

Dirac one-loop seesaw mechanism with a noninvertible fusion rule

Hiroshi Okada^{1,*} and Labh Singh^{2,†}

¹*Department of Physics, Henan Normal University, Xinxiang 453007, China*

²*Department of Physics and Astronomical Science, Central University of Himachal Pradesh, Dharamshala, Himachal Pradesh 176215, India*



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We propose a radiative Dirac neutrino mass model stabilized by a noninvertible fusion rule originating from a $Z_3 \times Z'_3$ gauging. The imposed symmetry forbids tree-level Yukawa couplings and ensures that neutrino masses are generated only at the one-loop level through the exchange of exotic fermions and inert scalars. This minimal framework simultaneously accommodates neutrino masses and mixings consistent with current oscillation data, while providing a viable dark matter candidate. We analyze lepton flavor violating processes and lepton anomalous magnetic moments, finding that all contributions remain well below present experimental bounds. In the dark matter sector, the bosonic singlet emerges as a promising candidate with relic density compatible with cosmological observations, whereas the fermionic option is strongly disfavored due to suppressed annihilation cross sections. Our study demonstrates that noninvertible fusion rules can serve as a powerful organizing principle for constructing minimal and phenomenologically consistent extensions of the Standard Model, linking neutrino physics and dark matter within a unified radiative framework.

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I. INTRODUCTION

The origin of neutrino masses and the nature of dark matter (DM) are two of the most significant pieces of evidence for physics beyond the Standard Model (SM). While Majorana neutrinos are often discussed in the context of the seesaw mechanism, Dirac neutrinos offer an equally compelling framework. In fact, the Dirac versus Majorana nature of neutrinos remains experimentally unresolved, particularly, the nonobservation of neutrinoless double beta decay ($0\nu\beta\beta$) plays a crucial role, as dictated by the Schechter-Valle black box theorem [1]. The theorem states that the observation of $0\nu\beta\beta$ would unambiguously imply the existence of a Majorana mass term for neutrinos, independent of the underlying mechanism. Consequently, the continued null results in $0\nu\beta\beta$ searches provide a strong motivation to consider Dirac type-neutrino scenario.

In conventional Dirac-type scenarios [2–44], first discussed in Ref. [45], the imposition of lepton number conservation or an equivalent symmetry is typically

required. This necessity arises from forbidding the Majorana mass term, and symmetries such as continuous $U(1)$ or discrete groups have often been employed for this purpose. Recently, however, a novel class of symmetries based on noninvertible fusion rules has been developed and begun to find applications in phenomenology [46–73]. Unlike conventional group-based symmetries, these noninvertible structures possess distinctive features that go beyond group-theoretic framework. In particular, they can universally preserve invariance only at the tree-level, with no guarantee of stability once loop corrections are taken into account. Exploiting this feature reduces the number of additional particles required and enables the construction of more minimal loop-level models.

On the other hand, sub-eV scale of neutrino mass suggests that it is unlikely to the other three fermion sectors in the SM, and we expect some different mechanisms to suppress the neutrino masses. One of the attractive scenarios is that the neutrino masses are generated at loop-level. This can be understood that the neutrino does not directly interact with the SM Higgs but indirectly couples to the SM Higgs via new particles. By identifying a new particle as the DM candidate, we can construct a novel model that explains the tiny neutrino mass and DM simultaneously. Since the DM is also neutral under the electric charge and is not directly detected yet, it would be natural to communicate the DM with the active neutrinos.

In this paper, we propose a Dirac loop-seesaw mechanism by introducing vectorlike neutral fermions $\psi_{L,R}$ together

*Contact author: hiroshi3okada@htu.edu.cn

†Contact author: sainilabh5@gmail.com

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with an inert boson sector consisting of an isospin singlet S and doublet $\eta \equiv [\eta^+, \eta_0]^T$. These new particles generate one-loop Dirac neutrino masses and simultaneously provide a viable DM candidate. To forbid direct tree-level interactions among the active neutrinos $L_L \equiv [\nu_L, \ell_L]^T$, the right-handed neutrinos N_R , and the SM Higgs $H \equiv [h^+, (v + h + iz)/\sqrt{2}]^T$, we impose a noninvertible fusion rule, specifically Z_3 gauging of $Z_3 \times Z'_3$, which is one of the minimal realizations to our model. Following the phenomenological framework adopted in recent studies, we treat the fusion rule symmetry as an effective tree-level organizing principle whose associated operators can be generated radiatively at the loop level, thereby inducing the Dirac neutrino term $\overline{L}_L H N_R$ as desired. This framework provides a minimal setup for one-loop Dirac neutrino masses, as can be seen in our main text. We then investigate the associated phenomenology, including lepton flavor violations (LFVs), the lepton anomalous magnetic dipole moment (lepton $g - 2$), and DM feature, in addition to the neutrino masses and mixing patterns, through a detailed numerical analysis. We emphasize that the renormalization properties of four-dimensional quantum field theories with noninvertible symmetries are not yet completely understood and remain under active investigation. In particular, the treatment of loop-induced divergences and the associated counterterms for operators forbidden at tree level has been investigated in recent recent studies, including spurion-based approaches [74,75], although a complete understanding has not yet been achieved. In this work, we adopt a phenomenological viewpoint and regard the noninvertible fusion rule as an effective organizing principle for constructing radiative Dirac neutrino mass models. Accordingly, our analysis should be interpreted as an effective field theory framework for exploring the phenomenological implications of noninvertible symmetries, rather than as a claim of a fully established ultraviolet-complete framework. In particular, we assume that higher-loop corrections to the neutrino mass are suppressed by additional loop factors and small couplings, as in standard perturbative estimates. However, we note that this assumption has not been rigorously justified in the presence of noninvertible symmetries, and a systematic treatment of higher-loop effects remains an important open problem for future work.

The remainder of this paper is organized as follows. In Sec. II we present the Dirac neutrino mass model setup based on the noninvertible fusion rule and describe the particle content and interactions. Section III discusses the phenomenological implications, including lepton flavor violation, lepton $g - 2$, and DM scenarios. Section IV provides the numerical analysis consistent with neutrino oscillation and cosmological data. Finally, Sec. V offers concluding remarks and discussion.

II. THE MODEL SETUP

In this section, we review how to construct the one-loop Dirac seesaw model with new particles imposing the fusion rule of Z_3 gauging of $Z_3 \times Z'_3$ [76].

