic in g

Regularization of the canonical terms proportional to g^2 concerns products of four fields. There are two kinds of them. One kind results from the constraint Eqs. (7) and (23). These terms involve an inverse of ∂^+. They are traditionally called seagulls. The other kind originates from the square of $G^{a\mu\nu}$ in the Lagrangian density. Both kinds are treated in the same way, described below using the example of a seagull.

Constraint Eqs. (8) and (23) relate gauge field component A^- to $1/\partial^{+2}$ acting on the color current J. The latter involves products of two fields. Therefore, when a color current is multiplied by the constrained A^-, the resulting density term is of the form $J(1/\partial^{+2})J$. Each J can be though of as creating or annihilating a gedanken gluon that carries the momentum associated with $1/\partial^{+2}$. The imagined gluon together with the current J form a vertex that is regulated by the factor $r_{12,3}$ in the same way as the three-gluon vertex is in Eq. (48). In the fermion seagulls, the gedanken quantum is a fermion rather than a boson. In the Hamiltonian term due to the product of commutators, $g^2[A^i, A^j]^a[A^i, A^j]^a$, each commutator is treated as a current, even though no constraints are involved and the inverse of ∂^+ is absent.

The regularization of terms proportional to g^2 is illustrated using the gluon seagull term,

$$H_{JJ} = \int dx^- d^2x^\perp \frac{1}{2}(J^{a+}_{\psi f} + J^{a+}_{Af}) \times \frac{-1}{\partial^{+2}}(J^{a+}_{Af} + J^{a+}_{\psi f}), \tag{53}$$

originating from the last density term in Eq. (16). It involves four terms, each of which is a product of four fields. All these products are regulated according to the same rule. It suffices to describe one product. We describe in detail the regularization of the term that involves only quark and antiquark operators,

$$H_{J_\psi J_\psi} = \int dx^- d^2x^\perp \frac{1}{2} J^{a+}_{\psi f} \frac{1}{(i\partial^+)^2} J^{a+}_{\psi f}. \tag{54}$$

The current at $x^+ = 0$ is

$$J^{a+}_{\psi f}(x) = -g\sum_{c_1,c_2}\sum_{\sigma_1,\sigma_2}\chi^\dagger_{c_1} T^a \chi_{c_2} \int [p_1 p_2] \times [\bar{u}_{p_1\sigma_1} b^\dagger_{p_1\sigma_1 c_1} e^{ip_1 x} + \bar{v}_{p_1\sigma_1} d_{p_1\sigma_1 c_1} e^{-ip_1 x}]\gamma^+ \times [u_{p_2\sigma_2} b_{p_2\sigma_2 c_2} e^{-ip_2 x} + v_{p_2\sigma_2} d^\dagger_{p_2\sigma_2 c_2} e^{ip_2 x}]. \tag{55}$$

The spinor products are,

$$\bar{u}_1\gamma^+ u_2 = \bar{v}_1\gamma^+ v_2 = 2\sqrt{p_1^+ p_2^+}\,\delta_{\sigma_1,\sigma_2}, \tag{56}$$

$$\bar{u}_1\gamma^+ v_2 = \bar{v}_1\gamma^+ u_2 = 2\sqrt{p_1^+ p_2^+}\,\delta_{\sigma_1,-\sigma_2}. \tag{57}$$

Hence the normal-ordered, regulated current is

$$:J^{a+}_{\psi f}: = -g\sum_{c,\sigma}\chi^\dagger_{c_1} T^a \chi_{c_2} \int [p_1 p_2] 2\sqrt{p_1^+ p_2^+}\,[:\{\}:]^r, \tag{58}$$

where the bracket $[:\{\}:]^r$ is

$$\begin{aligned} &b^\dagger_1 b_2 e^{i(p_1-p_2)x}[\theta_{1-2} r_{1,2(1-2)} + \theta_{2-1} r_{2,1(2-1)}]\delta_{\sigma_1,\sigma_2} \\ &- d^\dagger_2 d_1 e^{i(p_2-p_1)x}[\theta_{2-1} r_{2,1(2-1)} + \theta_{1-2} r_{1,2(1-2)}]\delta_{\sigma_1,\sigma_2} \\ &+ b^\dagger_1 d^\dagger_2 e^{i(p_1+p_2)x} r_{12,(1+2)}\delta_{-\sigma_1,\sigma_2} \\ &+ d_1 b_2 e^{-i(p_1+p_2)x} r_{12,(1+2)}\delta_{-\sigma_1,\sigma_2}. \end{aligned} \tag{59}$$

It carries the superscript r to indicate the insertion of the regulating factors. The gedanken gluon is assigned the mass m_g for evaluating its minus momentum. The Heaviside's θ function is $\theta_{i-j} = \theta(p_i^+ - p_j^+)$. Thus, the regulated quark seagull is

$$H^r_{J_\psi J_\psi} = \int_F \left[J^{a+}_{\psi f}\frac{1}{(i\partial^+)^2} J^{a+}_{\psi f}\right]^r \tag{60}$$

$$= \int_F :\,[:J^{a+}_{\psi f}:]^r \frac{1}{(i\partial^+)^2}[:J^{a+}_{\psi f}:]^r :. \tag{61}$$

The colon, a symbol of the normal ordering, is dropped below. We display an example of the term that transforms a quark-antiquark pair into another such pair,

$$H^{rq\bar{q}}_{J_\psi J_\psi} = \frac{1}{2} g^2 \sum_{1234\tilde{c}} T^{\tilde{c}}_{12} T^{\tilde{c}}_{34} \times \int [1234]\tilde{\delta}_{c.a}\{\ \}4\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}, \tag{62}$$

where

$$\begin{aligned} \{\ \} = {} & b^\dagger_1 d^\dagger_4 d_3 b_2 \theta_{1-2} r_{1,2(1-2)}\delta_{\sigma_1,\sigma_2} \\ & \times \frac{-2}{(p_3^+ - p_4^+)^2}\theta_{3-4} r_{3,4(3-4)}\delta_{\sigma_3,\sigma_4} \\ & + b^\dagger_1 d^\dagger_4 d_3 b_2 \theta_{2-1} r_{2,1(2-1)}\delta_{\sigma_1,\sigma_2} \\ & \times \frac{-2}{(p_3^+ - p_4^+)^2}\theta_{4-3} r_{4,3(4-3)}\delta_{\sigma_3,\sigma_4} \\ & + b^\dagger_1 d^\dagger_2 d_3 b_4 r_{12,(1+2)}\delta_{-\sigma_1,\sigma_2} \\ & \times \frac{2}{(p_3^+ + p_4^+)^2} r_{34,(3+4)}\delta_{-\sigma_3,\sigma_4}. \end{aligned} \tag{63}$$

3. Fully regulated bare Hamiltonian

Fully regulated QCD Hamiltonian with the gluon mass m_g and field ϕ, denoted by $H^{m_g r}_{\rm QCD}$, reads

$$H^{m_g r}_{\rm QCD} = \int d^2x^\perp dx^- \mathcal{H}^{m_g r}_{\rm QCD}(\psi_f, A_f, \phi), \tag{64}$$

where

$$\begin{aligned}\mathcal{H}^{m_g r}_{\rm QCD} = \mathcal{H}^{r}_{\rm QCD} &- \frac{1}{2} m_g^2 A^a_{f\mu} A^{a\mu}_f \\ &+ \frac{1}{2}\phi^a(-\partial^{\perp 2} + m_g^2)\phi^a \\ &+ \left[m_g \phi^a \frac{1}{\partial^+}(J^{a+}_{Af} + J^{a+}_{\psi f})\right]^r,\end{aligned} \tag{65}$$

and

$$\begin{aligned}\mathcal{H}^{r}_{\rm QCD} = \bar\psi_f \frac{\gamma^+(-\partial^{\perp 2} + m^2)}{2i\partial^+}\psi_f + \frac{1}{2}A^a_{f\mu}\partial^{\perp 2} A^{a\mu}_f \\ - [(J^{a\mu}_{\psi f} + J^{a\mu}_{Af})A^a_{f\mu}]^r + \frac{1}{2}g^2\left[\bar\psi_f \not{A}_f \frac{\gamma^+}{i\partial^+}\not{A}_f \psi_f\right]^r \\ - \frac{1}{4}g^2]A_{f\mu}, A_{f_\nu}]^a [A^\mu_f, A^\nu_f]^a]^r \\ - \frac{1}{2}\left[(J^{a+}_{\psi f} + J^{a+}_{Af})\frac{1}{\partial^{+2}}(J^{a+}_{Af} + J^{a+}_{\psi f})\right]^r.\end{aligned} \tag{66}$$

The regularization brackets $[\,]^r$ embrace all interaction terms. They are omitted to simplify notation.

IV. GLUON MASS AND SMALL x IN SCATTERING AMPLITUDES

Insertion of a mass term for gluons in the FF Hamiltonian of QCD leads to severe small-x divergences due to inverse of ∂^{+2}, e.g., see [4], Sec. IX A. In contrast, the Hamiltonian $H^{m_g r}_{\rm QCD}$ of Eq. (64), with density $\mathcal{H}^{m_g r}_{\rm QCD}$ given in Eq. (65), leads to the cancellation of these singularities despite that $m_g > 0$. The reason is that $H^{m_g r}_{\rm QCD}$ includes also a kinetic term for a color octet of scalar fields of the same mass and an interaction term that couples that octet field to the "good" color currents of quarks and transverse gluons, proportionally to the gluon mass.

A. Severe small-x singularity

To identify the source of the gluon severe small-x divergences, the current-gluon coupling term, i.e., the Hamiltonian term resulting from the third density term on the right-hand side of Eq. (66), is separated into its transverse and longitudinal parts,

$$-(J^{a\mu}_{\psi f} + J^{a\mu}_{Af})A^a_{f\mu} = (J^{aj}_{\psi f} + J^{aj}_{Af})A^{aj}_f - \frac{1}{2}(J^{a+}_{\psi f} + J^{a+}_{Af})A^{a-}_f. \tag{67}$$

Only the longitudinal part involves the inverse of ∂^+. Using Eq. (10) for A^{a-}_f and integrating by parts, the Hamiltonian density of Eq. (65) for fields that vanish at large distances is transformed to the equivalent form

$$\begin{aligned}\mathcal{H}^{m_g r}_{\rm QCD} = \bar\psi_f \frac{\gamma^+(-\partial^{\perp 2} + m^2)}{2i\partial^+}\psi_f + \frac{1}{2}A^{aj}_f(-\partial^{\perp 2} + m_g^2)A^{aj}_f + (J^{aj}_{\psi f} + J^{aj}_{Af})A^{aj}_f + \frac{1}{2}g^2\bar\psi_f A^j_f \gamma^j \frac{\gamma^+}{i\partial^+}\gamma^k A^k_f \psi_f \\ - \frac{1}{4}g^2[A^j_f, A^k_f]^a[A^j_f, A^k_f]^a - \frac{1}{2}(J^{a+}_{\psi f} + J^{a+}_{Af})\frac{1}{\partial^{+2}}(J^{a+}_{Af} + J^{a+}_{\psi f}) \\ + \frac{1}{2}\phi^a(-\partial^{\perp 2} + m_g^2)\phi^a + (\partial^\perp A^{a\perp}_f + m_g\phi^a)\frac{1}{\partial^+}(J^{a+}_{Af} + J^{a+}_{\psi f}).\end{aligned} \tag{68}$$

The superscripts j and k take only two values, 1 and 2, for the front transverse directions. The only terms leading to $1/\partial^{+2}$ or $1/x^2$ for gluons are

$$\mathcal{H}_{+1} = (\partial^\perp A^{a\perp}_f + m_g\phi^a)\frac{1}{\partial^+}\eta J^a_f, \tag{69}$$

$$\mathcal{H}_{+2} = -\frac{1}{2}\eta J^a_f \frac{1}{\partial^{+2}}\eta J^a_f, \tag{70}$$

where $\eta J^a_f = J^{a+}_{\psi f} + J^{a+}_{Af}$. The four-vector η has components $\eta^- = 2$, $\eta^+ = 0$ and $\eta^\perp = 0$. The term $\mathcal{H}_{+1}$ is proportional to g and the term $\mathcal{H}_{+2}$ to g^2. Integration of these two densities over the front yields the Hamiltonian terms denoted by H_{+1} and H_{+2}, respectively. Regularization factors in these terms are introduced according to the rules described in the previous section. The gluon severe small-x singularities occur due to H_{+2} and square of H_{+1}. It is shown in the next section that these two contributions cancel out in the transition amplitudes for quarks and gluons despite the presence of the mass parameter m_g.

B. Cancellation of $1/x^2$ in scattering amplitudes

Scattering of quarks and gluons in the femtouniverse [47] is described assuming that in the first approximation they propagate as free. Interactions are accounted for using an expansion in a series of powers of the coupling constant

g. Our discussion begins with the small-x divergences in the second-order scattering matrix operator [11],

$$T^{(2)} = H_I^{(1)} \frac{1}{P^- - H_f + i\epsilon} H_I^{(1)} + H_I^{(2)}. \tag{71}$$

Symbols $H_I^{(1)}$ and $H_I^{(2)}$ denote the Hamiltonian terms of order g and g^2, respectively. P^- denotes the initial-state eigenvalue of the free Hamiltonian H_f.

Consider the quark-antiquark scattering in which a pair $q\bar{q}$ turns into $q'\bar{q}'$. Dropping the overall momentum conservation $\tilde{\delta}$ function, using j_q and $j_{\bar{q}}$ to denote the quark currents, which are free from the gluon small-x singularities, denoting the four-momentum transfer from the quark to the antiquark by $k = q - q' = \bar{q}' - \bar{q}$, $k^+ > 0$, and omitting $i\epsilon$, one obtains the transition amplitude on shell of total P^- in the form

$$\langle q'\bar{q}'|T^{(2)}|q\bar{q}\rangle = j_{q\alpha} T^{\alpha\beta} j_{\bar{q}\beta}, \tag{72}$$

$$T^{\alpha\beta} = \frac{1}{k^+(P^- - H_f)} \sum_{\sigma=1}^{3} \epsilon_{k\sigma}^{\alpha} \epsilon_{k\sigma}^{*\beta} + \frac{\eta^\alpha \eta^\beta}{k^{+2}}. \tag{73}$$

The first term on the right-hand side of Eq. (73) comes from the exchange of gluons between the quarks, H_f denoting here the total free minus momentum of the particles in the intermediate state. The second term, proportional to $\eta^\alpha\eta^\beta$, is contributed by the seagull term, or H_{+2} corresponding to the density $\mathcal{H}_{+2}$ in Eq. (70). Both the exchange and seagull contributions diverge as $1/k^{+2}$ when $k^+ \to 0$. The sum over gluon polarizations extends from 1 to 3 because it includes the contribution of the longitudinal gluons described by the field operator $\hat{\phi}$ in Eq. (30),

$$\sum_{\sigma=1}^{3} \epsilon_{k\sigma}^{\alpha} \epsilon_{k\sigma}^{*\beta} = -g^{\alpha\beta} + (k_0^\alpha \eta^\beta + \eta^\alpha k_0^\beta)/k^+ + m_g^2 \eta^\alpha \eta^\beta / k^{+2}. \tag{74}$$

The first two terms above come from the transverse gluons, with $k_0^- = k^{\perp 2}/k^+$, and the third term comes from the longitudinal gluons. So, the amplitude is

$$\langle q'\bar{q}'|T^{(2)}|q\bar{q}\rangle = j_{q\alpha} \left[\frac{-g^{\alpha\beta} + \Pi^{\alpha\beta}}{k^2 - m_g^2} \right] j_{\bar{q}\beta}, \tag{75}$$

where

$$\Pi^{\alpha\beta} = (k_0^\alpha \eta^\beta + \eta^\alpha k_0^\beta)/k^+ + m_g^2 \eta^\alpha \eta^\beta / k^{+2} + (k^2 - m_g^2)\eta^\alpha \eta^\beta / k^{+2}. \tag{76}$$

The quark and antiquark currents in Eq. (75) are conserved, $k_\alpha j_q^\alpha = k_\alpha j_{\bar{q}}^\alpha = 0$. The formula $k_0^\alpha = k^\alpha + (k_0^- - k^-)\eta^\alpha/2$ implies $\Pi^{\alpha\beta} \equiv \Pi \eta^\alpha \eta^\beta$,

$$\Pi = (k_0^- - k^-)/k^+ + m_g^2/k^{+2} + (k^2 - m_g^2)/k^{+2}. \tag{77}$$

The key result is that Π vanishes. The terms $\sim 1/x^2$ cancel out completely. The regularization factors r do not interfere with the cancellation because they are the same in the gluon exchange and the seagull term. The resulting scattering amplitude has the covariant form of Eq. (75) with $\Pi^{\alpha\beta} = 0$. Moreover, the amplitude is free from any small-x singularity, not only the severe $1/x^2$. An alternative way of stating this result is that the diagrammatic rule (R8) of perturbative QCD in Appendix A of [28] continues to be valid even though the gluons are assigned the mass m_g.

