

Axion coupling accelerates the Universe through $\mathcal{PT}$ -symmetric phases

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(Received 5 January 2026; accepted 2 March 2026; published 24 March 2026)

The conjecture by two of the authors (N. E. M. and S. S.) that a $\mathcal{PT}$ -symmetric phase plays a role in understanding singular renormalization group (RG) flows for a Chern-Simons (CS) gauge theory of axions, has been reexamined and significantly improved. We have used the more complete Wetterich equation, which includes gravitational couplings in a systematic way from the start, to understand the emergence of this phase. The singular structure of the RG flows has persisted on including gravitational couplings, thereby offering further support to the conjecture that $\mathcal{PT}$ -symmetric phases of (repulsive) gravity characterize string-effective CS gravitational theories, where the axion is the massless string-model independent axion, which can also play a role of a totally antisymmetric torsion degree of freedom. This has suggested a novel interpretation of the currently observed acceleration of the expansion of the Universe in terms of such a phase at large (cosmological) scales.

DOI: [10.1103/11dp-b5cj](https://doi.org/10.1103/11dp-b5cj)

I. INTRODUCTION AND MOTIVATION

The observed late-time acceleration of the expansion of the Universe [1] is conventionally attributed to a small positive cosmological constant or dark energy. Recently, within the framework of a string-inspired Chern-Simons cosmological model, an alternative interpretation was proposed [2]. It was suggested that the late-time accelerated expansion of the Universe might be understood, as a change from a conventional Hermitian phase of gravity in the ultraviolet to a non-Hermitian $\mathcal{PT}$ -symmetric phase in the infrared. $\mathcal{PT}$ symmetry refers to parity ($\mathcal{P}$)-time-reversal ($\mathcal{T}$) symmetry and was first introduced in quantum mechanics and then later in quantum field theory (QFT) [2–6].¹ The conjecture of [2] is based on the presence of singularities in nonperturbative renormalization-group (RG) β functions of (massless) axion-electrodynamics couplings [7] that characterize the low-energy string theory cosmological model under consideration. We show that these singularities signal a transition from a conventional

Hermitian phase of gravity in the ultraviolet to a non-Hermitian but $\mathcal{PT}$ -symmetric phase in the infrared. The key advancement of the present work is to verify this conjecture by incorporating gravitational dynamics from the outset, using the Wetterich functional renormalization group (FRG) equation [8].

At such singular points of the nonperturbative RG flows, the theory becomes ill defined in the infrared. The conjecture of [2] is that the well-defined theory is a continuation of the Hermitian theory (in the sense of a phase transition) to a non-Hermitian but well-behaved, $\mathcal{PT}$ -symmetric theory, which, when coupled to gravity, makes gravity appear repulsive. On interpreting the infrared regions (where the singularity in the β function occurs) as corresponding to a late cosmological era of the Universe, this theory is conjectured [2] to relate to the observed late-era acceleration of an expanding Universe. In [2], the initial conjecture has been formulated in the context of flat-space-time axion electrodynamics. The role of gravity, and its RG flow, was considered later in that analysis, and only in the $\mathcal{PT}$ -symmetric phase, as a supplemental component. It is the purpose of the present work to incorporate gravity from the start and, thus, verify, in a rigorous way, the conjecture of [2]. We demonstrate that when gravity is included in low order truncations in the FRG framework, the singularities in the axion coupling persist. This supports our interpretation that the observed cosmological acceleration corresponds to a $\mathcal{PT}$ -symmetric phase of repulsive gravity at large scales, potentially resolving tensions in Λ Cold Dark Matter (Λ CDM) cosmology without invoking dark energy.

Our analysis is motivated by the string-inspired running vacuum model (StRVM) [9], a cosmological framework

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¹A brief but comprehensive discussion on the basic properties and features of this symmetry is given in Appendix A, for the benefit of the nonexpert readers, so as to place them into context.

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derived from low-energy string theory (cf. Appendix II). After compactification to $(3 + 1)$ dimensions, the effective action for the massless gravitational multiplet includes:

- (i) The metric $g_{\mu\nu}$ (graviton).
- (ii) A constant dilaton field.
- (iii) The Kalb-Ramond field $B_{\mu\nu}$, dual to a pseudoscalar axion $b(x)$.

The details of the compactification play no essential role in our analysis or conclusions in [2] and also here.

The conjecture is based on the presence of singularities in appropriate FRG β functions in the Chern-Simons electrodynamics part of (20). Specifically, the singularities appear [7] in the FRG β function of the renormalized coupling f_a^{-1} of the axion (pseudoscalar) field $b(x)$ to the Pontryagin anomaly term [10] (which we refer to as the *axion coupling*):

$$S_{\text{axion-Pontryagin-anomaly}} = \frac{1}{16\pi^2 f_a} \int d^4x \sqrt{-g} b(x) (-F_{\mu\nu} \tilde{F}^{\mu\nu}), \quad (1)$$

where, for the StRVM, the axion coupling is

$$f_a = \frac{1}{16\pi^2 A} = \frac{6}{\pi^2} \sqrt{\frac{3}{2}} \frac{\kappa}{\alpha'} = \frac{6}{\pi^2} \sqrt{\frac{3}{2}} \frac{M_s^2}{M_{\text{Pl}}}, \quad (2)$$

on using the definition of A in (8).

The singularities in the β function of f_a appear in the infrared regions of the (renormalized) Hermitian theory, making it ill defined [7]. The $\mathcal{PT}$ properties of the set of singularities and zeros of the beta function led to the conjecture that such an ill-defined theory, can be extended (beyond such singularities) to a $\mathcal{PT}$ -symmetric, non-Hermitian theory. At large cosmological scales (in the far infrared) [2], the $\mathcal{PT}$ phase is characterized by regular FRG-dressed axion couplings, but by a *repulsive* gravitational interaction, while the short-distance gravity is attractive.² There is a parallel with $\mathcal{PT}$ -symmetric quantum electrodynamics [13], where like charges *attract* and opposite charges *repel*, a feature of $\mathcal{PT}$ symmetry.

As already mentioned, in [2] the emergence of the $\mathcal{PT}$ phase was first investigated in a flat background. The situation is depicted concisely in Fig. 1. There is a singular behaviour of the FRG β function of the axion coupling f_a in the infrared, making the theory ill defined at large distances (left panel). Thus, it is not possible, by staying

within the Hermitian theory, to connect the interacting theory with the free theory at the trivial fixed point $g = 0$ smoothly, via an appropriate RG flow from the ultraviolet (UV) to the infrared fixed points.

These singularities render the Hermitian theory ill-defined beyond a critical RG scale. Following the approach of Refs. [11,12] for scalar field theories, we conjectured [2] that the resolution involves an analytic continuation of the coupling,

$$f_a \rightarrow i f_a, \quad (3)$$

which exhibits a well-defined regular behavior in the infrared [7]. The non-Hermitian theory is characterized by a singularity in the (trans)UV regime, a nontrivial UV fixed point, and a smooth running from the UV to the trivial infrared (attractive) fixed point at $\tilde{g} = 0$ (cf. right panels of Fig. 1). Here we followed the logic in [14] for ordinary Landau poles, inspired by the conjecture of [11], whereby the singularity is bypassed by an analytic continuation of the theory. The analytic continuation of the axion coupling f_a is a necessary condition for $\mathcal{PT}$ symmetry, as follows from the discussion in [5], for scalar field theories.

In [2], it has been stressed that Eq. (2), which is valid *only* for string-model-independent [or Kalb-Ramond (KR)] axions (cf. Appendix II), is *crucial* for the validity of the conjecture of [2]. Indeed, the latter relation implies that, upon the analytic continuation (3), which allows for the transition to the non-Hermitian theory, the gravitational coupling

$$\kappa \rightarrow i\tilde{\kappa}, \quad (4)$$

while the square of the string scale remains real.³ In view of (4) one, then, obtains, as explained in [2], a repulsive phase of gravity for the analytically continued phase of the axion electrodynamics model, when placed in a gravitational background,

$$G = \frac{\kappa^2}{8\pi} \rightarrow \tilde{G} = -\frac{\tilde{\kappa}^2}{8\pi} < 0, \quad (5)$$

where $G, \tilde{G}$ denote the respective Newton constants in the Hermitian and $\mathcal{PT}$ symmetric phases, respectively.

The original conjecture [2] was formulated in flat space time [8,15]. Gravitational effects were incorporated only *after* identifying the singularities, and only in the $\mathcal{PT}$ -symmetric phase. This raised the question: *Do the singularities persist when gravity is included from the start, accounting for the backreaction of gravitational*

²In [11,12] potential extensions of ill-defined Hermitian scalar field theories to well-defined $\mathcal{PT}$ -symmetric non-Hermitian Hamiltonians are conjectured, in a nongauge setting. We discuss the extension of this conjecture, in the presence of gravity, to explain the observed cosmological expansion of the Universe at large scales, without the need for introducing a dark energy component. We also assume this nontrivial phase change in the infrared region of the curved-space extension of axion electrodynamics [2].

³Even if one analytically continued M_s, M_s^2 remains real ($M_s^2 \in \mathbb{R}$). However, the physically correct approach is the one in which one does not analytically continue M_s , due to the requirement that the compactification radii in string models, which are proportional to M_s , must remain real [2].

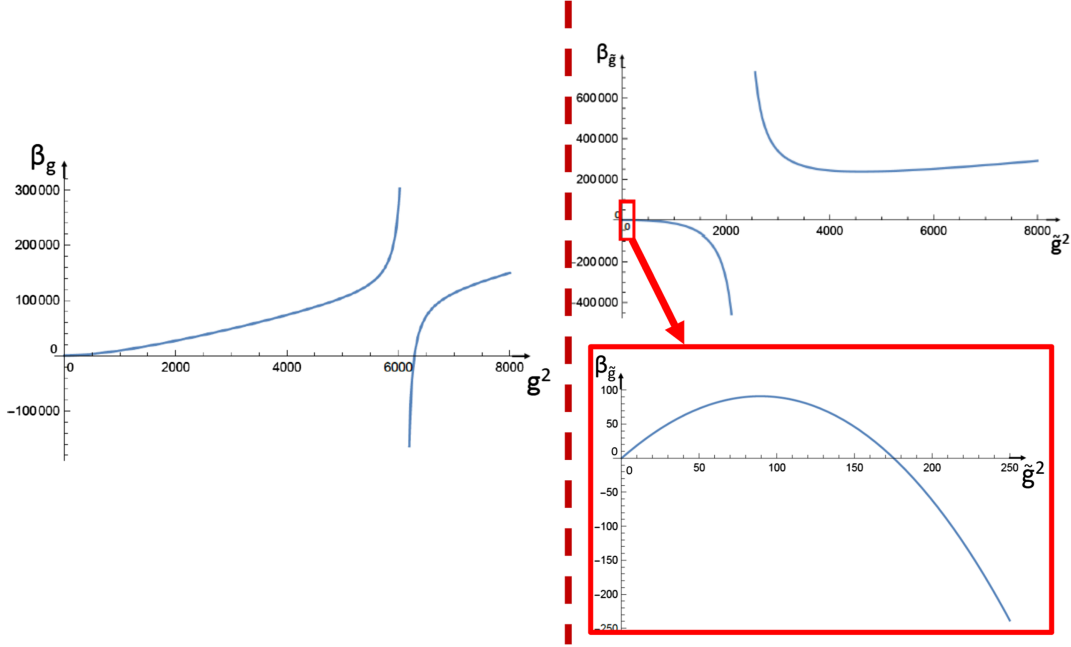


FIG. 1. Left panel: The FRG beta function $\beta_g \equiv \partial_t g^2$ of the Hermitian axion CS electrodynamics theory [7], exhibiting a singular behavior at a finite running coupling $g \equiv (4\pi^2 f_a)^{-1}$, as well as a fixed point behavior, characterized by the existence of a trivial fixed point at zero coupling, and a nontrivial one at strong coupling, $g_*^2 \gg 1$. Right Panels: Nonperturbative RG beta function for non-Hermitian case as a function of $\tilde{g}^2$, where $\tilde{g} = ig$. The behavior is qualitatively different from the Hermitian beta function [7]. The area included in the red rectangle includes a nontrivial fixed point (apart from the trivial one at zero coupling), which is indicated explicitly in the lower left panel. Figure synthesized from relevant graphs in [7,2].

fluctuations? To address this, we must use the Wetterich equation for theories coupled to dynamical gravity. This is a nontrivial extension because:

- (1) Quantum gravity is perturbatively nonrenormalizable, requiring careful truncation of the infinite tower of operators.
- (2) Diffeomorphism invariance imposes constraints on the RG flow.
- (3) The interplay between topological CS terms and wave-function renormalization is subtle.

The basic purpose of the current work, namely to incorporate the role of gravity from the start, and examine its implications for the Landau-pole behavior of the dynamical Abelian gauge Chern-Simons (CS) coupling and $\mathcal{PT}$ symmetry, is crucial for our earlier analysis. We shall demonstrate that the singularities in the FRG flow of the axion coupling, which led to the conjecture of [2], remain intact after the inclusion of gravity.

In the context of the StRVM we note that there are no uncanceled gravitational Chern Simons (gCS) anomaly terms of the form (7) in post-RVM-inflationary eras at large scales; in particular the late cosmic eras characterized by an accelerated expansion. Thus they will be ignored in this work.⁴ From the perspective of perturbative quantum fields

⁴Their presence would complicate the FRG equations in a way explained briefly in Sec. III C.

in four dimensions, the theory (6) is nonrenormalizable, thus, in [2] and here, we are forced to use a nonperturbative formulation, such as the one proposed by Wetterich [8,15], for the FRG framework [16,17], and revisit crucial nonrenormalization arguments on topological grounds.

The structure of the article is the following: The next Sec. II, reviews the essential elements of StRVM, which the conjecture of [2] is based upon, and motivates the curved space time axion-electrodynamics model in this context. Section II C further motivates the $\mathcal{PT}$ -symmetric framework [18] by addressing a known ambiguity in the appearance of CS gravity in (bosonic) string theory upon analytic continuation from Minkowski to Euclidean signature; this step is required to define a well-posed path integral for the low-energy gravitational effective action. In Sec. II D, we give, for completion, some elementary estimates on the magnitudes of couplings, within the context of our string-inspired model. Section III sets up the nonperturbative flow equations for Euclidean effective actions, introducing elements of the Wetterich formalism and the role of topology in constraining exact RG flows, used in our FRG analysis of axion electrodynamics with gravity, generalizing [2]. As mentioned in the introduction, this model is an essential part of the StRVM at late epochs [19,20], which are of interest to us here. Section IV introduces dynamical gravity into axion electrodynamics; within the StRVM framework we omit gCS anomaly terms. Section V computes the gravitational contributions to the

FRG β functions and anomalous dimensions of axion electrodynamics and presents our main results: under reasonable, physically motivated approximations (cf. Sec. VA), the Wetterich equation leaves the β -function singularities of the axion coupling found in [7] intact even in the presence of gravity (cf. Sec. VB), thereby supporting the conjecture of [2] that gravity may exhibit a repulsive phase at large scales. Finally, Sec. VI presents our conclusions and outlook. Appendix A reviews briefly the basic features of $\mathcal{PT}$ symmetry, mainly for introducing the nonexpert reader to the subject. Appendix B reviews the $\mathcal{PT}$ -symmetric aspects of non-Hermitian CS theories that underpin our approach. Appendix C derives the emergence of CS terms from anomaly-cancellation arguments and clarifies that the string-inspired reasoning of [2] extends beyond string theory. Appendix D reviews simple examples of the singular perturbative renormalization group [21] in field theory, where situations like the one conjectured in [2] might occur, but much simplified and in the absence of gravity. Its purpose is to demystify, as much as possible, the situation described in [2]. Appendix E introduces and illustrates key features of the FRG in a flat-space-time toy model of a real scalar field. Finally, Appendix F summarizes terminology and technical aspects of the nonperturbative FRG framework in our curved space time context, including discussion on: tensorial properties of gravitational tensor perturbations and their decomposition (Appendix F 1), FRG superdiagram rules (Appendix F 2), as well as local potential approximation (LPA') (Appendix F 3) and details of loop diagrams required for the calculation of anomalous dimensions (Appendix F 4), which we make use of in the main text.

II. THE STRING-INSPIRED LATE-EPOCH COSMOLOGY MODEL

The conjecture of [2] is based on a string-inspired cosmological model, which provides a concrete realization of a running vacuum model (RVM) cosmology [9,22–24]. For this reason the model is termed StRVM [19,20,25,26]. In this section, we review briefly the main features of the StRVM, which constrains the effective action, on which the analysis in [2] is reliant. For the postinflationary eras we are interested in here and in [2], since we seek explanations for the late-Universe observed acceleration, we emphasize the *cancellation* mechanism of the primordial *gravitational anomalies* that characterize the model. Below we shall explain this important point, which will facilitate the curved-space-time FRG analysis enormously, as we shall see.