A. Z_3 gauging of $Z_3 \times Z'_3$ fusion rule

Before discussing our model, we explain why the Z_3 gauging of $Z_3 \times Z'_3$ fusion rule provides a minimal realization and review its mathematical properties. For convenience, we hereafter adopt the short-hand notation FR to denote the Z_3 gauging of $Z_3 \times Z'_3$ fusion rule, unless its detailed features are explicitly discussed. In our framework, we introduce two types of $U(1)_Y$ singlet neutral fermions ψ_R and N_R , which share the same chirality. Since both these particles carry identical quantum numbers under the SM gauge symmetries, it is necessary to distinguish them by assigning different generators under the FR . Furthermore, to preserve the Dirac nature of neutrinos, Majorana mass terms such as $\overline{\psi}_R^c \psi_R$ and $\overline{N}_R^c N_R$ has to be forbidden. This requirement implies that any Z_2 gauging fusion rules are disfavored, as they possess the feature $\mathbb{I} \in g \otimes g$, which inevitably allows Majorana mass terms. Here, g denotes a generator belonging to a Z_2 gauging fusion rule. Consequently, we need to employ fusion rules that satisfy

$$\mathbb{I} \notin g \otimes g, \quad \mathbb{I} \notin g' \otimes g', \quad \mathbb{I} \notin g \otimes g',$$

where $g(g')$ are assigned to be ψ_R or N_R . Moreover, $g(g')$ has to satisfy the condition $\mathbb{I} \in g \otimes g^*(g' \otimes g'^*)$ in order to allow the construction of the Dirac mass term $\overline{\psi}_L \psi_R$. Taking all these requirements into account, one finds that the Z_3 gauging of $Z_3 \times Z'_3$ fusion rule provides one of the most minimal symmetry realizations.¹

Z_3 gauging of $Z_3 \times Z'_3$ fusion rule consists of the following generators $\{1, a, a^*, b, b^*\}$, their multiplication rules are as follows:

$$a \otimes a = a^*, \quad a^* \otimes a^* = a, \quad a \otimes a^* = \mathbb{I}, \quad (1)$$

$$b \otimes b = b^*, \quad b^* \otimes b^* = b, \quad b \otimes b^* = \mathbb{I} \oplus a \oplus a^*, \quad (2)$$

$$a \otimes b = b, \quad a^* \otimes b = b, \quad a \otimes b^* = b^*, \quad a^* \otimes b^* = b^*. \quad (3)$$

B. Model setup

Our model setup is as follows. We introduce three generations of vectorlike neutral fermions $\psi_{L,R}$ and right-handed neutral fermions N_R , together with an inert scalar sector consisting of an electroweak singlet scalar S and an inert scalar doublet $\eta \equiv (\eta^+, \eta^0)^T$. Neither S nor η

¹Equivalently, there is another possibility to adopt Z_3 gauging of Z_9 fusion rule [76].

TABLE I. Charge assignments of the fields under the $SU(2)_L \times U(1)_Y$ gauge symmetry and our FR; Z_3 gauging of $Z_3 \times Z'_3$.

	L_L	ℓ_R	ψ_L	ψ_R	N_R	H	η	S
$SU(2)_L$	2	1	1	1	1	2	2	1
$U(1)_Y$	$-\frac{1}{2}$	-1	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
FR	a^*	a^*	b	b	a	$\mathbb{I}$	b	b

acquires a vacuum expectation value. These inert scalars participate in the one-loop generation of Dirac neutrino masses and can also furnish a dark matter candidate. New particles of $(\psi_{L,R}, N_R, S, \eta)$ are respectively assigned by (b, a, b, b) under the FR. In addition, the SM lepton

doublets L_L and the charged-lepton singlets ℓ_R are assigned by a^* . We summarize our particle contents and their assignments in Table I. Then, the relevant Yukawa Lagrangian for the lepton sector is given by

$$-\mathcal{L} = y_i^\ell \bar{L}_{Li} H \ell_{Ri} + f_{i\alpha} \bar{L}_{Li} \tilde{\eta} \psi_{R\alpha} + g_{\alpha a} \bar{\psi}_{L\alpha} N_{Ra} S + M_{\psi_\alpha} \bar{\psi}_{L\alpha} \psi_{L\alpha} + \text{H.c.}, \quad (4)$$

where $\tilde{\eta} \equiv i\sigma_2 \eta^*$, σ_2 being the second Pauli matrix. Without loss of generality, here, y^ℓ, M_ψ are diagonal basis. We note that $\bar{\psi}_{L,R}^C \psi_{L,R} S$ is also allowed under our assignments, but this term does not affect our model. We list forbidden terms from the FR as follows:

$$\bar{L}_L \tilde{H} N_R, \quad \bar{L}_L \tilde{H} \psi_R, \quad \bar{L}_L \tilde{H} \psi_L^C, \quad \bar{N}_R^C \psi_R, \quad \bar{\psi}_R^C \psi_R, \quad \bar{\psi}_L^C \psi_L, \quad \bar{N}_R^C N_R, \quad \bar{N}_R^C N_R S^{(*)}, \quad (5)$$

$$H^\dagger \eta, \quad (H^\dagger \eta)^2, \quad (H^\dagger \eta) S, \quad S^2, \quad (S^*)^2. \quad (6)$$

C. Higgs potential

Our relevant Higgs potential is given by

$$V = -\mu_H^2 |H|^2 + \mu_S^2 |S|^2 + \mu_\eta^2 |\eta|^2 + \kappa (S^3 + \text{c.c.}) + \mu (H^\dagger \eta S^* + \text{c.c.}) + \lambda_0 (H^\dagger \eta S^2 + \text{c.c.}) + \lambda_H |H|^4 + \lambda_S |S|^4 + \lambda_\eta |\eta|^4 + \lambda_{HS} |H|^2 |S|^2 + \lambda_{H\eta} |H|^2 |\eta|^2 + \lambda'_{H\eta} |H^\dagger \eta|^2 + \lambda_{S\eta} |S|^2 |\eta|^2. \quad (7)$$

Instead of full analysis of the Higgs potential, here, we simply extract our relevant term μ that associates with the Dirac neutrino mass terms. Our model requires mixing between the singlet scalar S and the neutral component of the inert doublet, η^0 . This mixing is induced by the trilinear interaction $\mu (H^\dagger \eta S^* + \text{H.c.})$ after electroweak symmetry breaking and plays a crucial role in generating the one-loop Dirac neutrino masses. Then, we define the mass eigenvectors H_1 and H_2 in terms of flavor eigenvectors S and η_0 and its mixing θ ;

$$S = c_\theta H_1 + s_\theta H_2, \quad \eta_0 = -s_\theta H_1 + c_\theta H_2, \quad (8)$$

where $s_\theta \sim \mu v / (m_1^2 - m_2^2)$, $m_{1,2}$ are mass eigenvalues of $H_{1,2}$.