In order to extend the above reasoning to the transition amplitudes between intermediate states that are off shell of total P^-, we first note that the severe gluon singularities appear in perturbative calculations due to the operator,

$$T_{\text{singular}} = H_{+1} \frac{1}{P^- - H_f + i\epsilon} H_{+1} + H_{+2}, \tag{78}$$

which sums the exchange of a gluon between two color currents and the seagull. The divergences occur in the coefficients of the tensor $\eta^\alpha\eta^\beta$, contracted with the quark or gluon currents. The sum over transverse polarizations of the exchanged gluon contributes $k^{\perp 2}/k^{+2}$, as dictated by H_{+1}. The intermediate quanta of field ϕ contribute m_g^2/k^{+2}, also dictated by H_{+1}. Since the eigenvalues k^- of H_f for the quanta of fields A and ϕ are the same, $k_g^- = (k^{\perp 2} + m_g^2)/k^+$, the most singular contribution to the transition amplitude due to $k^+ \to 0$, takes the form

$$\langle T_{\text{singular}} \rangle = j_1 \left[\frac{k^{\perp 2}/k^{+2} + m_g^2/k^{+2}}{k^+(P^- - Q^- - k_g^-)} + \frac{1}{k^{+2}} \right] j_2, \tag{79}$$

where the second term in the bracket, $1/k^{+2}$, comes from H_{+2}. The sum of eigenvalues of H_f for other quanta in the same intermediate state, in which the exchanged gluons appear, is denoted by Q^-. Symbols j_1 and j_2 stand for the contractions of the quark or gluon currents with η. The three-momentum conservation $\tilde{\delta}$ function and $i\epsilon$ are omitted. The term $k^{\perp 2}/k^{+2}$ in the numerator in Eq. (79) is provided by the transverse gluons and the term m_g^2/k^{+2} by the longitudinal ones. The inverse of k^{+2} would produce the severe singularity for $k^+ \to 0$ if the free Hamiltonian H_f would include the mass m_g for transverse gluons and the longitudinal gluons were absent. When their contribution is included,

$$\langle T_{\text{singular}} \rangle = j_1 \frac{(P^- - Q^-)/k^+}{k^+(P^- - Q^-) - (k^{\perp 2} + m_g^2)} j_2. \tag{80}$$

The divergence $\sim 1/k^{+2}$ for $k^+ \to 0$ cancels out despite that $m_g > 0$. At the same time, $m_g^2 > 0$ regulates the infrared divergence due to $k^\perp \to 0$ when $k^+ \to 0$.

The same mechanism works to all orders of perturbation theory. Our demonstration begins with Eq. (69). Every numerator term associated with propagation of a transverse gluon, $(k^\perp/k^+)^2$, is accompanied by a term due to the propagation of a longitudinal gluon, $(m_g/k^+)^2$. The transverse-gluon exchange and the longitudinal-gluon exchange involve the same denominator factor with the same value of the gluon k_g^-. The exchange extending over one intermediate state always appears in addition to a contribution of the Hamiltonian term H_{+2}. Therefore, the cancellation in exchanges extending over one intermediate state occurs as described above. The cancellation holds no matter how many additional interactions precede or follow the exchange. When the gluon exchange extends over more than one intermediate state, the contribution of H_{+2} is absent. However, in that case, the gluon k_g^- appears in more than one denominator. The additional denominators provide additional powers of k^+ in the numerator, which eliminate the severe singularity $1/k^{+2}$.

C. Small x in self-interactions

In the amplitudes discussed in Sec. IV B, the severe small-x divergences of the gluon exchange and seagull terms cancel each other. But in the quark and gluon self-interactions, schematically illustrated in Fig. 1, the normal-ordered seagulls do not contribute. The second-order self-interaction sketched in Fig. 1 requires a counterterm. Including the counterterm, the self-interaction reads

$$\langle p'|p\rangle \frac{\Sigma^{(2)}}{p^+} = \langle p'|H_{+1}\frac{1}{P^- - Q^- - H_f}H_{+1}|p\rangle + \langle p'|p\rangle \frac{C^{(2)}}{p^+}, \tag{81}$$

where the counterterm contribution is denoted by $C^{(2)}$. The factor $\langle p'|p\rangle$ accounts for the state normalization. We obtain

$$\frac{\Sigma_q^{(2)}}{p^+} = \frac{g^2 C_F}{p^+}\int \frac{dx d^2k^\perp}{2(2\pi)^3}\left[\frac{N}{D} + 4\sigma_q^{(2)}\right] r_{qg,p}^2, \tag{82}$$

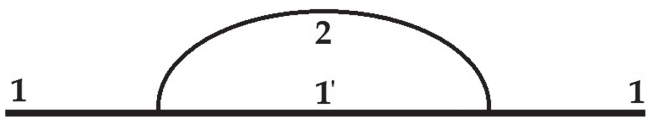


FIG. 1. Self-interaction of particle of type 1 via emission and absorption of particle of type 2.

$$N = 4\frac{k^{\perp 2} + (1-x)m_g^2}{x^2} + 2\frac{k^{\perp 2} + x^2 m_i^2}{1-x}, \tag{83}$$

$$D = x(1-x)p^+(P^- - Q^- - p_q^- - p_g^-), \tag{84}$$

where $4\sigma_q^{(2)}$ stands for the integrand of the counterterm. x is the fraction of the quark momentum p^+, carried by the emitted gluon, ranging between 0 and 1. $k^\perp = p_g^\perp - x p^\perp$. For N colors of quarks, $C_F = (N^2-1)(2N)$. The squared regularization factor, $r_{qg,q}^2$, comes from the double action of the Hamiltonian term H_{+1}. The severe small-x singularities are contained in

$$\frac{\Sigma_{q\,\mathrm{sing}}^{(2)}}{p^+} = \frac{g^2 C_F}{p^+}\int \frac{dx d^2k^\perp}{2(2\pi)^3} \times \left[\frac{4p_g^-/x}{x(P^- - Q^- - p_q^- - p_g^-)} + 4\sigma_q^{(2)}\right] r_{qg,q}^2. \tag{85}$$

p_q denotes the free four-momentum of a quark of mass m, after emitting and before absorbing a gluon. p_g denotes the free four-momentum of the gluon of mass m_g. It is visible that to counter the singularity $\sim 1/x^2$ one needs the counterterm integrand,

$$\sigma_q^{(2)} = 1/x^2. \tag{86}$$

Inclusion of the less singular terms than $4p_g^-/x$ shows that the small-x divergence $1/x^2$ and the ultraviolet quadratic transverse divergence in $\Sigma_q^{(2)}/p^+$ are simultaneously countered by the counterterm whose integrand is

$$\sigma_q^{(2)} = 1/x^2 + 1/[2(1-x)]. \tag{87}$$

Similarly, the second-order transverse-gluon self-interaction reads

$$\frac{\Sigma_{g\,\mathrm{sing}}^{(2)}}{p^+} = \frac{2g^2 C_A}{p^+}\int \frac{dx d^2k^\perp}{2(2\pi)^3}\left[\frac{N}{D} + \sigma_g^{(2)}\right] r_{gg,g}^2, \tag{88}$$

$$N = (k^{\perp 2} + m_g^2)\left[1 + \frac{1}{x^2} + \frac{1}{(1-x)^2}\right] - m_g^2\left[\frac{1}{2} + \frac{1}{x(1-x)}\right], \tag{89}$$

$$D = x(1-x)p^+(P^- - Q^-) - (k^{\perp 2} + m_g^2). \tag{90}$$

For N colors $C_A = N$. It is visible that the counterterm integrand,

$$\sigma_g^{(2)} = 1 + \frac{1}{x^2} + \frac{1}{(1-x)^2}, \tag{91}$$

removes the severe divergences of type $1/x^2$ for $x \to 0$. Inclusion of 1 in Eq. (91), secures simultaneous cancellation of the ultraviolet quadratic transverse divergence. The gluon self-interaction due to the intermediate quark pairs does not involve divergences $\sim 1/x^2$. The self-interaction of longitudinal gluons is free from such divergences as well. The remaining singularities are only logarithmic, as promised. In orders higher than second, the one-particle irreducible self-interactions require more complex expressions for the counterterms. Self-interactions in the renormalized bound-state eigenvalue problems of the Hamiltonian $H_{\rm QCD}^{m_g r}$ are discussed in the following sections.

V. HAMILTONIAN EIGENVALUE PROBLEM

Eigenstates of the Hamiltonian $H_{\rm QCD}^{m_g r}$ of Eq. (64) represent hadrons of QCD in the limit of lifting the regularization, $r \to 0$ and $m_g \to 0$. The eigenstates are the combinations

$$|h\rangle = \sum_i \psi_i |i\rangle, \tag{92}$$

where $|i\rangle$ are the basis states obtained by acting on the state $|0\rangle$ with products of creation operators introduced in Eqs. (28), (29), and (30). To describe a hadron, one needs to compute the relevant eigenvalue of the matrix $H_{ij}^{m_g r} = \langle i|H_{\rm QCD}^{m_g r}|j\rangle$ and evaluate the corresponding set of coefficients ψ_i. Analytic calculations appear not feasible because of the matrix size and complexity. To use computers, one needs to overcome several difficulties.

The Hamiltonian matrix has infinite size. One must replace it by some equivalent finite matrix. To begin with, one could limit the momenta and number of quarks and gluons in the basis states. However, setting such limits is not straightforward because the matrix elements $H_{ij}^{m_g r}$ diverge as functions of the momenta, e.g., see Eq. (46) for $k^\perp \to \infty$ or $x \to 0$. The diverging matrix elements dominate the dynamics. They need to be dealt with first thing to establish computational access to the finite quantities of physical interest [4,48,49], including the issue of explaining the quantum-binding mechanism for quarks and gluons. The singular interactions can change the number of quanta. Therefore, the states with quarks and gluons in fast relative motion are degenerate with states of less rapid but more numerous quanta. Traces of these complexities appear in QCD phenomenology. For example, applications of the parton model suggest that the gluon distribution in proton grows for small x and requires some mechanism of saturation. Eventually, one needs a method of estimating the accuracy of a computation.

To overcome the difficulties mentioned above, the QCD bound-state eigenvalue problems are formulated not in terms of the canonical creation and annihilation operators introduced in Eqs. (28), (29), and (30). Instead of b, d, and a, commonly denoted below by q, one uses operators b_s, d_s, and a_s, denoted by q_s. The latter are called the operators for effective particles. The parameter s can be thought about as a size of the particles. The canonical operators q correspond to the pointlike quanta, $q = q_{s=0}$. When $s \sim 1/m$, where m is a hadron mass, the effective particles are meant to correspond to the constituent quarks and gluons.

Evaluation of the QCD Hamiltonian in terms of operators q_s is carried out using the renormalization group procedure for effective particles (RGPEP) [12]. The procedure is designed to smooth out the singular canonical interactions. The smoothing is provided by the s-dependent vertex form factors obtained by solving the RGPEP equations. To describe how the gluon mass m_g and auxiliary field ϕ contribute to the effective bound-state dynamics, we need to recall the elements of the RGPEP.

A. Elements of the RGPEP

Quantum numbers of the phenomenological constituent quarks [3] are the same as those of the bare quarks in the canonical QCD, with the exception of the mass parameters. Therefore, the RGPEP for QCD assumes that the effective particles are unitarily related to the bare particles,

$$q_s = \mathcal{U}_s^\dagger q \mathcal{U}_s, \tag{93}$$

and the quantum numbers of q_s and q are the same. The Hamiltonian remains unchanged. Consequently, the products of operators q_s in the Hamiltonian have different coefficients, say c_s, from the coefficients c of the corresponding products of operators q, but

$$H(c_s, q_s) = H(c, q). \tag{94}$$

The renormalized expressions for the coefficients c_s are obtained by solving a first-order differential equation in s, called here the RGPEP equation, see Eq. (99) below. The initial condition at $s = 0$ is provided by the canonical coefficients, c, including the modifications resulting from the counterterms determined while solving the RGPEP equation, according to the rules of the similarity renormalization group procedure (SRG) [21,22]. Computation of the coefficients c_s is carried out using the operator

$$\mathcal{H} = H(c_s, q) = \mathcal{U}_s H(c_s, q_s) \mathcal{U}_s^\dagger, \tag{95}$$

whose definition implies that

$$\mathcal{H}' = [\mathcal{G}, \mathcal{H}], \tag{96}$$

where prime denotes the derivative d/ds^2; we use s^2 here for s to have dimension of length. The anti-Hermitian operator,

$$\mathcal{G} = \mathcal{U}'\mathcal{U}^\dagger, \tag{97}$$

is called the generator of the RGPEP. A whole class of generators can be considered [14,50]. Hamiltonian operators we discuss have the form $\mathcal{H} = \mathcal{H}_f + \mathcal{H}_I$, where $\mathcal{H}_f$ is given in Eq. (38). The commutator $[\mathcal{H}_f, \mathcal{H}_I]$ is anti-Hermitian and can be used as a generator of a unitary transformation,

$$\mathcal{G} = [\mathcal{H}_f, \mathcal{H}_I]. \tag{98}$$

Evolution in s with such generator tends to narrow the coefficients c_s in the sense that they only allow changes of eigenvalues of $\mathcal{H}_f$ that are limited by $\sim 1/s$. With this choice for the generator, Eq. (96) takes the double-commutator form,

$$\mathcal{H}' = [[\mathcal{H}_f, \mathcal{H}_I], \mathcal{H}], \tag{99}$$

closely resembling the Wegner flow equation for Hamiltonian matrices in condensed matter physics [51], see also [52–58]. The operator Eq. (99) serves the computation of the coefficients c_s and as such does not exactly reduce to finite matrix equations, because the particle operators do not. However, the double commutator structure still implies that $\mathcal{H}$ fulfills the cluster decomposition principle [40]; there is no generation of the disconnected interactions. Model studies of finite matrix approximations to the RGPEP equations with various generators show a need for alterations of the generator $\mathcal{G}$ of Eq. (98) when one seeks to achieve convergence of solutions for $\mathcal{H}$ using the weak-coupling expansion [15,59]. Here, we are concerned with the lowest orders of the expansion and no need arises for altering Eqs. (98) and (99).

B. Renormalized effective eigenvalue problem

Assuming a solution for $\mathcal{H}$ as a function of s is available, the QCD Hamiltonian is obtained in the form

$$H_s \equiv H(c_s, q_s) = \mathcal{U}_s^\dagger[\mathcal{H} = H(c_s, q)]\mathcal{U}_s. \tag{100}$$

The eigenvalue problem for hadrons as bound states of effective quarks and gluons reads

$$H_s|\psi\rangle = P^-|\psi\rangle, \tag{101}$$

$$|\psi\rangle = \sum_i \psi_i(s)|i\rangle_s, \tag{102}$$

where P^- is the eigenvalue. The basis states $|i\rangle_s$ are created by applying products of the creation operators q_s to the state $|0\rangle$. If one solved the RGPEP equation for H_s exactly and found exact eigenvalues and eigenstates of H_s, the coefficients $\psi_i(s)$ and basis states $|i\rangle_s$ would depend on s but the eigenvalues and corresponding eigenstates $|\psi\rangle$ would not [16]. Approximate calculations yield results varying with s. The magnitude of such variation indicates how large the theoretical error of an approximate calculation may be. Another measure is provided by the accuracy of obeying symmetries, such as the rotation symmetry.

VI. WEAK-COUPLING EXPANSION FOR H_s

Given the initial condition at $s = 0$ in the form of the regulated Hamiltonian $H_{\rm QCD}^{m_g\, r}$ of Eq. (64), the differential Eq. (99) can be solved order by order in the expansion of the coefficients c_s in the series of powers of the coupling constant g. Such expansion is valid when the coupling constant is made so small that all computed interaction terms are small. But solutions for the coefficients c_s may contain inverse powers or logarithms of the regularization parameter r. When one lifts the regularization, taking the limit $r \to 0$, such coefficients become infinite despite that s is finite and g small. Consequently, the corresponding matrix elements of H_s between the effective basis states of finite momenta diverge and the eigenvalue problem is ill defined. To formulate a soluble theory, the initial condition at $s = 0$, given by $H_{\rm QCD}^{m_g\, r}$ in Eq. (64), needs to be modified by adding terms, called counterterms, determined by the condition that the divergences in c_s for finite s are eliminated. The finite parts of the counterterms must be fixed by comparison with data, including the symmetries observed at s. Thus, the initial condition for solving the RGPEP equation is changed from $H_{\rm QCD}^{m_g\, r}$ of Eq. (64) to

$$H_{{\rm QCD}\,\mathcal{C}}^{m_g\, r} = \int dx^- d^2x^\perp \left(\mathcal{H}_{\rm QCD}^{m_g\, r} + \mathcal{C}_{\rm QCD}^{m_g\, r}\right). \tag{103}$$

The subscript $\mathcal{C}$ indicates the inclusion of the counterterms, denoted by $\mathcal{C}_{\rm QCD}^{m_g\, r}$. Perturbative solution of the RGPEP equation for H_s, including the counterterms in the initial condition at $s = 0$, becomes calculable order by order for very small coupling constants, though the radius of convergence is not known. To describe hadrons, one needs to extrapolate the weak-coupling expansion to the realistic values of the interaction strength [4]. The realistic magnitudes for large s ought to match the running interaction strength fitted to high-energy data within the same scheme. Outlines of the entire procedure, using simple models, are available in [15,16,59].

A. Solutions of lowest orders

This section illustrates the weak-coupling approach to solving the RGPEP equation for terms up to the third order. The second-order formulas are used later on to discuss the bound-state eigenvalue problems in QCD of heavy quarks including the mass m_g and field ϕ. The third-order solution shows how the RGPEP computations proceed beyond the second order, cf. [60].

