

Reciprocal symmetry and Koba-Nielsen-Olesen scaling violation in proton-proton collisions

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We analyze the charged particle multiplicity distributions in $p - p$ collisions and discuss the violation of the Koba-Nielsen-Olesen (KNO) scaling. We extract the deviations from the leading exponential behavior of the KNO scaled probability and identify a reciprocal symmetry $z \leftrightarrow 1/z$ in the KNO violating corrections observed in the ATLAS and CMS data at $\sqrt{s} = 7, 8, 13$ TeV. The symmetry imposes a local constraint on the multiplicity distribution at $n = \langle n \rangle$, namely $P'(\langle n \rangle) = -P(\langle n \rangle)/\langle n \rangle$, which we verify directly in the data. We use this constraint to extract the entanglement entropy from the well-measured region $n \simeq \langle n \rangle$, avoiding the large uncertainties associated with the distribution tail.

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I. INTRODUCTION

Charged particle production in high-energy collisions provides insight into the underlying QCD dynamics. The Koba-Nielsen-Olesen (KNO) scaling hypothesis [1,2] states that at high energies the multiplicity distribution depends only on the scaled variable $z = n/\langle n \rangle$. The experimental data for $p - p$ collisions from ATLAS [3] (reanalyzed in [4]) and CMS [5] show clear violations of this scaling.

Charged particle multiplicity distributions have recently been studied in the framework of quantum entanglement, following the proposal of Kharzeev and Levin [6,7] (see also [8]) that the partonic state probed in deep inelastic scattering (DIS) is a maximally entangled state with entropy related to final-state hadron multiplicity. This proposal has been examined through analyses of entanglement at subnucleonic scales [9], the H1 measurement [10], Balitsky-Fadin-Kuraev-Lipatov (BFKL)-based comparisons [11,12], diffractive DIS [13], QCD evolution [14], Monte Carlo studies [15], and dipole cascade models [16,17]. Several recent works have proposed alternatives or extensions, including a diffusion-scaling framework for high-multiplicity events [18], dipole evolution at high energy [19], and a Gaussian color-charge action at saturation [20].

In the present paper we identify a reciprocal symmetry $f_s(z) = f_s(1/z)$ in the KNO violating term, where $f_s(z)$ is

defined as the relative deviation of $\langle n \rangle P_n$ from the leading exponential e^{-z} . The symmetry holds in the range $1/3 < z < 3$ for the ATLAS and CMS data at $\sqrt{s} = 7, 8, 13$ TeV. As a direct consequence of the symmetry, the derivative of $f_s(z)$ vanishes at $z = 1$, which translates into a local relation $P'(\langle n \rangle) = -P(\langle n \rangle)/\langle n \rangle$ between the value of the multiplicity distribution and its derivative at $n = \langle n \rangle$. We verify this relation in the data at the few-percent level and use it to extract the entanglement entropy from the well-measured region $n \simeq \langle n \rangle$, avoiding the large uncertainties associated with the distribution tail.

II. DEVIATION FROM THE EXPONENTIAL DISTRIBUTION

The probability P_n describes the multiplicity distribution of the produced charged particles. The KNO scaling hypothesis [1,2] states that at high energies the probability is given by

$$P_n = \frac{1}{\langle n \rangle} \psi(z), \quad (1)$$

where $\psi(z)$ is a function of only one variable $z = n/\langle n \rangle$. Deviations from this simple scaling form are usually referred to as KNO scaling violation. The theoretical models and the experimental data for $p - p$ collisions by ATLAS and CMS show the leading exponential behavior of the scaled probability

$$\langle n \rangle P_n \simeq e^{-z} \left(1 + \mathcal{O}\left(\frac{1}{\langle n \rangle}\right) \right), \quad (2)$$

where $z = n/\langle n \rangle$ is the scaled variable that denotes the deviation from the average $\langle n \rangle$.