D. One-loop Dirac neutrino mass generation

As shown in the one-loop diagram in Fig. 1, the Dirac mass is induced via the mixing of S and H . The resultant Dirac neutrino mass matrix is given by

$$(m_\nu)_{ia} \simeq \frac{s_\theta c_\theta}{(4\pi)^2} \sum_\alpha f_{i\alpha} F(m_1, m_2, M_{\psi_\alpha}) g_{a\alpha}, \quad (9)$$

where the loop function $F(m_1, m_2, M_{\psi_\alpha})$ is defined as

$$F(m_1, m_2, M_{\psi_\alpha}) = M_{\psi_\alpha} \left[\frac{m_1^2 \ln(m_1^2/M_{\psi_\alpha}^2)}{m_1^2 - M_{\psi_\alpha}^2} - \frac{m_2^2 \ln(m_2^2/M_{\psi_\alpha}^2)}{m_2^2 - M_{\psi_\alpha}^2} \right]. \quad (10)$$

For our convenience to a numerical analysis, we redefine the dimensionless neutrino mass matrix $\tilde{m}_\nu$ by extracting overall parameters as follows:

$$(\tilde{m}_\nu)_{ia} \equiv \kappa_\nu \sum_\alpha \tilde{f}_{i\alpha} \tilde{F}(\tilde{m}_1, \tilde{m}_2, \tilde{M}_{\psi_\alpha}) \tilde{g}_{a\alpha}, \quad (11)$$

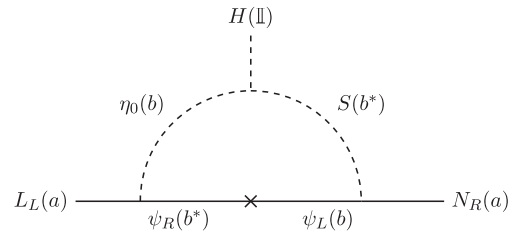


FIG. 1. One loop-induced Dirac neutrino masses. All vertices are invariant under FR. But one clearly finds that FR violates at one-loop, since $\mathbb{I} \otimes a \otimes a = a^* \neq \mathbb{I}$.

$$\tilde{F}(\tilde{m}_1, \tilde{m}_2, \tilde{M}_{\psi_\alpha}) \equiv \tilde{M}_\alpha \left[\frac{\tilde{m}_1^2 \ln(\tilde{m}_1^2/\tilde{M}_{\psi_\alpha}^2)}{\tilde{m}_1^2 - \tilde{M}_{\psi_\alpha}^2} - \frac{\tilde{m}_2^2 \ln(\tilde{m}_2^2/\tilde{M}_{\psi_\alpha}^2)}{\tilde{m}_2^2 - \tilde{M}_{\psi_\alpha}^2} \right], \quad (12)$$

$$\kappa_\nu \equiv \frac{s_\theta c_\theta M_{\psi_3} f_{33} g_{33}}{(4\pi)^2}, \quad (13)$$

where $f \equiv f_{33}\tilde{f}$, $g \equiv g_{33}\tilde{g}$, $M_\psi = M_{\psi_3}\tilde{M}_\psi$, $\tilde{m}_{1,2} \equiv m_{1,2}/M_{\psi_3}$, $f_{33}, g_{33}, M_{\psi_3}$ are respectively 3-3 components of f, g, M_ψ . From Eq. (13), f_{33} is not a independent parameter whose form is given by

$$f_{33} = \frac{(4\pi)^2}{s_\theta c_\theta M_{\psi_3} g_{33}} \kappa_\nu. \quad (14)$$

In our numerical analysis, we adopt the above form instead of f_{33} .

The neutrino mass eigenvalues $D_\nu \equiv \kappa_\nu \tilde{D}_\nu$ is diagonalized by $\tilde{D}_\nu = V_L^\dagger \tilde{m}_\nu V_R$. Since the charged-lepton mass matrix is diagonal basis, the observed mixing matrix is identical to V_L , which is denoted by U_{PMNS} . Then, κ_ν can be rewritten in terms of the solar mass squared difference Δm_{sol}^2 and dimensionless neutrino mass eigenvalues as follows:

$$\kappa_\nu^2 \equiv \frac{\Delta m_{\text{sol}}^2}{\tilde{D}_{\nu_2}^2 - \tilde{D}_{\nu_1}^2}. \quad (15)$$

In this case, the atmospheric mass squared difference Δm_{atm}^2 is given by

$$(\text{NH}): \Delta m_{\text{atm}}^2 \equiv \Delta m_{\text{sol}}^2 \frac{\tilde{D}_{\nu_3}^2 - \tilde{D}_{\nu_1}^2}{\tilde{D}_{\nu_2}^2 - \tilde{D}_{\nu_1}^2}, \quad (16)$$

$$(\text{IH}): \Delta m_{\text{atm}}^2 \equiv \Delta m_{\text{sol}}^2 \frac{\tilde{D}_{\nu_2}^2 - \tilde{D}_{\nu_3}^2}{\tilde{D}_{\nu_2}^2 - \tilde{D}_{\nu_1}^2}. \quad (17)$$

The sum of neutrino masses is constrained by the minimal standard cosmological model $\Lambda_{\text{CDM}} + \sum D_\nu$ that provides the upper bound $\sum D_\nu \leq 120$ meV [77,78], although it becomes weaker if the data are analyzed in

the context of extended cosmological models [79]. Recently, DESI and CMB data combination provides more stringent upper bound on the sum as $\sum D_\nu \leq 72$ meV [80]. Direct search for neutrino mass is done by the Karlsruhe Tritium Neutrino (KATRIN) experiment [81], which is the first sub-eV sensitivity on $m_{\nu_e}^2 = (0.26 \pm 0.34)$ eV² at 90% CL. Here, $m_{\nu_e}^2$ is defined by

$$\kappa_\nu^2 [(\tilde{D}_{\nu_1} \cos \zeta_{13} \cos \zeta_{12})^2 + (\tilde{D}_{\nu_2} \cos \zeta_{13} \sin \zeta_{12})^2 + (\tilde{D}_{\nu_3} \sin \zeta_{13})^2]. \quad (18)$$

$\zeta_{12,23,13}$ are mixing angles for U_{PMNS} in the standard parametrization. Each of mixing angles are obtained by

$$\begin{aligned} \sin \zeta_{13} &= |(U_{\text{PMNS}})_{13}|, & \tan \zeta_{23} &\equiv \frac{(U_{\text{PMNS}})_{23}}{(U_{\text{PMNS}})_{33}}, \\ \tan \zeta_{12} &\equiv \frac{(U_{\text{PMNS}})_{12}}{(U_{\text{PMNS}})_{11}}. \end{aligned} \quad (19)$$

In numerical analysis, we will adopt NuFit 6.1 [82] to experimental ranges of neutrino observables in addition to the above constraints of neutrino masses.