In the series expansion of the operator $\mathcal{H}$ in powers of g,

$$\mathcal{H} = \mathcal{H}_f + \mathcal{H}^{(1)} + \mathcal{H}^{(2)} + \mathcal{H}^{(3)} + O(g^4), \quad (104)$$

where the terms $\mathcal{H}^{(n)}$ are proportional to g^n. The operator $\mathcal{H}^{(n)}$ obeys the equation obtained by equating coefficients of g^n on both sides of Eq. (99). For the terms of first three orders, besides $\mathcal{H}^{(0)} = \mathcal{H}_f$, the equations read

$$\mathcal{H}^{(1)'} = [[\mathcal{H}_f, \mathcal{H}^{(1)}], \mathcal{H}_f], \quad (105)$$

$$\mathcal{H}^{(2)'} = [[\mathcal{H}_f, \mathcal{H}^{(2)}], \mathcal{H}_f] + [[\mathcal{H}_f, \mathcal{H}^{(1)}], \mathcal{H}^{(1)}], \quad (106)$$

$$\mathcal{H}^{(3)'} = [[\mathcal{H}_f, \mathcal{H}^{(3)}], \mathcal{H}_f] + [[\mathcal{H}_f, \mathcal{H}^{(2)}], \mathcal{H}^{(1)}] + [[\mathcal{H}_f, \mathcal{H}^{(1)}], \mathcal{H}^{(2)}]. \quad (107)$$

In general, the derivative of $\mathcal{H}^{(n)}$ only involves operators $\mathcal{H}^{(k)}$ with $k < n$, which facilitates solving for $\mathcal{H}$ order by order, to all orders.

The solution for terms of the first order reads

$$\mathcal{H}^{(1)} = f_{LR}\mathcal{H}_0^{(1)}, \quad (108)$$

where the subscript 0 refers to the initial condition at $s = 0$, and

$$f_{LR} = e^{-(s\Delta_{LR})^2}, \quad (109)$$

$$\Delta_{LR} = p_L^- - p_R^-. \quad (110)$$

The subscript L refers to the product of creation operators in a term in $\mathcal{H}$, standing on the term's left-hand side, and the subscript R refers to the product of annihilation operators in a term, standing on its right-hand side. Using the notation introduced in Eqs. (41)–(45) for an interaction term $\mathcal{H}_i$, action of f_{LR} on any term in Eq. (108), and on any other operator at any value of s, is defined by

$$f_{LR}\mathcal{H}_i = \left[\prod_{m=1}^{c_i}\int [p_m] a^\dagger_{p_m}\right]\left[\prod_{n=1}^{a_i}\int [q_n] a_{q_n}\right] \times \tilde{\delta}_{c.a} h_i(\bar{p}, \bar{q}) f^s_{\bar{p},\bar{q}}. \quad (111)$$

In words, the action of f_{LR} on $\mathcal{H}_i$ implies insertion of the factor $f^s_{\bar{p},\bar{q}}$ in the integrand of $\mathcal{H}_i$. The solution for terms of the second order, obtained using solutions for the first-order terms, reads

$$\mathcal{H}^{(2)} = f_{LR}\mathcal{H}_0^{(2)} + (f_{LR} - f_{LI}f_{IR})\Delta_{LIR}\mathcal{H}_0^{(1)}\mathcal{H}_0^{(1)}. \quad (112)$$

The subscript I refers to the labels of the annihilation operators in $f_{LI}\mathcal{H}_0^{(1)}$, or to the labels of the creation operators in $f_{IR}\mathcal{H}_0^{(1)}$,

$$\Delta_{LIR} = \frac{\Delta_{LI} - \Delta_{IR}}{\Delta_{LI}^2 + \Delta_{IR}^2 - \Delta_{LR}^2}. \quad (113)$$

When the denominator in Δ_{LIR} vanishes, the difference of the form factors, $f_{LR} - f_{LI}f_{IR}$, vanishes too. The solution for the third-order terms reads

$$\begin{aligned}\mathcal{H}^{(3)} = {} & (f_{LR} - f_{LI}f_{IR})\Delta_{LIR} \\ & \times [\Delta_{IJR}\mathcal{H}_0^{(1)}\mathcal{H}_0^{(1)}\mathcal{H}_0^{(1)} + \mathcal{H}_0^{(1)}\mathcal{H}_0^{(2)}] \\ & + (f_{LR} - f_{LJ}f_{JR})\Delta_{LJR} \\ & \times [\Delta_{LIJ}\mathcal{H}_0^{(1)}\mathcal{H}_0^{(1)}\mathcal{H}_0^{(1)} + \mathcal{H}_0^{(2)}\mathcal{H}_0^{(1)}] \\ & - (f_{LR} - f_{LI}f_{IJ}f_{JR})\Delta_{LIJR}\mathcal{H}_0^{(1)}\mathcal{H}_0^{(1)}\mathcal{H}_0^{(1)}, \end{aligned} \quad (114)$$

where

$$\Delta_{LIJR} = \frac{(\Delta_{LJ} - \Delta_{JR})\Delta_{LIJ} + (\Delta_{LI} - \Delta_{IR})\Delta_{IJR}}{\Delta_{LI}^2 + \Delta_{IJ}^2 + \Delta_{JR}^2 - \Delta_{LR}^2}. \quad (115)$$

Subscripts I and J refer to the intermediate configurations of the particle operators involved in the interaction terms. Solutions for the Hamiltonian terms of orders higher than third are obtained following the pattern shown by these examples.

B. Counterterms

Computation of the counterterms in the initial Hamiltonian, $H^{m_g r}_{\mathrm{QCD}\mathcal{C}}$, is outlined above Eq. (103). It is illustrated below in the case of the quark self-interaction, which contributes to the quark bound-state dynamics. The self-interaction appears first in the second-order effective Hamiltonian in Eq. (112). The right-hand side contains a product of two first-order terms, $\mathcal{H}_0^{(1)}\mathcal{H}_0^{(1)}$. Consider the part of that product corresponding to Fig. 1, where a quark 1 emits and absorbs gluons 2. The part involves the product $b_1^\dagger a_2 b_{1'}\, b_{\tilde{1}}^\dagger a_{\tilde{2}}^\dagger b_{\tilde{1}'}$. Commuting $b_{1'}$ to the right of $b_{\tilde{1}}^\dagger$ and a_2 to the right of $a_{\tilde{2}}^\dagger$ yields $\delta_{1'\tilde{1}}\delta_{2\tilde{2}}$, according to the commutation rules specified in Eq. (31). Integration $\int[\tilde{1}\,\tilde{2}\,\tilde{1}']$ results in the operator $\hat{C} = \int[1] c_1 \hat{b}_1^\dagger \hat{b}_1$, with the coefficient

$$c_1 = \frac{g^2 C_F}{p_1^+}\int \frac{dx d^2k^\perp}{2(2\pi)^3}(f_{LR} - f_{LI}f_{IR})\frac{N}{D} r^2_{qg,p}. \quad (116)$$

The numerator N and denominator D are given in Eqs. (83) and (84) with $p = p_1$ and $P^- - Q^- = p_1^-$. Integration variables x and $k^\perp$ are defined below Eq. (84). The color factor C_F equals 4/3 for $SU(3)$. The operators $\hat{b}_1^\dagger$ and $\hat{b}_1$ carry the same quantum numbers 1. Therefore, Eq. (109) implies $f_{LR} = 1$. In contrast, the momenta p_2 and $p_{1'}$ of the contracted quark and gluon operators yield $f_{LI} = f_{IR} = \exp\{-(s/p_1^+)^2[m_1^2 - (p_2 + p_{1'})^2]^2\}$, which results in

$$f_{LI}f_{IR} = \exp\{-2(sD/p_1^+)^2/[x(1-x)]^2\}. \quad (117)$$

This factor vanishes for small x and large $k^\perp$. Therefore, the integrand involving $f_{LI}f_{IR}$ contributes a finite quantity to the coefficient $c_1(s>0)$ in the limit $r \to 0$.

The only source of severe small-x singularity in the coefficient c_1 is the term $4(k^{\perp 2}+m_g^2)/x^2$ in the numerator N, in the integrand with $f_{LR}=1$. This term is also singular as a function of $k^\perp \to \infty$. Both singularities are regulated by the factor $r_{qg,p}^2$. When the regularization is lifted by going to the limit $r \to 0$, the quark self-interaction diverges to negative infinity proportionally to $1/r$. But the solution for $\mathcal{H}^{(2)}$ in Eq. (112) contains also the term $f_{LR}\mathcal{H}_0^{(2)}$, where $\mathcal{H}_0^{(2)}$ is the initial condition for solving the RGPEP equation. The initial condition can be set to contain a part canceling the divergent dependence of the self-interaction on r in the effective Hamiltonian. The diverging integrand is already calculated in Sec. IV C. We can remove that divergence from the effective Hamiltonian by including an opposite term, called the self-interaction counterterm, in $\mathcal{H}_0^{(2)}$. The inclusion amounts to the replacement of the integrand $f_{LR}N/D$ in the expression for the effective Hamiltonian by $f_{LR}[N/D+4\sigma_q^{(2)}]$, where $\sigma_q^{(2)}$ is given in Eq. (87). This is how the RGPEP explains the guess made in the scattering theory in Sec. IV C. The resulting integrand,

$$f_{LR}[N/D+4\sigma_q^{(2)}] = -(4m_1^2+2m_g^2)/D, \quad (118)$$

integrates in the limit $r \to 0$ to a logarithmically divergent term,

$$-\delta m_{1\,\mathrm{ln}}^2 = -\frac{g^2 C_F}{(2\pi)^2} m_1^2[\ln(2r^2m_1^2)+\gamma_E/2]. \quad (119)$$

Consequently, the Hamiltonian $\mathcal{H}_0^{(2)}$ is further supplied with a logarithmic mass-squared counterterm $\delta m_{1\,\mathrm{ln}}^2$. This means that not only the self-interaction divergence is removed from the effective Hamiltonian with finite s but also that a choice is made for the finite part of the counterterm. The choice made here is adopted to obtain the result that the quark masses appearing in the eigenvalue problems for the finite second-order effective Hamiltonian, such as in Eq. (125) in Sec. VII B, are the ones introduced in H_f in Eq. (38). A similar analysis can be carried out for the gluon self-interaction counterterms, but we do not discuss the eigenvalue equations for gluonia in this work.

VII. GLUON MASS AND SMALL x IN HEAVY QUARKONIA

To discuss the cancellation of gluon singularities $\sim 1/x^2$ in the QCD eigenvalue problems with $m_g > 0$, we consider the theory with only heavy quarks, i.e., the theory in which Λ_{QCD} of the RGPEP scheme is set to values much smaller than the quark masses, cf. [6]. In such a theory, the running coupling constant for the parameter $s \lesssim 1/m$, where m is the quark mass, can be made sufficiently small for using the weak-coupling expansion of Sec. VI to approximately compute the Hamiltonian H_s in Eq. (100).

A. Dynamics in dominant component

According to Eqs. (101) and (102), the quarkonium eigenstate of the Hamiltonian H_s has infinitely many components with various numbers of the effective quarks, antiquarks, and gluons, cf. [16]. However, the interaction terms that change the number of effective particles include a small coupling constant and the vertex factors f^s, which implies that a quarkonium eigenstate of the lowest mass is dominated by its quark-antiquark component,

$$|P\rangle = \int [12]\psi_P^s(1,2)|12\rangle_s. \quad (120)$$

The effective basis states, $|12\rangle_s = b_{s1}^\dagger d_{s2}^\dagger|0\rangle$, and the wave function $\psi_P^s(1,2)$ correspond to the RGPEP evolution parameter s. The Hamiltonian eigenstate is labeled by P to indicate that $|P\rangle$ is simultaneously an eigenstate of the operators $\hat{P}^+$ and $\hat{P}^\perp$ with the eigenvalues P^+ and $P^\perp$, respectively. In other words, the matrix elements of the Hamiltonian H_s between the effective quark-antiquark basis states are assumed to form the central part of the window matrix whose eigenstates approximately represent quarkonia in the theory; the concept of window Hamiltonians in the RGPEP is described in detail in Sec. 4.3 of [12]. Its essence is that one identifies a small subspace of the space of states, called the window, in which the eigenstate in question can be approximated well by neglecting its components outside that subspace. The assumption of dominance of the two-body component is successfully used in interpreting quarkonium data [3,61–67]. In our approach, the two-body component can, as a first approximation, be described by solutions to the eigenvalue problem for the two-body part of the operator H_s whose form is

$$H_{Q\bar{Q}} = \int [122'1']_s \langle 12|H_s|1'2'\rangle_s b_{s1}^\dagger d_{s2}^\dagger d_{s2'} b_{s1'}. \quad (121)$$

The integration includes summing over the discrete quantum numbers. Without losing generality, the wave function is given the form

$$\psi_P^s(1,2) = P^+\tilde{\delta}_{12.P}\psi_{12}^s(x_1,k_{12}^\perp), \quad (122)$$

where $\tilde{\delta}_{12.P}$ constrains the quarks to carry together P^+ and $P^\perp$, cf. Eq. (42). The relative momentum variables x_1 and $k_{12}^\perp$ are defined as in the case of an interaction term in Eq. (46). The variables are defined in the same way no matter what two-body system one considers. The eigenvalue

equation for the wave function reads

$$\int [1'2']_s \langle 12|H_s|1'2'\rangle_s \psi^s_P(1',2') = E\psi^s_P(1,2), \quad (123)$$

where $E(P^+,P^\perp) = (P^{\perp 2}+M^2)/P^+$ is the eigenvalue, and M denotes the quarkonium mass, $E(M,0^\perp) = M$. In the leading order of the weak-coupling expansion, $H_s = H_f + H_s^{(2)}$, where $H_s^{(2)}$ is obtained from $\mathcal{H}^{(2)}$ of Eq. (112) using Eq. (100). Hence,

$$ {}_s\langle 12|H_s^{(2)}|1'2'\rangle_s = \langle 12|\mathcal{H}^{(2)}|1'2'\rangle. \quad (124)$$

This direct relationship between the effective-particle matrix elements of H_s and matrix elements of $\mathcal{H}$ is a general feature of solutions to the RGPEP equation.

All matrix elements of all Hamiltonian terms acting in the effective two-body space include the factor $\tilde{\delta}_{12.1'2'}$, which universally secures conservation of P^+ and $P^\perp$ by the dynamics. Unless it needs to be mentioned, this δ function is omitted. We also point out that there are no first-order terms in Eq. (124), corresponding to $\mathcal{H}^{(1)}$ in Eq. (104) because they change the number of quanta. Such changes in the effective dynamics are addressed in Sec. IX, after the pertinent analysis of the dynamics conserving the number of effective particles.

B. $Q\bar{Q}$ Hamiltonian matrix

According to Eqs. (112) and (124), the $Q\bar{Q}$ Hamiltonian matrix in Eq. (123) has matrix elements

$$ {}_s\langle 12|H_s|1'2'\rangle_s = \langle 12|[\mathcal{H}_f + f_{LR}\mathcal{H}_r^{(2)} - f_{LI}f_{IR}\Delta_{LIR}\mathcal{H}_0^{(1)}\mathcal{H}_0^{(1)}]|1'2'\rangle, \quad (125)$$

where

$$\mathcal{H}_r^{(2)} = \mathcal{H}_0^{(2)} + \Delta_{LIR}\mathcal{H}_0^{(1)}\mathcal{H}_0^{(1)}. \quad (126)$$

In Eq. (125), the factor $f_{LI}f_{IR}$ suppresses all singularities generated by the interactions. The factor f_{LR} does not produce such suppression for the singularities in $\mathcal{H}_r^{(2)}$. Instead, these singularities cancel out. The divergent quark self-interactions, resulting from $\Delta_{LIR}\mathcal{H}_0^{(1)}\mathcal{H}_0^{(1)}$ in $\mathcal{H}_r^{(2)}$, are canceled by the self-interaction counterterms introduced in $\mathcal{H}_0^{(2)}$, as described in Sec. VI B. The singular gluon-exchange term in $\mathcal{H}_r^{(2)}$, generated by $\Delta_{LIR}\mathcal{H}_0^{(1)}\mathcal{H}_0^{(1)}$ in Eq. (126), diverges like $1/x^2$, where x is the fraction carried by the exchanged gluon of the bound-state momentum P^+. In that case, the opposite divergence is contributed by the gluon seagull term in $\mathcal{H}_0^{(2)}$. Details of the latter cancellation are described in the next section.

C. Cancellation of $1/x^2$ in dominant component

We now describe how the singularities $\sim 1/x^2$ in the gluon-exchange parts of $\Delta_{LIR}\mathcal{H}_0^{(1)}\mathcal{H}_0^{(1)}$ and $\mathcal{H}_0^{(2)}$, both contained in $\mathcal{H}_r^{(2)}$, cancel each other in the leading-order bound-state eigenvalue problem for the quarkonium dominant component. The cancellation occurs despite that m_g is not zero.