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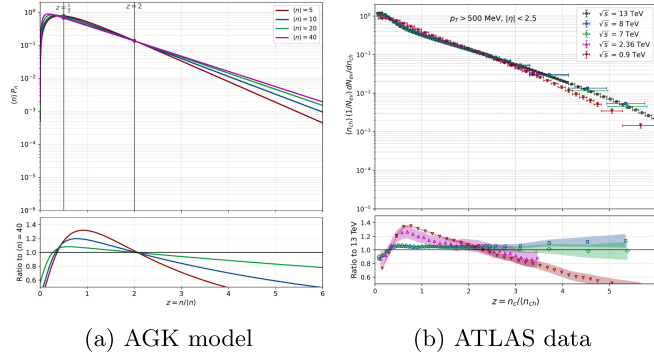


FIG. 1. Figure shows the plot of $\langle n \rangle P_n$ as a function of the scaled variable $z = n/\langle n \rangle$ for the AGK model (a) and the corresponding plot of the ATLAS experimental data (b) for the charged multiplicity distribution at p - p collisions at $\sqrt{s} = 0.9, 2.36, 7, 8, 13$ TeV for $p_t > 500$ MeV and $|\eta| < 2.5$. In both plots we use the averages $\langle n \rangle$ calculated in [4]. The intersection points $z = 2$ and $z = 1/2$ found in the AGK model are also clearly visible in the ATLAS experimental data.

It is useful to introduce a function $f_s(z)$, which measures the deviation from the exponential distribution

$$\langle n \rangle P_n = e^{-z}(1 + f_s(z)). \quad (3)$$

If the function $f_s(z)$ is a function of only the variable z , then the KNO scaling holds exactly. This is not the case for p - p collisions, where the experimental data show that the KNO scaling is violated because $\langle n \rangle P_n$ versus z curves are quite different for different energies and other parameters as depicted in Fig. 1(b).

In the right plot of Fig. 1 we used the averages $\langle n \rangle$ calculated in [4] that are listed in Table I.

The probability P_n was calculated by Mueller [21] for the color dipole model using a Markov chain equation, and was later used by Kharzeev and Levin [6] for studying entanglement in p - p scattering. The Mueller-Kharzeev-Levin (MKL) model is a pure geometric distribution given by

$$P_n^{\text{Mueller}} = e^{-\alpha Y}(1 - e^{-\alpha Y})^{n-1}, \quad (4)$$

TABLE I. Averages of the charged multiplicity distribution in p - p collisions at different energies for $p_t^{\text{min}} > 500$ MeV and $|\eta| < 2.5$ as calculated in [4].

$\sqrt{s}$, (TeV)	$\langle n \rangle$
13	14.66 ± 0.04
8	12.25 ± 0.03
7	11.98 ± 0.05
2.36	8.66 ± 0.51
0.9	6.53 ± 0.03

where α and Y are the coupling constant and the rapidity, respectively.

An extended version of the MKL model was developed by the authors [22] by implementing the relative weights found by Abramovsky, Gribov, and Kancheli (AGK) [23] in the Markov chain evolution equation. To avoid the negative probabilities arising from the negative AGK weights in Minkowski space, the evolution was reformulated in Euclidean space [22]. The resulting AGK model is slightly different from the pure geometric distribution

$$P_n^{\text{AGK}} = \left(1 - \frac{2^n}{3^n}\right) e^{-\alpha Y} (1 - e^{-\alpha Y})^{n-1} (1 + 2e^{-\alpha Y}). \quad (5)$$

The two models are very close to each other and numerically describe the experimental data in a very similar way, even though the AGK model better describes the experimental values for the moments C_q [22]. However, there is a qualitative difference between the two models when it comes to the corrections to the leading e^{-z} behavior. Both models exhibit a universal fixed point of the scaled probability $\langle n \rangle P_n$ at $z = 2$, where curves at different $\langle n \rangle$ intersect. As recently shown by the authors [24], this fixed point is a generic feature of any unitary model that approximates a memoryless distribution at high energy, with KNO-violating corrections of order $1/\langle n \rangle^2$ near $z = 2$, reflecting the maximal entanglement of the final states. Besides this, the AGK-based model reveals another approximate intersection point at $z = 1/2$, which is also present in the experimental data as shown in Fig. 1(a). The intersection point $z = 1/2$ is absent in the MKL model and other popular models. The charged multiplicity distribution ATLAS data for p - p collisions show evidence for the approximate intersection point at $z = 1/2$ of the $\langle n \rangle P_n$ curves at different energies as illustrated in Fig. 1. The two intersection points $z = 1/2$ and $z = 2$ are themselves reciprocal partners, suggestive of a residual symmetry that becomes more pronounced in the data, as we discuss next.