III. PHENOMENOLOGY

A. Lepton flavor violations and lepton $g-2$

Branching ratio of $\ell_i \rightarrow \ell_j \gamma$ and lepton $g-2$ are arisen from the same coupling of $f_{i\alpha}$ at the one-loop level, and their forms are given by

$$\text{BR}(\ell_i \rightarrow \ell_j \gamma) \approx \frac{48\pi^3 \alpha_{\text{em}}}{G_F^2 m_{\ell_i}^2} C_{ij} (|a_{Lij}|^2 + |a_{Rij}|^2), \quad (20)$$

$$\Delta a_{\ell_i} \approx -\frac{m_{\ell_i}}{2} (a_{Lii} + a_{Rii}), \quad (21)$$

where $\ell_1 \equiv e$, $\ell_2 \equiv \mu$, $\ell_3 \equiv \tau$, $\alpha_{\text{em}} \approx 1/137$ is the fine structure constant, and $G_F \approx 1.17 \times 10^{-5}$ GeV⁻² is the Fermi constant. C_{ij} is defined by $C_{21} \approx 1$, $C_{31} \approx 0.1784$, $C_{32} \approx 0.1736$, otherwise C_{ab} is zero. Here $a_{L,R}$ are given by

$$a_{Lij} = -m_{\ell_i} \sum_{\alpha=1}^3 \frac{f_{i\alpha} f_{\alpha j}^\dagger}{(4\pi)^2} \int [dx]_3 \frac{xz}{(x+y)m_\eta^2 + zM_{\psi_\alpha}^2 - xzm_{\ell_i}^2 - yzm_{\ell_j}^2} \quad (22)$$

$$\approx -m_{\ell_i} \sum_{\alpha=1}^3 \frac{\tilde{f}_{i\alpha} \tilde{f}_{\alpha j}^\dagger |f_{33}|^2}{(4\pi)^2 M_{\psi_3}^2} \int [dx]_3 \frac{xz}{(x+y)\tilde{m}_\eta^2 + z\tilde{M}_{\psi_\alpha}^2}, \quad (23)$$

$$a_{Rij} = -m_{\ell_j} \sum_{\alpha=1}^3 \frac{f_{i\alpha} f_{\alpha j}^\dagger}{(4\pi)^2} \int [dx]_3 \frac{yz}{(x+y)m_\eta^2 + zM_{\psi_\alpha}^2 - xzm_{\ell_i}^2 - yzm_{\ell_j}^2} \quad (24)$$

$$\approx -m_{\ell_j} \sum_{a=1}^3 \frac{\tilde{f}_{ia} \tilde{f}_{aj}^\dagger |f_{33}|^2}{(4\pi)^2 M_{\psi_3}^2} \int [dx]_3 \frac{yz}{(x+y)\tilde{m}_\eta^2 + z\tilde{M}_{\psi_a}^2}, \quad (25)$$

where $[dx]_3 \equiv dx dy dz \delta(1-x-y-z)$. The current upper bounds on $\text{BR}(\ell_a \rightarrow \ell_b \gamma)$ are given at 90% CL as [83–86]

$$\text{BR}(\mu \rightarrow e \gamma) < 1.5 \times 10^{-13}, \quad \text{BR}(\tau \rightarrow e \gamma) < 3.3 \times 10^{-8}, \quad \text{BR}(\tau \rightarrow \mu \gamma) < 4.2 \times 10^{-8}. \quad (26)$$

The lepton $g-2$ receives the following experimental measurements [87–90]:

$$\Delta a_e = (3.41 \pm 1.64) \times 10^{-13}, \quad \Delta a_\mu = (39 \pm 64) \times 10^{-11}. \quad (27)$$

Here, Δa_τ is not relevant since it is less precise compared with $\Delta a_{e,\mu}$. Therefore, we neglect analysis of Δa_τ in the numerical analysis.

B. Dark matter analysis

In this scenario, the model admits two viable DM candidates: bosonic state S and a fermionic state ψ_1 . We notice that η_0 cannot be a DM candidate since it interacts with the Z -boson, which induces sizable spin-independent elastic scattering cross sections off nuclei. The corresponding scattering rates exceed the stringent bounds imposed by current direct detection experiments, thereby excluding η_0 as a phenomenologically acceptable DM candidate.

1. Bosonic DM case: S state

Since S does not couple to Standard Model fields through Yukawa interactions, its only portal to the visible sector arises *via* scalar interactions in the Higgs potential. In such a case, detailed analysis is already explored, and the correct relic density of DM is obtained near the half of Higgs mass by the resonant enhancement [91] without conflict with the direct detection bounds. Therefore,

$$m_S \approx \frac{m_h}{2} \approx 63 \text{ GeV}. \quad (28)$$

In our numerical analysis, we adopt this fixed mass for $m_S < M_{\psi_1}$.

Here, it must be noted that S can decay into H and η through one-loop level, when this process is kinematically allowed. The corresponding terms are κ and λ_0 and the decay rate (Γ_s) is found as

$$\Gamma_s = \frac{|\mu_{SH\eta}|^2}{16\pi m_S^2} \sqrt{1 - 4 \frac{m_h^2}{m_S^2}}, \quad (29)$$

$$\mu_{SH\eta} = -\frac{\kappa \lambda_0^*}{16\pi^2} \left[2 + \frac{1+r}{1-r} \ln(r) \right], \quad (30)$$

where $r \equiv m_S^2/\Lambda^2$, Λ being cutoff scale, putting $m_\eta \sim m_h$ for simplicity. In order to satisfy that DM has to be longer

the age of Universe $\mathcal{O}(10^{41}) \text{ GeV}^{-1} \lesssim \Gamma_s^{-1}$, the following condition under $m_S = (m_h - 1)/2$ has to be satisfied.

$$\Lambda = 10^5 \text{ GeV}, \quad \kappa \lambda_0^* \lesssim 6.47 \times 10^{-18} \text{ GeV}, \quad (31)$$

$$\Lambda = 10^{10} \text{ GeV}, \quad \kappa \lambda_0^* \lesssim 2.31 \times 10^{-18} \text{ GeV}, \quad (32)$$

$$\Lambda = 10^{19} \text{ GeV}, \quad \kappa \lambda_0^* \lesssim 1.07 \times 10^{-18} \text{ GeV}. \quad (33)$$

2. Fermionic DM case: ψ_1 state

Although the fermionic DM can decay into two Higgs pair plus lepton through the Yukawa coupling f and the same diagram as the D decay process, we suppose that the fermionic dark matter candidate is effectively stable on cosmological timescales since its lifetime exceeds the age of the Universe once the S decay process is suppressed enough as discussed in the bosonic DM section. Then, the mass of X is given by M_{ψ_1} . Since X does not directly couple to quark sector, we can neglect the constraints of direct detection searches. The relevant annihilation processes are $X\bar{X} \rightarrow \ell_i \bar{\ell}_j (\nu_\sigma \bar{\nu}_\rho)$ and they are s-wave dominant due to the Dirac fermion DM. The thermally averaged cross section in the limit of massless final states is given by

$$\begin{aligned} \langle \sigma v \rangle (X\bar{X} \rightarrow \nu_\sigma \bar{\nu}_\rho) & \approx \frac{|f_{33}|^4}{M_{\psi_3}^2} \frac{\tilde{M}_{\psi_1}^2}{32\pi} \frac{c_\theta^2(\tilde{m}_1^2 + \tilde{M}_{\psi_1}^2) + s_\theta^2(\tilde{m}_2^2 + \tilde{M}_{\psi_1}^2)}{(\tilde{m}_1^2 + \tilde{M}_{\psi_1}^2)^2 (\tilde{m}_2^2 + \tilde{M}_{\psi_1}^2)^2} \\ & \times \sum_{\sigma,\rho=1}^3 |\tilde{f}'_{\sigma 1} \tilde{f}'_{1\rho}|^2, \end{aligned} \quad (34)$$