Omitting the momentum-conserving $\tilde{\delta}$ function, the exchange matrix elements $\langle 12|f_{LR}\mathcal{H}_r^{(2)}|1'2'\rangle$ form the matrix

$$h^f = j_{1\mu}h^{\mu\nu}j_{2\nu}, \quad (127)$$

where the superscript f indicates inclusion of the factor f_{LR} in the tensor $h^{\mu\nu}$. The quark and antiquark currents, obtained from Eq. (17), are denoted by j_1 and j_2, and

$$h^{\mu\nu} = f_{LR}(d^{\mu\nu}\Delta_{LIR} + \eta^\mu\eta^\nu/q^{+2})r_{\bar{1},\underline{1}g}r_{\bar{2},\underline{2}g}. \quad (128)$$

This formula is illustrated in Fig. 2 using diagrams. The two upper diagrams correspond to the gluon exchange interaction described by the term containing $d^{\mu\nu}$ in the bracket of Eq. (128). The lower diagram corresponds to the term containing $\eta^\mu\eta^\nu$ in the bracket, which is due to the seagull. Besides the labels 1, 1′, 2, and 2′, the notation we use includes labels $\bar{1}$, $\underline{1}$, $\bar{2}$, and $\underline{2}$, defined by the equations $\bar{1} = 1'\theta(p_{1'}^+ - p_1^+) + 1\theta(p_1^+ - p_{1'}^+)$, $\underline{1} = 1'\theta(p_1^+ - p_{1'}^+) + 1\theta(p_{1'}^+ - p_1^+)$ for the quarks and by similar equations for the antiquarks. There are two momentum transfers, one for the quark and another one for the antiquark,

$$q_1 = p_{\bar{1}} - p_{\underline{1}}, \qquad q_2 = p_{\bar{2}} - p_{\underline{2}}, \quad (129)$$

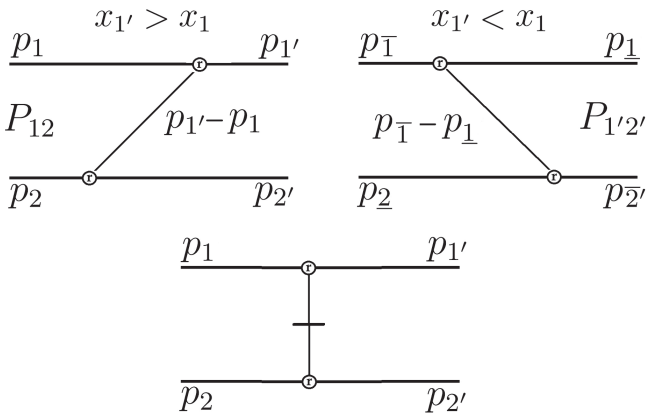


FIG. 2. Notation for the Hamiltonian matrix elements $\langle 12|f_{LR}\mathcal{H}_r^{(2)}|1'2'\rangle$ involving two complementary orderings of the gluon exchange and the seagull term. Small circles with letter r symbolize the regularization factors $r_{\bar{1},\underline{1}g}$ and $r_{\bar{2},\underline{2}g}$ in the tensor $h^{\mu\nu}$ in Eqs. (127) and (128).

where p refers to the four-momentum of a quark of mass m_1 or an antiquark of mass m_2. We use

$$\rho_1 = m_g^2 - q_1^2, \qquad \rho_2 = m_g^2 - q_2^2. \tag{130}$$

For the components $+$ and $\perp$, $q_1 = q_2 \equiv q$. The minus components of q_1 and q_2 differ when $p_1^- + p_2^- \neq p_{1'}^- + p_{2'}^-$. According to Eq. (113),

$$\Delta_{LIR} = -\frac{1}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right). \tag{131}$$

The tensor $d^{\mu\nu}$ multiplying Δ_{LIR} in Eq. (128),

$$d^{\mu\nu} = -g^{\mu\nu} + \eta^\mu \eta^\nu (\rho_1 + \rho_2)/(2q^{+2}), \tag{132}$$

results from summing over three polarizations of the exchanged gluon, two transverse and one longitudinal. The latter is contributed by the quanta of the auxiliary field ϕ.

The term $-g^{\mu\nu}$ on the right-hand side of Eq. (132) leads to the quark-antiquark Coulomb potential with the Breit–Fermi spin dependence, described by the matrix

$$h^C = j_{1\mu} f_{LR} (-g^{\mu\nu}) \Delta_{LIR} j_{2\nu} r_{\bar{1},\underline{1}g} r_{\bar{2},\underline{2}g}. \tag{133}$$

The potential is slightly altered to the Yukawa form with the gluon mass parameter m_g. The long-distance Coulomb form is recovered when $m_g \to 0$. The Coulomb term is also altered at large momentum transfers, by the factor f_{LR} of Eq. (109) [37,45,68],

$$f_{LR} = e^{-[s(\rho_1 - \rho_2)/q^+]^2}. \tag{134}$$

The alteration occurs because the effective particles of size s do not interact at short distances as the pointlike particles do. Since the Coulomb term is finite, one can lift the regularization in it, sending $r \to 0$. Thus, the product of the regularization factors $r_{\bar{1},\underline{1}g} r_{\bar{2},\underline{2}g}$ is replaced by 1. The resulting matrix, denoted by h_C, is called the effective Coulomb interaction.

The second term on the right-hand side of Eq. (132) is obtained using the quark current conservation, as described in the case of Eq. (76) in Sec. IV B. That term diverges like $1/x^2$ when $x = q^+/P^+$ tends to 0. Similar divergence appears in the seagull term $\sim \eta^\mu \eta^\nu$, the second term in the bracket in Eq. (128). The terms $\sim 1/x^2$ in the tensor $h^{\mu\nu}$ sum up to

$$h_\eta^{\mu\nu} = f_{LR}\left[\frac{(\rho_1 + \rho_2)^2}{4\rho_1\rho_2} - 1\right]\frac{\eta^\mu \eta^\nu}{q^{+2}} r_{\bar{1},\underline{1}g} r_{\bar{2},\underline{2}g}. \tag{135}$$

Using the identity $\rho_1 - \rho_2 = (x_{1'} - x_1)(\mathcal{M}_{12}^2 - \mathcal{M}_{1'2'}^2)$, one arrives at the expression

$$h_\eta^{\mu\nu} = f_{LR}\frac{(\mathcal{M}_{12}^2 - \mathcal{M}_{1'2'}^2)^2}{\rho_1\rho_2}\frac{\eta^\mu \eta^\nu}{4P^{+2}} r_{\bar{1},\underline{1}g} r_{\bar{2},\underline{2}g}, \tag{136}$$

which demonstrates that the interaction is free from the small-x divergences, as promised. We denote the matrix involving $j_{1\mu} h_\eta^{\mu\nu} j_{2\nu}$ by h^η. The regularization can be lifted in h^η as in the effective Coulomb interaction.

The product $\rho_1\rho_2$ in the denominator of h^η in Eq. (136) suggests that, in the limit $m_g \to 0$, a singularity $\sim 1/q^4$ is obtained. However, the difference $\mathcal{M}_{12}^2 - \mathcal{M}_{1'2'}^2$ tends to 0 when q^+ and $q^\perp$ tend to zero. Therefore, the net dependence of h^η on $q \to 0$ in the limit $m_g \to 0$ is of the type $1/q^2$ instead of $1/q^4$. The former is integrable in the eigenvalue equation with the measure $dq^+ d^2q^\perp$.

The result in Eq. (136) implies that the need for the counterterm identified in Eq. (36) of [6] is eliminated, cf. [13], simplifying the formalism. Instead of the counterterm, the divergence is countered by the contribution of quanta of the auxiliary field ϕ for any value of m_g. Moreover, the cancellation mechanism is rooted in using m_g together with ϕ, which is set up already at the stage of defining the bare theory Hamiltonian in Eq. (21). It thus appears simpler to implement than the program of computing counterterms in a sequence of successive approximations [22].

The factor $\eta^\mu \eta^\nu$ implies that h^η is diagonal in the quark spins. As such, it could have a classical limit. However, it does not have any counterpart among classical interactions because it vanishes for $|\vec{k}_{12}| = |\vec{k}_{1'2'}|$, which is the on-shell condition for conservative classical limits of quantum interactions. The magnitude of matrix elements of h^η for $|\vec{k}_{12}| \neq |\vec{k}_{1'2'}|$ is much smaller than the magnitude of the Coulomb interaction matrix elements for the same momentum transfers, roughly by the factor $\sim (k/m)^2$, where k refers to the typical relative momentum of the quarks in a quarkonium and m to the scale of their masses. As a result,

$$h^f = h^C + h^\eta, \tag{137}$$

where the matrix h^η can be neglected in comparison with h^C in the case of nonrelativistic relative motion of quarks.

D. Finite $Q\bar{Q}$ eigenvalue problem with $m_g > 0$

The bound-state eigenvalue problem in Eq. (123), for the effective quark-antiquark Hamiltonian matrix of Eq. (125), written in terms of the wave function $\psi_{12}^s(x_1, k_{12}^\perp)$, defined in Eq. (122) and abbreviated to φ_k for simplifying the notation, omitting the discrete quark labels, has the form

$$\sum_{i=1}^{2} \frac{m_i^2 + k_{12}^{\perp 2} + \sigma_i}{p_i^+}\varphi_k + \int_{k'} \frac{H_{kk'}}{P^+}\varphi_{k'} = \frac{M^2}{P^+}\varphi_k. \tag{138}$$

Subscript $i = 1$ corresponds to the quark and $i = 2$ to the antiquark. The self-interaction terms σ_i result solely from

$-\langle 12|f_{LI}f_{IR}\Delta_{LIR}\mathcal{H}_0^{(1)}\mathcal{H}_0^{(1)}|1'2'\rangle$, since $\Sigma_i + \delta m_{i\ln}^2 = 0$ as a consequence of the adjustment of the mass-squared counterterms derived in Sec. VI B, see Eqs. (118) and (119). The gluon momentum variables in the self-interactions are $x = p_g^+/p_i^+$, ranging from 0 to 1, and $k^\perp = p_g^\perp - x p_i^\perp$, which is unlimited. We obtain

$$\sigma_i = \frac{g^2 C_F}{16\pi^3}\int \frac{dx d^2k^\perp}{x(1-x)}\frac{j_{i\mu}d^{\mu\nu}j_{i\nu}^*}{p_i^+\Delta p_i^-}e^{-2(s\Delta p_i^-)^2}, \tag{139}$$

where $j_{1\mu} = \bar{u}_{p_1}\gamma_\mu u_{p_q}$ or $j_{2\mu} = \bar{v}_{p_q}\gamma_\mu v_{p_2}$, and

$$\Delta p_i^- = p_g^- + p_q^- - p_i^- = \rho_i/p_g^+, \tag{140}$$

$$d^{\mu\nu} = -g^{\mu\nu} + \eta^\mu\eta^\nu\rho_i/p_g^{+2}, \tag{141}$$

with $\rho_i = m_g^2 - (p_i - p_q)^2$. The kernel $H_{kk'}$ is

$$H_{kk'} = h_{kk'}^f + h_{kk'}^{ff}, \tag{142}$$

where $h_{kk'}^f$ denotes the matrix elements of h^f in Eq. (137). The kernel $h_{kk'}^{ff}$ comes from the gluon-exchange in $-\langle 12|f_{LI}f_{IR}\Delta_{LIR}\mathcal{H}_0^{(1)}\mathcal{H}_0^{(1)}|1'2'\rangle$ of Eq. (125). The superscript ff indicates the inclusion of the factors $f_{LI}f_{IR}$. We have

$$h_{kk'}^{ff} = -g^2 C_F e^{-s^2(\rho_1^2+\rho_2^2)/p_g^{+2}}\frac{\rho_1+\rho_2}{2\rho_1\rho_2} \times j_{1\mu}\left[-g^{\mu\nu} + \eta^\mu\eta^\nu\frac{\rho_1+\rho_2}{2p_g^{+2}}\right]j_{2\nu}, \tag{143}$$

with ρs defined in Eq. (130). The color factor $C_F = 4/3$, the same as in the self-interaction terms σ_i, is obtained for $Q\bar{Q}$ singlets of the color $SU(3)$. The integral over k' denotes

$$\int_{k'} = \int \frac{dx_{1'}d^2k_{1'2'}^\perp}{16\pi^3 x_{1'}x_{2'}} = \int \frac{d^3k'}{16\pi^3}\frac{\mathcal{M}_{1'2'}}{E_{1'}E_{2'}}, \tag{144}$$

where $E_{i'} = \sqrt{m_{i'}^2 + k'^{\perp 2} + k'^{z2}}$, and the components of the three-vector $\vec{k}'$ are $k'^\perp = k_{1'2'}^\perp$, $k'^z = x_{1'}p_{2'}^- - x_{2'}p_{1'}^-$. In the rest frame of constituents $1'$ and $2'$, the quark carries a three-momentum $\vec{k}'$, and the antiquark carries $-\vec{k}'$. Analogous change of variables applies for the quarks carrying momenta p_1 and p_2 in the same kernel, defining the relative momentum $\vec{k}$. Such relative momenta can be defined in any two-body system.

VIII. $Q\bar{Q}$ DYNAMICS FOR $m_g \to 0$

The eigenvalue problem in Eq. (138) involves effective interaction terms depending on the mass parameter m_g, which is meant to be sent to 0 at the end of a calculation. The functions f_{LI} and f_{IR}, obtained by solving the RGPEP Eq. (99), behave as

$$\exp[-(s\Delta p^-)^2] = \exp[-(s\rho/p_g^+)^2]. \tag{145}$$

Their dependence on ρ/x, where $x = p_g^+/P^+$, makes them decrease as m_g^2/x increases. They vanish faster than any power of x when the gluon x tends to 0. The inverse powers of x are thus harmless in their presence. However, when $m_g \to 0$, the small-x divergences reappear as logarithms of m_g, for the divergences $\sim 1/x^2$ cancel out. We show, keeping track of the details, that the logarithms of m_g cancel out as well, but only in the eigenvalue equation for the colorless quark-antiquark states. Further useful features of the limit $m_g \to 0$ are discussed in subsequent sections.

A. Terms with $-g^{\mu\nu}$

The integrand of the self-interaction σ_i in Eq. (139) and the gluon exchange kernel $h_{kk'}^{ff}$ in Eq. (143) are sums of two terms, one with $-g^{\mu\nu}$ and another one with $\eta^\mu\eta^\nu$. One can write

$$\sigma_i = \sigma_i^g + \sigma_i^\eta, \qquad h_{kk'}^{ff} = h_{kk'}^{ffg} + h_{kk'}^{ff\eta}. \tag{146}$$

The division facilitates the relativistic analysis that follows. All the terms with $-g^{\mu\nu}$ are finite when $m_g \to 0$. In the case of self-interaction,

$$\frac{\sigma_i^g}{p_i^+} = \frac{g^2 C_F}{(4\pi)^2}\frac{m_i^2}{p_i^+}\int_0^\infty dz\left[1 - \frac{6}{z+1} + \frac{1}{(z+1)^2}\right]e^{-(az)^2}, \tag{147}$$

where $a = \sqrt{2}sm_i^2/p_i^+$, the variable $z = (E_i^2 - m_i^2)/m_i^2$, and $E_i = E_{qi} + E_g$, according to the choice of integration variables introduced in Eq. (144). The integral converges for $s > 0$ and, for small s, yields the leading term $(p_i^+/m_i^2)\sqrt{2\pi}/(4s)$, which contributes to Eq. (138) through

$$\frac{\sigma_1^g}{p_1^+} + \frac{\sigma_2^g}{p_2^+} = \frac{g^2 C_F}{(4\pi)^2}\frac{\sqrt{2\pi}}{2s}. \tag{148}$$

There are also contributions depending on s logarithmically. The magnitude of the self-interaction terms can be estimated in the quarkonium rest frame assuming a nonrelativistic relative motion of the quarks with equal masses, $m_1 = m_2 = m$, for $s \sim 1/m$. In these circumstances, the complete result on the right-hand side in Eq. (148), including the integral over z of all terms in Eq. (147), is on the order of $-\alpha m/3$, not exceeding about 10% reduction of the bound-state mass eigenvalue for $\alpha \sim 0.3$ favored by the phenomenology of quarkonia [2,3]. In the part of $h_{kk'}^{ff}$ containing $-g^{\mu\nu}$, in Eq. (143),

$$h^{ffg}_{kk'} = g^2 C_F e^{-s^2(\rho_1^2+\rho_2^2)/q^{+2}} \frac{\rho_1+\rho_2}{2\rho_1\rho_2} j_1^\mu j_{2\mu}, \quad (149)$$

the product of quark currents is finite for the quark states with finite invariant masses. The ρs depend only on the quark momenta. Large values of ρs are suppressed by the factors ff, not indicated. In comparison with the effective Coulomb interaction, Eqs. (133) and (134), the term $h^{ffg}_{kk'}$ is smaller by the factor of $\exp[-2s^2\rho_1\rho_2/q^{+2}]$. It is included in the effective dynamics in combination with the finite terms containing $\eta^\mu\eta^\nu$ in Sec. VIII D.

B. Terms with $\eta^\mu\eta^\nu$

The terms proportional to the tensor $\eta^\mu\eta^\nu$ in σ_i of Eq. (139) and in the kernel $h^{ff}_{kk'}$ of Eq. (143), are

$$\sigma_i^\eta = \frac{g^2 C_F}{16\pi^3} \int \frac{dx d^2k^\perp}{x(1-x)} \frac{j_i^+ j_i^{+*}}{x p_i^{+2}} e^{-2(s\rho_i/p_g^+)^2}, \quad (150)$$

$$h^{ff\eta}_{kk'} = -g^2 C_F \frac{(\rho_1+\rho_2)^2}{4\rho_1\rho_2} \frac{j_1^+ j_2^+}{p_g^{+2}} e^{-s^2(\rho_1^2+\rho_2^2)/p_g^{+2}}. \quad (151)$$

The self-interaction does not depend on the quark spin. It is a number with the engineering dimension of momentum squared, and it multiplies the wave function φ_k. The kernel $h^{ff\eta}_{kk'}$ is dimensionless but it is integrated together with the wave function $\varphi_{k'}$ using the measure of Eq. (144), whose engineering dimension is momentum squared. The result of integration is independent of the quark spin. We obtain

$$\sigma_i^\eta \varphi_k = \frac{g^2 C_F}{16\pi^3} \varphi_k \int dx d^2k^\perp \frac{4}{x^2} e^{-2(s\rho_i/p_g^+)^2}, \quad (152)$$

$$\int_{k'} h^{ff\eta}_{kk'} \varphi_{k'} = -g^2 C_F \int \frac{dx_{1'} d^2k^\perp_{1'2'}}{16\pi^3 x_{1'} x_{2'}} \frac{(\rho_1+\rho_2)^2}{4\rho_1\rho_2} \times \frac{4\sqrt{x_1 x_{1'} x_2 x_{2'}}}{(x_{1'}-x_1)^2} e^{-s^2(\rho_1^2+\rho_2^2)/p_g^{+2}} \varphi_{k'}. \quad (153)$$

These interactions contribute to the eigenvalue problem the sum

$$S = \left(\frac{\sigma_1^\eta}{p_1^+} + \frac{\sigma_2^\eta}{p_2^+}\right)\varphi_k + \frac{1}{P^+}\int_{k'} h^{ff\eta}_{kk'} \varphi_{k'}. \quad (154)$$

To show that the sum is finite in the limit $m_g \to 0$, we introduce $\sigma^\eta_{12} = \sigma^\eta_1/x_1 + \sigma^\eta_2/x_2$ and write

$$P^+ S = \sigma^\eta_{12}\varphi_k + \int_{k'} h^{ff\eta}_{kk'}\varphi_{k'} = \left(\sigma^\eta_{12} + \int_{k'} h^{ff\eta}_{kk'}\right)\varphi_k + \int_{k'} h^{ff\eta}_{kk'}(\varphi_{k'} - \varphi_k), \quad (155)$$

where the wave function φ_k is a vector with four components corresponding to the four possible combinations of the quark spins. The kernel $h^{ff\eta}_{kk'}$ is a matrix proportional to $I_{4\times4}$. The wave function with argument $\vec{k}$ is subtracted from the wave function with argument $\vec{k}'$ under the integral over $\vec{k}'$. Simultaneously, the integral of the kernel over k' is multiplied by the wave function φ_k and added to the self-interaction terms.