Moreover, the ATLAS data reveal an intriguing reciprocal symmetry $z \leftrightarrow 1/z$ of the KNO violating term $f_s(z)$ in the range $1/3 < z < 3$ for energies $\sqrt{s} = 2.36, 7, 8, 13$ TeV, as we show in the next Section.

III. RECIPROCAL SYMMETRY $z \leftrightarrow 1/z$

It is convenient to rewrite the function $f_s(z)$ defined in (3) as the relative deviation of the measured distribution from the leading exponential distribution

$$f_s(z) = \frac{\langle n \rangle P_n - e^{-z}}{e^{-z}}. \quad (6)$$

The function $f_s(z)$ generally depends on both z and the average $\langle n \rangle$; if it depended only on z , the KNO scaling would hold exactly. The function $f_s(z)$ is constructed to measure the relative deviation from the

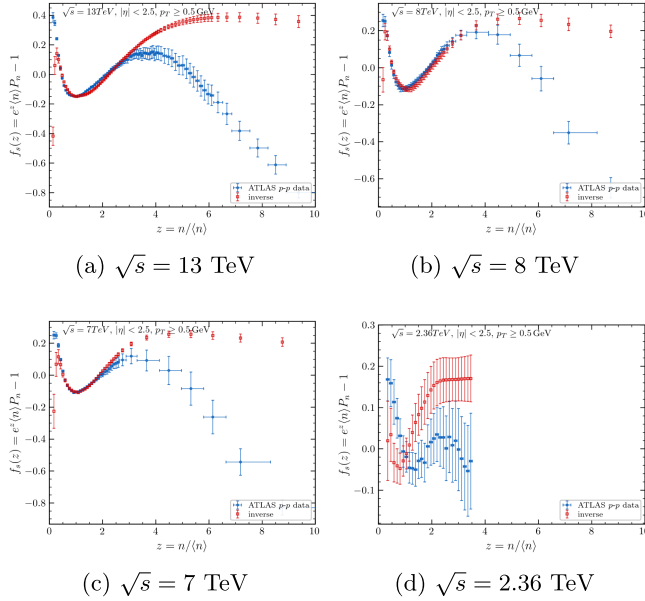


FIG. 2. ATLAS data plots for $f_s(z)$ defined in (6) at $\sqrt{s} = 2.36, 7, 8, 13$ TeV, $p_t > 500$ MeV and $|\eta| < 2.5$. The blue points represent $f_s(z)$ as a function of z , and the red points represent $f_s(1/z)$ of the inverse variable $1/z$, illustrating the reciprocal symmetry $z \leftrightarrow 1/z$ in the range $1/3 < z < 3$ at $\sqrt{s} = 7, 8, 13$ TeV. The averages $\langle n \rangle$ used are those calculated in [4] and given in Table I. The plot at $\sqrt{s} = 2.36$ TeV demonstrates that the reciprocal symmetry does not hold at lower energies.

memoryless exponential behavior e^{-z} , i.e., from the geometric (memoryless) distribution that arises in the color-dipole picture [6,9,25].

The ATLAS and CMS experimental data for the charged multiplicity distribution at $\sqrt{s} = 7, 8$, and 13 TeV show evidence for a remarkable reciprocal symmetry of the KNO violating term $f_s(z)$. The reciprocal symmetry $f_s(z) = f_s(1/z)$ holds well in a wide range of fluctuations from the average $\langle n \rangle$, namely for $1/3 < z < 3$, as shown in Fig. 2 for the ATLAS data at $\sqrt{s} = 7, 8$, and 13 TeV. For $\sqrt{s} = 2.36$ TeV the reciprocal symmetry is much less pronounced. We include the $\sqrt{s} = 2.36$ TeV plot to illustrate that the symmetry builds up with energy.