$$\langle \sigma v \rangle (X\bar{X} \rightarrow \ell_i \bar{\ell}_j) \approx \frac{|f_{33}|^4}{M_{\psi_3}^2} \frac{\tilde{M}_{\psi_1}^2}{32\pi (\tilde{m}_\eta^2 + \tilde{M}_{\psi_1}^2)^2} \sum_{\sigma,\rho=1}^3 |\tilde{f}_{i1} \tilde{f}_{1j}^\dagger|^2, \quad (35)$$

where $f'_{a1} \equiv \sum_{i=1}^3 (U_{\text{PMNS}}^\dagger)_{ai} f_{i1}$. The value of cross section to satisfy the observed relic density of DM is approximately

given by

$$1.776 \lesssim \frac{\langle \sigma v \rangle \times 10^9}{\text{GeV}^{-2}} \lesssim 1.970, \quad (36)$$

at 2σ confidence level (CL) that corresponds to $\Omega h^2 = 0.1196 \pm 2 \times 0.0031$ [92]. Spin independent scattering cross section can be found at one-loop level via gauge bosons such as photon and Z boson. However, it can be negligible when the Yukawa couplings are of the order 10^{-3} .

IV. NUMERICAL ANALYSIS

In this section, we perform a $\Delta\chi^2$ numerical analysis. Here, we assume Gaussian distribution with four observables; $[s_{12}, s_{13}, \Delta m_{\text{sol}}^2, \Delta m_{\text{atm}}^2]$ from NuFit 6.1 [82], as follows:

$$\Delta\chi^2 = \sum_i \left(\frac{O_i^{\text{obs}} - O_i^{\text{th}}}{\delta O_i^{\text{exp}}} \right)^2, \quad (37)$$

where $O_i^{\text{obs(th)}}$ is observed (theoretically obtained) value of corresponding observables and δO_i^{exp} expresses experimental error. Four degrees of freedom lead us to $\Delta\chi^2 = 4.72$ at 1σ , $\Delta\chi^2 = 9.72$ at 2σ , $\Delta\chi^2 = 16.25$ at 3σ , $\Delta\chi^2 = 24.50$ at 4σ , and $\Delta\chi^2 = 34.56$ at 5σ , respectively. It is notable that we do not consider Dirac phase denoted by δ_{CP} and s_{23} as output parameters. Because any values of δ_{CP} are allowed by experiments at 3σ CL. While s_{23} is deviated from the Gaussian distribution and we independently impose to be the following experimental constraint

$$0.432 \leq s_{23}^2 \leq 0.587 \quad \text{for NH}, \quad (38)$$

$$0.437 \leq s_{23}^2 \leq 0.590 \quad \text{for IH}, \quad (39)$$

within the range of 3σ CL in NuFit 6.1. In a numerical analysis, we plot the allowed parameter space with green (red) points within $5(3)\sigma$ CL. In addition to the allowed regions in NuFit 6.1, we take into account the recent experimental result from JUNO [93] at the 1σ CL:

$$\sin^2\theta_{12} = 0.3092 \pm 0.0087, \quad \frac{\Delta m_{\text{sol}}^2}{10^{-5} \text{ eV}^2} = 7.50 \pm 0.12. \quad (40)$$

In our numerical analysis below, blue circles are shown when both experimental constraints are satisfied.

We randomly select our viable input parameters as follows:

$$[\tilde{f}_{11,12,13,21,22,23,31,32}, \tilde{g}_{11,12,13,21,22,23,31,32}, g_{33}] = [10^{-3}, \sqrt{4\pi}], \quad s_\theta = [-1, 1], \quad (41)$$

$$M_{\psi_3} = [10, 10^5] \text{ GeV}, \quad [\tilde{M}_{\psi_1}, \tilde{M}_{\psi_2}, \tilde{m}_1, \tilde{m}_2] = [10^{-5}, 0], \quad (42)$$

where we set $m_\eta = m_2$ simply to evade the constraint of oblique parameters.

A. NH

1. Bosonic DM: S state

In case of bosonic DM, we impose the following condition

$$\tilde{m}_1 \leq 1.2 \times (\tilde{M}_{\psi_1}, \tilde{m}_2). \quad (43)$$

where 1.2 is roughly evaluated by suppressing coannihilation cross sections [94].

Fig. 2, we show the allowed regions of $\tilde{m}_{1,2}$ (upper left panel), $\tilde{M}_{\psi_{1,2}}$ (upper right panel), $|f_{33}| - |g_{33}|$ (lower left panel), and $\sum D_\nu - |\kappa_\nu|$ in meV unit (lower right panel), respectively. The dashed horizontal line in the lower right plot represents the upper bound on $\sum D_\nu \leq 72$ meV. The red color is within 5σ CL, which corresponds to $\Delta\chi^2 \approx 34.56$, while the blue circle satisfies the recent experimental result of JUNO in addition to the NuFit 6.1 within 5σ CL. One straightforwardly finds that the allowed parameter space is drastically reduced one the JUNO results are taken into account. JUNO results suggest that $\sum D_\nu$ is localized at nearby 59–64 meV, which is totally safe for the experimental upper bounds from DESI.

In Fig. 3, we show the allowed regions of branching ratios of $\tau \rightarrow \mu\gamma$ and $\mu \rightarrow e\gamma$ (upper left panel), $\tau \rightarrow e\gamma$ and $\mu \rightarrow e\gamma$ (upper right panel), lepton $g-2$ of Δa_μ and Δa_e (lower left panel), and $\sum D_\nu - \delta_{CP}$ in meV unit (lower right panel), respectively. All the color legends and the line are the same as the ones in Fig. 2. These figures show that the cLFVs and lepton $g-2$ are far below the experimental upper bounds. Therefore, they are unlikely to be tested in the foreseeable future.

2. Fermionic DM: ψ_1 state

In case of fermionic DM, we impose the following condition

$$\tilde{M}_{\psi_1} \leq 1.2 \times (\tilde{M}_{\psi_2}, \tilde{m}_1, \tilde{m}_2). \quad (44)$$

where 1.2 is rough estimation to suppress the coannihilation cross sections.