A. String-inspired gravitational model for the primordial universe

Our StRVM cosmology in the early (preinflationary) universe eras is based [19,20,25,26] on the massless gravitational multiplet of superstrings [27,28], consisting of gravitons, dilatons, and antisymmetric (KR) fields, the

latter being dual, in $(3+1)$ -dimensions, to the so-called string-model independent axion, or KR axion $b(x)$.⁵ A constant dilaton, is a basic assumption of the StRVM.⁶ Retaining, for the moment, gauge fields [which, for strings [27,28], are in general non-Abelian (and denoted by bold-face field strengths)], the pertinent action, which describes the dynamics in the early Universe, reads [30],⁷

$$S_{b,\text{grav}} = \int d^4x \sqrt{-g} \left[\frac{R}{2\kappa^2} + \frac{1}{2} (\partial_\mu b)(\partial^\mu b) - \frac{1}{4} \mathbf{F}_{\mu\nu} \mathbf{F}^{\mu\nu} + S_{\text{CS}} + \dots \right], \quad (6)$$

where g is the determinant of the metric $g_{\mu\nu}$, and S_{CS} is a four-space-time-derivative topological Chern-Simons term [30],

$$S_{\text{CS}} = A \int d^4x \sqrt{-g} b(x) (R_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma} - \mathbf{F}_{\mu\nu} \tilde{\mathbf{F}}^{\mu\nu}), \quad (7)$$

with

$$A \equiv \sqrt{\frac{2}{3}} \frac{\alpha'}{96\kappa}. \quad (8)$$

Here $\kappa = M_{\text{Pl}}^{-1}$ (in units $\hbar = c = 1$) is the $(3+1)$ -dimensional gravitational coupling (inverse of the reduced Planck mass $M_{\text{Pl}} = 2.4 \times 10^{18}$ GeV), $\alpha' = M_s^{-2}$ is the *Regge slope* of the string [27], and M_s is the string mass scale. The axionlike particle (ALP) $b(x)$ is independent of the model used for compactification [31].⁸

The interaction terms (7) arise in string theory from the Green-Schwarz anomaly cancellation modification [29] of the antisymmetric tensor field strength $H_{\mu\nu\rho}$ by appropriate Lorentz (gravitational) Ω_{3L} and gauge Ω_{3Y} CS three-forms (see Appendix C for details):

⁵The duality is expressed via (21) below, where the three-form H is the field strength of the spin-1 KR field $B_{\mu\nu} = -B_{\nu\mu}$ [27,28], modified by the Green-Schwarz anomaly cancellation counterterms [29], see (9) and subsequent discussion.

⁶In realistic string compactifications (e.g., flux compactifications) the dilaton is often stabilized by fluxes or nonperturbative effects such as gaugino condensation. Stabilization implies that the dilaton sits at a minimum of an effective potential.

⁷Our conventions and definitions used throughout this work are: signature of metric $(+, -, -, -)$, Riemann Curvature tensor $R^\lambda_{\mu\nu\sigma} = \partial_\nu \Gamma^\lambda_{\mu\sigma} - \partial_\sigma \Gamma^\lambda_{\mu\nu} + \Gamma^\rho_{\mu\sigma} \Gamma^\lambda_{\rho\nu} - (\nu \leftrightarrow \sigma)$, Ricci tensor $R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$, and Ricci scalar $R_{\mu\nu} g^{\mu\nu}$. We also work in units $\hbar = c = 1$.

⁸A string-model independent axion is distinct from compactification ALPs, which are abundant in string theory, and depend on details of the string model and its compactification. In the current article we shall restrict our attention to the KR axion $b(x)$ field, and its anomalous coupling (7).

$$H(x) = \mathbf{d}B(x) + \frac{\alpha'}{8\kappa}(\Omega_{3L}(x) - \Omega_{3Y}(x)),$$

$$\Omega_{3L} = \omega^a{}_c(x) \wedge d\omega^c{}_a(x) + \frac{2}{3}\omega^a{}_c(x) \wedge \omega^c{}_d(x) \wedge \omega^d{}_c(x),$$

$$\Omega_{3Y}(x) = \mathbf{A}(x) \wedge \mathbf{dA}(x) + \frac{2}{3}\mathbf{A}(x) \wedge \mathbf{A}(x) \wedge \mathbf{A}(x). \quad (9)$$

For coordinate independence and topological considerations, we use the language of differential forms [10], with $\wedge$ denoting an exterior product among appropriate one-forms. The one-form $\omega^a{}_b(x)$ denotes the torsion-free spin connection (Latin indices $a, b, c = 1, \dots, 4$ denote tangent-space indices at a space time point x), and $A(x)$ are generic non-Abelian gauge-field one-forms that characterize strings. The $\Omega_{3L}(\Omega_{3Y})$ are Lorentz (Yang-Mills) CS three-forms [10].

The (massless) axionlike field $b(x)$ in (6) arises in a path-integral formalism [30,32], as a pseudoscalar Lagrange multiplier, associated with the implementation of the Bianchi-identity constraint. This constraint is derived on taking the exterior derivative on both sides of (9), which in component form reads,

$$\varepsilon_{abc}{}^\mu H^{abc}{}_{;\mu} = \frac{\alpha'}{32\kappa} \sqrt{-g}(R_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma} - \mathbf{F}_{\mu\nu} \tilde{\mathbf{F}}^{\mu\nu}), \quad (10)$$

with the semicolon (;) denoting the gravitational covariant derivative with respect to the torsion-free connection. The interaction (7), emerges after path-integrating out the field strength $H_{\mu\nu\rho}$ with respect to the action (6).

In (7) the symbol $\widetilde{\dots}$ denotes the dual of the gauge field strength and Riemann curvature, respectively,

$$\tilde{F}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}, \quad \tilde{R}_{\alpha\beta\gamma\delta} = \frac{1}{2} R_{\alpha\beta}{}^{\rho\sigma} \varepsilon_{\rho\sigma\gamma\delta}, \quad (11)$$

with $\varepsilon_{\mu\nu\rho\sigma}$ the covariant Levi-Civita tensor [10]. The gauge field strength in (6), (7) could be non-Abelian, but in our considerations below, and in [2], we take them to be Abelian, associated with electromagnetic interactions, for simplicity. The gravitational theory (6), (7) is a CS gravity [33,34]. In effective low-energy limits of string theories, the Lagrangian (6) is part of an infinite set of (higher mass-dimension) terms, which play a role, dependent on the energy scale under consideration.

Despite their similarities in form language, the two anomaly terms in (7) are different in their contributions to the stress energy tensor $T_{\mu\nu}$. The gauge CS term [last term on the right-hand side of the equality (7)] is topological since it is independent of the space time metric. Hence its contribution to $T_{\mu\nu}$ vanishes. In contrast, the gravitational CS (gCS) term [first term on the right-hand side of the equality (7)] gives a nonzero contribution to the stress tensor, a Cotton-like symmetric but traceless term $C_{\mu\nu}$ [33],

$$C_{\mu\nu} = -\frac{1}{2}[(\partial^\beta b)\tilde{R}_{\alpha\mu\beta\nu} + (\partial^\beta b)\tilde{R}_{\alpha\nu\beta\mu}]^{\alpha}, \quad (12)$$

which yields the following modified Einstein (graviton) equation of motion,

$$G_{\mu\nu} = \kappa^2 T_{\mu\nu}^{(b)} - 2\kappa^2 A C_{\mu\nu} \equiv \kappa^2 T_{\mu\nu}^{\text{improved}}. \quad (13)$$

Here $T_{\mu\nu}^{(b)}$ is the stress energy-momentum tensor associated with the kinetic term of the massless axion-KR field in the action $T_{\mu\nu}^{(b)} = \nabla_\mu b \nabla_\nu b - \frac{1}{2} g_{\mu\nu} (\nabla b)^2$. The property of the Cotton-like tensor [33]: $C_{\mu\nu}{}^{;\mu} = \frac{1}{8} (\partial_\nu b) R^{\alpha\beta\rho\sigma} \tilde{R}_{\alpha\beta\rho\sigma}$, implies that the naive covariant conservation of the axion-matter stress tensor fails, due to a nontrivial exchange of energy between the axion and the gravitational anomaly. This does not imply any pathological behavior of the underlying gravity theory, since diffeomorphism invariance is kept intact, due to the (covariant) conservation of the improved stress-tensor $T_{\mu\nu}^{\text{improved}}$.

As already mentioned, the conjecture in [2] is motivated by the StRVM framework, in which the dynamics of the primordial, preinflationary Universe is described *only* by dilaton, graviton, and antisymmetric tensor fields. No other fields (except perhaps instanton configurations of non-Abelian primordial gauge field that appear in the string-theory spectrum [35], which, however, are not relevant for our purposes here) are present as external fields in this primordial, preinflationary phase. Thus, the pertinent action for this epoch is given by (6) upon setting

$$\mathbf{F}_{\mu\nu} = 0. \quad (14)$$

Inflation in this theory, which is of running vacuum model type [9,22–24], is due to condensates of the gravitational CS terms (7), due to chiral primordial gravitational waves. The latter imply a vacuum energy density proportional to the fourth power of the Hubble parameter H^4 of the primordial string Universe. The condensate has imaginary parts [35], which imply a finite-lifetime inflation. To obtain phenomenologically relevant durations of inflation, of order [36] $\mathcal{O}(50\text{--}60)$ e-foldings one needs [35] $M_s \sim 0.2M_{\text{Pl}}$.

B. Matter fields in postinflationary eras and gravitational-CS-anomaly cancellation

Matter and gauge fields are generated by the decay of the RVM unstable vacuum at the exit phase of inflation. Matter includes massless chiral fermions, which are assumed such that they cancel the primordial gravitational anomalies. The generation of chiral fermionic matter, though, will generate its own gravitational and gauge anomalies, via nonconservation of the chiral current $J^{5\mu} = \sum_i \bar{\psi}_i \gamma^\mu \gamma^5 \psi_i$, where i is an index running over the chiral fermionic species ψ_i of the respective string effective theory.

Thus, the effective gravitational theory at the end of the RVM inflationary phase will be schematically given by [19],

$$S_{\text{total}} = S_{b,\text{grav}} + S_{b,F} + \dots \quad (15)$$

where $S_{b,\text{grav}}$ is given in (6), and the chiral fermionic part of the action $S_{b,F}$ interacting with the b KR axion reads (we suppress the chirality index in the fermions for notational brevity) [19,32],

$$S_{b,F} = S_{F,\text{kin}} + \int d^4x \sqrt{-g} \frac{\kappa}{2} \sqrt{\frac{3}{2}} b J^{\mu}_{\mu} + \dots, \quad (16)$$

where $S_{b,\text{kin}}$ are the fermion kinetic terms in the presence of torsion-free gravity, J^{μ}_{μ} is the torsion-free gravitational covariant derivative, and the $\dots$ denote four-fermion $J^{\mu}_{\mu} J^{\mu}_{\mu}$ terms.⁹

For late eras that we are interested in in [2] and here, it suffices to assume that only the U(1) Abelian gauge anomaly term coupled to the axion b field remains in the effective axion. Heuristically, which suffices for our purposes, on assuming N_f chiral fermionic degrees of freedom, we would obtain in the low-energy field theory the (one-loop exact) anomaly equation [39,40],

$$J^{\mu}_{\mu} = \frac{N_f}{96\pi^2} R_{CS} + \frac{e^2}{8\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu}, \quad (18)$$

where e is the electron charge.

Consistency of the *postinflationary theory*, which is of interest to us in [2] and here, requires [20,25,26] that at most chiral global anomalies may remain uncanceled. Cancellation of gravitational anomalies in (15), then, would require

$$N_f = 192\pi^2 \sqrt{\frac{8}{3}} \frac{A}{\kappa} = \frac{16}{3} \pi^2 \left(\frac{M_{\text{Pl}}}{M_s} \right)^2, \quad (19)$$

⁹The interaction term with its specific coupling arises from the fact that to this order in derivatives, the (totally antisymmetric in its covariant indices) field strength $H_{\mu\nu\rho}$ of the antisymmetric tensor field plays the role of torsion, with a generalized connection [30,37]

$$\tilde{\Gamma}^{\mu}_{\nu\rho} = \Gamma^{\mu}_{\nu\rho} + \frac{\kappa}{\sqrt{3}} g^{\mu\sigma} H_{\sigma\nu\rho}, \quad (17)$$

with $\Gamma^{\mu}_{\nu\rho}$ the torsion-free Christoffel connection. The coupling of b to chiral fermions in (16) is precisely dictated by this role, after the appropriate dualization procedure, i.e., implementing the Bianchi identity (10) in the presence of fermions, and passing to the dual picture in terms of the KR axion field b , during which the $J^5 - J^5$ also arise [32]. Such four-fermion terms are characteristic of theories with torsion [30,38]. Appropriate integration by parts has also taken place, in order to cast the b -fermion interactions in the form (16) [32].

where in the last equality we used (8) for the case of string theory.^{10,11} As mentioned above, for the condensate-induced-inflationary model of [35,41], we have $M_s/M_{\text{Pl}} \sim 0.2$, which, on account of (19), would imply an $N_f \sim \mathcal{O}(10^3)$, in agreement with standard string theory models, used also in gravitational leptogenesis scenarios [42].¹²

The late eras cosmology of StRVM, then, would encompass a standard gCS-anomaly-free axion electrodynamics for the KR field $b(x)$ in curved space time,

$$S_{b,A_\nu,\text{grav}} = \int d^4x \sqrt{-g} \left[\frac{R}{2\kappa^2} + \frac{1}{2} (\partial_\mu b)(\partial^\mu b) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + Ab F_{\mu\nu} \tilde{F}^{\mu\nu} \right], \quad (20)$$

where A_μ is the photon field, and the coefficient A is given in (8).

The Euclidean version of the effective theory (20) is the basic theory that was used in [2] and here, to formulate the conjecture that the observed late-era acceleration of our Universe corresponds to a phase transition to a $\mathcal{PT}$ symmetric, but repulsive phase of gravity. Under the action of a FRG, the coupling A develops singularities in its RG flow in the infrared, in the presence of gravity, handling of which led to the aforementioned phase transition, with the $\mathcal{PT}$ -symmetric theory being characterized by finite FRG flows in the infrared, but with matter in this theory feeling a repulsive gravity. This leads to the $\mathcal{PT}$ -symmetric-universe acceleration at late eras [large cosmological (infrared) length scales], which in [2] has been conjectured to correspond to the observed accelerated expansion phase of our Universe, thereby alleviating the need for the assumption of a dominant dark energy component.

C. $\mathcal{PT}$ symmetry and analytic continuation in string theory

Before proceeding with our main claim, namely that the curved space time action (20) is characterized by singular FRG flows of the axion coupling, as its flat-space-time counterpart [7], which admit a continuation into a $\mathcal{PT}$ symmetric (repulsive gravity) phase, we shall argue in this

¹⁰The gCS cancelation pertains to large (cosmological) scales. Nonetheless, it is understood that locally in the post inflationary Universe, e.g., in the neighborhood of rotating black holes, gCS anomaly terms are nontrivial. Such local phenomena are not relevant to our discussion here.

¹¹It is understood that the choice of appropriate string scales M_s that satisfy (19) are such that N_f is a positive integer.