A. Gaussian parametrization

The reciprocal symmetry $f_s(z) = f_s(1/z)$ is automatically respected by any even function of $\ln z$, since

$\ln(1/z) = -\ln z$. The simplest such function with finite support around $z = 1$ is a Gaussian in $\ln z$, which we adopt as a convenient parametrization

$$f_s^{\text{fit}}(z) = a + be^{-c(\ln z - \mu)^2}, \quad (7)$$

where μ measures the departure from exact reciprocal symmetry: $\mu = 0$ corresponds to the exactly symmetric case. The Gaussian in $\ln z$ serves here as a convenient phenomenological parametrization that makes the symmetry manifest; any even function of $\ln z$ would respect the symmetry equally. The dynamical origin of both the symmetry itself and the specific shape of $f_s(z)$ is left for future investigation.

The fit results in the range $1/3 < z < 3$ are shown in Table II and compared to the data in Figs. 3 and 4. The smallness of the shift parameter μ for $\sqrt{s} = 7, 8, 13$ TeV reflects the fact that the reciprocal symmetry holds well at those energies, while at $\sqrt{s} = 2.36$ TeV $|\mu|$ is much larger because the symmetry is much less pronounced.

The reciprocal symmetry is not observed outside the range $1/3 < z < 3$, suggesting that the dynamics governing the tails of the distribution differ from those near the average $\langle n \rangle$. Recent work [18] has proposed that high-multiplicity events depart from KNO scaling and are better described by a diffusion-scaling framework, consistent with the picture that the tail and the central region are governed by distinct dynamical regimes. This is also supported by the fact that theory reproduces the experimental moments C_2 and C_3 much better than C_4 and C_5 , which capture the tail of the multiplicity distribution.

Possible candidates for the additional dynamics in the tail include multiparton interactions and collective motion. Our AGK-based model exhibits the intersection points at $z = 1/2$ and $z = 2$ but does not exhibit the full reciprocal symmetry, which we attribute to the absence of Pomeron loops in the model (i.e., the omission of dipole merging terms in the color-dipole language).

B. Reciprocal transformation of $\langle n \rangle P_n$

It is instructive to reformulate the reciprocal symmetry $z \leftrightarrow 1/z$ of $f_s(z)$ as a transformation for the KNO scaled probability $\langle n \rangle P_n$. Using (3) we write

$$[\langle n \rangle P_n](z) = e^{-z}(1 + f_s(z)) \quad (8)$$

TABLE II. Best-fit parameters for $f_s^{\text{fit}}(z) = a + be^{-c(\ln z - \mu)^2}$ restricted to the range $1/3 < z < 3$. The fitted function is compared to the corresponding experimental curves in Figs. 3 and 4.

Energy ($\sqrt{s}$)	a	b	c	μ
13 TeV	0.59623 ± 0.15575	-0.74619 ± 0.15432	0.36615 ± 0.09314	-0.01574 ± 0.00549
8 TeV	0.71888 ± 0.15820	-0.83440 ± 0.15721	0.35141 ± 0.07807	-0.02889 ± 0.00401
7 TeV	0.76508 ± 0.31955	-0.86774 ± 0.31800	0.30193 ± 0.12938	0.03357 ± 0.00702