1. Self-interaction

We now discuss the resulting self-interaction terms including the kernel integrals, which multiply the wave function φ_k on the right-hand side of Eq. (155),

$$\tilde{\sigma} = \tilde{\sigma}_1 + \tilde{\sigma}_2 = \sigma^\eta_{12} + \int_{k'} h^{ff\eta}_{kk'}. \quad (156)$$

Integration of the kernel over $\vec{k}'$ is split into two parts corresponding to the two self-interaction terms, $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$. The quark self-interaction $\tilde{\sigma}_1$, illustrated with the upper left diagram of Fig. 3, is paired with the integral of the kernel $h^{ff\eta}_{kk'}$ over $p^+_{1'}$ from 0 to p^+_1, illustrated by the upper right diagram in the same figure. The integral in the antiquark self-interaction, shown graphically in the lower left diagram, is paired with the integral of the kernel $h^{ff\eta}_{kk'}$ over $p^+_{2'}$ from 0 to p^+_2, represented by the lower right diagram. The integrals of the quark self-interaction and the paired kernel part behave the same way for $m_g \to 0$ as the integrals of the antiquark self-interaction and its paired kernel part. Only the quark terms are described in more detail. Using Eqs. (152) and (153), the sum of the pertinent integrals is

$$\tilde{\sigma}_1 = \frac{\sigma^\eta_1}{x_1} + \int_{k'} h^{ff\eta}_{kk'}\theta(x_1 - x_{1'}) \quad (157)$$

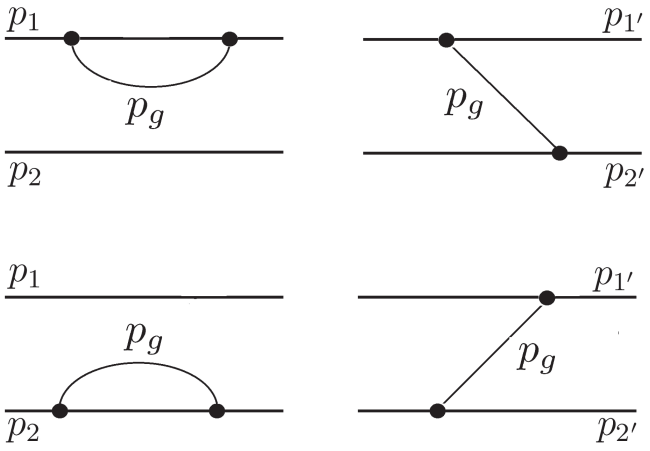


FIG. 3. Diagrams illustrating the self-interaction terms σ^η_i of Eq. (152) together with the associated parts of the interaction kernel $h^{ff\eta}_{kk'}$ of Eq. (153), arranged according to Eq. (155). The black blobs represent the RGPEP vertex factors f_{LI} and f_{IR}.

$$= \frac{G}{x_1}\int_0^1 dx d^2k^\perp \frac{4}{x^2} e^{-2(s\rho_1/p_g^+)^2} - G\int_0^{x_1} \frac{dx_{1'}d^2k^\perp_{1'2'}}{x_{1'}x_{2'}} \frac{(\rho_1+\rho_2)^2}{4\rho_1\rho_2} \times \frac{4\sqrt{x_1x_{1'}x_2x_{2'}}}{(x_{1'}-x_1)^2} e^{-s^2(\rho_1^2+\rho_2^2)/p_g^{+2}}, \quad (158)$$

with $G = g^2C_F/(16\pi^3)$. The integrals over transverse momenta extend over the entire transverse plane. The limit of $\tilde{\sigma}_1$ when $m_g \to 0$ is found by changing variables in the exchange integral,

$$x_g = x_1 - x_{1'} = xx_1, \qquad p_g^\perp = xk_{12}^\perp + k^\perp, \quad (159)$$

$$x_{q'} = x_{1'} = (1-x)x_1, \qquad p_{q'}^\perp = (1-x)k_{12}^\perp - k^\perp, \quad (160)$$

$$x_{\bar{q}'} = x_{2'} = x_2 + xx_1, \qquad p_{\bar{q}'}^\perp = -k_{12}^\perp + p_g^\perp. \quad (161)$$

The Jacobian is x_1, and

$$\int_0^{x_1} \frac{dx_{1'}d^2k^\perp_{1'2'}}{x_{1'}x_{2'}} = \int_0^1 \frac{dxd^2k^\perp}{(1-x)(x_2+xx_1)}. \quad (162)$$

In the exchange integral, ρ_1 and ρ_2 are the ones defined in Eq. (130). Proceeding in analogy to Eqs. (135) and (136), using the identity $\rho_1 - \rho_2 = -xx_1(\mathcal{M}_{12}^2 - \mathcal{M}_{1'2'}^2)$, we obtain

$$x_1\tilde{\sigma}_1/G = \int_0^1 dx d^2k^\perp \frac{4}{x^2} e^{-2(s\rho_1/p_g^+)^2} - \int_0^1 \frac{dxd^2k^\perp\sqrt{x_2}}{\sqrt{(x_2+xx_1)(1-x)}} e^{-s^2(\rho_1^2+\rho_2^2)/p_g^{+2}} \times \left[\frac{4}{x^2} + \frac{x_1^2(\mathcal{M}_{12}^2 - \mathcal{M}_{1'2'}^2)^2}{\rho_1\rho_2}\right]. \quad (163)$$

The discussion following Eq. (136) applies here as well, and the only terms potentially generating a divergence when $x \to 0$ for $m_g = 0$ are the ones with $4/x^2$, denoted by $\tilde{\sigma}_{1x}$. We substitute $q^\perp = k^\perp/\sqrt{x}$ and obtain

$$x_1\tilde{\sigma}_{1x}/G = \int_0^1 dx d^2q^\perp \frac{4}{x} e^{-s_1^2(\rho_1^2+\rho_2^2)/x^2} \times \left[e^{-s_1^2(\rho_1^2-\rho_2^2)/x^2} - \frac{1}{\sqrt{(1+xx_1/x_2)(1-x)}}\right], \quad (164)$$

where $s_1 = s/p_1^+$ and

$$\rho_1/x = \frac{q^{\perp 2} + m_1^2 x}{1-x} + \frac{m_g^2}{x}, \quad (165)$$

$$\rho_2/x = \frac{(q^\perp + \sqrt{x}k_{12}^\perp/x_2)^2 + m_2^2 x x_1^2/x_2^2}{1+xx_1/x_2} + \frac{m_g^2}{x}. \quad (166)$$

The factor $4/x$ in Eq. (164) appears to produce a logarithmic divergence in the limit $m_g \to 0$, since in that limit the terms m_g^2/x are eliminated from ρ_1/x and ρ_2/x. But when $m_g = 0$, for fixed x_1 and $k_{12}^\perp$, in the limit $x \to 0$, $\rho_1/x \to q^{\perp 2} + O(xm^2)$, $\rho_2/x \to q^{\perp 2} + O(\sqrt{x}m^2)$, and the exponential function inside the square bracket approaches 1 when $x \to 0$. The approach to 1 is secured because the exponential function outside the square bracket limits the range of integration over $q^\perp$. The second term in the square bracket in Eq. (164) approaches 1, too. Consequently, the square bracket vanishes when $x \to 0$, and the logarithmic divergence is absent. The self-interactions $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are finite in the limit $m_g \to 0$.

Assuming equal quark masses, m, $p_i^+ \sim x_i m$, $s \sim 1/m$ and neglecting $k_{12}^\perp$, a calculation shows that $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ amount in the eigenvalue problem to modifying the mass m by 5% or less when x_1 ranges from 0.2 to 0.8. Outside that range, the wave function φ_k is expected to be negligible.

2. Exchange

Now we describe the second term on the right-hand side of Eq. (155),

$$h^{ffη}\Delta\varphi = \int_{k'} h_{kk'}^{ffη}(\varphi_{k'} - \varphi_k), \quad (167)$$

in the limit $m_g \to 0$. Quark spins are conserved. The integration over $x_{1'}$ and $k_{1'2'}^\perp$ is carried out using the variables z and $q^\perp$,

$$x_{1'} = x_1 + z, \qquad k_{1'2'}^\perp = k_{12}^\perp + q^\perp, \quad (168)$$

for fixed x_1 and $k_{12}^\perp$. We introduce $s_P = s/P^+$ and write

$$h^{ffη}\Delta\varphi = \frac{-g^2C_F}{2(2\pi)^3}\int_{-x_1}^{x_2} dz \int d^2q^\perp e^{-s_P^2(\rho_1^2+\rho_2^2)/z^2} \times \frac{(\rho_1+\rho_2)^2}{4\rho_1\rho_2}\frac{4\sqrt{x_1x_{1'}x_2x_{2'}}}{x_{1'}x_{2'}z^2}(\varphi_{k'} - \varphi_k). \quad (169)$$

Using $p_1^\perp = x_1P^\perp + k_{12}^\perp$, $p_{1'}^\perp = x_{1'}P^\perp + k_{1'2'}^\perp$ etc., and introducing

$$q^\perp = \sqrt{|z|}k^\perp, \quad (170)$$

we obtain

$$\frac{\rho_1}{|z|} = \frac{m_g^2}{|z|} + \frac{m_1^2|z|/x_1^2 + (k^\perp - k_{12}^\perp s_z\sqrt{|z|}/x_1)^2}{1+z/x_1}, \quad (171)$$

$$\frac{\rho_2}{|z|} = \frac{m_g^2}{|z|} + \frac{m_2^2|z|/x_2^2 + (k^\perp + k_{12}^\perp s_z\sqrt{|z|}/x_2)^2}{1 - z/x_2}, \quad (172)$$

where s_z denotes the sign of z. The integral in Eq. (169) is finite for $m_g > 0$ because the exponential factor suppresses the divergence due to the factor $1/z^2$. We inspect how the integral behaves for m_g set to 0. The measure $d^2q^\perp$ provides a factor $|z|$, $d^2q^\perp = |z|d^2k^\perp$, and turns the singularity $\sim 1/z^2$ to $\sim 1/|z|$. The wave function difference $\varphi_{k'} - \varphi_k$ vanishes in the limit of z and $q^\perp$ tending to 0. But Eq. (170) implies that $q^\perp$ vanishes like $\sqrt{|z|}k^\perp$ when $|z|$ tends to 0, provided $k^\perp$ is limited. The latter is the case thanks to the exponential factor. Hence, the difference $\varphi_{k'} - \varphi_k$ eliminates the singularity $\sim 1/|z|$, and the exchange term is finite in the limit $m_g \to 0$.

C. Summary of the limit $m_g \to 0$

The renormalized effective two-body dynamics, encoded in the matrix elements ${}_s\langle 12|H_s^{(2)}|1'2'\rangle_s$ of Eq. (124), has the form of Eq. (138), with the specified self-interactions σ_i of Eq. (146) and kernel $H_{kk'}$ of Eq. (142),

$$\sum_{i=1}^{2}\frac{m_i^2 + k^{\perp 2} + \sigma_i^g + \sigma_i^\eta}{x_i}\varphi_k^s + \int_{k'}(h_{kk'}^{fg} + h_{kk'}^{f\eta} + h_{kk'}^{ffg} + h_{kk'}^{ff\eta})\varphi_{k'}^s = M^2\varphi_k^s. \quad (173)$$

The self-interactions σ_i^g are finite and in the limit $m_g \to 0$ contribute a small amount to the eigenvalue in comparison to the quark masses, see Sec. VIII A. The self-interaction term σ_i^η and the exchange term with the kernel $h_{kk'}^{ff\eta}$, i.e., the last term under the integral, are analyzed together in Sec. VIII B. The spin-conserving terms are the only ones involving logarithms of the gluon mass, m_g. It is shown in Secs. VIII B 1 and VIII B 2 that the logarithms of m_g in these terms cancel out. The term $h_{kk'}^{fg}$ provides the Coulomb interaction including the associated spin-dependent factors. The small-x singularities due to the gluon-exchange and seagull terms in $h_{kk'}^{f\eta}$ are shown in Sec. VII C to cancel out in the presence of $m_g > 0$, leaving behind a small correction when $m_g = 0$, Eq. (136) and h^η in Eq. (137). The kernel $h_{kk'}^{ffg}$ is suppressed in comparison to the Coulomb term, as explained in Sec. VIII A. The parameter s is reinstated in the wave function labeling as a reminder that Eq. (173) is obtained by solving the RGPEP Eq. (101), using the weak-coupling expansion of second-order.

D. Nonrelativistic approximation to $Q\bar{Q}$ interactions

Previous sections show that the relativistic eigenvalue problem of Eq. (173) has a finite limit when $m_g \to 0$. We now identify the main features of the dynamics for $m_g = 0$. To describe the motion of quarks in a quarkonium, we use their relative momenta, $\vec{k}$ and $\vec{k}'$, defined below Eq. (144). The average quark mass, $(m_1 + m_2)/2$, is denoted by m. We discuss the effective dynamics at $s \sim 1/m$. The vertex factors limit the difference between $|\vec{k}|$ and $|\vec{k}'|$ that the interactions can cause. The difference does not exceed the scale set by $1/s \sim m$. The smallest value of the invariant mass of quarks 1 and 2, $2m$, corresponds to $\vec{k} = 0$, or $k^\perp = 0$ and $x_1 = x_{10} = m_1/(m_1 + m_2)$, $x_{20} = m_2/m_1 + m_2)$. Since the invariant mass of the quarks rises with the magnitude of their relative momentum, and the off-diagonal matrix elements of the interaction involve a small coupling constant, the quark relative momentum in an eigenstate with a low mass eigenvalue does not exceed the scale of m [12]. The small term h^η in h^f in Eq. (137) is neglected. The quarkonium mass eigenvalue is written in the form $M = 2m + B$, in which B denotes the binding energy considered small in comparison to $2m$. Consequently, M^2 is approximated by $4m(m + B)$, neglecting B^2. The quarks' invariant mass reads

$$\mathcal{M}_{12}^2 = \frac{m_1^2 + k^{\perp 2}}{x_1} + \frac{m_2^2 + k^{\perp 2}}{x_2} \quad (174)$$

$$= \left[m_1 + m_2 + \frac{\vec{k}^2}{2\mu} + O(k^4/m^3)\right]^2, \quad (175)$$

where μ denotes the reduced mass for the quarks. Neglecting $B^2/(4m)$ in Eq. (173), one obtains

$$\frac{\vec{k}^2}{2\mu}\varphi_k^s + \int_{k'} V_{kk'}^C\varphi_{k'}^s + [V\varphi^s]_k = B\varphi_k^s, \quad (176)$$

with

$$4m[V\varphi^s]_k = \left(\frac{\sigma_1}{x_1} + \frac{\sigma_2}{x_2}\right)\varphi_k^s + \int_{k'} h_{kk'}^{ff}\varphi_{k'}^s. \quad (177)$$

The term with matrix $V^C = h^C/(4m)$ is the Coulomb interaction stemming from Eq. (133). The term $[V\varphi^s]_k$ is described below.