¹²We remark, for completion, that StRVM also involves postinflationary gravitational leptogenesis, via unconventional scenarios, however, entailing spontaneous Lorentz and CPT violation [19,32], as a consequence of the gCS condensates during the RVM inflationary era. Thus, StRVM stands a chance of providing a phenomenologically realistic cosmology.

section, that the $\mathcal{PT}$ -symmetric version arises naturally in axion physics within the framework of string-inspired effective actions. As a result of analytic continuation in the context of string effective theories, an argument has been made in [18], motivating the potential existence of a non-Hermitian axion-gravitational-CS-anomaly interaction. In Euclidean space there is a known ambiguity [43] in analytically continuing the CS effective action back to Minkowski space time; this stems from properties of the Levi-Civita tensor. In string theory [27,28,44,45], after compactification to $(3+1)$ -dimensional space times, the KR axion is dual to the field strength $H_{\mu\nu\rho}$ of the anti-symmetric tensor field $B_{\mu\nu}$. In [2], we have fixed the dilaton to be a constant, and so [30],

$$\partial_\mu b = \frac{1}{\sqrt{2}} \varepsilon_{\mu\nu\rho\sigma} H^{\nu\rho\sigma}. \quad (21)$$

To lowest order in derivatives, the dynamics of $B_{\mu\nu}$ [which is a $U(1)$ gauge field in the closed string sector [27], with a gauge transformation (C2)] is described by the kinetic term,

$$S_{\text{H-field}} = - \int d^4x \sqrt{-g} \frac{1}{6} H_{\mu\nu\rho} H^{\mu\nu\rho}. \quad (22)$$

Because of (21), the Euclidean path integration over the H -field-strength results in a *Euclidean* (E) string effective action of the form [18],

$$S_{\text{eff B}}^{(\text{E})} \ni \int d^4x \sqrt{g^{(\text{E})}} \frac{1}{12} \varepsilon_{\mu\nu\rho\lambda}^{(\text{E})} \varepsilon^{\mu\nu\rho\sigma(\text{E})} \partial^\lambda b \partial_\sigma b, \quad (23)$$

with a $b(x)$ term, quadratic in space-time derivatives, and dependent on the covariant Levi-Civita tensor density,

$$\varepsilon_{\mu\nu\rho\sigma} = \sqrt{-g} \epsilon_{\mu\nu\rho\sigma}, \quad \varepsilon^{\mu\nu\rho\sigma} = \frac{\text{sgn}(g)}{\sqrt{-g}} \epsilon^{\mu\nu\rho\sigma}. \quad (24)$$

The notation (E) in (23) indicates that this quantity is evaluated in a metric of Euclidean signature. If we *first* use the following property of the Levi-Civita tensor density in four dimensional space time with the Euclidean metric,

$$\varepsilon_{\mu\nu\rho\lambda}^{(\text{E})} \varepsilon^{\mu\nu\rho\sigma(\text{E})} = +6\delta_\lambda^\sigma, \quad (25)$$

where δ_λ^σ denotes the Kronecker delta, and *then* analytically continue to Minkowski space time, we obtain a *real* effective action for the dynamical field $b(x)$ (which plays the role of the KR gravitational axion). Alternatively, if one *first* analytically continues (23) to a Minkowski-signature space time, and *then* uses the Minkowski version of (25)

$$\varepsilon_{\mu\nu\rho\lambda} \varepsilon^{\mu\nu\rho\sigma} = -6\delta_\lambda^\sigma. \quad (26)$$

The minus sign on the right-hand side is due to the dependence of the contravariant Levi-Civita tensor

density (24) on the (Minkowski) signature of the metric tensor. The resulting effective action is nonunitary, since the b field has the wrong-sign kinetic term with respect to the Einstein-Hilbert term in the effective action. In [18] a field redefinition

$$b(x) \rightarrow i\tilde{b}(x), \quad (27)$$

was suggested as a way of correcting the wrong-sign kinetic term of the KR axion. However then any gauge/gravitational anomaly term S_{CS} is non-Hermitian but $\mathcal{PT}$ symmetric when written in terms of $\tilde{b}$.

Nonetheless, as already stated in the introduction of the current article, this by itself is not sufficient to guarantee continuation of the gravitational coupling (4). One needs the specific relation of the axion coupling to κ , (2), provided in the case of the string-model independent axion [30]. It is for this model, which also lies at the heart of the StRVM, for which the conjecture of [2] is valid. The relation (4) gives context to the conjecture in [2] on the emergence of a non-Hermitian $\mathcal{PT}$ -symmetric phase of S_{CS} since we see that the theory in the $\mathcal{PT}$ phase is a possible outcome of our derivation.

D. Phenomenology: String-theory estimate of coupling magnitudes

In the string-inspired effective action (20), the axion-photon coupling is set by A [(8) (cf. (B8) in Appendix B for our normalizations)],

$$g_{\text{mix,k}} \sim 3.4 \times 10^{-2} \frac{M_{\text{Pl}}}{M_s^2}, \quad (28)$$

up to numerical factors, where M_s is the string scale. At an RG scale $k \simeq M_s$ (we remark that in an effective low-energy string-inspired theory the M_s acts as the UV cutoff), the corresponding dimensionless coupling $[g_{b\gamma}(k) = g_{\text{mix,k}}]$ is

$$g_{b\gamma}(M_s) \sim 3.4 \times 10^{-2} \frac{M_{\text{Pl}}}{M_s}, \quad g_{b\gamma}^2(M_s) \sim 1.15 \times 10^{-3} \left(\frac{M_{\text{Pl}}}{M_s} \right)^2. \quad (29)$$

The dimensionless Newton coupling [(86)] is

$$g_N(M_s) \sim \left(\frac{M_s}{M_{\text{Pl}}} \right)^2. \quad (30)$$

Consequently, the mixed term is generically suppressed at the level of,

$$g_N g_{b\gamma}^2 \sim 1.15 \times 10^{-3} \left(\frac{M_s}{M_{\text{Pl}}} \right)^2 \left(\frac{M_{\text{Pl}}}{M_s} \right)^2 \sim \mathcal{O}(10^{-3}), \quad (31)$$

independently of the relative magnitude of M_s/M_{Pl} .

For representative values of the string scale (playing the role of k) one finds,¹³

$$M_s \sim 10^{18} \text{ GeV}: \quad g_N \sim 1, \quad g_{b\gamma}^2 \sim 10^{-3}, \quad g_N g_{b\gamma}^2 \sim 10^{-3}, \quad (32)$$

$$M_s \sim 10^{17} \text{ GeV}: \quad g_N \sim 10^{-3}, \quad g_{b\gamma}^2 \sim 6.7, \quad g_N g_{b\gamma}^2 \sim 10^{-3}, \quad (33)$$

$$M_s \sim 10^{16} \text{ GeV}: \quad g_N \sim 10^{-5}, \quad g_{b\gamma}^2 \sim 11.6, \quad g_N g_{b\gamma}^2 \sim 10^{-3}. \quad (34)$$

We work in a controlled small-mixing approximation, retaining the leading axion-photon and pure gravitational contributions to the anomalous dimensions, while neglecting mixed terms of order $g_N g_{\text{mix}}^2$.

We stress, for the benefit of the reader, that our string-model independent (KR) axion $b(x)$, which enters the analysis of [2] and the current article, is massless. Such massless KR axions play the role of totally antisymmetric torsion degrees of freedom in string theory [30], to the order in target-space derivative expansion of the effective action we are working on. Thus, the standard axion phenomenology/cosmology [46] does not apply.

However, mechanisms for the KR axion to acquire a mass during the postinflationary epoch, do exist within the StRVM [25,47]. We remark that the presence of Landau-pole-like singularity in the axion couplings persists in the massive axion case [7], and thus, the conjecture of [2] could apply. However, the fixed-point structure of the massive axion electrodynamics is different in the presence of axion mass.

III. THE FUNCTIONAL RENORMALIZATION GROUP (FRG)

The functional renormalization group (FRG) provides a powerful nonperturbative framework for studying quantum field theories. In this section, we introduce the essential concepts needed to understand our analysis of axion electrodynamics coupled to gravity. We focus on the aspects most relevant to our system: the Wetterich flow equation, topological constraints from shift symmetry, and the LPA'.

In conventional perturbative renormalization, one integrates out high-energy modes in momentum shells, constructing effective theories valid at progressively lower scales. The Wilsonian RG [48,49] provides a framework for understanding the scale dependence of quantum field theories. In RG, one integrates out quantum fluctuations in momentum shells from a UV scale Λ down to a lower scale k , thereby constructing a sequence of scale-dependent effective actions $S_k(\Phi)$. The functional renormalization group makes this procedure exact and systematic through the introduction of a scale-dependent effective action

$\Gamma_k[\Phi]$, where k is an infrared cutoff scale and Φ collectively denotes all fields in the theory.

The effective average action $\Gamma_k[\Phi]$ interpolates between the bare (classical) action at the UV cutoff Λ and the full quantum effective action in the infrared,

$$\Gamma_k \xrightarrow{k \rightarrow \Lambda} S_{\text{bare}}, \quad \Gamma_k \xrightarrow{k \rightarrow 0} \Gamma_{\text{quantum}}. \quad (35)$$

This interpolation is achieved by adding a momentum-dependent regulator term to the action that suppresses fluctuations with momenta $p^2 \lesssim k^2$ while leaving modes with $p^2 \gtrsim k^2$ unaffected.

The evolution of Γ_k with the RG scale $t \equiv \ln(k/\Lambda)$ is governed by the *Wetterich equation* [7,8,50–52],

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2} \text{STr}[(\Gamma_k^{(2)}[\Phi] + R_k)^{-1} \partial_t R_k], \quad (36)$$

where,

- (i) $\Gamma_k^{(2)}[\Phi]$ is the Hessian (second functional derivative) of Γ_k with respect to the fields Φ , evaluated at a field configuration (typically constant backgrounds);
- (ii) R_k is the infrared regulator, a momentum-dependent “mass term” that implements the scale cutoff (see Appendix E);
- (iii) STr denotes a supertrace: a functional trace over momenta, a sum over internal field indices, and a minus sign for fermionic or ghost degrees of freedom.

We are now ready to proceed with our main study of the phase structure of the curved-space axion-electrodynamics effective action (20). To this end, we first remark that the Wilsonian RG [48,49] provides a framework for understanding the scale dependence of quantum field theories. In RG, one integrates out quantum fluctuations in momentum shells from a UV scale Λ down to a lower scale k , thereby constructing a sequence of scale-dependent effective actions $S_k(\Phi)$. Couplings evolve as high momentum modes are eliminated. This procedure usually requires a sharp momentum cutoff, which can obscure gauge and diffeomorphism invariance. It is also cumbersome since it requires frequent reparametrizations and is not framed in terms of the one-particle irreducible (1PI) effective action. By contrast the Wetterich equation [8] evolves the effective *average* action $\Gamma_k[\Phi]$ —a scale-dependent version of the 1PI effective action—which includes the effects of quantum fluctuations

¹³The value expected in the context of StRVM, in order to have phenomenologically acceptable duration of inflation, appears in the second row [35].

with momenta $p^2 \gtrsim k^2$ while suppressing those below k^2 via a smooth regulator function $R_k(p^2)$ (see Appendix E). This formulation (an example of the FRG) remains exact even in strongly coupled regimes; it allows systematic truncations, which are approximations that go beyond perturbation theory (e.g., derivative expansions) that retain nontrivial quantum information, while being computationally tractable.

Although (36) resembles a one-loop formula [50], all vertices and propagators are fully dressed (coming from $\Gamma_k^{(2)}$) so no approximation is made. Boundary conditions are implemented via the limiting behaviors of R_k . In order to make the approach tractable approximations need to be made. General relativity and its modifications are conventionally nonrenormalizable theories but still predictive for $E \ll M_{\text{Pl}}$. Because of nonrenormalizability quantum gravity generates an infinite tower of higher-dimensional operators [as noted in the effective field theory (EFT) point of view]. If one tries to absorb loop corrections into a single running of the Newtonian coupling constant, after making truncations of the operators mixing under renormalization, there is typically no unique, universal β function [53]. We will not deal with the difficult issue of truncation dependence of beta function fixed points. Such issues are related to a full theory of quantum gravity; our considerations do not claim an understanding of this issue.

The universal content of FRG lies in the fixed point values and critical exponents of dimensionless ratios such as Gk^2 [51]. In asymptotic safety it is fixed points controlling the $k \rightarrow \infty$ behavior that is studied [51]. Another approach [53–55] (in terms of on-shell couplings) is more suited for $k \rightarrow 0$. The two approaches complement each other [51]. Our purpose is to consider topological

gravitational theories, which are not usually considered within the FRG framework; there are symmetry considerations, which lead to nonrenormalization of couplings. This nonrenormalization is important in extending the analysis in [2]. Our calculations are based on those in [7,52,56,57]; however the effect of the gravitational CS term on the axion is not required explicitly following arguments in [19,20,26] and discussed in Sec. II. In order to familiarize the general reader with the machinery of the functional renormalization group we will first examine a toy example for a single component scalar field theory in Appendix E; we indicate how the ideas extend to more general theories of (36). The Appendix introduces the LPA' [58], which we use in our Chern-Simons theory. LPA' is the simplest framework incorporating wave-function renormalization, essential for discussing anomalous dimensions and couplings of the renormalization in terms of threshold functions, a feature common to the more complex theories that we consider.

A. Renormalization and topology

Topology and symmetry constrain the solutions of the Wetterich equation in gauge theories. The language of differential forms [10,59] is useful in topological settings [60] and allows a more compact exposition without the need for explicit factors of the Levi-Civita tensor and (torsionless) metrics. A part of our action coincides with *axion electrodynamics* [written in terms of a $U(1)$ gauge field with a field strength 2-form F and a 0-form axion field b]; the associated average 1PI action is

$$\Gamma_k = \int_M \left\{ \frac{1}{2} Z_{A,k} F \wedge *F + \frac{1}{2} Z_{b,k} db \wedge *db + \frac{g_{\text{mix},k}}{4} b F \wedge F - V(b) *1 \right\}, \quad (37)$$

(and equivalently in index notation

$$\Gamma_k = \int d^4x \sqrt{g} \left[\frac{Z_{A,k}}{4} F_{\mu\nu} F^{\mu\nu} + \frac{Z_{b,k}}{2} \partial_\mu b \partial^\mu b + \frac{g_{\text{mix},k}}{4} b F_{\mu\nu} \tilde{F}^{\mu\nu} - V(b) \right], \quad (38)$$

where, for generality, we have included a potential $V(b)$ (which will be taken to be zero on requiring shift-symmetry, implied by our string-theory based framework). For our full system, there are additional contributions $\Delta\Gamma_k$: the gravitational CS term,¹⁴ the dynamical CS term with the axion coupling and the Einstein-Hilbert action. Explicitly

$$\Delta\Gamma_k = \frac{i\theta_k}{384\pi^2} \int_M R^{ac} \wedge R_{ac} + \frac{ig_{N,k}}{192\pi^2} \int_M b R^{dc} \wedge R_{dc} - \frac{1}{16\pi G_k} \int_M R *1 \quad (39)$$

(and equivalently in index notation

$$\Delta\Gamma_k = \int d^4x \sqrt{g} \left[\frac{i\theta_k}{384\pi^2} R_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma} + \frac{ig_{N,k}}{192\pi^2} b R_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma} - \frac{1}{16\pi G_k} R \right]. \quad (40)$$

¹⁴This term, because of StRVM, does not contribute to our model [19,20,25,26], but is included in this discussion in case one wishes to consider more general CS gravity models.

Here the curvature 2-form in the orthonormal frame is $R_b^a = d\omega_b^a + \omega_c^a \wedge \omega_b^c$ (where the one-form ω_c^a is the spin connection) or equivalently $R_b^a = \frac{1}{2} R_{b\mu\nu}^a dx^\mu \wedge dx^\nu$ in terms of the Riemann tensor; R is the Ricci scalar.

For all gauge fields (including gravity) we also need to add *gauge fixing* terms $\Delta\Gamma_{\text{GF}}$ and *Fadeev-Popov ghost* terms in order to eliminate nonphysical degrees of freedom but they are not relevant for our arguments based on topology. Furthermore, to avoid overburdening our exposition, we do not explicitly consider non-Abelian gauge fields since our analysis extends to this case in a straightforward way.

In the absence of $V(b)$ the effective action is invariant under the constant shift $b(x) \rightarrow b(x) + \zeta$ since $\int bR \wedge R \rightarrow \int (b + \zeta)R \wedge R$ and $\int bF \wedge F \rightarrow \int (b + \zeta)F \wedge F$; the resulting change is $\zeta \int (F \wedge F + R \wedge R)$. Owing to the topological nature of these terms

$$\zeta \int F \wedge F = \zeta \int d(A \wedge F) = 0 \quad (41)$$

and

$$\zeta \int R \wedge R = \zeta \int d\left(\omega \wedge R - \frac{1}{2}\omega \wedge \omega \wedge \omega\right) = 0. \quad (42)$$

We use here the general result that for an oriented d -dimensional manifold M (with boundary ∂M) and $u \in \Omega^{d-1}(M)$, we have $\int_M du = \int_{\partial M} u$. If u vanishes on ∂M then $\int_M du = 0$. In our case the boundary of our space time manifold is at infinity and our fields are assumed to fall off sufficiently fast for boundary contributions to vanish.

This shift symmetry has implications for renormalization of the relevant coupling constant $\tilde{g}$ (a generic notation, not to be confused with gravitational metric or coupling). If $g^{\text{CS}} \rightarrow g^{\text{CS}} + \delta\tilde{g}$ under renormalization we can introduce a multiplicative field renormalization,

$$b(x) = \zeta b'(x) \quad (43)$$

where $\zeta = g^{\text{CS}}/(g^{\text{CS}} + \delta\tilde{g})$ so that

$$(\tilde{g} + \delta\tilde{g}) \int bR \wedge R = \tilde{g} \int b'R \wedge R.$$

Hence

$$Z[\tilde{g} + \delta\tilde{g}] = \int Db e^{-S[b;\tilde{g}+\delta\tilde{g}]} = \int Db' e^{-S[b';\tilde{g}]} = Z[\tilde{g}], \quad (44)$$

which implies that $\delta\tilde{g} = 0$, i.e., $\tilde{g}$ is not renormalized. As noted in [7] the physical observables are in terms of

$g_{\text{mix},R}^2 = \frac{g_{\text{mix},k}^2}{Z_A^2 Z_b}$ and the running of $g_{\text{mix},R}$ is due to running of Z_A and Z_b , but *not* $g_{\text{mix},k}$. A similar analysis implies the nonrenormalization of the gravitational (and any non-Abelian) CS coupling.