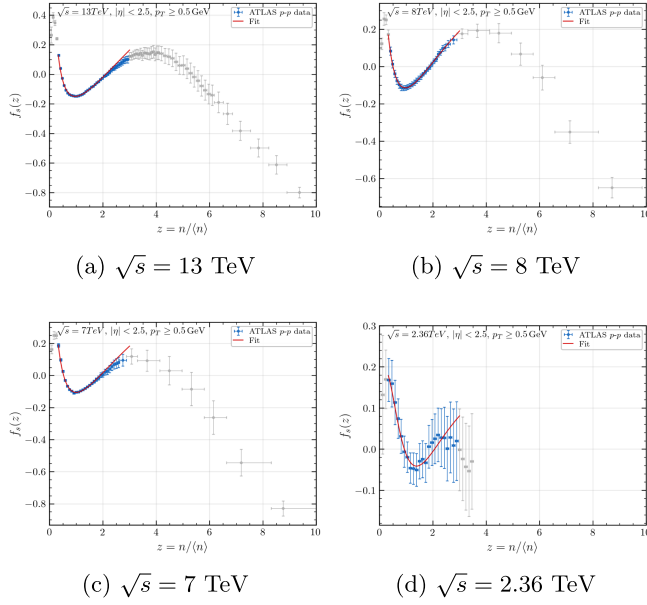


FIG. 3. Gaussian fit (red line) of the deviation function $f_s(z)$ defined in (6) against z for the ATLAS data (blue points) for $p-p$ collisions at $\sqrt{s} = 2.36, 7, 8, 13$ TeV, $p_t > 500$ MeV and $|\eta| < 2.5$. The Gaussian of $\ln z$ given by (7) shows good agreement with the experimental data and reflects the reciprocal symmetry $z \leftrightarrow 1/z$ in the range $1/3 < z < 3$ for $\sqrt{s} = 7, 8, 13$ TeV. The plot at $\sqrt{s} = 2.36$ TeV is presented to illustrate that the reciprocal symmetry is less pronounced at lower energies and builds up at higher energies. The grey dots represent data points beyond the fitting range.

and its reciprocal expression

$$[\langle n \rangle P_n] \left(\frac{1}{z} \right) = e^{-1/z} \left(1 + f_s \left(\frac{1}{z} \right) \right). \quad (9)$$

Provided the reciprocal symmetry holds, $f_s(z) = f_s(1/z)$, we obtain

$$[\langle n \rangle P_n](z) = e^{1/z-z} [\langle n \rangle P_n] \left(\frac{1}{z} \right). \quad (10)$$

The resulting transformation of the scaled probability $\langle n \rangle P_n$ versus the original experimental data is depicted in Fig. 5. The transformed scaled probability $e^{1/z-z} [\langle n \rangle P_n](1/z)$ reasonably reproduces the experimental data for $[\langle n \rangle P_n](z)$. We emphasize that no fit function is involved in Fig. 5: the curves contain only the experimental data points and their transformed counterparts using the transformation given by (10).

C. Reciprocal symmetry in CMS data

We extend the reciprocal symmetry analysis to the CMS experimental data [5]. The corresponding plots for $f_s(z)$ and $\langle n \rangle P_n$ at $\sqrt{s} = 7$ TeV and $\sqrt{s} = 2.36$ TeV are shown in Fig. 6. The CMS measurements used here correspond to