1. Interaction $V\varphi^s$

The interaction $V\varphi^s$ of Eq. (177) in the limit $m_g \to 0$, is discussed following the steps made in Sec. VIII B, where it is shown that the terms $\sim \ln m_g$ cancel out in the full relativistic formula. Knowing that the singular terms cancel out, we can identify elements of the remaining finite terms. We take advantage of the simplifications occurring when one uses the nonrelativistic approximation to describe the relative motion of quarks. Following Sec. VIII B, we write $[V\varphi^s]_k$ as a sum of two parts corresponding to the upper and

lower diagrams in Fig. 3,

$$[V\varphi^s]_k = [V\varphi^s]^1_k + [V\varphi^s]^2_k, \tag{178}$$

where, using $z = x'_1 - x_1$,

$$4m[V\varphi^s]^1_k = \frac{\sigma_1}{x_1}\varphi^s_k + \int_{k'} h^{ff}_{kk'}\varphi^s_{k'}\theta(-z), \tag{179}$$

$$4m[V\varphi^s]^2_k = \frac{\sigma_2}{x_2}\varphi^s_k + \int_{k'} h^{ff}_{kk'}\varphi^s_{k'}\theta(z). \tag{180}$$

We focus on $[V\varphi^s]^1_k$. Details of $[V\varphi^s]^2_k$ are similar. The self-interactions σ_i are given in Eq. (139). They preserve the quark spins. We have

$$\sigma_i = G\int \frac{dxd^2k^\perp}{1-x} e^{-2(s\rho_i/p_g^+)^2} \times \frac{j_{i\mu}[-g^{\mu\nu} + \eta^\mu\eta^\nu\rho_i/p_g^{+2}]j^*_{i\nu}}{\rho_i}, \tag{181}$$

where $G = g^2C_F/[2(2\pi)^3]$, $x = p_g^+/p_i^+$ ranges from 0 to 1, $k^\perp = p_g^\perp - xp_i^\perp$ is unlimited, and $\rho_i = -(p_i - p_q)^2$. The dominant parts of the quark currents are $j_i^+ = 2\delta_{\rm spin}\sqrt{p_i^+p_q^+}$ and $j_i^- = 2\delta_{\rm spin}m_i^2/\sqrt{p_i^+p_q^+}$, being on the order of the quark mass. The other parts of the currents are on the order of the quark relative momenta [28]. Neglecting the momenta in comparison to the masses, the self-interactions amount to the multiplication of the wave function φ^s_k by

$$\sigma_i = 4G\int \frac{dxd^2k^\perp}{1-x}\frac{-m_i^2 + p_i^+p_q^+\rho_i/p_g^{+2}}{\rho_i} e^{-2(s\rho_i/p_g^+)^2}. \tag{182}$$

In the case of $[V\varphi^s]^1_k$, Eq. (143) for $h^{ff}_{kk'}$ implies

$$\int_{k'} h^{ff}_{kk'}\varphi^s_{k'}\theta(-z) = -G\int_0^{x_1}\frac{dx_{1'}d^2k^\perp_{1'2'}}{x_{1'}x_{2'}}\frac{\rho_1+\rho_2}{2\rho_1\rho_2} \times j_{1\mu}d^{\mu\nu}j_{2\nu}e^{-(s\rho_1/p_g^+)^2-(s\rho_2/p_g^+)^2}\varphi^s_{k'}, \tag{183}$$

where, neglecting the transverse relative momenta in comparison to the masses, the dominant term in the product of currents conserves the quark spins and reads

$$j_{1\mu}d^{\mu\nu}j_{2\nu} = 4m_1m_2\left[-1 + \frac{\sqrt{x_1x_{1'}x_2x_{2'}}}{2m_1m_2}\frac{\rho_1+\rho_2}{z^2}\right]. \tag{184}$$

From Eqs. (171) and (172) for $m_g = 0$, with accuracy to terms bilinear in the quark relative momenta,

$$\rho_1 = \rho_2 = \vec{q}^2, \vec{q} = [k^\perp_{1'2'} - k^\perp_{12}, z(m_1+m_2)]. \tag{185}$$

Consequently,

$$\int_{k'} h^{ff}_{kk'}\varphi^s_{k'}\theta(-z) = -G\int_0^{x_1}\frac{dx_{1'}d^2k^\perp_{1'2'}}{x_{1'}x_{2'}}\frac{1}{\rho} \times j_{1\mu}d^{\mu\nu}j_{2\nu}e^{-2(s\vec{q}^2/p_g^+)^2}\varphi^s_{k'}, \tag{186}$$

where

$$j_{1\mu}d^{\mu\nu}j_{2\nu} = 4m_1m_2\left(-1 + \frac{\vec{q}^2}{q^{z2}}\right). \tag{187}$$

The last equation follows from the approximations $x_1 = x_{1'} = x_{10}$ and $x_2 = x_{2'} = x_{20}$. Now we use Eqs. (182), (186), and the relations $|q^z| = xm_1 = |z|(m_1+m_2)$, which hold in the leading approximation neglecting the quark relative momenta in comparison to their masses. Equation (179) is obtained in the form

$$4m[V\varphi^s]^1_k = \frac{4G}{x_1}\int_0^1 dxd^2k^\perp m_1^2\left[\frac{1}{q^{z2}} - \frac{1}{(1-x)\vec{q}^2}\right] \times e^{-2(s\vec{q}^2/p_g^+)^2}\varphi^s_k - 4G\int_0^{x_1}\frac{dx_{1'}d^2k^\perp_{1'2'}}{x_{1'}x_{2'}}m_1m_2\left(\frac{1}{q^{z2}} - \frac{1}{\vec{q}^2}\right) \times e^{-2(s\vec{q}^2/p_g^+)^2}\varphi^s_{k'}. \tag{188}$$

We change the integration variable $k^\perp$ to $q^\perp$, $p_q^\perp = p_1^\perp - q^\perp$, $k^\perp_{1'2'} = k^\perp_{12} - q^\perp$, and $x_{1'} = x_1 + z = x_1 - |z|$. We also substitute $p_g^+ = |z|P^+$ and introduce $s_p = s/P^+$. Using $q^z = xm_1 = |z|(m_1+m_2)$, $dxm_1 \to dq^z$ and $dx_{1'}(m_1+m_2) \to dq^z$. For $s = 1/m$ and $P^+ = 2m$,

$$e^{-2(2ms_p\vec{q}^2/q^z)^2} = e^{-2(\vec{q}^2/mq^z)^2} = e^{-(2/\cos^2\theta)(q/m)^2}, \tag{189}$$

which implies that q/m can be limited to values smaller than 1, validating the nonrelativistic approximation. Extending the integrals freely beyond the mass scale because of the exponential functions quick falloff, and approximating $x_i \sim x_{i'} \sim x_{i0}$, we arrive at

$$[V\varphi^s]^1_k = -2G\int d^3q\theta(q^z)\left(\frac{1}{q^{z2}} - \frac{1}{\vec{q}^2}\right) \times e^{-2(2ms_p\vec{q}^2/q^z)^2}(\varphi^s_{k'} - \varphi^s_k). \tag{190}$$

The interaction $[V\varphi^s]^2_k$ provides the complementary part corresponding to $\theta(-q^z)$, and the combined result is

$$[V\varphi^s]_k = -g^2C_F\int\frac{d^3q}{(2\pi)^3}\left(\frac{1}{q^{z2}} - \frac{1}{\vec{q}^2}\right) \times e^{-2(2ms_p\vec{q}^2/q^z)^2}(\varphi^s_{k'} - \varphi^s_k). \tag{191}$$

This result matches Eq. (67) in [6], obtained there using an additional counterterm. In our approach, such counterterm is not needed, which we explain in Sec. VII C below Eq. (136). Expanding the wave function in a Taylor series in $\vec{q}$,

$$\varphi^s_{\vec{k}+\vec{q}} - \varphi^s_{\vec{k}} = \frac{1}{2} q^i q^j \frac{\partial}{\partial k^i}\frac{\partial}{\partial k^j}\varphi^s_{\vec{k}} + O(q^4), \tag{192}$$

one obtains the quadratic term,

$$[V\varphi^s]_k = -t^{ij}\frac{\partial}{\partial k^i}\frac{\partial}{\partial k^j}\varphi^s_k, \tag{193}$$

$$t^{ij} = G\frac{\pi^{\frac{3}{2}}}{12(\sqrt{8}ms_p)^3}\delta^{ij}. \tag{194}$$

The eigenvalue Eq. (176) thus takes the form

$$\frac{\vec{k}^2}{2\mu}\varphi^s_k + \int_{k'} V^C_{kk'}\varphi^s_{k'} - \frac{1}{2}\mu\omega^2\Delta_k\varphi^s_k = B\varphi^s_k, \tag{195}$$

including the rotation invariant harmonic oscillator potential with

$$\omega^2 = \frac{\alpha C_F}{48\sqrt{2\pi}}\frac{(P^+/s)^3}{m_1 m_2(m_1+m_2)^2}. \tag{196}$$

Thus, our approach to QCD of heavy quarks reproduces the frequency obtained in Eq. (73) of [6].

The effective Hamiltonian eigenvalue Eq. (195), containing a superposition of the Coulomb and harmonic oscillator potentials, and similar equations derived for three-quark states, are known to lead to reasonable values of masses of ground and lowest excited mesons and baryons made of heavy quarks [17]. For example, after fitting the masses of quarks c and b to the heavy meson mass spectrum, the predicted masses of baryons $\Omega_{ccc}(3/2+)$ and $\Omega_{ccc}(3/2-)$ are about 4797 and 5103 MeV/c^2, respectively [17]. Recent lattice computations [25] produce 4793 and 5094 MeV/c^2, correspondingly. On one hand, this agreement is surprising because of the crude approximations made in the second-order RGPEP approach. On the other hand, it is remarkable because the simple RGPEP calculation not only predicts the baryon masses but also provides expressions for their wave functions. There exist independent arguments for expecting that an effective FF theory may involve a quadratic potential [69].

2. $Q\bar{Q}$ component and quark confinement

The harmonic oscillator potential in Eq. (195) is obtained at short quark-antiquark distances after the cancellation of small-x logarithmic divergences is accounted for in the eigenvalue problem for the Hamiltonian of Eq. (121). The diverging terms involve the logarithms $\ln(sm_g^2/P^+)^{-1}$ and color factors. The cancellation occurs in the globally colorless states of the effective quark and antiquark, because the color factors in the quark self-interactions and in the gluon exchange terms are opposite to each other, $4/3$ and $-4/3$, respectively. But in the color-octet effective quark-antiquark states, the gluon exchange color factor is $1/6$ instead of $-4/3$ while the quark self-interaction terms stay the same. Therefore, the logarithms of the gluon mass do not cancel out, and the color-octet quarkonium mass eigenvalues tend to infinity when $m_g \to 0$. Moreover, if instead of the quark-antiquark states one considers states with the quantum numbers of a single quark, the diverging quark self-interaction term has no second-order Hamiltonian term to cancel with. These results are in agreement with the absence in the Hamiltonian spectrum of single-quark eigenstates with finite mass eigenvalues, while the colorless quarkonia have finite masses. One can thus hope that the RGPEP scheme for computing renormalized, scale-dependent Hamiltonians in QCD of heavy quarks, understood as a limit of a theory with $m_g > 0$ when $m_g \to 0$, is a candidate for deriving realistic quantum mechanical eigenvalue problems for hadrons. We wish to stress at this point that independently of the arguments given in this paper, defining confinement as the absence of colored states with finite masses requires some parameter to carry out a mathematically well defined, finite calculation before one considers the infinite masses of colored states as the result of taking that parameter to its suitable limit. The remaining discussion concerns some aspects of our approach, such as the cancellation of small-x singularities in the quarkonium components including effective gluons in addition to the heavy quark pair, and comparison with QED [70], where the cancellation of the photon small-x singularities does not result in any residual harmonic force between electrons. Some of these aspects have been addressed in the past in various other ways [4,5,18,19]. The discussion that follows is focused on approaching QCD through the limit $m_g \to 0$.

The interaction terms discussed above appear in the renormalized Hamiltonian for the heavy effective quarks corresponding to the scale parameter $s \sim 1/m$, instead of the quarks of bare QCD, corresponding to $s = 0$. Interactions of the effective quarks at a scale s cause only limited changes of the FF free energy, the limit being $\sim 1/s$. In contrast, the interaction terms in the bare QCD Hamiltonian can directly produce arbitrarily large changes of the FF energy, and thus, they can reach quark-antiquark states that are degenerate in energy with the multiparticle states. Consequently, the bare interactions are able to generate significant eigenstate components with many bare quanta, producing that way a complex image of the hadronic structure. In the alternative representation in terms of the effective particles, the probability of the multiparticle components is diminished. This feature is of interest because the phenomenology of heavy quarkonia assures us that the quark-antiquark component provides a

physically reasonable approximation [2,3]. Nevertheless, the effective Hamiltonian computed using the RGPEP still includes the interactions changing the number of the effective quanta, albeit they are suppressed by the vertex factors with a small FF-energy width when s is large.

The effective interactions discussed above are merely of the second order in the expansion in a series of powers of the coupling constant g. The coupling constant is plainly assumed to be small. However, in the FF phenomenology of heavy hadrons, one uses $\alpha_s = g_s^2/(4\pi)$ that depends on s, e.g., see [17]. The realistic coupling constant α_s for $s \sim 1/m$ is on the order of $1/3$. Such value corresponds to the running coupling constant measured at the electroweak mass scale. Rigorous extrapolation of the second-order picture to the realistic values of the coupling constant requires a demonstration that the terms of orders higher than second in the expansion in powers of g_s involve relatively small operator coefficients. The expectation that for $s \sim 1/m$ the quarkonium components other than the quark and antiquark are relatively small needs to be verified using Hamiltonians computed in the RGPEP of orders higher than second [16].

The rotation invariant effective quadratic potential is the first term in the Taylor expansion in Eq. (192). The sum of the Taylor series yields an effective quark-antiquark potential varying logarithmically with $\vec{r}$ at large distances [6]. But the rotation symmetry is increasingly broken as the distance increases toward and above the quarkonium radius. Namely, at large distances, the coefficient of $\ln|r^{\perp}|$ for $r^z = 0$ is twice larger than the coefficient of $\ln|r^z|$ for $r^{\perp} = 0$. This result matches the results obtained in [18,19]. The reason for the rotation symmetry breaking to increase with the interquark distance is likely to be the fact that the quark-antiquark component of large potential energy is degenerate with the components containing additional gluons. Therefore, one may expect the $Q\bar{Q}$ component with large potential energy to be strongly mixed by the interactions with the components containing additional gluons [20]. The mechanism of breaking and the gluon cure for recovering the rotation symmetry need to be assessed using the RGPEP, by computing the renormalized Hamiltonian terms including effective gluons at scales $s \sim 1/m$ in the limit $m_g \to 0$. The effective gluons may develop masses that grow as s increases.

The second-order seed of confinement cannot appear in QED. But the second-order QCD interactions differ from the analogous second-order QED interactions between the charged fermions by replacement of e^2 with $(4/3)g^2$ only. The issue is that the second-order RGPEP additions to the Coulomb interaction between a fermion and an antifermion are canceled in QED when one accounts for the interactions that couple the fermion-antifermion and fermion-antifermion-boson components of a bound state. Thus, the second-order quark-confinement seed can be valid in QCD and absent in QED if the coupling between the components $Q\bar{Q}$ and $Q\bar{Q}G$ is suppressed in QCD (G denotes an effective gluon). The suppression may result from the FF energy gap generated by the non-Abelian interactions in the three-body component. There are no such interactions between photons and charged fermions. Hence, there is no such gap in QED. In the next section, we provide an initial discussion of the interaction terms in the quark-antiquark-gluon components of quarkonia using our RGPEP approach with m_g and ϕ in the limit of $m_g \to 0$.

IX. SMALL-x SINGULARITIES IN COMPONENTS WITH EFFECTIVE GLUONS

The quarkonium two-body component is coupled to the three-body component containing an effective gluon in addition to the quark pair,

$$|Q_{s1}\bar{Q}_{s2}G_{s3}\rangle = b^{\dagger}_{s1}d^{\dagger}_{s2}a^{\dagger}_{s3}|0\rangle, \tag{197}$$

and to the states with more effective gluons. States including additional quark-antiquark pairs are assumed so heavy that they do not significantly contribute to the ground and low excited quarkonium eigenstates. They can be included in precise RGPEP calculations. All these components correspond to the basis states $|i\rangle_s$ in Eq. (102), while their coefficients $\psi_i(s)$ are provided by the eigenstate wave functions.

The RGPEP effective eigenvalue problem differs from the equations derived using the Tamm–Dancoff approximation [71,72]. That approximation requires counterterms to the regularization effects, including the counterterms depending on the Fock sector [73,74]. In contrast, the renormalized Hamiltonian H_s is finite and independent of the regularization parameter $r \to 0$. The effective interaction terms in H_s include the vertex form factors $f^s_{\bar{p},\bar{q}}$ computed according to Eqs. (108) to (111). The eigenvalue problem for H_s with finite s does not require any counterterms. Nevertheless, its eigenstates have components with different numbers of effective constituents,

$$|\psi\rangle = |2\rangle_s + |3\rangle_s + \cdots, \tag{198}$$

where dots indicate the components $|n\rangle_s$ with $n > 3$. Below, we demonstrate the cancellation of the gluon small-x singularities among the Hamiltonian terms acting in the effective-particle three-body component.

A. Effective interactions in the three-body component

According to Eq. (112), using abbreviated notation for the states of effective particles, $|Q_{s1}\bar{Q}_{s2}G_{s3}\rangle \equiv |123\rangle_s$, and taking advantage of an identity analogous to Eq. (124),

$${}_s\langle 123|H^{(2)}_s|1'2'3'\rangle_s = \langle 123|\mathcal{H}^{(2)}|1'2'3'\rangle, \tag{199}$$

the Hamiltonian matrix elements in the $Q\bar{Q}G$ component of a quarkonium are

$$ {}_s\langle 123|H_s|1'2'3\rangle_s = \langle 123|[\mathcal{H}_f + f_{LR}\mathcal{H}_r^{(2)} - f_{LI}f_{IR}\Delta_{LIR}\mathcal{H}_0^{(1)}\mathcal{H}_0^{(1)}]|1'2'3'\rangle, \quad (200) $$

where $\mathcal{H}_r^{(2)} = \mathcal{H}_0^{(2)} + \Delta_{LIR}\mathcal{H}_0^{(1)}\mathcal{H}_0^{(1)}$. The matrix elements of the operator $f_{LR}\mathcal{H}_r^{(2)}$ include the self-interactions and the gluon exchanges, both due to $\Delta_{LIR}\mathcal{H}_0^{(1)}\mathcal{H}_0^{(1)}$ and the self-interaction counterterms and seagulls in $\mathcal{H}_0^{(2)}$. These terms involve singularities $\sim 1/x^2$. Matrix elements of $f_{LI}f_{IR}\Delta_{LIR}\mathcal{H}_0^{(1)}\mathcal{H}_0^{(1)}$ do not diverge for states of finite total P^+ when $s > 0$, as long as $m_g > 0$. Our description of the interactions in the effective three-body component parallels the case of two-body component discussed in Sec. VII.