The above argument relies on shift symmetry and is appropriate for our case since the axion does not have a potential term (including a mass term). For nonvanishing mass it is possible to relax this requirement [7] by using $U(1)$ gauge invariance and Bose symmetry, which allows us to write the full three-point function as

$$\Gamma_{k,bAA}^{\mu\nu}(p_b, p_1, p_2) = i g_{\text{mix},k} \mathcal{F}_k(s, t, u) \epsilon^{\mu\nu\rho\sigma} p_{1\rho} p_{2\sigma} \quad (45)$$

where, in the form factor $\mathcal{F}_k$, s , t , and u are the Mandelstam variables defined by

$$s = (p_1 + p_2)^2, \quad t = (p_2 + p_b)^2, \quad u = (p_b + p_1)^2.$$

In a derivative expansion (at the basis of our effective action) any $\mathcal{F}_k$ is analytic at small external momenta, i.e., $F(s, t, u) = F_0 + O(p^2/k^2)$ for generic external momentum p . From our previous topological arguments any F_0 generated through higher order loops gives a vanishing contribution to the running of $g_{\text{mix},k}$.

Consequence for RG flow. The nonrenormalization of $g_{\text{mix},k}$ means that the running of the dimensionless coupling g_{mix} comes entirely from wave-function renormalization. Defining

$$g_{\text{mix}}(k) \equiv k g_{\text{mix},R}(k) = k \frac{g_{\text{mix},k}}{\sqrt{Z_b(k)} Z_A(k)}, \quad (46)$$

we obtain

$$\partial_t g_{\text{mix}} = \left(1 + \frac{1}{2}\eta_b + \eta_A\right) g_{\text{mix}}, \quad (47)$$

where the anomalous dimensions are

$$\eta_b = -\partial_t \ln Z_b, \quad \eta_A = -\partial_t \ln Z_A. \quad (48)$$

This result is central to our analysis: the fate of g_{mix} under RG flow is determined by the anomalous dimensions of the axion and photon fields. Singular behavior in η_b or η_A translates directly into singular behavior of $\beta_{g_{\text{mix}}}$.

1. Gravitational Chern-Simons term

A similar argument applies to the gravitational CS term,

$$\Gamma_k \supset \int d^4x \sqrt{g} \frac{i\theta_k}{384\pi^2} R_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma}, \quad (49)$$

where θ_k is the (dimensionless) gravitational CS coupling. Diffeomorphism invariance and the Bianchi identity for the

Riemann tensor ensure that θ_k does not run. *However*, as discussed in Sec. III C within the StRVM framework *this term is absent* in the postinflationary era due to anomaly cancellation by chiral fermions. We therefore omit it from our analysis.

B. The LPA'

To solve the Wetterich equation in practice, we must truncate the infinite tower of operators that appear in the effective action. The *local potential approximation* (LPA) is the simplest nontrivial truncation, retaining only a field-dependent potential $U(b)$ and assuming canonical kinetic terms (i.e., $Z_b = Z_A = Z_h = 1$ for all k).

The *modified local potential approximation* (LPA') improves on this by allowing scale-dependent wave-function renormalizations,

$$\Gamma_k = \int d^4x \sqrt{g} \left[\frac{Z_b(k)}{2} (\nabla b)^2 + \frac{Z_A(k)}{4} F_{\mu\nu} F^{\mu\nu} + U_k(b) + \dots \right], \quad (50)$$

where the ellipsis denotes gravitational and appropriate Chern-Simons terms. In our case, shift symmetry requires $U_k(b) = 0$, so the truncation reduces to running wave functions and couplings.

1. Regulator choice and anomalous dimensions

In LPA' [61], the infrared regulator is chosen to respect the running wave functions,

$$R_{k,X}(p^2) = Z_X(k) k^2 r_X\left(\frac{p^2}{k^2}\right), \quad X \in \{b, A, h\}, \quad (51)$$

where $r_X(y)$ is a dimensionless shape function. A convenient choice is the *Litim regulator* [62],

$$r_X(y) = \left(\frac{1}{y} - 1\right) \Theta(1 - y), \quad (52)$$

which provides a sharp cutoff at $p^2 = k^2$ and leads to simple analytical expressions for loop integrals. Taking the scale derivative of Eq. (51), we find

$$\partial_t R_{k,X}(p^2) = Z_X(k) k^2 [(2 - \eta_X) r_X(y) - 2y r'_X(y)], \quad (53)$$

where we used $\partial_t Z_X = -\eta_X Z_X$ and $\partial_t y = -2y$. The factor $(2 - \eta_X)$ is crucial: it introduces the anomalous dimensions into the right-hand side of the Wetterich equation, leading to a *self-consistent* system of equations for the η 's.

2. Self-consistency and singularities

Inserting Eq. (53) into the Wetterich equation (36) and projecting onto the two-point functions of b and A ,

we obtain coupled equations of the form

$$\eta_b = f_b(g_{\text{mix}}, g_N; \eta_b, \eta_A), \quad \eta_A = f_A(g_{\text{mix}}, g_N; \eta_b, \eta_A), \quad (54)$$

where f_b and f_A are functions determined by loop integrals (see F 4).

These equations are typically *linear* in the η 's, allowing us to write

$$M(g_{\text{mix}}, g_N) \begin{pmatrix} \eta_b \\ \eta_A \end{pmatrix} = \begin{pmatrix} s_b(g_{\text{mix}}, g_N) \\ s_A(g_{\text{mix}}, g_N) \end{pmatrix}, \quad (55)$$

where M is a 2×2 matrix and s_b, s_A are source terms.

a. The origin of singularities. Singularities in the anomalous dimensions (and hence in the beta functions) occur when the matrix M becomes noninvertible,

$$\det M(g_{\text{mix}}, g_N) = 0. \quad (56)$$

At such points, the system of equations has no unique solution, and the theory becomes ill defined. In the flat-space-time analysis of Ref. [7], such a singularity was found at a finite value of g_{mix} , signaling a Landau-pole obstruction to the UV completion of Hermitian axion electrodynamics.

Our goal in Sec. VA is to determine whether the inclusion of gravitational fluctuations removes this singularity (by modifying M) or whether it persists, confirming the conjecture of Ref. [2].

C. Comments on the incorporation of the axion-gravitational-anomaly interactions

The StRVM model of cosmology motivates the approach in [2]. In view of the cancelation of the gCS anomalies in post RVM-inflationary eras of the Universe, such terms have not been incorporated in our FRG analysis. In this subsection we briefly discuss the potential role of gCS anomalous interactions with the KR axion, as far as the aforementioned FRG flows are concerned. In view of the increase in complexity, and the aforementioned motivation within the StRVM framework, we shall not present any analysis of such terms here, but we shall outline potential ways of dealing with them, in more general situations in which they are present.

First of all we remark that, in general, the inclusion of non-perturbative FRG methods to treat the renormalization of the coefficients of the gravitational and gauge CS terms in (7) is expected to lead to different FRG flows for these two couplings. This is a first issue of concern, given that the anomalous terms (7) are already quantum effects arising from the addition of appropriate RG counterterms in string theory, to ensure cancelation of gauge and gravitational anomalies. As such, in a full string theory, they should be

exact. This is the meaning of the introduction of the Lagrange multiplier KR field b in the effective string-inspired path integral, to implement the Bianchi constraint (10), following the exact redefinition (9) of the antisymmetric-tensor field strength, $H_{\mu\nu\rho}$, as discussed in the introduction. Truncating the full string theory to a gravitational field theory, and using a QFT, rather than string theory, nonperturbative FRG, based on the Wetterich equation (36), is therefore not fully appropriate for the string-inspired case at hand.

If the gCS interaction in the Wetterich truncated formalism is characterized by finite FRG behavior of the pertinent coupling in the analytically continued $\mathcal{PT}$ -symmetric theory, in a similar manner to the axion coupling in the flat-space-time axion electrodynamics [7], then the conclusions of [2] remain intact, without the need to perform an explicit string theory extension of the Wetterich formalism, which is currently not available. If, on the other hand, the contribution of the gravitational CS terms is such that the $\mathcal{PT}$ symmetric theory is characterized by a singular FRG flow of the gCS coupling in the infrared regime, then the work of [2] will be invalidated, since, in contrast to the flat case, the gravitationally anomalous theory would still be characterized by singular flows, and thus the continuation of [2] would be meaningless within this truncated field-theoretic FRG approach.

To ensure the string-theory considerations of Green-Schwarz mechanism [29] are respected by our truncated FRG analysis, we propose below to impose the constraint that the CS terms (7) remain intact under FRG, that is, the couplings of the gCS and gauge CS anomaly terms are renormalized in the same way, so their FRG flows remain the same, thus reducing the dimensionality of the coupling space. In other words, one is compelled to use an appropriately *constrained Wetterich equation* to tailor the FRG to our string-inspired system (7).

For completeness we note that a similar argument [63] can be made for the nonrenormalization of the coupling for $bR\tilde{R}$ in the presence of a potential for b . We shall discuss gravity and its quantization later. The 1PI vertex for one axion and two gravitons [after gauge fixing and transverse traceless (TT) projection] is

$$\Gamma_{bhh}^{\mu\nu,\rho\sigma}(p_b, p_1, p_2) = i g_{N,k} \epsilon^{\alpha\beta\gamma\delta} p_{1\alpha} p_{1\beta} p_{2\gamma} p_{2\delta} T^{\mu\nu,\rho\sigma}(p_1, p_2) \times F(s, t, u, \dots) \quad (57)$$

where $T^{\mu\nu,\rho\sigma}$ is rank-2 transverse projector such that $p_{1\mu} T^{\mu\nu,\rho\sigma} = 0$ or $p_{2\rho} T^{\mu\nu,\rho\sigma} = 0$ [cf. (40)]. The diffeomorphism and local Lorentz Ward identity [63] requires the vertex to be transverse in each graviton leg. In a derivative expansion $F = F_0 + O(p^2/k^2)$ which is related to a constant b and by our previous arguments there is no running of the coupling constant. For phenomenological reasons (based on condensates in the running vacuum model [9])

this term involving the Pontryagin density will not be part of our minimal model for including the effect of gravity.

D. Dependence of the axion anomalous dimension η_b on the photon anomalous dimension η_A in axion electrodynamics

The classical action for axion electrodynamics is,

$$S = \int d^4x \left[\frac{1}{4} Z_A F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} Z_b \partial_\mu b \partial^\mu b + \frac{g_{\text{mix},k}}{4} b F_{\mu\nu} \tilde{F}^{\mu\nu} \right]; \quad (58)$$

we functionally differentiate the Wetterich equation twice with respect to the axion fields $b(p)$, $b(-p)$ (in Fourier space), and project onto the kinetic term (i.e., isolate the coefficient of p^2). The term $bF\tilde{F}$ generates a vertex between one axion and two photons. Therefore, one-loop diagrams contributing to η_b involve internal photon lines, where

$$\eta_{b(A)} = -\frac{\partial}{\partial t} Z_{b(A)}, \quad t \equiv \ln k, \quad (59)$$

denote the corresponding quantum field b or even in absence of a direct axion loop, photon dynamics can induce nontrivial flow for η_b . The structure is universal for dimension-5 operators of the type $bF\tilde{F}$, and can be seen in general settings in axion electrodynamics: η_b sees η_A via the loop diagrams generated by the $bF\tilde{F}$ interaction. This coupling manifests in FRG through $\partial_t R_k^A$, which carries η_A . The result is nonperturbative, and essential for accurately describing the renormalization of axion-photon systems [7],

$$\eta_b = \frac{g_{\text{mix}}^2}{6(4\pi)^2} \left(2 - \frac{\eta_A}{4} \right), \quad (60)$$

$$\eta_A = \frac{g_{\text{mix}}^2}{6(4\pi)^2} \left(4 - \frac{1}{4} (\eta_b + \eta_A) \right). \quad (61)$$

In recent work [2,7] the renormalization flow of the coupling¹⁵

$$g_{\text{mix}}^2 = \frac{g_{\text{mix},k}^2}{Z_A^2 Z_b} \quad (62)$$

has been investigated. In the massless b case the flow of g_{mix} is given by

$$\partial_t g_{\text{mix}}^2 = 2g_{\text{mix}}^2 \frac{13g_{\text{mix}}^4 - 8064\pi^2 g_{\text{mix}}^2 - 147456\pi^4}{g_{\text{mix}}^4 - 384\pi^2 g_{\text{mix}}^2 - 147456\pi^4}. \quad (63)$$

¹⁵Note that as we shown above and stated in [2,7], $g_{\text{mix},k}$ is nonrenormalizable, and hence not dependent on the energy scale k , while g_{mix} gain dependence on the energy scale k by the flow of the fields.

The Landau pole structure of this flow led to speculation connecting Hermitian theory to $\mathcal{PT}$ -symmetric theory, which led to the possibility of a repulsive gravity phase in the $\mathcal{PT}$ -symmetric theory [2]. Since gravity was not included in the theory that showed interesting features at the Landau pole, our purpose is to investigate whether the *Landau pole* features remain *when gravity is included*.

IV. INTRODUCTION OF GRAVITY INTO THE AXION MODEL

In a four dimensional space time the quantum theory of gravity is highly nontrivial; as a conventional field theory it is nonrenormalizable and needs to respect the exact Ward identities of diffeomorphism invariance. This sets it apart from the scalar theory discussed in Appendix E. In the FRG approach we integrate out field fluctuations by wavelength. To discuss wavelengths we need a background metric denoted by $\bar{g}$ (which can be flat). The FRG cuts off modes with momentum below a scale k . For a tensor field, like the space time metric, momentum is defined using a Laplacian built using the background. Gravity has gauge redundancy (diffeomorphisms). To quantize the theory, as is common to gauge theories, it is necessary to gauge fix and add ghosts (using a background connection). We have two options, one being to use a (slightly) curved background (and the heat kernel method [64]) to project to terms (in the effective action) or use a flat background but extract couplings from fluctuation vertices (e.g., two- and three-point functions at some external momentum). Implementing gauge invariance consistently in approximate formulations is one of the difficulties of the method. The fluctuation $h_{\mu\nu}$ around the background decomposes into spin-2, spin-1, and spin-0 parts [irreducible decompositions with respect to $SO(3)$ [65]]. If we perform the York decomposition of the symmetric tensor fluctuation $h_{\mu\nu}$ around a fixed background metric $\bar{g}_{\mu\nu}$ and decompose the full metric $g_{\mu\nu}$, any symmetric tensor fluctuation $h_{\mu\nu}(x)$ can be uniquely decomposed as (cf. Appendix F 1),

$$h_{\mu\nu} = h_{\mu\nu}^{\text{TT}} + \bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu + \left(\bar{\nabla}_\mu \bar{\nabla}_\nu - \frac{1}{4} \bar{g}_{\mu\nu} \bar{\nabla}^2 \right) \sigma + \frac{1}{4} \bar{g}_{\mu\nu} h, \quad (64)$$

where: $h_{\mu\nu}^{\text{TT}}$ is the TT tensor component, h and σ are scalars, and ξ_μ is a vector. In a rigorous treatment the renormalization of these different pieces of the metric are connected owing to diffeomorphism invariance.¹⁶ Using a background (including a flat one) is not about claiming that space time really has that background. The background is a useful scaffold [50] that organizes gauge symmetry and the

background-covariant regulator gives a soft, gauge-compatible infrared mass to all spin sectors (and so avoids infrared divergences).

The metric is treated in the background-field formalism with a linear split,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu}, \quad \kappa = \sqrt{32\pi G_N}. \quad (65)$$

Ignoring Chern-Simons terms in (39) for the moment, the gravity ansatz below is represented by an Einstein-Hilbert action, background-covariant gauge fixing term, and Fadeev-Popov ghosts [16,66]

$$\Gamma_k^{\text{grav}} = -\frac{1}{16\pi G_N} \int d^4x \sqrt{g} R + \frac{1}{2\alpha_{\text{gf}}} \int d^4x \sqrt{\bar{g}} \bar{g}^{\mu\nu} F_\mu[h; \bar{g}] F_\nu[h; \bar{g}] + \Gamma_k^{\text{ghost}} \quad (66)$$

where the gauge-fixing functional is chosen as

$$F_\mu[h; \bar{g}] = \left(\delta_\mu^\alpha \bar{g}^{\nu\beta} - \frac{1 + \beta_{\text{gf}}}{4} \delta_\mu^\nu \bar{g}^{\alpha\beta} \right) \bar{\nabla}_\nu h_{\alpha\beta}. \quad (67)$$

Here $\bar{\nabla}$ is the covariant derivative with respect to the background metric $\bar{g}_{\mu\nu}$, and α_{gf} , β_{gf} are gauge parameters. In the “standard gravity” analysis we consider the Landau (de Donder) gauge,

$$\alpha_{\text{gf}} \rightarrow 0, \quad \beta_{\text{gf}} = 0. \quad (68)$$

In this gauge, the TT and a scalar term in the decomposition remain. Γ_k^{ghost} is the Faddeev-Popov ghost action obtained from the gauge-fixing condition F_μ .