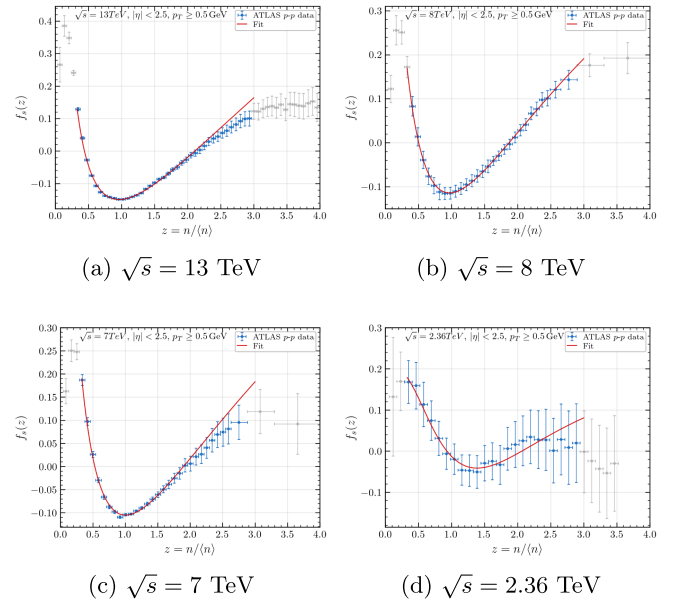


FIG. 4. Zoomed Gaussian fit (red line) for $z \leq 4$ of the deviation function $f_s(z)$ defined in (6) for the ATLAS data (blue points) for $p-p$ collisions at $\sqrt{s} = 2.36, 7, 8, 13$ TeV, $p_t > 500$ MeV, and $|\eta| < 2.5$. The Gaussian of $\ln z$ given by (7) shows good agreement with the experimental data and reflects the reciprocal symmetry $z \leftrightarrow 1/z$ in the range $1/3 < z < 3$ for $\sqrt{s} = 7, 8, 13$ TeV. The plot at $\sqrt{s} = 2.36$ TeV illustrates that the reciprocal symmetry is less pronounced at lower energies and builds up at higher energies. The grey dots represent data points beyond the fitting range.

$p_t > 500$ MeV and $|\eta| < 2.4$, and we use the corresponding average multiplicities $\langle n \rangle$. As in the ATLAS data, the reciprocal symmetry is observed at $\sqrt{s} = 7$ TeV in the central window $1/3 < z < 3$, while at $\sqrt{s} = 2.36$ TeV it is much less pronounced.

IV. LOCAL CONSEQUENCE OF THE RECIPROCAL SYMMETRY

The reciprocal symmetry $f_s(z) = f_s(1/z)$ implies a simple, fit-independent local relation at $z = 1$. Differentiating the symmetry relation gives

$$f'_s(z) = -\frac{1}{z^2} f'_s \left(\frac{1}{z} \right), \quad (11)$$

which at $z = 1$ reduces to

$$f'_s(1) = -f'_s(1) \Rightarrow f'_s(1) = 0. \quad (12)$$

Thus the reciprocal symmetry forces the derivative of $f_s(z)$ to vanish at $z = 1$.

On the other hand, $f_s(z)$ defined in (6) reads

$$f_s(z) = -1 + e^z \langle n \rangle P(z \langle n \rangle), \quad (13)$$

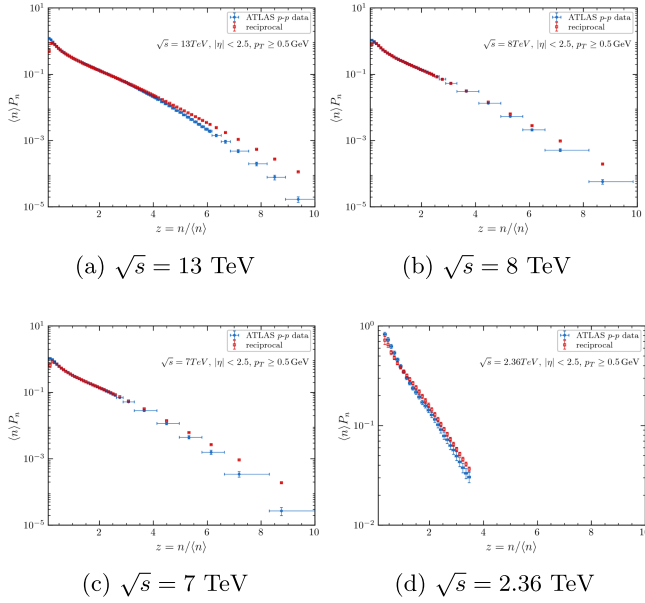


FIG. 5. ATLAS data plots for $p-p$ collisions at $\sqrt{s} = 2.36, 7, 8, 13$ TeV, $p_t > 500$ MeV, and $|\eta| < 2.5$. The blue points represent $\langle n \rangle P_n$ as a function of z , and the red points represent the transformed probability $e^{1/z-z}[\langle n \rangle P_n](1/z)$ given in (10), illustrating the reciprocal symmetry $z \leftrightarrow 1/z$ of $f_s(z)$ in the range $1/3 < z < 3$ at $\sqrt{s} = 7, 8, 13$ TeV. The averages $\langle n \rangle$ are those calculated in [4] and given in Table I.