Fig. 4, we show the allowed regions of $\tilde{m}_{1,2}$ (upper left panel), $\tilde{M}_{\psi_{1,2}}$ (upper right panel), $|f_{33}| - |g_{33}|$ (lower left panel), and $\sum D_\nu - |\kappa_\nu|$ in meV unit (lower right panel), respectively. All the color legends and the line are the same as the ones in Fig. 2. Specifically, there is a strong correlation between $|g_{33}|$ and $|f_{33}|$, which is different from

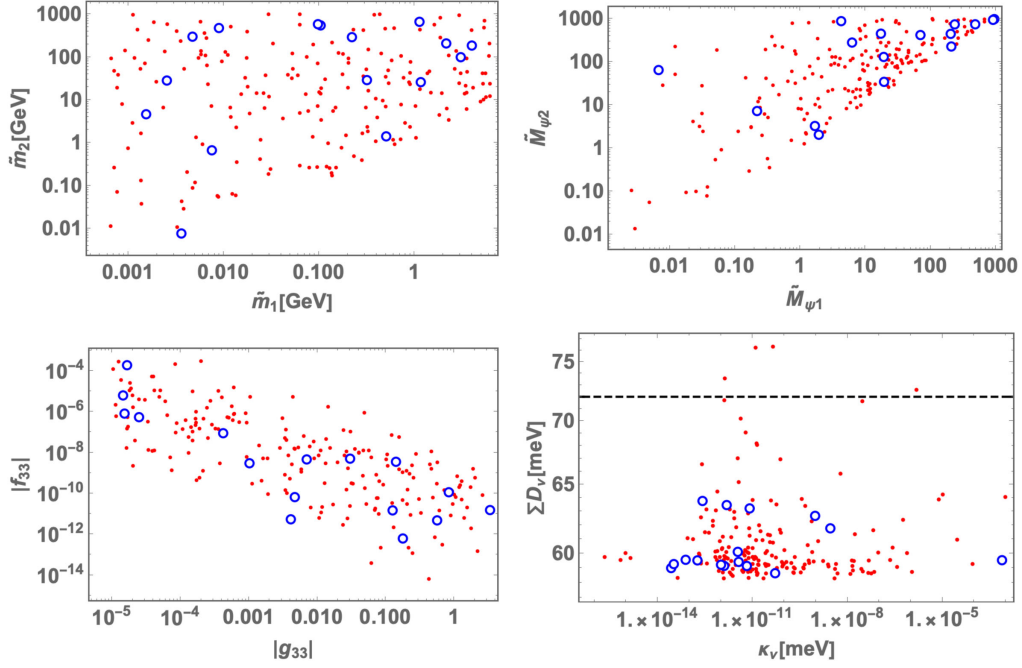


FIG. 2. The allowed regions of $\tilde{m}_{1,2}$ (upper left panel), $\tilde{M}_{\psi_{1,2}}$ (upper right panel), $|f_{33}| - |g_{33}|$ (lower left panel), and $\sum D_\nu - |\kappa_\nu|$ in meV unit (lower right panel), respectively. The dashed horizontal line in the lower right plot represents the upper bound on $\sum D_\nu \leq 72$ meV. As we explained in the first part of this section, the red color is within 5σ CL. The blue circle satisfies the recent experimental result of JUNO in addition to the Nufit 6.1.

the case of BDM in NH. JUNO results suggest that $\sum D_\nu$ is localized at nearby 59–68 meV, which is still safe for the experimental upper bounds from DESI.

In Fig. 5, we show the allowed regions of branching ratios of $\tau \rightarrow \mu\gamma$ and $\mu \rightarrow e\gamma$ (upper left panel), $\tau \rightarrow e\gamma$ and $\mu \rightarrow e\gamma$ (upper right panel), lepton $g-2$ of Δa_μ and Δa_e

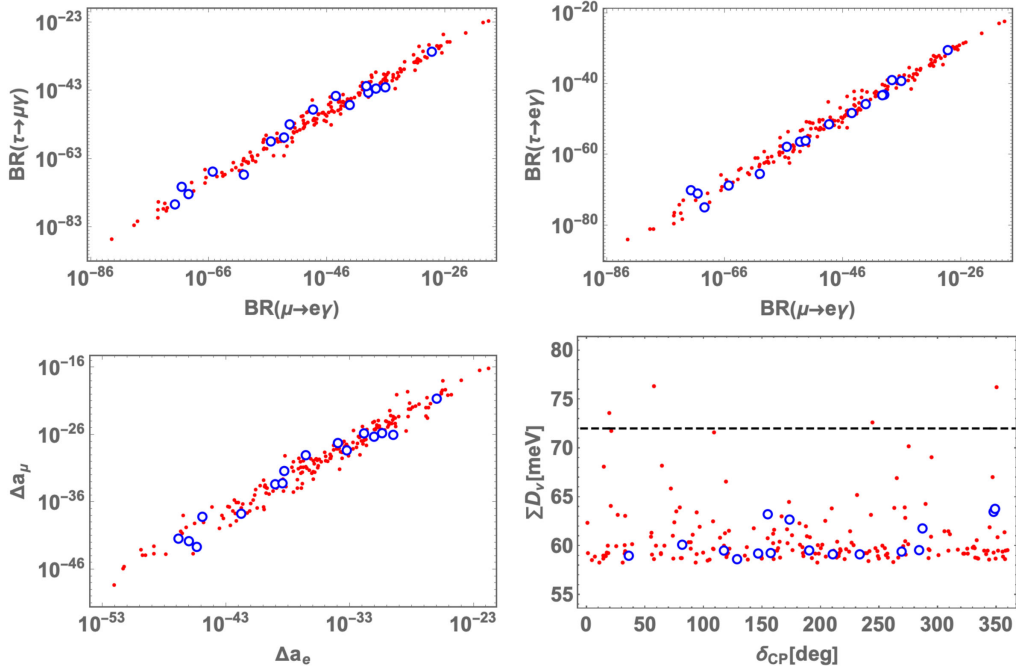


FIG. 3. Allowed regions of branching ratios of $\tau \rightarrow \mu\gamma$ and $\mu \rightarrow e\gamma$ (upper left panel), $\tau \rightarrow e\gamma$ and $\mu \rightarrow e\gamma$ (upper right panel), lepton $g-2$ of Δa_μ and Δa_e (lower left panel), and $\sum D_\nu - \delta_{CP}$ in meV unit (lower right panel), respectively. All the color legends and the line are the same as the ones in Fig. 2.