In this truncation we do *not* include an explicit cosmological constant term, in order to keep the setting close to perturbative gravity about flat space. Gauge fixing $F_\mu[h; \bar{g}]$ and the ghost action Γ_k^{ghost} are constructed to be covariant under background diffeomorphisms, i.e., transformations generated by a vector field ξ^μ acting as $\delta \bar{g}_{\mu\nu} = \mathcal{L}_\xi \bar{g}_{\mu\nu}$, with corresponding transformations on $h_{\mu\nu}$ and the matter fields. In this way, the effective average action Γ_k is invariant under *background* diffeomorphisms, while the full diffeomorphism (“split”) symmetry between g and $\bar{g}$ is broken by the gauge fixing and by the IR regulator R_k . The contributions of the various bits of the decomposition can also be obtained by using projection operators. The IR regulator R_k in the Wetterich equation and the gauge-fixing term break full diffeomorphism invariance at intermediate scales k .

We simplify the analysis through a single-metric truncation so that we use one set of running couplings/wave-function factors [i.e., one $Z_N(k)$] for the whole metric sector; no separate Z ’s for different fluctuation irreducible components and no distinction between background and fluctuation couplings. It is still possible to

¹⁶Another use of such a decomposition is to monitor the efficiency of an approximate calculation in maintaining diffeomorphism invariance [57].

project observables using a specific spin component such as spin-2. In Fourier transform momentum space labeled by momentum k_μ the spin-2 projection operator is [66]

$$P_{\mu\nu\rho\sigma}^{(2)} \equiv \frac{1}{2}(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}) - \frac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma}, \quad (69)$$

where $\theta_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$ is the transverse projector. In particular the graviton anomalous dimension can be expressed in terms of this projection. We define the second functional derivative (Hessian)

$$\Gamma_{hh}^{(2)\mu\nu\rho\sigma}(x, y) \equiv \frac{\delta^2 \Gamma_k}{\delta h_{\mu\nu}(x) \delta h_{\rho\sigma}(y)} \quad (70)$$

with all fields set to 0 after taking the functional derivatives. In momentum space we have

$$\begin{aligned} & \tilde{\Gamma}_{hh}^{(2)\mu\nu\rho\sigma}(p)(2\pi)^4 \delta^{(4)}(p+q) \\ &= \int \int d^4x d^4y e^{-ip \cdot x - iq \cdot y} \Gamma_{hh}^{(2)\mu\nu\rho\sigma}(x, y). \end{aligned} \quad (71)$$

We can consider $P^{(2)}$ and Γ_{hh}^2 as rank-4 tensors. On using the projection,

$$\eta_h = - \frac{P_{\mu\nu\rho\sigma}^{(2)}(p) \frac{\partial}{\partial p^2} \partial_\tau \tilde{\Gamma}_{hh}^{(2)\mu\nu\rho\sigma}(p)|_{p^2=k^2}}{P_{\alpha\beta\gamma\delta}^{(2)}(q) \frac{\partial}{\partial q^2} \tilde{\Gamma}_{hh}^{(2)\alpha\beta\gamma\delta}(q)|_{q^2=k^2}}. \quad (72)$$

In the de Donder gauge, $h_{\mu\nu}^{\text{TT}}$ and a scalar survive. The renormalization group flow of the effective action Γ_k is split into matter and gravitational parts,

$$\Gamma_k = \Gamma_k^{\text{matter}} + \Gamma_k^{\text{grav}}. \quad (73)$$

We follow the approach of [52]. The matter content consists of,

- (a) the real pseudoscalar massless axion field b ,
- (b) an Abelian gauge field A_μ with field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

The truncation for the matter sector is

$$\begin{aligned} \Gamma_k^{\text{matter}} = & \int d^4x \sqrt{g} \left[\frac{Z_b}{2} g^{\mu\nu} \partial_\mu b \partial_\nu b + \frac{\lambda}{4} b^4 \right. \\ & \left. + \frac{Z_A}{4} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} \right], \end{aligned} \quad (74)$$

where

- (1) Z_b and Z_A are wave-function renormalizations, with anomalous dimensions (59);
- (2) λ is the quartic scalar coupling; the requirement of shift symmetry will restrict us to $\lambda = 0$;
- (3) the gauge coupling is encoded via Z_A .

The matter action Γ_k^{matter} is written in a manifestly covariant form, ensuring invariance under space time diffeomorphisms (via the use of $g^{\mu\nu}$, $\sqrt{g}$, and covariant derivatives) and shift symmetry when $\lambda = 0$. An Abelian gauge-fixing term and (decoupled) Abelian ghosts are used in the gauge sector when computing the anomalous dimension η_A . However the IR regulator R_k and the gauge-fixing term break full diffeomorphism invariance at intermediate scales k . The results in [52] show some dependence on the regulator function of the beta functions.

We shall only concern ourselves with the simple but nontrivial model in (39) augmented with (66) and (74) in order to get an indication on the possible role of gravity on the Landau pole structure found in our earlier work [2]. In this model we keep, for the gravitational degrees of freedom, a projection onto the contribution from the transverse-traceless part $h_{\mu\nu}^{\text{TT}}$.¹⁷

Thus, we simplify our study in the gravitational sector by assuming,

- (a) One universal Z_N , i.e., project onto TT flows only. A single metric truncation uses one set of running couplings/wave-function factors and assumes no distinction between background and fluctuation couplings. We project observables using a spin-2 component; however we do not assign it its own independent Z . In pure Einstein-Hilbert gravity, the only propagating degrees of freedom are the spin-2 helicities. The scalar sector in the York split is either pure gauge or

¹⁷It is of course not a consistent gauge-fixed formulation of full quantum gravity, but suffices for our purposes in this work. We shall outline below, in a little detail, the next steps for a more complete analysis. Once we have decomposed the fluctuation $h_{\mu\nu}$ according to (64), the quadratic part of the (gauge-fixed) effective average action splits into four *separate* blocks, each capturing one spin sector. A rigorous procedure needs to introduce distinct wave-function renormalizations for each sector,

$$Z_{\text{TT}}(k), \quad Z_\xi(k), \quad Z_\sigma(k), \quad Z_{h(\text{Tr})}(k), \quad (75)$$

where we used the symbol $Z_{h(\text{Tr})}(k)$ for the wave function renormalization pertaining to the trace part h of $h_{\mu\nu}$ in the York decomposition (64), to avoid confusion with Z_h [cf. (77), below], representing the TT graviton wave function renormalization. The corresponding anomalous dimensions are [57],

$$\begin{aligned} \eta_{\text{TT}} &= -\partial_t \ln Z_{\text{TT}}, & \eta_\xi &= -\partial_t \ln Z_\xi, \\ \eta_\sigma &= -\partial_t \ln Z_\sigma, & \eta_h &= -\partial_t \ln Z_{h(\text{Tr})}. \end{aligned} \quad (76)$$

There will be constraints from diffeomorphism invariance; since our aim is to get a simple analysis of the possible role of quantum gravity we shall ignore all the anomalous dimensions except for η_{TT} . Keeping nothing but $h_{\mu\nu}^{\text{TT}}$ breaks Becchi-Rouet-Stora-Tyutin (BRST) invariance and so, as we have noted, our approach is phenomenological. It is, of course, possible to go beyond this approximation, but at the price of much more complicated expressions, and for our purposes their inclusion will not alter the basic conclusions.

constrained. The Hessian is already minimal and block diagonal in the de Donder gauge on flat space. The single-metric approximation is the identification that a single Z_N rescales the metric sector and is good for a flat background (in the de Donder gauge). We retain the minimally coupled matter fields the axion and Abelian gauge field used in our earlier analysis [2].¹⁸

- (b) We do not consider issues of consistency that have been examined in terms of modified Ward identities (mWI) [67], and derived from first principles using the transformation properties of the path integral and the regulator. In various cases the mWI reduce to the true Ward identities in the limit $k \rightarrow 0$. However truncations (like finite-dimensional truncations of the effective action) do not automatically satisfy mWIs. We adapt results, which as a first approximation, use single-field approximations where background and fluctuation fields are identified. Within the above framework, our analysis uses the work on the renormalization group flow equations in [7,52,56,57,57]. As discussed in Sec. III C, we ignore the axion-gravitational-anomaly interaction, since we work in the StRVM framework [19,20,26].

Nonetheless, such simplifications suffice to support our main point, namely that the singularity structure in the RG β function of the axion coupling in flat space times [7], which prompted the conjecture of [2] for a regularization provided by a $\mathcal{PT}$ -symmetric repulsive phase of gravity, is verified in this more rigorous (but still incomplete) treatment, which includes RG flows of the gravitational sector of the theory.

Below will give a brief overview of our approach. Our work adopts the background-field functional renormalization group (effective average action) and a minimal Einstein-Hilbert truncation for gravity together with an axion b and an Abelian gauge field A_μ [16]. Although the background field method (BFM) is intuitively useful for studying gravity in curved spacetimes, it remains a computational tool even when the background is flat. In standard quantization, fixing a gauge breaks the symmetry of the Lagrangian. With BFM we can distinguish between two types of transformations: quantum and background transformations. By choosing a background-covariant gauge fixing, beta functions will appear in terms of coordinate invariant objects like the Riemann tensor.

The key interaction in the matter sector is the operator $bF_{\mu\nu}\tilde{F}^{\mu\nu}$. In a derivative expansion this local vertex is not renormalized: the scale derivative of its bare coefficient

vanishes. Consequently, the running of the renormalized axion-photon coupling is entirely encoded in the wave-function renormalizations of b and A_μ . So we phrase the problem in terms of anomalous dimensions,

$$\begin{aligned}\eta_b &= -\frac{\partial}{\partial t} \ln Z_b, & \eta_A &= -\frac{\partial}{\partial t} \ln Z_A, \\ \eta_h &= -\frac{\partial}{\partial t} \ln Z_h, & t &\equiv \ln k,\end{aligned}\quad (77)$$

the first two of which we have previously defined precisely in flat space times (cf. Sec. III D). In our one-metric approximation we identify $Z_N = Z_h$, which should be understood from now on [see (85), below, and Appendix F 4], so the gravitational anomalous dimension can be interchangeably denoted by η_N or η_h .

On the gravity side, rather than rebuilding the metric/ghost Hessians *ab initio*, we import the standard Einstein-Hilbert (EH) result for the pure gravitational contribution to the graviton anomalous dimension.

This leads to the following recipe:

- (1) Take η_b^{CS} and η_A^{CS} from [7]
- (2) Add Einstein-Hilbert minimal coupling pieces linearly

$$\begin{aligned}\eta_b &\approx \eta_b^{\text{CS}}(g_{\text{mix}}) + A_s g_N, \\ \eta_A &\approx \eta_A^{\text{CS}}(g_{\text{mix}}) + A_V g_N\end{aligned}\quad (78)$$

where A_s and A_V are standard EH coefficients for a minimally coupled scalar and vector (usually calculated in the presence of a cosmological constant and incorporating Faddeev-Popov ghosts [57]). Here η_b^{CS} and η_A^{CS} satisfy (60) and (61).

So we are assuming no “interference” diagrams where a graviton correction dresses the axion photon loop (or vice versa) This simplification ignores the explicit incorporation of the anomalous dimension of Faddeev-Popov ghosts, and mutual feedback of η_A , η_b , and η_h . We give arguments deriving the conditions for the validity of this procedure in Appendix F 3.

With this input, the coupled problem reduces to determining η_b , η_A , and η_h [17]. The Wetterich equation is one loop in topology, but because the regulator (in LPA') carries the running wave-function factors, $\partial_t R_{X,k} \propto (2 - \eta_X)$, the anomalous dimensions also appear on the right-hand side. Projecting the flow of the two-point functions (axion, photon, and TT graviton) yields three algebraic equations that can be solved self-consistently. In this first approximation we keep only the leading dependence on the small couplings—schematically, terms linear in g_N and in g_{mix}^2 , where g_{mix} is the dimensionless CS coupling built from the scale independent bare coefficient $g_{\text{mix},k}$ [cf. (62)]. This is discussed in Appendix F 3. The resulting expressions for η_b and η_A contain the familiar CS contributions from axion

¹⁸If we were to add higher-derivative terms such as R^2 and $R_{\mu\nu}R^{\mu\nu}$ to the action, the scalar/conformal sector in the York decomposition becomes dynamical and would have its own running. In our simple case χ , although gauge invariant, does not correspond to an independent propagating mode. This makes our model the natural one for considering fluctuations around flat backgrounds.

photon loops and minimal-coupling gravity corrections taken from EH analysis [52].

This yields a transparent two-coupling RG system. The Newton coupling runs with

$$\partial_t g_N = (2 + \eta_h) g_N. \quad (79)$$

The CS coupling runs with

$$\partial_t g_{\text{mix}} = \left(1 + \eta_A + \frac{1}{2} \eta_b\right) g_{\text{mix}}. \quad (80)$$

Conceptually this scheme is attractive for four reasons. First, the system of equations respects the protected local topological structure of the CS interaction. Second, it leverages established EH results for the pure gravity sector. It has simple limits: as $g_N \rightarrow 0$. Most importantly this scheme is an RG-improved one-loop calculation: although the diagrams are one loop in topology, the self-consistent appearance of η 's inside $\partial_t R_k$ leads to a form of resummation known as LPA' [17], which is a refinement of the local potential approximation. For additional tractability and simplicity we can linearize the LPA' in small couplings $C \equiv \frac{g_{\text{mix}}^2}{16\pi^2} \ll 1$ and $g_N \ll 1$. What remains is a simple sum of sectorwise one-loop contributions.

We summarize the key elements of the FRG framework used in our analysis:

- (1) The Wetterich equation provides an exact, non-perturbative flow equation for the effective average action Γ_k .
- (2) Topological constraints (shift symmetry, Bianchi identities) ensure that certain couplings, such as $g_{\text{mix},k}$, are not renormalized directly. Their running arises entirely from wave-function renormalizations.
- (3) LPA' is a systematic truncation that retains running wave functions, while neglecting higher-derivative operators. It leads to self-consistent equations for anomalous dimensions, which can exhibit singularities.
- (4) Gravity is incorporated via the background field method and a single-metric approximation, allowing us to compute the graviton anomalous dimension η_h and its backreaction on the matter sector.

A. Calculation of renormalization group flows

1. Flow of Z_b

To compute $\partial_t Z_b$, we take two functional derivatives of the Wetterich equation (36) with respect to $b(p)$ and $b(-p)$, expand for small external momentum p^2 , and project onto the coefficient of p^2 . The loop diagrams contributing to $\partial_t \Gamma_k^{(2)}$ for b include:

- (a) Scalar self-loops, which generate a piece proportional to η_b itself (only if b has a self-interaction; in our minimal axion case, there is none).

- (b) Graviton loops, where one internal graviton line couples to two external b legs via the minimal kinetic term $\sqrt{g} Z_b g^{\mu\nu} \partial_\mu b \partial_\nu b$.

2. Contributions of η_N to the axion anomalous dimension η_b

Since the anomalous dimension of the axion field, $\eta_b = -\partial_t \ln Z_b(k)$, is determined by computing the momentum-dependent part of the axion two-point function's flow,

$$\partial_t \Gamma_b^{(2)}(p, -p) \sim \eta_b p^2 + \dots, \quad (81)$$

even in the absence of self-interactions, gravitational fluctuations contribute to this running. When the metric fluctuation $h_{\mu\nu}$ is dynamical, loops involving internal gravitons produce a contribution to $\partial_t Z_b$.

Let us consider the minimally coupled scalar (or axion) field b part of the action,

$$\Gamma_k^{\text{matter}} = \frac{Z_b(k)}{2} \int d^4 x \sqrt{g} g^{\mu\nu} \partial_\mu b \partial_\nu b. \quad (82)$$

Expanding this in metric fluctuations $h_{\mu\nu}$, we obtain interaction vertices between two axions and one or two gravitons. These allow one-loop diagrams with an internal graviton line and external axions. The flow of the axion two-point function depends on loop integrals involving $(\Gamma^{(2)} + R_k)^{-1} \cdot \partial_t R_k \cdot \partial_t R_k^{\text{grav}}$ contains a factor of $(\eta_N - 2)$, because $R_k^{\text{grav}} \sim Z_N(k)$. Thus, the diagram evaluating the axion wave-function renormalization receives contributions (from the right-hand side of the Wetterich equation) proportional to η_N . As noted earlier, in a simple truncation, one introduces a single $Z_N(k)$ (identified with Z_h) multiplying the Einstein-Hilbert action,

$$\Gamma_k^{\text{EH}} = \frac{Z_N(k)}{16\pi} \int d^4 x \sqrt{g} (-R + 2\Lambda). \quad (83)$$

The graviton propagator and the regulator R_k both involve $Z_N(k)$, so the flow equation includes,

$$\partial_t R_k^{\text{grav}} \sim (\eta_N - 2) R_k^{\text{grav}}. \quad (84)$$

Therefore, every loop with a graviton internal line brings a piece proportional to η_N . In the case of the axion, the diagrams that dress the kinetic term involve exactly this structure.