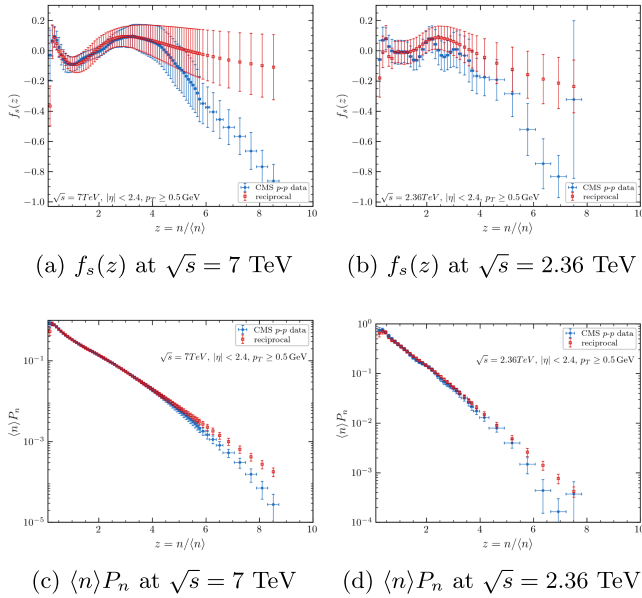


FIG. 6. Reciprocal symmetry in the CMS data [5] at $\sqrt{s} = 7$ TeV and $\sqrt{s} = 2.36$ TeV with $p_t > 500$ MeV and $|\eta| < 2.4$. Top row: $f_s(z)$ in blue and $f_s(1/z)$ in red. Bottom row: $\langle n \rangle P_n$ in blue and the transformed probability $e^{1/z-z}[\langle n \rangle P_n](1/z)$ in red. As in the ATLAS data, the reciprocal symmetry is observed at the higher energy in the range $1/3 < z < 3$ and is much less pronounced at $\sqrt{s} = 2.36$ TeV.

TABLE III. Values of the ratio $\rho = -\langle n \rangle P'(\langle n \rangle) / P(\langle n \rangle)$ defined in (16), obtained from a local Gaussian fit (7) of $f_s(z)$ to the ten data points closest to $z = 1$. The deviation of ρ from unity stays below $\sim 2.3\%$ for $\sqrt{s} = 7, 8, 13$ TeV, providing a direct experimental confirmation of the relation $P'(\langle n \rangle) = -P(\langle n \rangle)/\langle n \rangle$ in (15).

Energy	$\langle n \rangle$	a	b	c	μ	ρ
13 TeV	14.66	0.2982	-0.4479	0.6418	-0.0331	0.9776
8 TeV	12.25	0.2826	-0.4009	0.8518	-0.0228	0.9824
7 TeV	11.98	0.2858	-0.3903	0.7345	0.0017	1.0011

where we treat $n = z\langle n \rangle$ as a continuous variable. Differentiating with respect to z ,

$$f'_s(z) = \langle n \rangle e^z [P(z\langle n \rangle) + \langle n \rangle P'(z\langle n \rangle)]. \quad (14)$$

Setting $z = 1$ and using (12) gives a direct relation between the value and the derivative of $P(n)$ at $n = \langle n \rangle$,

$$P'(\langle n \rangle) = -\frac{1}{\langle n \rangle} P(\langle n \rangle). \quad (15)$$

The relation (15) is a falsifiable, fit-independent prediction of the reciprocal symmetry. To test it directly, we define the dimensionless ratio

$$\rho \equiv -\langle n \rangle \frac{P'(\langle n \rangle)}{P(\langle n \rangle)}, \quad (16)$$

which equals unity if (15) holds. We extract ρ from the experimental data using the local Gaussian fit $P(n) = \frac{1}{\langle n \rangle} e^{-n/\langle n \rangle} (1 + f_s^{\text{fit}}(n/\langle n \rangle))$ with f_s^{fit} from (7), restricting the fit to the ten data points closest to $z = 1$. The resulting values of ρ are listed in Table III: at $\sqrt{s} = 7, 8$, and 13 TeV, ρ equals unity at the level of a few percent, providing a direct experimental confirmation of the local relation (15) and hence of the reciprocal symmetry in the vicinity of $z = 1$.