B. Cancellation of $1/x^2$ in terms $f_{LR}\mathcal{H}_r^{(2)}$

One source of the divergences $\sim 1/x^2$ in $f_{LR}\mathcal{H}_r^{(2)}$ are the self-interactions. These are canceled by the self-interaction counterterms. These counterterms act in all sectors of the virtual Fock space equally, as a consequence of the RGPEP operator calculus, see Sec. VI B. Therefore, the self-interaction divergences in the three-body component are countered in the same way as in the two-body component. Divergences $\sim 1/x^2$ appear in $f_{LR}\mathcal{H}_r^{(2)}$ also due to the gluon exchange and gluon seagull terms. These are illustrated graphically in Fig. 4. The gluon exchange terms proportional to $g^{\mu\nu}$ are finite. The singularities $\sim 1/x^2$ appear in the terms proportional to $\eta^\mu\eta^\nu$ only. They cancel out within the three subsets of terms illustrated by the three rows of panels in Fig. 4. The cancellation occurs despite the assignment of mass m_g to gluons thanks to the contribution of quanta of the auxiliary field ϕ.

The mechanism of cancellation of singularities $\sim 1/x^2$ in the case of the quark-antiquark gluon exchange and the corresponding seagull term, panels c in Fig. 4, is the same as the one described in Sec. VII C, Eqs. (135) and (136). The quark-antiquark interactions in the three-body and two-body components differ only by the color factor, $-1/6$ instead of $4/3$, and thus do not require further discussion.

The interaction of a gluon with a quark or an antiquark in the three-body space, illustrated by panels a_1, a_2, b_1, and b_2 in Fig. 4, results from the gluon exchange, caused by the double action of the terms stemming from $-(J^{a\mu}_{\psi f} + J^{a\mu}_{Af})A^a_{f\mu}$ in $\mathcal{H}_{\rm QCD}$ in Eq. (16) and from $m_g\phi^a\frac{1}{\partial^+}(J^{a+}_{Af} + J^{a+}_{\psi f})$ in Eq. (21). The currents $J^{a\mu}_{\psi f}$ and $J^{a\mu}_{Af}$ are defined in Eqs. (17) and (18). The divergences $\sim 1/x^2$ in these exchanges are canceled by the seagull terms originating from the last Hamiltonian density term in Eq. (16). In terms of the operators, the singularity $1/x^2$ in $f_{LR}\Delta_{LIR}f_r\mathcal{H}_0^{(1)}f_r\mathcal{H}_0^{(1)}$ is canceled by the seagull $f_{LR}f_rf_rH_{\rm sgq}$, where

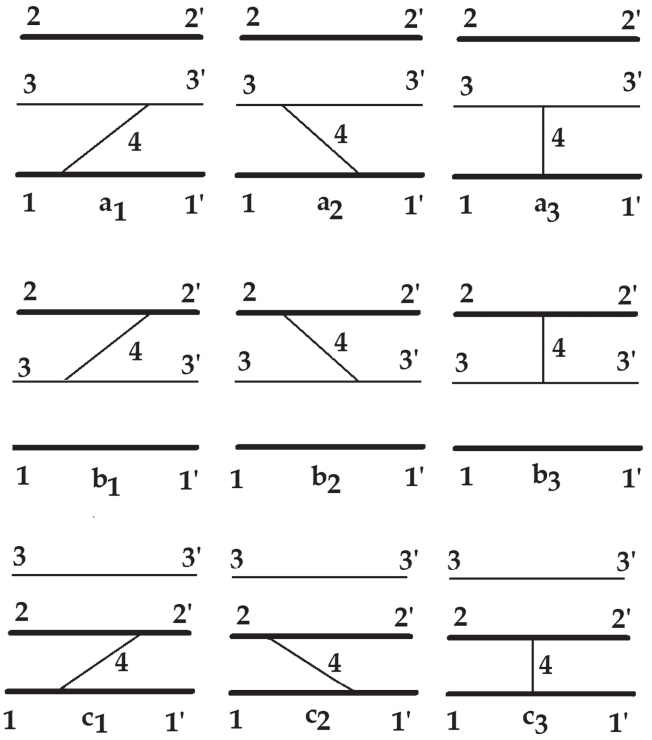


FIG. 4. Illustration of the Hamiltonian $Q\bar{Q}G$ matrix elements $\langle 123|f_{LR}\mathcal{H}_r^{(2)}|1'2'3'\rangle$ in Eq. (200), involving two complementary orderings of an exchange of a gluon 4 between a gluon 3 and a quark 1, panels a_1, a_2, and an associated seagull interaction between the gluon 3 and the quark 1, panel a_3. Panels b_1, b_2, b_3 illustrate the matrix elements of the analogous interactions of the gluon 3 with antiquark 2. Exchange of the gluon 4 between the quark 1 and antiquark 2 and the associated seagull are illustrated in panels c_1, c_2, c_3. Cancellation of the small-x singularities among the interactions illustrated by the triple panels a, b, and c is described in the text.

$$ \mathcal{H}_0^{(1)} \equiv -\int (J^{a\mu}_{\psi f} + J^{a\mu}_{Af})\left(A^a_{f\mu} + \frac{m_g\eta_\mu}{\partial^+}\phi^a\right), \quad (201) $$

$$ H_{\rm sgq} = -\int J^{a+}_{Af}\frac{1}{\partial^{+2}}J^{a+}_{\psi f}, \quad (202) $$

and $\int = \int d^2x^\perp dx^-$ for $x^+ = 0$. Ignoring pair creation and annihilation, the operator $\mathcal{H}_0^{(1)}$ is a sum of four terms,

$$ \mathcal{H}_0^{(1)} \equiv \mathcal{H}_{\psi A} + \mathcal{H}_{\psi\phi} + \mathcal{H}_{AA} + \mathcal{H}_{A\phi}. \quad (203) $$

Including sums over discrete quantum numbers in the integration sign, one has

$$\mathcal{H}_{\psi A} = -\int \left[:J^{a\mu}_{\psi f} A^{a}_{f\mu}: \right]^r \tag{204}$$

$$\begin{aligned} &= g\int [11'4]\tilde{\delta}_{c.a} r_{\bar{1},\underline{1}4} \chi_1^\dagger T^4 \chi_{1'} \bar{u}_1 \gamma_\nu u_{1'} \\ &\times \left(\varepsilon_4^\nu b_1^\dagger b_{1'} a_4 + \varepsilon_4^{\nu *} a_4^\dagger b_1^\dagger b_{1'} \right) \\ &- g\int [22'4]\tilde{\delta}_{c.a} r_{\bar{2},\underline{2}4} \chi_{2'}^\dagger T^4 \chi_2 \bar{v}_{2'} \gamma_\nu v_2 \\ &\times \left(\varepsilon_4^\nu d_2^\dagger d_{2'} a_4 + \varepsilon_4^{\nu *} a_4^\dagger d_2^\dagger d_{2'} \right), \end{aligned} \tag{205}$$

$$\mathcal{H}_{\psi\phi} = -\int \left[:J^{a\mu}_{\psi f} \frac{m_g \eta_\mu}{\partial^+} \phi^a: \right]^r \tag{206}$$

$$\begin{aligned} &= g\int [11'\tilde{4}]\tilde{\delta}_{c.a} r_{\bar{1},\underline{1}\tilde{4}} \chi_1^\dagger T^{\tilde{4}} \chi_{1'} \bar{u}_1 \gamma^+ u_{1'} \\ &\times \frac{m_g}{p_{\tilde{4}}^+} \left(b_1^\dagger b_{1'} a_{\tilde{4}} + a_{\tilde{4}}^\dagger b_1^\dagger b_{1'} \right) \\ &- g\int [22'\tilde{4}]\tilde{\delta}_{c.a} r_{\bar{2},\underline{2}\tilde{4}} \chi_{2'}^\dagger T^{\tilde{4}} \chi_2 \bar{v}_{2'} \gamma^+ v_2 \\ &\times \frac{m_g}{p_{\tilde{4}}^+} \left(d_2^\dagger d_{2'} a_{\tilde{4}} + a_{\tilde{4}}^\dagger d_2^\dagger d_{2'} \right), \end{aligned} \tag{207}$$

$$\mathcal{H}_{AA} = -\int \left[:J^{a\mu}_{Af} A^{a}_{f\mu}: \right]^r \tag{208}$$

$$\begin{aligned} &= g\int [343'] \Big[r_{34,3'} a_3^\dagger a_4^\dagger a_{3'} Y_{343'} \tilde{\delta}_{34.3'} \\ &+ r_{3,43'} a_3^\dagger a_4 a_{3'} Y^*_{3'43} \tilde{\delta}_{3.43'} \Big], \end{aligned} \tag{209}$$

$$Y_{343'} \equiv i f^{c_3 c_4 c_{3'}} [\varepsilon_3^* \varepsilon_4^* \varepsilon_{3'} p_3 - \varepsilon_3^* \varepsilon_{3'} \varepsilon_4^* (p_{3'} + p_3)], \tag{210}$$

$$\mathcal{H}_{A\phi} = -\int \left[:J^{a\mu}_{Af} \frac{m_g \eta_\mu}{\partial^+} \phi^a: \right]^r \tag{211}$$

$$\begin{aligned} &= g m_g \int [433']\tilde{\delta}_{c.a} r_{43,3'} i f^{c_4 c_3 c_{3'}} \\ &\times \frac{p_3^+ + p_{3'}^+}{p_4^+} \varepsilon_3^* \varepsilon_{3'} \tilde{a}_4^\dagger a_3^\dagger a_{3'} + \text{H.c.}. \end{aligned} \tag{212}$$

The relevant seagull terms are contained in the operator of Eq. (53). From Eqs. (58) and (59), we obtain the gluon-quark and gluon-antiquark seagull terms,

$$\begin{aligned} H_{\text{sgq}} = -g^2 &\int [33'11'] \tilde{\delta}_{c.a} \delta_{\text{spins}} r_{\bar{1},\underline{1}(\bar{1}-\underline{1})} r_{\bar{3},\underline{3}(\bar{3}-\underline{3})} \\ &\times \frac{(p_3^+ + p_{3'}^+) 2\sqrt{p_1^+ p_{1'}^+}}{(p_{3'}^+ - p_3^+)^2} \chi_1^\dagger T^a \chi_{1'} i f^{abc} a_{3b}^\dagger a_{3'c} b_1^\dagger b_{1'}, \end{aligned} \tag{213}$$

$$\begin{aligned} H_{sg\bar{q}} = +g^2 &\int [33'22'] \tilde{\delta}_{c.a} \delta_{\text{spins}} r_{\bar{3},\underline{2}(\bar{2}-\underline{2})} r_{\bar{3},\underline{3}(\bar{3}-\underline{3})} \\ &\times \frac{(p_3^+ + p_{3'}^+) 2\sqrt{p_2^+ p_{2'}^+}}{(p_{3'}^+ - p_3^+)^2} \chi_{2'}^\dagger T^a \chi_2 i f^{abc} a_{3b}^\dagger a_{3'c} d_2^\dagger d_{2'}, \end{aligned} \tag{214}$$

corresponding to the panels a_3 and b_3 in Fig. 4. We focus on the gluon 3 interaction with the quark 1 and omit discussion of the gluon 3 interaction with the antiquark 2. Cancellation in the latter works in the same way. Since the sign and color factors for exchanges and seagulls in the panels a and b are the same, $-3/2$, the cancellation rests on the momentum and spin factors in Eqs. (204) to (214). The divergence $\sim 1/x^2$, with $p_4^+ = |x| P^+$, comes from the squares of ε_4^- and m_g/p_4^+ in the exchange and from $1/p_4^{+2}$ in the seagull. The gluon exchanges illustrated in the panel a of Fig. 4 include two complementary orderings, one for positive and another one for negative $x = x_{1'} - x_1$, both producing the divergence $\sim 1/x^2$. It suffices to consider the cancellation for one sign of x. We describe details of the case $x_1 > x_{1'}$. It resembles the case described in Sec. VII C for the two-body component, see the panels upper left and lower in Fig. 2.

Using the abbreviation $\theta_{1-1'} = \theta(p_1^+ - p_{1'}^+)$, the operator generating the exchange of the transverse gluon 4, a quantum of the field A, between the gluon 3 and quark 1, reads

$$\begin{aligned} H_A = g^2 \sum_4 &\int [11'33'] \theta_{1-1'} \frac{\tilde{\delta}_{13.1'3'}}{p_4} \Delta_{LIR} \chi_1^\dagger T^4 \chi_{1'} \\ &\times \bar{u}_1 \gamma_\nu u_{1'} i f^{c_3 c_4 c_{3'}} Y_{131'3'} b_1^\dagger b_{1'} a_3^\dagger a_{3'}, \end{aligned} \tag{215}$$

where $Y_{131'3'} = 2\varepsilon_4^\nu (\varepsilon_3^* \varepsilon_4^* \varepsilon_{3'} p_3 + \varepsilon_4^* \varepsilon_{3'} \varepsilon_3^* p_4 - \varepsilon_3^* \varepsilon_{3'} \varepsilon_4^* p_{3'})$. The singularity $1/x^2$ comes from the last term in $Y_{131'3'}$. The regularization factors r are not displayed here and below to simplify the formulas. They are the same in all terms of panel a and can be reinstated according to Eqs. (204) to (214) or (63). The operator contributing the exchange of the field-ϕ quanta is

$$\begin{aligned} H_\phi = g^2 &\int [11'33'] \theta_{1-1'} \frac{\tilde{\delta}_{13.1'3'}}{p_4^+} \Delta_{LIR} \chi_1^\dagger T^4 \chi_{1'} \bar{u}_1 \gamma_\nu u_{1'} \\ &\times \frac{m_g^2 \eta^\nu}{p_4^{+2}} i f^{c_4 c_3 c_{3'}} (p_3^+ + p_{3'}^+) \varepsilon_3^* \varepsilon_{3'} b_1^\dagger b_{1'} a_3^\dagger a_{3'}. \end{aligned} \tag{216}$$

The factor Δ_{LIR} enters the operators according to Eq. (200). In precise analogy with Sec. VII C, where we discuss the quark-antiquark interaction, we now have for the gluon-quark interaction

$$\frac{1}{p_4^+} \Delta_{LIR} = -\frac{1}{2} \left(\frac{1}{\rho_1} + \frac{1}{\rho_3} \right), \tag{217}$$

where $\rho_3 = m_g^2 - (p_3 - p_{3'})^2$. When p_4^+ tends to 0,

$$\frac{1}{p_4^+}\Delta_{LIR} \to \frac{-1}{p_4^{\perp 2} + m_g^2}. \tag{218}$$

The gluon-quark seagull term with $\theta_{1-1'}$ reads

$$H_s = -g^2 \int [33'11'] \tilde{\delta}_{c.a}\delta_{\text{spins}}\theta_{1-1'} \frac{(p_3^+ + p_{3'}^+)2\sqrt{p_1^+ p_{1'}^+}}{(p_{3'}^+ - p_3^+)^2} \times \chi_1^\dagger T^a \chi_{1'} i f^{abc} b_1^\dagger b_{1'} a_{3b}^\dagger a_{3'c}. \tag{219}$$

The common factor in Eqs. (215), (216), and (219), is

$$CF = g^2 \sum_4 \int [11'33'] \theta_{1-1'} \frac{\tilde{\delta}_{13.1'3'}}{p_4} \chi_1^\dagger T^4 \chi_{1'} \times \bar{u}_1 \gamma_\nu u_{1'} i f^{c_3 c_4 c_{3'}} a_3^\dagger b_1^\dagger b_{1'} a_{3'}. \tag{220}$$

Using CF, one can write the sum of terms involving $1/x^2$ in $f_{LR}\mathcal{H}_r^{(2)}$ in Eq. (200) as

$$f_{LR}(H_A + H_\phi + H_s) \tag{221}$$

$$= f_{LR} CF \left[\frac{1}{p_4^+}\Delta_{LIR}(L_A + L_\phi) + L_s\right], \tag{222}$$

where

$$L_A = 2\sum_4 \varepsilon_4^\nu [-\varepsilon_3^* \varepsilon_{3'} \varepsilon_4^* p_{3'}], \tag{223}$$

$$L_\phi = -\frac{m_g^2 \eta^\nu}{p_4^{+2}}(p_3^+ + p_{3'}^+)\varepsilon_3^* \varepsilon_{3'}, \tag{224}$$

$$L_s = -\varepsilon_3^* \varepsilon_{3'} \frac{p_3^+ + p_{3'}^+}{p_4^{+2}} \eta^\nu. \tag{225}$$

Since $\varepsilon_4 = \eta \varepsilon_4^-/2 + \varepsilon_4^\perp$, the small-$p_4^+$ singularity of L_A comes solely from $\varepsilon_4^- = 2\varepsilon_4^\perp p_4^\perp / p_4^+$, and $\varepsilon_4^* p_{3'} = \varepsilon_4^* p_3$, the singular part of transverse gluon exchange is only

$$L_A \to -\varepsilon_3^* \varepsilon_{3'} \frac{p_4^{\perp 2}}{p_4^{+2}} \eta^\nu (p_{3'}^+ + p_3^+). \tag{226}$$

All terms diverging as $1/p_4^{+2}$ for $p_4 \to 0$ add up to

$$f_{LR}(H_A + H_\phi + H_s)^{\text{div}} \tag{227}$$

$$= f_{LR} CF(-\varepsilon_3^* \varepsilon_{3'})\eta^\nu \frac{p_{3'}^+ + p_3^+}{p_4^{+2}} \times \left\{\frac{-1}{p_4^{\perp 2} + m_g^2}[(p_4^{\perp 2})_A + (m_g^2)_\phi] + (1)_s\right\}, \tag{228}$$

where the subscripts indicate the terms origin. The curly bracket is 0. This result completes our demonstration that the severe small-x singularities cancel out in the effective $Q_1\bar{Q}_2$ and $Q_1\bar{Q}_2 G_3$ components of the quarkonium eigenvalue problem for the renormalized Hamiltonian of QCD for heavy quarks, when the gluons are assigned the mass m_g and the Hamiltonian includes a coupling to the auxiliary field ϕ. The terms remaining in $f_{LR}\mathcal{H}_r^{(2)}$ after the cancellation can be computed using the analogy between the interactions in the three-body component arranged pairwise as shown in Fig. 4 and the two-body interactions derived in Sec. VII C, Eqs. (133), (135), and (136), replacing ρ_2 by ρ_3. The resulting terms have finite limits as functions of the finite constituent momenta when m_g tends to 0.