V. CONTRIBUTIONS OF GRAVITY TO BETA FUNCTIONS AND ANOMALOUS DIMENSIONS

Before further calculations it is useful to summarize the definition of our model in terms of a single-metric Einstein-Hilbert truncation for gravity. The scale-dependent effective action reads

$$\begin{aligned} \Gamma_k[g; A, b] = & -\frac{Z_h(k)}{16\pi G_0} \int d^4x \sqrt{g} (R - 2\Lambda_k) \\ & + \int d^4x \sqrt{g} \left[\frac{Z_A(k)}{4} F_{\mu\nu} F^{\mu\nu} + \frac{Z_b(k)}{2} \partial_\mu b \partial^\mu b \right. \\ & \left. + \frac{i g_{\text{mix},k}}{4} b F_{\mu\nu} \tilde{F}^{\mu\nu} \right], \end{aligned} \quad (85)$$

with gauge fixing [de Donder for gravity, Landau for U(1)] and standard Faddeev-Popov ghost terms understood. We use the dimensionless Newton coupling and cosmological constant (which in our model we will later take to vanish)

$$g_N \equiv k^2 G_k, \quad \lambda \equiv \Lambda_k / k^2, \quad (86)$$

and anomalous dimensions (77).

The *dimensionless* axion-photon CS coupling is

$$\begin{aligned} g_{\text{mix}} & \equiv k g_{\text{mix},k} Z_b^{-1/2} Z_A^{-1}, \\ \Rightarrow \quad \beta_{g_{\text{mix}}} & \equiv \partial_t g_{\text{mix}} = \left(1 + \eta_A + \frac{1}{2} \eta_b \right) g_{\text{mix}}, \end{aligned} \quad (87)$$

reflecting the nonrenormalization of the *local* $bF\tilde{F}$ vertex. The Newton coupling runs as

$$\begin{aligned} \beta_{g_N} & = (2 + \eta_h) g_N, \\ \eta_h^{\text{grav}}(g_N, \lambda) & \equiv H(g_N, \lambda) = \frac{g_N B_1(\lambda)}{1 - g_N B_2(\lambda)}, \end{aligned} \quad (88)$$

where $B_{1,2}$ encode the usual EH traces (metric + ghosts) for the chosen gauge/regulator. While the York decomposition is a convenient way to organize fluctuation components, our projections are implemented with Barnes-Rivers spin projectors on flat background [66]; no explicit York identities are required for what follows. All pure-gravity effects enter via $H(g_N, \lambda)$ and the standard minimal-coupling matter dressing.

A. Results incorporating the role of gravity in LPA'

We next lead up to results step by step based on earlier sections and previous work [2]. The Wetterich equation, although exact, is a complex functional integrodifferential equation that cannot be solved without approximation. We introduced an approximate solution of a scalar theory in flat space time in (E9) of Appendix E, known as LPA' (cf. also Appendix F 3). If $Z_k = 1$ for all scales, then anomalous dimensions are zero, an approach known as the local potential approximation. The LPA' is a minimal but crucial refinement that addresses this drawback, by reintroducing a scale-dependent wave function renormalization factor for all fields.

In the LPA' formulation the regulator function gets an η dependence on using the form

$$R_{k,X}(p^2) = Z_X(k) k^2 r_X(y) \quad (89)$$

where $y = \frac{p^2}{k^2}$ and $r(y)$ has forms such as [62] $r_X(y) = \Theta(1-y)(\frac{1}{y} - 1)$ and X can denote h , b , A and ghost fields. From this we can deduce that

$$\partial_t R_{k,X}(p^2) = Z_X k^2 [(2 - \eta_X) r_X(y) - 2y r'_X(y)]. \quad (90)$$

In Sec. IV we have discussed the considerable complications that arise as soon as we wish to incorporate quantum gravity in a systematic way into the Wetterich formulation in the presence of gauge and matter (axion) fields, as well as topological interactions. Because of the subsequent technical nature of the discussion we will recall some of the key ideas to reconnect with the discussion in the earlier sections. The bare axion-U(1) CS interaction $g_{\text{mix},k}$ has mass dimension $[g_{\text{mix},k}] = -1$. The associated renormalized coupling is $g_{\text{mix},R} = Z_b^{-1/2} Z_A^{-1} g_{\text{mix},k}$. In FRG it is the flow of dimensionless couplings such as $g_{\text{mix}}(k) = k g_{\text{mix},R}(k)$ that is studied. We have

$$\frac{\partial}{\partial t} \log g_{\text{mix}} = 1 + \frac{1}{2} \eta_b + \eta_A \quad (91)$$

where $\eta_b = -\partial_t \ln Z_b$ and $\eta_A = -\partial_t \log Z_A$. Because shift-symmetry forbids any direct $bF\tilde{F}$ vertex renormalization, all running of g_{mix} comes from the two wave-function factors. We will decompose each anomalous dimension into matter-only and gravity induced pieces; so (in weak coupling) $\eta_b^{(\text{matter})} \propto g_{\text{mix}}^2$, and $\eta_b^{(\text{gravity})} \propto g_N = G(k) k^2$. Similarly $\eta_A = \eta_A^{(\text{matter})} + \eta_A^{(\text{grav})}$. The quantities $\eta_A^{(\text{matter})}$ and $\eta_b^{(\text{matter})}$ come from axion-photon loops and are $O(g_{\text{mix}}^2)$, whilst $\eta_{b,A}^{(\text{grav})}$ comes from 1-loop graviton exchange into the b or A propagator. As we commented earlier these calculations are scheme (choice of regulator) and gauge dependent.

In the spirit of (92) and (93), below,

$$\partial_t \gamma^2 = (2 + 2\eta_A + \eta_b) \gamma^2, \quad (92)$$

$$\int_{\gamma_0^2}^{\gamma^2} \frac{dy}{y(2 + \frac{5}{3(4\pi)^2} y)} = \int_{t_0}^t dt, \quad (93)$$

where $\gamma = k g_{\text{mix},R}$ and we simplified the nonperturbative expressions arising from (60) and (61); so we will first consider a perturbative approach to the effect of gravity.¹⁹ We consider the flow (in $d = 4$) of g_{mix}

$$\partial_t g_{\text{mix}}^2 = (2 + 2\eta_A + \eta_b) g_{\text{mix}}^2 - \mathcal{B}_{g_N} g_{\text{mix}}^2. \quad (94)$$

¹⁹The analog of (60) and (61) in the presence of gravity is given in the relations (77), which are computed in [57].

where $g_N = G$ and $\mathcal{B}(= 6\frac{\Phi_1^{(0)}}{\pi})$ is given in terms of a threshold function $\Phi_n^p(w)$ defined by

$$\Phi_n^p(w) = \frac{1}{\Gamma(n)} \int_0^\infty dz z^{n-1} \frac{R^{(0)}(z) - z \frac{d}{dz} R^{(0)}(z)}{(z + R^{(0)}(z) + w)^p}$$

and $R_k(p^2) = k^2 R^{(0)}(z = \frac{p^2}{k^2})$. This shows explicitly a scheme dependence of the coefficient $\mathcal{B}$, which we keep arbitrary. We shall take the flow of $g_N(= G)$ in the Einstein-Hilbert truncation as

$$\partial_t g_N = (2 + \eta_N(g_N))g_N, \quad (95)$$

where η_N has the form

$$\eta_N(g_N) = \frac{B_1 g_N}{1 - B_2 g_N} \quad (96)$$

and so, to lowest order in g_N ,

$$\partial_t g_N \approx (2 + B_1 g_N)g_N. \quad (97)$$

We consider

$$\partial_t g_{\text{mix}}^2 = (C(t) + A g_{\text{mix}}^2) g_{\text{mix}}^2 \quad (98)$$

where $C(t) = 2 - B g_N(t)$. As a first approximation $g_N(t)$ will be taken to be a constant, at a fixed point say. For Λ some reference momentum scale

$$\int_{g_{\text{mix}}^2(\Lambda)}^{g_{\text{mix}}^2(k)} dx \frac{1}{x(C + Ax)} = \log \frac{k}{\Lambda} \quad (99)$$

and so

$$g_{\text{mix}}^2(k) = \frac{C g_{\text{mix}}^2(\Lambda) (k/\Lambda)^C}{C + A g_{\text{mix}}^2(\Lambda) - A g_{\text{mix}}^2(\Lambda) (k/\Lambda)^C}. \quad (100)$$

A Landau pole k_p has to satisfy

$$k_p = \Lambda \left[\frac{C + A g_{\text{mix}}^2(\Lambda)}{A g_{\text{mix}}^2(\Lambda)} \right]^{1/C}. \quad (101)$$

[When $C \leq 0$ and $(C + A g_{\text{mix}}^2(\Lambda)) < 0$, there is no Landau pole but such values of C are not physical.]

1. Coupled flow of Newton constant and dynamical CS coupling g_{mix}

Our analysis will be based on the coupled equations based on the simplest Einstein-Hilbert truncation, which we recall,

$$\partial_t g_N = (2 + \eta_N(g_N))g_N \quad (102)$$

and to simplify the analysis, $\eta_N(g_N) \approx -E g_N$ with $E > 0$ (and a value that is scheme dependent). So

$$g_N(t) = \frac{g_N^*}{1 + (\frac{g_N^*}{g_N(0)} - 1) \exp(-2t)} \quad (103)$$

where $g_N^* = 2/E$. Writing $x(t) = g_{\text{mix}}^2(t)$ we have

$$\frac{dx}{dt} = (a(t) + Ax)x \quad (104)$$

with $a(t) \equiv 2 - B g_N(t)$. This is a Riccati-type equation that we rewrite as

$$\frac{dy}{dt} + a(t)y = -A \quad (105)$$

where $y(t) = 1/x(t)$. The solution for $y(t)$ is

$$y(t) = \left(I(0)y(0) - A \int_0^t I(s) ds \right) / I(t), \quad (106)$$

where $I(t) = \exp(\int^t a(s) ds)$, is an integrating factor determined by

$$\int_0^t a(s) ds = 2t - \frac{B g_N^*}{2} \left[\log \left(\frac{e^{2t} + \alpha}{1 + \alpha} \right) \right] \quad (107)$$

and $1 + \alpha \equiv g_N^*/g_N(0)$. The condition for a Landau pole is

$$\frac{1}{g^2(0)} = A \int_0^t ds \exp \left[\int_0^s ds' a(s') \right]. \quad (108)$$

In general the Landau pole is associated with a singularity of the beta function.

We need now to incorporate gravity into the *nonperturbative* framework in [7] by following (78), which we have discussed in the context of the Wetterich equation. This allows us to investigate whether the Landau pole persists in the presence of gravity. So we write

$$\eta_b = A_0 \left(2 - \frac{1}{4} \eta_A \right) - a \tilde{g}_N, \quad (109)$$

$$\eta_A = A_0 (4 - \eta_b - \eta_A) - \tilde{b} g_N, \quad (110)$$

with

$$A_0 = \frac{g_{\text{mix}}^2}{6(4\pi)^2}. \quad (111)$$

Rewriting (109)–(110) in linear form yields

$$\eta_b + \frac{A_0}{4} \eta_A = 2A_0 - a \tilde{g}_N, \quad (112)$$

$$A_0\eta_b + (1 + A_0)\eta_A = 4A_0 - \tilde{b}g_N. \quad (113)$$

This can be written compactly as

$$\begin{aligned} M(A_0) \begin{pmatrix} \eta_b \\ \eta_A \end{pmatrix} &= \begin{pmatrix} 2A_0 - a\tilde{g}_N \\ 4A_0 - \tilde{b}g_N \end{pmatrix}, \\ M(A_0) &= \begin{pmatrix} 1 & \frac{A_0}{4} \\ A_0 & 1 + A_0 \end{pmatrix}. \end{aligned} \quad (114)$$

The determinant of the coefficient matrix is

$$\Delta(A_0) \equiv \det M(A_0) = 1 + A_0 - \frac{1}{4}A_0^2. \quad (115)$$

Hence the system develops a singularity when

$$\Delta(A_0) = 0 \quad \Leftrightarrow \quad 1 + A_0 - \frac{1}{4}A_0^2 = 0. \quad (116)$$

Solving the system explicitly gives

$$\eta_A = \frac{4(-2A_0^2 + 4A_0 + A_0a\tilde{g}_N - \tilde{b}g_N)}{-A_0^2 + 4A_0 + 4}, \quad (117)$$

$$\eta_b = \frac{4A_0^2 + 8A_0 - 4a\tilde{g}_N - 4A_0a\tilde{g}_N + A_0\tilde{b}g_N}{-A_0^2 + 4A_0 + 4}, \quad (118)$$

where the common denominator is proportional to $\Delta(A_0)$. Therefore both anomalous dimensions diverge when the condition (116) is met and so the beta function is singular.

Importantly, the gravitational couplings g_N and $\tilde{g}_N$ enter Eqs. (109)–(110) only as inhomogeneous source terms and do *not* modify the coefficient matrix $M(A_0)$. Consequently, within this truncation, the position of the self-consistency pole is identical to that of axion electrodynamics without gravity.

Consequently the conjecture made for the singularities of the beta function for g_{mix} discussed in [2] in the presence of gravity is valid. The above result allows one to apply the conjectural discussion on the transition of the system to a repulsive phase of gravity at large scales, which may have important cosmological implications, as far as the interpretation of the currently observed acceleration of the Universe is concerned.

2. Connection with LPA' approach

As discussed in more detail in Appendix F 3, we can regard our calculation above as a linearized approximation of LPA'. We define the small parameters $C = \mathcal{O}(g_{\text{mix}}^2)$ and $D = \mathcal{O}(g_N)$. Then $\eta_b, \eta_A = \mathcal{O}(C) + \mathcal{O}(D)$. In η_h the terms proportional to $-\eta_b$ and $-\eta_A$ come multiplied by $D\tilde{\Phi}$, hence they are $\mathcal{O}(D\eta_{b,A}) = \mathcal{O}(CD) + \mathcal{O}(D^2)$. In the *linearized*

LPA' we retain only terms *linear* in C and in D and discard mixed/higher orders.

We still solve the coupled (η_b, η_A) system to linear order using the equations above, and use

$$\beta_{g_{\text{mix}}} = \left(1 + \eta_A + \frac{1}{2}\eta_b\right)g_{\text{mix}}, \quad \beta_{g_N} = (2 + \eta_h)g_N, \quad (119)$$

with $\eta_h (= \eta_N) \simeq DB_1/(1 - DB_2)$. This approximation *relies only on published* matter LPA' ([7]) and EH-sector inputs ([52,57]), and defers genuinely new $\mathcal{O}(CD)$ back-reactions to a subsequent iteration.

We will now give a more explicit discussion. The LPA' equation is in our case a set of three coupled inhomogeneous linear equations written as,

$$(1 - \tilde{U})\eta = S \quad (120)$$

where

$$\eta = \begin{pmatrix} \eta_b \\ \eta_F \\ \eta_N \end{pmatrix}, \quad (121)$$

$\mathbb{1}$ is the three-dimensional unit column vector, S is an η -independent three-dimensional column vector arising from the η independent terms in (E17), and $\tilde{U}$ is a 3×3 matrix derived from (E19) based on the chosen average action. Our argument will depend on the order of magnitude of the matrix elements rather than detailed expressions. In the simplification that we have considered $\tilde{U}$ is $\mathbb{U}$ where

$$\mathbb{U} = \begin{pmatrix} \tilde{U}_{bA} & 0 \\ 0 & \tilde{U}_{NN} \end{pmatrix} \quad (122)$$

and $\tilde{U}_{bA}$ is a 2×2 matrix. In the general case we can write

$$\mathbb{M} = \mathbb{1} - \tilde{U} = \begin{pmatrix} \mathcal{A} & \tilde{b} \\ \tilde{c}^T & \tilde{d} \end{pmatrix} \quad (123)$$

where

$$\mathcal{A} \equiv \begin{pmatrix} 1 - U_{bb} & -U_{bA} \\ -U_{Ab} & 1 - U_{AA} \end{pmatrix}, \quad (124)$$

$$\tilde{b} = \begin{pmatrix} -U_{bN} \\ -U_{AN} \end{pmatrix}, \quad (125)$$

and

$$\tilde{c}^T = (-U_{Nb} \quad -U_{NA}). \quad (126)$$

The matrix elements U_{bN}, U_{AN}, U_{Nb} , and U_{NA} are $O(g_N g_{\text{mix}}^2)$. We define the mixing parameter ϵ as the ratio

$$\epsilon = \frac{g_N g_{\text{mix}}^2}{g_{\text{mix}}^2}. \quad (127)$$

On defining Δ_{bAN} to be $\det(\mathbb{M})$, it is straightforward to show that

$$\Delta_{bAN} = \tilde{d} \det \mathcal{A} - \tilde{c}^T \text{adj}(\mathcal{A}) \tilde{b}, \quad (128)$$

where $\text{adj}(\mathcal{A})$ denotes the adjoint of $\mathcal{A}$, assuming that $\mathcal{A}$ is invertible. The term $\tilde{c}^T \text{adj}(\mathcal{A}) \tilde{b}$ is at most of $O(\epsilon^2)$ since $\text{adj}(\mathcal{A})$ is at most of order 1. Hence our simplified analysis, which considered only the zeros of $\det \mathcal{A}$, hence the singularity of the beta function persists and its location is shifted by $O(\epsilon^2)$.