V. EXTRACTING THE ENTANGLEMENT ENTROPY

The local relation (15) suggests an alternative route for extracting the entanglement entropy from the experimental data without resorting to global fits of the multiplicity distribution. The entropy associated with the multiplicity distribution is given by

$$S = -\sum_{n=1}^{\infty} P(n) \ln P(n), \quad (17)$$

which coincides with the von Neumann entropy of the partonic density matrix when the latter is diagonal in the

Fock basis [6]. For both the MKL and AGK models, the large- $\langle n \rangle$ expansion of S takes the form [6,10,12,22]

$$S \simeq \ln \langle n \rangle + 1 + \mathcal{O}\left(\frac{1}{\langle n \rangle}\right), \quad (18)$$

so that for large $\langle n \rangle$ the average multiplicity is approximately related to S by

$$\langle n \rangle \simeq e^{S-1}. \quad (19)$$

We emphasize that the relation (19) is model dependent: it follows from the leading exponential behavior of $\langle n \rangle P_n$, which is a generic feature of the MKL and AGK models but holds only approximately in the data. Combining (19) with the local consequence (15) of the reciprocal symmetry gives

$$P'(e^{S-1}) \simeq -e^{1-S} P(e^{S-1}). \quad (20)$$

The expression (20) relates the local properties of the production probability in the vicinity of $z = 1$ (i.e., at $n \simeq \langle n \rangle$) to the global quantity S . Measurements of P_n have small uncertainties near $n \simeq \langle n \rangle$, in contrast to the large experimental errors of the distribution tail at $n \gg \langle n \rangle$ that can significantly affect S extracted from a global fit. The relation (20) therefore provides a methodology for extracting S from the well-measured central region, complementary to a global-fit determination, and allows for a more direct comparison with the entanglement entropy S calculated from parton distribution functions in the Kharzeev-Levin framework [6,7], from BFKL-evolved unintegrated gluon distributions [11,12], or from dipole cascade models [16,17]. The comparison between partonic and hadronic entropy is sensitive to the choice of charged-vs-total multiplicity and to the kinematic window [12,14], both of which constitute important systematics for any such

extraction at the LHC. A detailed numerical extraction of S along these lines is left for future work.

VI. CONCLUSION

We have identified a reciprocal symmetry $z \leftrightarrow 1/z$ in the KNO violating term $f_s(z)$ in the range $1/3 < z < 3$, observed in the ATLAS and CMS charged-multiplicity data at $\sqrt{s} = 7, 8, 13$ TeV. The symmetry is well described by a Gaussian in $\ln z$ and becomes more pronounced with increasing collision energy. As a direct consequence of the symmetry, the derivative of $f_s(z)$ vanishes at $z = 1$, which translates into the local relation $P'(\langle n \rangle) = -P(\langle n \rangle)/\langle n \rangle$. We have verified this relation in the data by computing the ratio $\rho \equiv -\langle n \rangle P'(\langle n \rangle)/P(\langle n \rangle)$ and finding $\rho = 1$ at the few-percent level at all three high energies. Combining this local relation with the large- $\langle n \rangle$ approximation $\langle n \rangle \simeq e^{S-1}$ from the MKL/AGK models allows the entanglement entropy to be extracted from the well-measured region $n \simeq \langle n \rangle$, avoiding the large uncertainties associated with the distribution tail. The dynamical origin of the reciprocal symmetry is left for future investigation. In particular, it would be interesting to explore its possible connection to the conformal structure of high-energy QCD, to the diffusion-scaling alternative to KNO recently proposed in [18], and to the small- x QCD origins of multiplicity distributions explored in [19,20].

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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