C. $Q\bar{Q}G$ interactions with factors $f_{LI}f_{IR}$

The pertinent Hamiltonian terms in Eq. (200) are $-f_{LI}f_{IR}\Delta_{LIR}\mathcal{H}_0^{(1)}\mathcal{H}_0^{(1)}$. They are illustrated in Fig. 5. As in the case of the quark-antiquark component discussed in Sec. VIII, these terms are finite for $m_g > 0$ despite the presence of factors $1/x_4^2$ or $1/x_4$, because for $x_4 \to 0$ the factors f_{LI} and f_{IR} depend on the square of p_4^- exponentially,

$$f_{LI} \sim f_{IR} \sim e^{-(sp_4^-)^2} = e^{-[(s/P^+)(p_4^{\perp 2} + m_g^2)/x]^2}. \tag{229}$$

This behavior of f_{LI} and f_{IR} extends to gluon exchanges in all components of the quarkonium eigenstate. Further, as in Sec. VIII, the effective interactions in Fig. 5 are finite when $m_g \to 0$ for fixed $|p_4^\perp|$. But for $|p_4^\perp| \lesssim m_g$, the argument of the exponential becomes $-[(sm_g^2/P^+)/x_4]^2$, and the region of small $x_4 \equiv x$ is suppressed only for $m_g > 0$. The small-x singularities translate to the singular dependencies of the interaction terms on m_g as in the two-body component. Below, we outline the argument that the singular dependencies on m_g cancel out in the globally color-singlet states $|Q_{s1}\bar{Q}_{s2}G_{s3}\rangle$. The outline closely follows the analysis carried out in Sec. VIII B. We focus attention on the terms with $\eta^\mu \eta^\nu$ in the sum over gluon polarizations, since terms with $g^{\mu\nu}$ are finite and small.

Interactions illustrated in Fig. 5 involve only subsystems of two particles, or a single particle in the case of self-interactions. The quark and antiquark self-interactions are the same as in the component $Q_{s1}\bar{Q}_{s2}$. The row c of panels features the gluon self-interaction, which does not appear in the component $Q_{s1}\bar{Q}_{s2}$. Singularity of a self-interaction in a panel row cancels out with the singularities of two exchanges in the same row. The color factors of the three terms in each of the rows, sum to 0. To demonstrate the cancellation, it suffices to show that the momentum singularities of the interactions in one row are the same.

We describe details in the case of row a. Following the case of the two-body component, the integral over

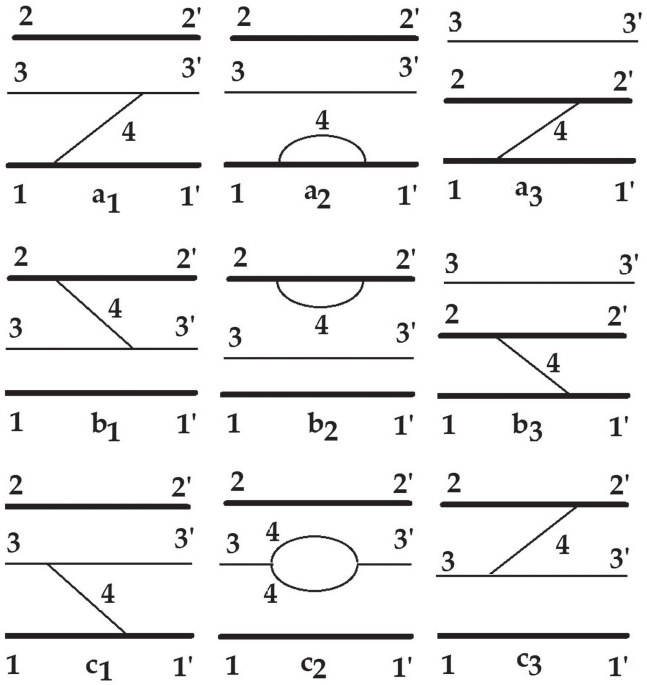


FIG. 5. Illustration of the Hamiltonian $Q\bar{Q}G$ matrix elements $\langle 123|f_{LI}f_{IR}\Delta_{LIR}\mathcal{H}_0^{(1)}\mathcal{H}_0^{(1)}|1'2'3'\rangle$ in Eq. (200). Panels a_1 and a_3 correspond to an exchange of a gluon 4 between the quark 1 and gluon 3 or quark 1 and quark 2, which together cancel the logarithmic dependence of the self-interaction of the quark 1 shown in panel a_2 on the gluon mass m_g. Panels b_1 and b_3 correspond to an exchange of a gluon 4 between the antiquark 2 and gluon 3 or antiquark 2 and quark 1, which together cancel the logarithmic dependence on m_g in the self-interaction of the antiquark 2 shown in panel b_2. Panels c_1 and c_3 illustrate the exchanges of gluon 4 between the gluon 3 and quark 1 and antiquark 2, which together cancel the logarithm of m_g in the self-interaction of gluon 3 shown in panel c_2. Details are described in the text.

transverse momentum $q^\perp$ carried by the gluon in panel a_1 is rewritten in terms of $k^\perp = q^\perp/\sqrt{x}$, where $x = p_4^+/(p_1^+ + p_3^+)$, cf. Eq. (170). This way the power of x in the denominator is reduced by 1, leaving only the logarithmic singularity to deal with. The gluon exchange of panel a_1 is integrated with the wave function $\varphi_{1'23'}$. Following the example of two-body Eq. (155), we integrate the gluon exchange kernel, denoted here by h^{ff}, with $\varphi_{1'23'} - \varphi_{123}$, instead of just with $\varphi_{1'23'}$. The difference $\varphi_{1'23'} - \varphi_{123}$ vanishes for $p_4^+ \to 0$, the reason being explained in the last paragraph of Sec. VIII B 2. Hence, the integral of the difference with the kernel h^{ff} has a finite limit when $m_g \to 0$. Simultaneously, we add the integral of φ_{123} with h^{ff} to the self-interaction in panel a_2. It remains to show that the singularities due to small p_4^+ in the integrands of the self-interaction and the kernel h^{ff} are equal to each other. Our argument proceeds *a la* Eq. (164). But the equality in the three-body case cannot occur as it does in the two-body case because the $Q\bar{Q}$-component gluon-exchange kernel contains a product of two quark currents. In contrast, the $Q\bar{Q}G$ component gluon exchange in panel a_1 involves one quark current and one gluon current. The key to the cancellation of $\ln m_g$ is that in the terms with tensor $\eta^\mu\eta^\nu$ the fermion and boson currents provide the factors $2(p_1^+ p_{1'}^+)^{1/2}$ and $p_3^+ + p_{3'}^+$, respectively. When the gluon p_4^+ vanishes, i.e., the gluon $x \to 0$, the product of these factors equals $2p_1^+ 2p_3^+$, which is the same as the product of two fermion currents, or two boson currents. Since the kernel h^{ff} has the color factor $-3/2$ and the self-interaction has $4/3$, one needs to take into account the color factor $1/6$ for the gluon exchange in the octet pair to obtain the cancellation. The subsystem of quark 1 and antiquark 2 in the three-body component is analyzed as in Sec. VIII B, except for the color factors that differ. Analysis of the interactions illustrated in the panel row b in Fig. 5 proceeds as in the case of the panel row a. In panel row c, the cancellation between two kernel integrals, each with a quark and a gluon current, and the product of two gluon currents is expected because of the equality of products of these currents when $p_4^+ \to 0$. The color factors mentioned above, leading to the cancellation of $\ln m_g^{-1}$, are obtained for colorless $Q\bar{Q}G$ states. If a three-body state is not a color singlet, the diverging self-interactions are not canceled. They grow logarithmically as functions of $1/m_g$ to infinity when $m_g \to 0$, signifying that only the color singlets may lead to finite bound-state mass eigenvalues.

In the integrands including the exchange kernel and wave function difference, such as $h^{ff}(\varphi_{1'23'} - \varphi_{123})$ above, one can apply the Taylor expansion to the difference, as suggested by the case of the two-body component discussed in Sec. VIII D 1. For gluons, the nonrelativistic approximation does not apply. Nevertheless, in the transverse directions, one can proceed using the expansion analogous to Eq. (192), substituting $k'^\perp = k^\perp + q^\perp$. Thus, the kernels involving $\eta^\mu\eta^\nu$, which are even in $q^\perp$, can generate transverse harmonic oscillator potentials with explicit formulas for their frequencies. However, the three-body eigenvalue problem is more complex than the two-body one, and the author has not analyzed its longitudinal dynamics enough to discuss it in this paper. The transverse oscillators are expected to contribute mass squared gaps. The corresponding eigenfunctions should fall off when the constituents relative momenta become extreme, since the Hamiltonian part H_f grows extremely for such momenta.

D. Interactions changing the number of gluons

The bound-state effective Fock space components are coupled by the interactions that change the number of constituents. These interactions can be considered perturbations since they include the small coupling constant and RGPEP factors f^s, the latter significantly reducing the strength of the effective emission and absorption processes

in comparison to their canonical counterparts. On one hand, the coupling constant extrapolation to realistic values at $s \sim 1/m$ increases the importance of the mixing of the Fock sectors in the eigenvalue problems for hadrons. On the other hand, the effective potentials are expected to further weaken the mixing. For example, one can write the three-body wave function of the component $Q_s\bar{Q}_sG_s$ in a quarkonium as a combination of the oscillator wave functions and seek coefficients of such combination numerically from the eigenvalue condition for a superposition of the two-body and three-body components. The harmonic frequencies would contribute to the masses squared of the two-body subsystems in the three-body system. The three-body component invariant mass would thus increase in comparison to the free invariant mass. The probability amplitude for the component would decrease.

If the interactions between the effective gluon and quarks in the $Q\bar{Q}G$ component were absent, the transitions from the two-body to three-body component and back would cancel the interaction terms proportional to $f_{LI}f_{IR}$ in Eq. (125). The cancellation mechanism would be analogous to the cancellation mechanism between the self-interaction of an electron and emission of low energy photons [70]. In QCD, the situation is different precisely because gluons carry the color charge and constantly interact with quarks. Photons carry no electric charge and behave as free in the presence of electrons, neglecting effects such as the photon-photon interaction. There is no second-order effective oscillator potential at short distances between the electron and positron in positronium.

In the Fock sector with two gluons, there would be an additional interaction between the gluons themselves, besides the gluon-quark and gluon-antiquark interactions. The suppression effects associated with the effective harmonic interactions would appear for all components. Dominance of the quark-antiquark component can be modeled by giving the gluon in the $Q\bar{Q}G$ component a sizable mass m_G and forgetting all components with more than one gluon. The quarkonium eigenstate is thus replaced by two components, the dominant quark-antiquark component and the quark-antiquark-gluon component in which the gluon is assigned the ansatz mass m_G. The three-body component can then be eliminated, assuming it is small. Such modeling yields reasonable results for the masses of quarkonia and baryons in QCD with only heavy quarks [6,17]. In the approach with the gluon mass m_g and auxiliary field ϕ, the components coupled to the dominant one in a hadron are open to study, including the issue of evaluating the effective mass for gluons as constituents of a heavy hadron. Excited states are expected to have increased probability of components with gluons. Extension of the study to light quarks requires inclusion of the running coupling constant in the Hamiltonian, which means extension of the RGPEP weak-coupling expansion for the Hamiltonian to fourth order. The reason is that the running appears first in the third order. The running third-order vertex needs to be combined with another interaction, bringing the order to fourth. The sixth order would be required for precisely including effects of the running coupling constant squared, which corresponds to two loops in summing over contributing quanta. In any case, the self-interactions proportional to just g^2 in the second-order calculation are expected to grow as g_s^2 when s increases, likely to produce running quark and gluon masses. The oscillator frequencies are also going to increase as g_s increases. The coherent evolution of various terms could thus define an effective low-energy Hamiltonian of QCD with the characteristic elements: the Coulomb and harmonic oscillator effective potentials, constituent masses, large coupling combined with narrow vertex factors, and finite eigenvalues only for colorless states in the limit $m_g \to 0$.

X. DISCUSSION

The RGPEP with m_g and ϕ facilitates a formulation of the Hamiltonian eigenvalue problem for hadrons in QCD by simplifying the treatment of divergences stronger than logarithmic and providing useful logarithms of m_g. The logarithms lead to a computational definition of confinement as a feature of the Hamiltonian spectrum. Namely, the eigenvalues corresponding to colorless eigenstates have finite limits when $m_g \to 0$, while the eigenvalues for colored eigenstates tend to infinity. This definition is not new, but it is here arrived at by solving the RGPEP equation for the derivative of the Hamiltonian operator with respect to the scale parameter s. The initial condition at $s = 0$ is provided by the canonical QCD. The auxiliary field ϕ secures cancellation of the singularities $\sim 1/x^2$ that the gluon mass m_g leads to. Despite the limitations of the presented calculations, they do suggest a form of the first approximation to an effective theory for QCD hadrons. At the scales s corresponding to the binding mechanism, the renormalized effective Hamiltonian structure includes the free part, the Coulomb potentials, and the harmonic oscillator potentials. The potentials are coefficients of products of creation and annihilation operators of the effective quarks and gluons corresponding to the scale s.

The heuristic picture is derived in QCD of heavy quarks only and in the weak-coupling expansion for the running Hamiltonian including terms merely up to the second order. In that order, the coupling constant does not run. However, the approximate calculation suggests that the running of the coupling constant may extend the approximate picture to light quarks. As the coupling constant grows and the growth accelerates as s increases, the quark and gluon self-interactions proportional to g_s^2 grow as well, hopefully leading to the masses on the order of $1/3$ of the nucleon mass. The effective potentials would increase in strength, too, but their momentum range would be limited by the strengthened form factors f^s. The potentials would hence limit the

relative motion of the constituents. Such coherence of the interaction terms, if found, would simplify calculations of the renormalized effective Hamiltonian in fourth and higher orders of the weak coupling expansion, adjusting the RGPEP generator introduced in Sec. V A to most efficiently take the running of the coupling constant into account.

The Hamiltonian evolution with the scale parameter s relates description of a hadron in its center of mass system (CMS) in terms of constituents and in the infinite momentum frame (IMF) in terms of the QCD partons. The RGPEP form factor,

$$f^s = e^{-(s\Delta p^-)^2} = e^{-[(s/P^+)\Delta\mathcal{M}^2/x]^2}, \tag{230}$$

has the invariant mass width depending on the ratio s/P^+. Suppose one sets s to a value on the order of the inverse of $\Lambda_{\rm QCD}$ of the proper magnitude for the RGPEP scheme. In the CMS of a hadron, P^+ would equal the hadron mass m. The effective invariant mass squared width of the RGPEP vertex factor would be $m\Lambda_{\rm QCD}$. The renormalized effective Hamiltonian would be expressed in terms of the creation and annihilation operators corresponding to constituent quarks. Their wave functions would describe the relative motion with limited momenta. But in the IMF, where $P^+ \to \infty$, the width of f^s would become infinite, equivalent to $s = 0$, and f^s would become 1. The effective Hamiltonian interaction terms would appear equal to the canonical QCD Hamiltonian interaction terms. The wave functions would correspond to the parton model picture of the hadron. Since the invariant mass squared width is proportional to x, the small-x partons could only interact with small invariant mass changes, perhaps freezing in a saturated configuration. In other words, interactions of the quarks and gluons with large p^+ are little evolved in the RGPEP. Identified as partons, they interact according to the canonical QCD. By comparison, interactions of the quarks and gluons with small p^+, on the order of a hadron mass or its sizable fraction, are evolved a lot in the RGPEP towards the picture of constituent models, cf. [75,76].

The examples of effective Hamiltonian terms derived using our method suggest a way of thinking about the issue of vacuum effects in the FF of Hamiltonian dynamics. Such effects are extensively discussed in the literature, e.g., see [26,77–80]. In the FF of dynamics, one is inclined to expect that the vacuum modes are associated with momentum $p^+ = 0$, for the vacuum cannot carry momentum. Since the singularities for $p^+ \to 0$ require regularization and the associated divergences need counterterms, one may expect that the finite parts of these counterterms may be associated with the vacuum effects considered in the IF of dynamics [4]. However, the IF vacuum effects are primarily considered for the purpose of deriving the associated quark and gluon interactions. We note that the RGPEP does yield the effective interactions for $s > 0$ that grow with the distance, such as between a quark and an antiquark. Such interactions are obtained even if no vacuum effects are included at $s = 0$, which is the case in our approach with m_g and ϕ. We can hence hypothesize that the effective interactions that are associated with the vacuum in the IF of dynamics may appear in the FF of dynamics as a result of the RGPEP evolution of the effective Hamiltonian at $s > 0$, while no vacuum-induced interactions are included in the initial condition at $s = 0$. If so, instead of studying the vacuum effects through the regularization and computation of the counterterms, one may study the effective interactions using the RGPEP at $s > 0$ in the theory with $m_g \to 0$.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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