B. Implications for Landau-pole analyses

In the Hermitian (CP -even) axion electrodynamics, the nonperturbative FRG flow for the dimensionless Chern–Simons coupling g_{mix} develops a finite-scale singularity in $\beta_{g_{\text{mix}}}$ (“Landau-type” obstruction). Defining the $\mathcal{PT}$ -symmetric theory by $g_{\text{mix}} \rightarrow i\tilde{g}_{\text{mix}}$ yields a non-Hermitian but $\mathcal{PT}$ -invariant flow in which the problematic segment of the Hermitian flow is replaced by a regular evolution for $\tilde{g}_{\text{mix}}$ that is UV safe (asymptotically free or controlled by a UV fixed point, depending on the scheme). Thus the Landau-type obstruction in the Hermitian branch is “resolved” by switching to the $\mathcal{PT}$ branch at the singular scale.

The prescription advocated above—the continuation to a $\mathcal{PT}$ -symmetric theory at that singularity—still remains consistent once gravity is included and η_b, η_A, η_h are coupled in a full LPA' in the region of parameter space where the linearized approximation can be justified. We recapitulate that within LPA', the axion-photon subsystem produces a self-consistency denominator from the $(2 - \eta)$ feedback in the Wetterich trace. Solving the coupled (η_b, η_A) equations yields the matrix inverse

$$\mathbb{M}(C)^{-1} \sim 1/\Delta(C), \quad (129)$$

with, schematically,

$$\Delta(C) = 1 + CA_A^{(2)} - C^2 A_b^{(1)} A_A^{(1)}, \quad C \equiv g_{\text{mix}}^2/(16\pi^2), \quad (130)$$

where $\eta_A = A_A^{(1)}(2 - \eta_A) + A_A^{(2)} + (\text{gravity pieces})$. The locus $\Delta(C) = 0$ induces a divergence of η_b, η_A , and hence of $\beta_g = (1 + \eta_A + \frac{1}{2}\eta_b)g$, at a finite RG scale. This is the Landau-type singularity in the Hermitian theory. The continuation is motivated by using the *Symanzik construction* of Green functions for $\mathcal{PT}$ theories [2,4,68].

Moreover also analytic continuation in the complex coupling plane can relate the $\mathcal{PT}$ partition function to the real part of the Hermitian one across the cut, $\ln Z_{\mathcal{PT}}(g) = \text{Re} \ln Z(\lambda = -g + i0^+)$, so that Schwinger-Dyson and FRG equations on the two sides match in their domain of validity. This is not a theorem in $3 + 1\text{D}$ but a well-motivated *conjecture* [11,12] based on scalar field models.²⁰

Our aim has been to check whether the $\mathcal{PT}$ evasion of the Landau pole can even be maintained in a naïve treatment of the perturbative gravitational theory. Including the graviton sector (single-metric EH truncation) adds,

- (i) the pure-EH denominator $(1 - DB_2(\lambda))$ in η_h with $D \equiv g_N$,

and

- (ii) minimal-coupling dressings that modify the coefficients $A^{(\cdot)}(\lambda)$ entering $\Delta(C)$.

Crucially, as we saw in VA 2, gravity *does not remove* the axion-photon self-consistency pole: the singularity surface $\Delta(C) = 0$ persists (albeit shifted quantitatively), and the only additional singular set is the standard nonphysical EH pole $1 - DB_2(\lambda) = 0$ in the Newton sector. Therefore the qualitative obstruction that motivates the $\mathcal{PT}$ switch in the matter-only analysis *also* appears in the coupled (η_b, η_A, η_h) system.

VI. CONCLUSIONS AND OUTLOOK

In the current article we have extended the FRG study of flat space time (massless) axion electrodynamics by including gravitational interactions, but not gravitational anomalies. The model fits within the StRVM framework, a string-inspired framework for cosmology, which yields a running-vacuum model for the evolution of the Universe. Such models, although consistent with the vast majority of the large-scale cosmological data available at present, predict observable deviations from ΛCDM cosmological framework. The deviations are capable of alleviating the persisting cosmic tensions [1] in the data measuring the Hubble parameter but also the matter growth at modern eras [24,69].

In our model for massless axions interacting through topological terms with Abelian gauge fields and Einstein gravity, we considered truncations in the York decomposition that allowed us to keep within a small set of couplings. In particular we restricted ourselves to project onto η_{TT} . In addition, of course, there are issues of ignoring higher curvature and in general space-time-derivative terms. An effective string-inspired theory of gravity consists formally of an infinity of such terms, and their resummability is of course beyond control, at least at

²⁰The difficulties of establishing the conjecture for gravitational theories are significant. The continuation from Minkowski metric to Euclidean metric is ambiguous and this applies to the $\mathcal{PT}$ inner product. The conformal mode of gravity is phantom and the role of $\mathcal{PT}$ symmetry in interpreting this is not clear.

present. Such terms complicate enormously the FRG analysis in the presence of gravity.

Fortunately, in our context of StRVM cosmology [19,20,25,26] and in the absence of dynamical dilatons, the quadratic terms in Riemann curvature can always be cast in a ghost-free Gauss-Bonnet (GB) form (within the context of perturbative string theory [37,70] using the matching of S-matrix amplitudes and the requirements of world-sheet conformal invariance). The GB terms are total derivatives in $(3 + 1)$ -dimensions, hence for constant dilatons, and up to fourth order in spatial derivatives in the string effective low-energy actions, the only nontrivial higher curvature terms are the CS anomaly terms (7). In the absence of gravitational anomalies, due to their cancellation in the postinflationary Universe in the StRVM framework, one is left essentially with axion electrodynamics models in the presence of gravity, (20), as we discussed in Sec. II; chiral U(1) global anomalies remain in this framework. The entire cosmological evolution of the StRVM Universe, from inflation to the current era, can be sufficiently described (from a phenomenological viewpoint) by restricting ourselves to terms of at most fourth order in derivatives [20,26,41,69]. Hence we need not consider higher order terms in our analysis to formulate our conjecture on the $\mathcal{PT}$ -symmetric repulsive phase interpretation of the observed acceleration of the Universe [2] within the CS theory.

In the presence of nonconstant dilatons, quadratic Riemann curvature terms should be included, which would complicate the FRG analysis. Furthermore the inclusion of a pseudoscalar potential has also been ignored since shift symmetry is crucial for restricting ourselves to an Abelian gauge anomaly term coupled to the axion; this term does not contain instantons and the maintenance of the shift symmetry is guaranteed. For generic UV complete effective theories of gravity outside the framework of StRVM, more complete truncations would be needed in order to support the conjecture of [2].

The conjecture of a $\mathcal{PT}$ -symmetric, repulsive gravity phase interpretation of the currently observed acceleration of the Universe, if true, is consistent with the swampland conjecture [71–75] on the incompatibility of string theory with the Λ CDM framework (characterized by a positive cosmological constant). Moreover, for such a novel interpretation, the recent data [76–78], pointing to a nonconstant relaxing dark energy, as well as other cosmological data [1,36], including gravitational-wave astronomy [79], need to be rethought, and reanalyzed. The fact that in our approach, the early (inflation) and late acceleration phases of the Universe are due to entirely different physical mechanisms is intriguing, especially because of the involvement of $\mathcal{PT}$ -symmetric non-Hermitian physics involved in the late cosmological eras. This calls for entirely novel interpretations of the available data, and perhaps unconventional probes as well. In this work we do not deal with such important phenomenological issues, which are postponed for future works.

We stress that the massless KR axion discussed in [2] and here, plays the role of torsion in space time geometry. We also remark that, although the Landau-pole-like singularities in the axion electrodynamics can persist in the *massive-axion* case, and thus the conjecture of [2] might be valid, nonetheless the fixed-point structure, and the incorporation of gravity in the pertinent FRG studies, is modified in a nontrivial way. We hope to be able to return to this case in a future work.

There are two different “run with scale” ideas being conflated in the literature. In EFT of gravity one can compute loop corrections and define a scale-dependent $G(\mu)$, but its running is scheme/process dependent and not directly an observable. In the framework of asymptotic safety, k is identified as a coarse-graining scale, which facilitates the study of the flow of the effective average action. The central object is the dimensionless Newton coupling. We comment on a potential contribution of our work to this ongoing debate on the RG running of the (dimensionful) gravitational constant with a scale. (For a selected partial literature on this see [53–55,80–82].) In the context of asymptotic safety in gravity [16,83], the running of Newton’s “constant” G is with respect to a momentum scale that is assumed physical. For asymptotically safe extensions of the Standard Model of particle physics, it is claimed²¹ that on allowing k to probe trans-Planckian momenta (see [66,86–90] for example), the existence of a fixed point implies predictivity: only finitely many relevant directions correspond to free parameters, while the remaining couplings are fixed by the requirement of reaching the fixed point. This is also the case in [7] and in our current work; one can see which couplings of the model are free parameters, and which can be predicted; one can attribute a physical meaning to the gravitational contributions to this type of running. In this way, it is argued that the objections of [53,80] on a potential physical running of G , which essentially pertain to conventional RG-scale-running below the Planck scale, can be avoided.

The arguments in [54,55,80] against gravitational contributions to gauge couplings, suggested in [91], and elaborated further in [92,93], relate to the fact that such a running is due to contributions of the effective action, which can be eliminated by appropriate field redefinitions. These leave the physical S matrix of the theory invariant, and hence do not contribute to physical quantities. In our approach, the above issues are not relevant for the main conclusion, namely the conjecture of a phase transition of the Universe to a $\mathcal{PT}$ -symmetric phase in the infrared. Here we have discussed the contributions of gravity to an FRG study of axion electrodynamics, in which the running of couplings is with respect to a renormalization-group/cutoff scale k , within an asymptotic safety scenario. However, for our

²¹See, e.g., [84,85] and references therein.

purposes here and in [2], the issue of a physical running of G , which lies at the heart of the aforementioned debate, is not of central importance. The central issue of our approach is the presence of singularities of the RG flow of a flat-space-time dimensionful coupling, the axion coupling. As we have argued above, this singularity structure of flat-space axion electrodynamics remains unaffected by the presence of gravity.

In the current article and in [2], we have identified the infrared energy scale at which the dimensionful axion-coupling RG flow becomes infinite: a phase transition to a $\mathcal{PT}$ -symmetric phase occurs, at the physical cosmological length scale at which the Universe enters an accelerated expansion. At such points of the nonperturbative FRG flow, the corresponding RG β function of the axion coupling diverges. The (renormalized) scattering S matrix of the model breaks down due to this singularity, even at flat space times. In this case, the Landau pole in the RG flow prompted us in [2] to conjecture the continuation of the theory to a new phase with $\mathcal{PT}$ symmetry and repulsive gravity. This phase has well-defined RG flows for the axion couplings in the infrared regime of physical momenta. The repulsive nature of gravity in the $\mathcal{PT}$ -symmetric phase is actually independent of the question as to whether a running of G is physical or not (which lies at the heart of the debate discussed in, e.g., [53,80]). This also does not invalidate the arguments of [54,55], given that the S matrix in our case is ill defined in the $\mathcal{PT}$ -broken phase at the singularities of the axion-coupling nonperturbative FRG flow. The singularities are deemed to have physical significance. Since this work relies on the conjecture of [11], it is necessary to investigate further the nature of the analytic continuation across the singular surface encountered in coupling space, the $\mathcal{PT}$ -metric that replaces the standard Hermitian inner product and the gauge/diffeomorphism consistency of the continued theory. These are all nontrivial issues. The robustness of our findings, implied by this naive gravitational calculation, makes this an interesting and necessary program.

Finally we close with a speculation on the physical applications of our $\mathcal{PT}$ -symmetric cosmic-acceleration approach to realistic cosmology. As our considerations do not require the existence of a dark energy component in the late-epoch energy budget of the Universe (or a cosmological constant, for that matter), it may perhaps find support from alternative-to-dark-energy theories for the late eras of the Universe, such as those in [94,95], and references therein. In this respect, as one potential test of our approach, one might consider following the suggestion of [96], employing the study of large-scale galaxy correlations. Another test is the proper use of late-universe gravitational waves in an attempt to indicate the onset of our $\mathcal{PT}$ phase transition, which might leave imprints in their profiles. These are topics that we hope to consider in the future.

ACKNOWLEDGMENTS

N. E. M. thanks the University of Valencia and its Theoretical Physics Department for a visiting Research Professorship supported by the program *Atracción de Talento* INV25-01-15. The work of L. C. is supported by a PhD scholarship from the King's-China Scholarship Council. The work of N. E. M. and S. S. is supported in part by the UK Engineering and Physical Sciences Research Council (EPSRC) and Science and Technology Facilities research Council (STFC) under the Research Grants No. EP/V002821/1 and No. ST/X000753/1, respectively. N. E. M. also acknowledges participation in the COST Association Actions CA21136 “Addressing observational tensions in cosmology with systematics and fundamental physics (CosmoVerse)” and CA23130 “Bridging high and low energies in search of Quantum Gravity (BridgeQG).”

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: BASIC FEATURES OF $\mathcal{PT}$ SYMMETRY

Global discrete $\mathcal{PT}$ symmetry is mostly considered in the context of flat space time, but classically it naturally translates, to curved space time.²² Parity $\mathcal{P}$ is a discrete diffeomorphism of the space time manifold M with a composition $\mathcal{P} \circ \mathcal{P} = id_M$, where id_M is the identity mapping. Moreover $\mathcal{P}$ is an isometry of the metric. As usual we get a familiar representation of $\mathcal{P}$ by choosing charts and an atlas for M . In a chart we choose $x: M \rightarrow \mathbb{R}^4$ and $\mathcal{P}: x^\mu \rightarrow \bar{x}^\mu = \mathcal{P}^\mu(x)$; in a particular case we may have $\mathcal{P}^\mu(x) = (x^0, -x^1, -x^2, -x^3)$. The operator $\mathcal{T}$ is handled in a similar way (provided the manifold is time orientable). In *quantum theory* $\mathcal{T}$ is an antilinear operator on the space of fields. In flat space time with Lorentzian signature $(p, q) = (3, 1)$, the real orthonormal-frame group $O(3, 1)$ has four disconnected pieces (its cosets) labeled by orientation and time orientation. In Euclidean signature $(p, q) = (4, 0)$ the frame group is $O(4)$ and has only proper or improper rotations. Although in general relativity the vierbein bundle is a principal $O(3, 1)$ bundle [59], in any small enough patch of the atlas, the frame bundle can be trivialized. Every local $O(3, 1)$ transformation is written as $\Lambda(x) = P^{\epsilon_P(x)} \mathcal{T}^{\epsilon_T(x)} L(x)$ where $L(x)$ is a *proper orthochronous* transformation; P_P^ϵ and T_T^ϵ change the space and time orientation for $L(x)$ depending on $\epsilon_P, \epsilon_T \in \{0, 1\}$.

A new form of quantum mechanics emerges where the role of Hermiticity is replaced by $\mathcal{PT}$ symmetry [6]. Non-Hermitian Hamiltonians H satisfying $[H, \mathcal{PT}] = 0$ can, remarkably, have entirely real spectra [97,98] for a large

²²We will be considering gravity linearized around a flat background. Minkowski (Lorentzian) manifold does allow charts and an atlas, which is a notion related to topology and smoothness.

family of Hamiltonians; the phase of H is determined by the domain of definition of H on a Hilbert space or the class of allowed paths in a path integral. Just as in the evaluation of the asymptotic behavior of real integrals, where the method of steepest descents is used involving the complex plane, the path integral considers complex paths using a generalization (of the method of steepest descents) known as Lefschetz thimbles [99]. This has led to much interest in constructing unitary field theories based on $\mathcal{PT}$ symmetry (see, for example, [100,101]) in Euclidean path integrals. The equivalence of canonical and path integral formulations of $\mathcal{PT}$ -symmetric theories is not settled even though there are many indications that calculations of Green functions can be done using either formulation [68]. In particular path integral formulations are often simpler to implement. An explicit determination of new inner products associated with $\mathcal{PT}$ symmetry is not required for calculations in some $\mathcal{PT}$ -symmetric scalar field theories [4]. Our considerations are for physics beyond the standard model of particle physics (BSM) and $\mathcal{PT}$ symmetry is an example of BSM. Such theories are commonly implemented using path integrals in four space time dimensions. We assume, for our purposes in this work, that the path integral approach is valid for $\mathcal{PT}$ symmetry. Given this starting point, it becomes possible to investigate the role of renormalization in $\mathcal{PT}$ quantum field theory and make connections with cosmology, and this is attempted in [2], and here, within the context of the StRVM, reviewed in Sec. II.