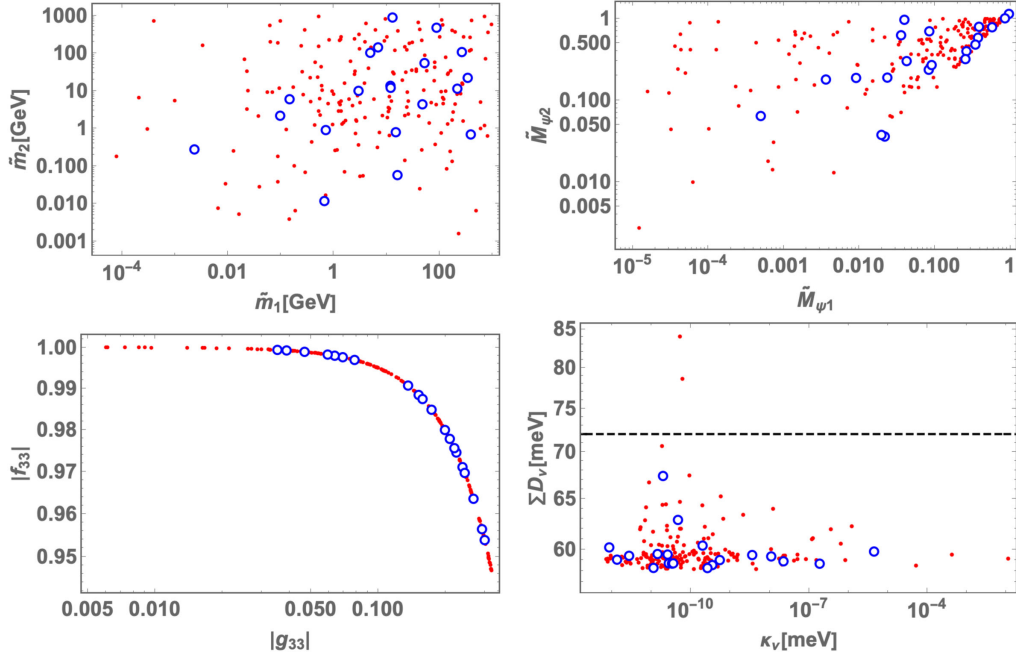


FIG. 4. The allowed regions of $\tilde{m}_{1,2}$ (upper left panel), $\tilde{M}_{\psi 1,2}$ (upper right panel), $|f_{33}| - |g_{33}|$ (lower left panel), and $\sum D_\nu - |\kappa_\nu|$ in meV unit (lower right panel), respectively. All the color legends and the line are the same as the ones in Fig. 2.

(lower left panel), and $\sum D_\nu - \delta_{CP}$ in meV unit (lower right panel), respectively. All the color legends and the line are the same as the ones in Fig. 2. These figures show that the predicted cLFV branching ratios and lepton $g-2$ are similar to the case of BDM, therefore it is difficult to test experimentally.

In Fig. 6, we show the allowed region of the DM cross section to explain the relic density in terms of DM mass. All the color legends and the line are the same as the ones in Fig. 2. This plot indicates that fermionic DM candidate is disfavored since the obtained cross section is much lower

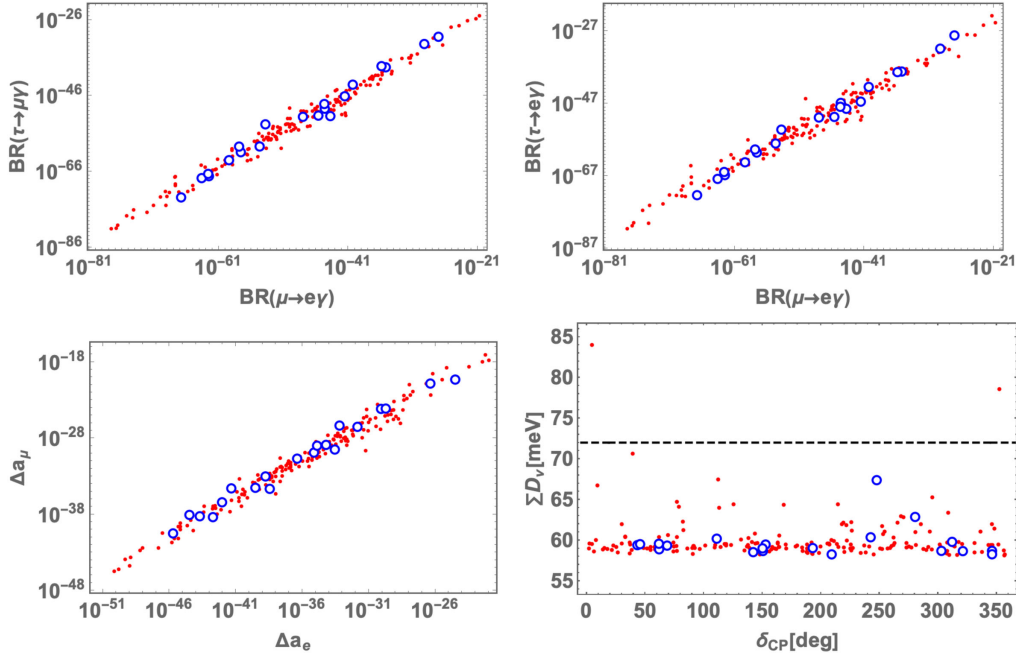


FIG. 5. Allowed regions of branching ratios of $\tau \rightarrow \mu \gamma$ and $\mu \rightarrow e \gamma$ (upper left panel), $\tau \rightarrow e \gamma$ and $\mu \rightarrow e \gamma$ (upper right panel), lepton $g-2$ of Δa_μ and Δa_e (lower left panel), and $\sum D_\nu - \delta_{CP}$ in meV unit (lower right panel), respectively. All the color legends and the line are the same as the ones in Fig. 2.

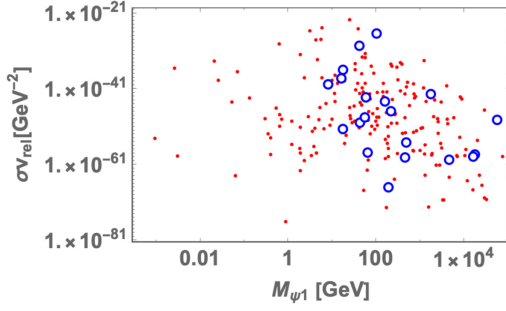


FIG. 6. Allowed region of the DM cross section to explain the relic density in terms of DM mass. All the color legends and the line are the same as the ones in Fig. 2.

than the required cross section $\sim 10^{-9} \text{ GeV}^{-2}$ to explain the relic density.

B. IH

It was found that unlike NH, the IH scenario does not yield a unique solution unless the parameter space is constrained. Therefore, rather than performing a global analysis as done for NH, we present a specific benchmark (BP) for IH.

1. Bosonic DM: S state

Table II lists the BP that minimizes $\Delta\chi^2$, where our DM is fixed to be half of the SM Higgs mass $\sim 63 \text{ GeV}$ and the resulting minimum value is approximately 0.200. The corresponding values for $\tilde{g}$ and $\tilde{f}$ are also provided as follows:

$$\tilde{g} \times 10^3 \approx \begin{pmatrix} -0.890644 + 1.97046i & -229.461 + 164.759i & -1.46252 + 1.0415i \\ 114.175 - 22.8643i & 1.34085 + 42.0388i & -1326.2 - 1491.42i \\ -209.66 - 876.698i & -437.105 + 40.0709i & 10^3 \end{pmatrix}, \quad (45)$$

$$\tilde{f} \times 10^3 \approx \begin{pmatrix} -1009.05 - 1888.65i & -3.17909 - 3.85315i & -1.63715 - 0.329819i \\ 3.26515 - 152.467i & 171.427 - 88.4997i & -29.766 - 109.796i \\ -9.45895 - 4.10588i & 25.7098 - 265.766i & 10^3 \end{pmatrix}. \quad (46)$$

Consequently, the benchmark point simultaneously accommodates neutrino oscillation data and all phenomenological constraints. Since the bosonic singlet mass is fixed near the Higgs-resonance region, the singlet scalar remains a viable dark matter candidate in the IH case as well.

2. Fermionic DM: ψ_1 state

Table III lists the BP that minimizes $\Delta\chi^2$. The resulting minimum value is approximately 0.148. The corresponding values for $\tilde{g}$ and $\tilde{f}$ are also provided as follows:

$$\tilde{g} \times 10^3 \approx \begin{pmatrix} 75.3899 - 79.9135i & -1.10263 + 0.0792865i & 0.0115587 - 0.0801839i \\ -7.77716 + 26.6837i & -459.569 - 3588.96i & -2.9311 + 11.6813i \\ 0.0249052 + 0.108523i & 8.89349 + 16.6716i & 10^3 \end{pmatrix}, \quad (47)$$

TABLE II. BF dataset in case of bosonic DM in IH, where $\Delta\chi^2 \approx 0.200$.

Parameter	BF	Parameter	BF	Parameter	BF
$\tilde{M}_{\psi_1}$	164	$\tilde{M}_{\psi_2}$	142	$M_{\psi_3}/[\text{GeV}]$	92.7
$\tilde{m}_1$	0.676	$\tilde{m}_2$	49.6	s_θ	0.0273
$g_{33} \times 10^5$	$-9.51 + 4.78i$	$f_{33} \times 10^7$	$11.0 + 5.55i$	$\kappa_\nu^2/[\text{GeV}^2]$	4.41×10^{-24}
...	...	Δa_e	5.59×10^{-30}	Δa_μ	9.12×10^{-27}
$\text{BR}(\mu \rightarrow e\gamma)$	1.87×10^{-37}	$\text{BR}(\tau \rightarrow e\gamma)$	1.35×10^{-43}	$\text{BR}(\tau \rightarrow \mu\gamma)$	1.66×10^{-41}
s_{12}	0.554	s_{23}	0.752	s_{13}	0.150
Δm_{sol}^2	$7.52 \times 10^{-5} \text{ eV}^2$	Δm_{atm}^2	$2.51 \times 10^{-3} \text{ eV}^2$	δ_{CP}	29.5°
$\sum D_\nu$	112 meV	...	...	...	...

TABLE III. BF dataset in case of fermionic DM in IH, where $\Delta\chi^2 \approx 0.148$.

Parameter	BF	Parameter	BF	Parameter	BF
$\tilde{M}_{\psi_1}$	2.14×10^{-5}	$\tilde{M}_{\psi_2}$	0.961	$M_{\psi_3}/[\text{GeV}]$	4.35×10^3
$\tilde{m}_1$	2.41×10^{-5}	$\tilde{m}_2$	3.54	s_θ	0.127
$g_{33} \times 10^4$	$-3.85 - 1.05i$	$f_{33} \times 10^9$	$-8.82 + 2.40i$	$\kappa_\nu^2/[\text{GeV}^2]$	1.59×10^{-22}
$\frac{\langle\sigma v_{\text{rel}}\rangle}{\text{GeV}^{-2}}$	2.90×10^{-35}	Δa_e	4.21×10^{-34}	Δa_μ	1.97×10^{-30}
$\text{BR}(\mu \rightarrow e\gamma)$	2.52×10^{-48}	$\text{BR}(\tau \rightarrow e\gamma)$	2.10×10^{-50}	$\text{BR}(\tau \rightarrow \mu\gamma)$	2.96×10^{-50}
s_{12}	0.553	s_{23}	0.697	s_{13}	0.150
Δm_{sol}^2	$7.54 \times 10^{-5} \text{ eV}^2$	Δm_{atm}^2	$2.50 \times 10^{-3} \text{ eV}^2$	δ_{CP}	173°
$\sum D_\nu$	99.2 meV	...	...	...	...

$$\tilde{f} \times 10^3 \approx \begin{pmatrix} 1832.83 + 2268.43i & -97.301 + 383.696i & -1.19209 + 1.64211i \\ 7.70775 - 0.953411i & -26.8426 + 38.648i & -900.194 - 499.272i \\ 30.3331 - 16.0167i & 5.87214 + 38.3581i & 10^3 \end{pmatrix}. \quad (48)$$

As shown in this Table III, the cross section required to account for the observed relic density of the FDM case is significantly smaller than the expected value of $\sim 10^{-9} \text{ GeV}^{-2}$. Consequently, the FDM cannot be the dominant component of DM in the IH case either.

V. CONCLUSION AND DISCUSSION

In this paper, we have constructed a radiative Dirac neutrino mass model based on a noninvertible fusion rule derived from Z_3 gauging of $Z_3 \times Z'_3$ fusion rule. The proposed framework forbids tree-level Yukawa couplings and ensures that neutrino masses arise only at the one-loop level, thereby realizing a minimal Dirac seesaw mechanism. The model introduces vectorlike neutral fermions and inert scalar fields, which simultaneously generate neutrino masses and provide a viable DM candidate. Phenomenological analyses demonstrate that:

- (1) The model is consistent with current neutrino oscillation data, including NuFit 6.1 and JUNO constraints.
- (2) Lepton flavor violating processes such as $\mu \rightarrow e\gamma$ remain well below experimental bounds.
- (3) Contributions to the lepton anomalous magnetic moments are small, ensuring compatibility with precision measurements.
- (4) The bosonic singlet S emerges as a promising DM candidate, while the fermionic option is strongly disfavored due to insufficient annihilation cross sections.

The significance of this work lies in demonstrating that noninvertible fusion rules can serve as a powerful organizing principle for constructing minimal and phenomenologically viable extensions of the Standard Model. By forbidding Majorana mass terms while allowing loop-

induced Dirac masses, the framework provides a natural explanation for the smallness of neutrino masses and establishes a direct connection to DM physics. Although the model successfully satisfies all current experimental constraints, its testability remains limited. The predicted branching ratios for charged lepton flavor violation and the contributions to lepton $g-2$ are far below present sensitivities, making near-term experimental verification challenging. Nevertheless, the bosonic DM candidate offers a concrete scenario that could be probed through precision cosmology and collider searches. Future directions include exploring collider signatures of the inert scalar and exotic fermions, investigating possible connections to baryogenesis, and examining alternative fusion rules. These extensions may broaden the phenomenological implications and provide new avenues for experimental tests.

In conclusion, the proposed Dirac one-loop seesaw model with noninvertible fusion rules offers a consistent and minimal framework that simultaneously addresses neutrino masses and DM, while remaining fully compatible with current experimental data. It establishes a foundation for further theoretical developments and motivates future experimental efforts to probe the underlying symmetry structures beyond the Standard Model.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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