$\mathcal{PT}$ symmetry can arise due to renormalization in quantum systems [21]. This was first noticed in a non-relativistic toy field theory known as the Lee model [102]. This early model was quantum mechanical and invented in order to study the phenomenon of renormalization rigorously (which was facilitated by removing crossing symmetry). In this model, starting from a Hermitian Hamiltonian, renormalization forced the development of an imaginary coupling. Before the advent of $\mathcal{PT}$ symmetry, this model was abandoned as being unphysical. In the important work of Ref. [103] it was shown that the introduction of a $\mathcal{PT}$ inner product led to a reinterpretation of the Lee model in a new $\mathcal{PT}$ -symmetric phase. We shall now give a discussion of the emergence of $\mathcal{PT}$ symmetry in some simple standard field theoretic contexts [104]. Such studies are precursors of our work on a gravitational Chern-Simons model. [2,4].

APPENDIX B: $\mathcal{PT}$ SYMMETRY IN CHERN-SIMONS THEORIES

By choosing four-dimensional diagonal matrices for $\mathcal{P}$ and $\mathcal{T}$ (and noting the pseudovector nature of magnetic fields) we find for electrodynamics that under $\mathcal{PT}$, $E_i \rightarrow E_i$, $B_i \rightarrow B_i$. Hence, under $\mathcal{PT}$,

$$F\tilde{F} = \vec{E} \cdot \vec{B} \xrightarrow{\mathcal{PT}} \vec{E} \cdot \vec{B}, \quad (\text{B1})$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, with A_μ the U(1) gauge field, and the Hodge dual $\tilde{F}$ is defined by

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}. \quad (\text{B2})$$

So the interaction term

$$abF\tilde{F} \xrightarrow{\mathcal{PT}} -abF\tilde{F}, \quad (\text{B3})$$

for pseudoscalar b and real coupling α , is Hermitian but not $\mathcal{PT}$ symmetric. This should be contrasted with the case of pure imaginary coupling $i\alpha$

$$iabF\tilde{F} \xrightarrow{\mathcal{PT}} iabF\tilde{F}, \quad (\text{B4})$$

which is non-Hermitian but $\mathcal{PT}$ symmetric. b is taken to be *even* under $\mathcal{T}$, which is required when it is the dual of the Kalb-Ramond field strength. We have two possible actions:

$$S_E[\alpha \in \mathbb{R}] = \int d^4x \left[\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + i \frac{\alpha}{4} b F_{\mu\nu} \tilde{F}_{\mu\nu} \right], \quad (\text{B5})$$

$$S_E[\alpha \rightarrow i\alpha', \alpha' \in \mathbb{R}] = \int d^4x \left[\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \frac{\alpha'}{4} b F_{\mu\nu} \tilde{F}_{\mu\nu} \right]. \quad (\text{B6})$$

In the work of [2] it was suggested that singularities in the renormalization group flow [4,12] and the requirements of a finite partition function led to a phase change from Hermitian to $\mathcal{PT}$ phase. There is an analog with conventional phase transitions, where for finite volume the partition function Z is analytic in couplings. In the thermodynamic/continuum limit it splits into branches (phases) selected by different saddles/contours (Lefschetz thimbles). In $\mathcal{PT}$ symmetric models the same phenomenon appears as Stokes transitions; as couplings are varied the set of thimbles that makes the non-Hermitian but $\mathcal{PT}$ -invariant path integral jumps. Heuristically the renormalized theory defines $\beta(g) = \mu \frac{dg}{d\mu}$ from renormalization functions built out of Green's functions. Here we recall the argument of [4,68], which states that Green's functions can be calculated using path integrals without including any additional measure factor. On approaching a Stokes line certain counterterms can diverge and diagrammatic resummations switch dominant saddles (or thimbles). Z exists on *both* sides but the dominant thimbles (and thus the effective free-energy branch) differ. The Stokes set (singularities) is where these thimbles reconnect or rearrange. In finite volume this is perfectly smooth; in the limit of infinite volume limit the rearrangement is nonanalytic. These arguments are framed in Euclidean space.

Moreover similar considerations apply if the Abelian $F\tilde{F}$ is replaced by the non-Abelian or gravitational analogs.

These theories are conventionally nonrenormalizable and the set of possible interactions in theory space, which are connected by renormalization, is arbitrarily large, unless there are energy-scale, symmetry, or topological reasons for restricting the set.

Next we will show how the $\mathcal{PT}$ -symmetric version arises naturally in (massless) axion physics. We first define the theory in Minkowski signature $(+, -, -, -)$. For a massless pseudoscalar b ,

$$\mathcal{L}_M = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{g_{\text{mix}}}{4}bF_{\mu\nu}\tilde{F}^{\mu\nu}, \quad \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}. \quad (\text{B7})$$

In the StRVM model [19,20,26], which the conjecture of [2] is based upon, one has [cf. (7)],

$$g_{\text{mix}} = \sqrt{\frac{2}{3}} \frac{1}{24} \frac{M_{\text{Pl}}}{M_s^2}. \quad (\text{B8})$$

With standard $\mathcal{P}$ and $\mathcal{T}$ assignments (and b a pseudoscalar), we recall that $\mathcal{PT}: F_{\mu\nu}\tilde{F}^{\mu\nu} \mapsto -F_{\mu\nu}\tilde{F}^{\mu\nu}$. Hence, for real g_{mix} the interaction is Hermitian but not $\mathcal{PT}$ invariant; if $g_{\text{mix}} \rightarrow ig'_{\text{mix}}$ with $g'_{\text{mix}} \in \mathbb{R}$, the interaction becomes non-Hermitian but $\mathcal{PT}$ invariant.

For the FRG we work in Euclidean signature. Under Wick rotation $t \rightarrow -ix_4$,

$$S_M = \int d^4x \mathcal{L}_M \longrightarrow S_E = \int d^4x_E \mathcal{L}_E, \quad e^{-S_E} = \int \mathcal{D}A_\mu \mathcal{D}b e^{iS_M}, \quad (\text{B9})$$

and the Lagrangian reads

$$\mathcal{L}_E = +\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{ig'_{\text{mix}}}{4}bF_{\mu\nu}\tilde{F}_{\mu\nu}, \quad \tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}. \quad (\text{B10})$$

Thus a real Minkowski coupling α becomes an imaginary coefficient $i\alpha$ in S_E . Equivalently, the $\mathcal{PT}$ choice $g_{\text{mix}} = ig'_{\text{mix}}$ (Lorentzian) appears with a real coefficient $+g'_{\text{mix}}$ in S_E .

APPENDIX C: MOTIVATION FOR CHERN-SIMONS MODEL FROM ANOMALY CANCELLATION AND THE ROLE OF SHIFT SYMMETRY

The action in (7) arises in superstring theory [27,28,44,45], after *compactification* to four space-time dimensions; the bosonic ground state of the closed-string sector consists of *massless* fields in the so-called *gravitational multiplet*, which contains $\phi(x)$ a spin-0 (scalar) dilaton, $g_{\mu\nu}(x)$ a spin-2 traceless symmetric tensor

field, identified as the $(3+1)$ -dimensional graviton, and $B_{\mu\nu}(x)(=-B_{\nu\mu}(x))$ a spin-1 antisymmetric tensor gauge field known as the KR field, related to a pseudoscalar axion $b(x)$. We set the four-dimensional dilaton field to be a constant, $\phi(x) = \phi_0$, which fixes the string coupling

$$g_s = \exp(\phi) = \exp(\phi_0). \quad (\text{C1})$$

Since there are consistent solutions of four-dimensional string theory with such a configuration, we shall adopt this treatment of the dilaton. Associated with the KR B field there is a $U(1)$ gauge symmetry of the closed-string $(3+1)$ -dimensional target-space-time effective-field-theory action,

$$B_{\mu\nu}(x) \rightarrow B_{\mu\nu}(x) + \partial_{[\mu}\theta_{\nu]}(x), \quad \mu, \nu = 0, \dots, 3, \quad \theta_\mu(x) \in \mathbb{R}, \quad (\text{C2})$$

where the brackets $[\dots]$ denote antisymmetrization of the respective indices, and a suitable three-form field strength $\mathcal{H}$ of the B field [32].

We are also influenced by M theory *phenomenology* and anomaly cancellation (in higher dimensional space time), for inclusion of Kalb-Ramond fields in orbifold compactifications of M theory. Such compactifications are an essential ingredient in models beyond the SM of particle physics. SM has *chiral* fermions, ψ say, with action in curved space time given by $S[\psi, e] = \int d^4x e \bar{\psi} i \gamma^\mu D_\mu \psi$ where D_μ is the spinor covariant derivative and (in terms of vierbeins) $e = \det e_\mu^a$. Even though the classical action is invariant under local Lorentz transformations and diffeomorphisms, the quantum theory may not be. The reason is that the path integral measure for a chiral fermion is not invariant under such transformations. This leads to an anomalous variation of the effective action Γ . In four-dimensional *chiral* gauge theories triangle diagrams can generate gauge and mixed gauge-gravitational anomalies. If left uncanceled, these spoil the consistency of the quantum theory. A compact way to remove such anomalies is to introduce an antisymmetric two-form field $B_{\mu\nu}$ (the *Kalb-Ramond* field) and couple it through a Green-Schwarz (GS) counterterm. Below we briefly sketch summarize how this mechanism might work in a stand-alone effective-field-theory setting.

For simplicity we first consider a theory with Abelian and non-Abelian gauge interactions. In the context of anomaly cancellation involving a chiral fermion coupled to a non-Abelian gauge group $\mathcal{G}$, we shall use T^a to refer to the generators of the group $\mathcal{G}$ in some representation $\mathcal{R}$, under which the chiral fermions transform.

In the SM the gauge group is $SU(3)_c \times SU(2)_L \times U(1)_Y$. For a chiral $U(1)_Y$ with fermions of charges q_i the one-loop variation of the effective action reads

$$\delta_\alpha \Gamma_{1\text{-loop}} = \frac{1}{24\pi^2} \left(\sum_i q_i^3 \right) \int \alpha F \wedge F + \frac{1}{24\pi^2} \left(\sum_i q_i \text{Tr} T^2 \right) \int \alpha \text{Tr}(G \wedge G), \quad (\text{C3})$$

where F is the $U(1)_Y$ field strength, $\alpha(x)$ is the infinitesimal parameter of the $U(1)_A$ gauge transformation, and G is the field strength for the appropriate non-Abelian group $\mathcal{G}$. The two coefficients measure, respectively, the cubic $U(1)_A^3$ and mixed $U(1)_A$ - $\mathcal{G}^2$ anomalies. In SM the assignments q_i are such that the coefficients of the anomalous terms vanish, necessary for the consistency of the theory. However there are additional $U(1)$ axial symmetries²³ in compactifications from higher dimensional theories [to be denoted by $U(1)'$], and these need to be canceled in a different way due to Green and Schwarz [29] because of the necessary constraints. This new form of cancellation (in higher dimensions) is mathematically complicated and involves the descent equations [106], which are a mathematical tool used to analyze anomalies in gauge or gravitational theories, especially in the context of cohomological approaches. These equations arise from the observation that gauge anomalies in d dimensions can be related to characteristic classes (like the Chern character) in $d+2$ dimensions. Schematically we start with a closed, gauge invariant form $\text{Tr} F^{n+1}$ in $2n+2$ dimensions. Using the descent procedure we write $\text{Tr} F^{n+1} = d\Omega_{2n+1}^{(0)}$ and under a gauge variation

$$\delta \Omega_{2n+1}^{(0)} = d\Omega_{2n}^{(1)} \quad (\text{C4})$$

where $\Omega_{2n}^{(1)}$ is the anomaly polynomial, a local anomalous term.

Suppose we have a chiral fermion in $4D$, coupled to a gauge field A . Its anomaly polynomial [39,107] is a six-form in $6D$,

$$I_6 = \text{Tr}(F^3). \quad (\text{C5})$$

This is a closed and gauge-invariant six-form on a fictitious $6D$ space, often called the anomaly polynomial. We now use the descent procedure [108] to write $I_6 = d\Omega_5^{(0)}$ and $\delta_\alpha \Omega_5^{(0)} = d\Omega_4^{(1)}(\alpha)$. For $U(1)$ we have $\Omega_5^{(0)} = A \wedge F \wedge F$. Under a gauge variation $\delta A = d\alpha$ and so $\delta \Omega_5^{(0)} = d(\alpha F \wedge F) = d\Omega_4^{(1)}$. To cancel the anomaly or reproduce this variation, one introduces a scalar field $b(x)$, which plays the role of an axion or Green-Schwarz field in $4D$. We construct the $4D$ dynamical CS term

$$S_{\text{CS}} = \int_{M^4} b(x) F \wedge F \quad (\text{C6})$$

or for gravity (in form language, for economy in notation): $S_{\text{CS}}^{\text{grav}} = \int_{M^4} b(x) \text{Tr}(R \wedge R)$. It is thus plausible that a symmetry based on a shift of the axion field $b(x)$ can be used to cancel the anomalous terms in (C3) through adding to the action a term

$$S_{\text{GS}} \sim c_L \int b(x) \text{Tr}(G \wedge G). \quad (\text{C7})$$

Without further justification, we consider for our purposes in this work the role of shift symmetry in nonrenormalization of CS couplings in the main text.

APPENDIX D: THE ROLE OF SINGULAR FLOWS IN SIMPLE MODELS

We give simple examples of singular perturbative renormalization group flows in field theory and their potential connection with $\mathcal{PT}$ -symmetric phase transitions.

Consider a single real scalar field in four dimensions:

$$\mathcal{L} = \frac{1}{2} (\partial\phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4, \quad \lambda > 0. \quad (\text{D1})$$

For weak coupling in four dimensions the beta function (at one loop [4]) is positive,

$$\beta(\lambda) \equiv \frac{d\lambda}{d \ln k} = b\lambda^2 + \mathcal{O}(\lambda^3), \quad b > 0 \left(b = \frac{3}{16\pi^2} \text{ for } \phi^4 \right). \quad (\text{D2})$$

Integrating between a reference scale μ and k gives

$$\frac{1}{\lambda(k)} = \frac{1}{\lambda(\mu)} - b \ln \frac{k}{\mu}. \quad (\text{D3})$$

The coupling blows up at the Landau scale

$$k_L = \mu \exp \left(\frac{1}{b\lambda(\mu)} \right). \quad (\text{D4})$$

Let the *bare* coupling be $\lambda_0 \equiv \lambda(\Lambda)$ at the cutoff Λ , and the *renormalized* coupling be $\lambda_R \equiv \lambda(\mu)$ at fixed μ . Evaluating (D3) at $k = \mu$ and $k = \Lambda$ yields the matching relation

$$\frac{1}{\lambda_R} = \frac{1}{\lambda_0} + b \ln \frac{\Lambda}{\mu}. \quad (\text{D5})$$

Solving for the bare coupling as a function of Λ at fixed $\lambda_R > 0$,

²³Heterotic $E_8 \times E_8$ or type-II orientifold models [105] generically yield four to eight Abelian generators after compactification. Exactly one linear combination is anomalous.

$$\lambda_0(\Lambda) = \frac{1}{\frac{1}{\lambda_R} - b \ln(\Lambda/\mu)}. \quad (\text{D6})$$

Define the denominator

$$D(\Lambda) \equiv \frac{1}{\lambda_R} - b \ln \frac{\Lambda}{\mu}. \quad (\text{D7})$$

Then:

- (i) For Λ close to μ , $D(\Lambda) > 0$ and $\lambda_0 > 0$.
- (ii) $D(\Lambda)$ crosses zero at $\Lambda = k_L$:

$$D(k_L) = 0 \quad \Leftrightarrow \quad \ln \frac{\Lambda}{\mu} = \frac{1}{b\lambda_R}. \quad (\text{D8})$$

At that point $\lambda_0 \rightarrow \pm\infty$.

- (iii) For $\Lambda > k_L$, $D(\Lambda) < 0$ and hence

$$\lambda_0(\Lambda) < 0. \quad (\text{D9})$$

As $\Lambda \rightarrow \infty$,

$$\lambda_0(\Lambda) \simeq -\frac{1}{b \ln(\Lambda/\mu)} \xrightarrow{\Lambda \rightarrow \infty} 0^-. \quad (\text{D10})$$

Thus, keeping a fixed positive λ_R while sending the cutoff beyond the Landau scale forces the bare coupling λ_0 to become negative [it does not pass smoothly through zero; it diverges at $\Lambda = \Lambda_L$ (the Landau pole²⁴) and reappears negative for $\Lambda > \Lambda_L$ where $\Lambda_L = \mu_0 \exp(\frac{16\pi^2}{3\lambda(\mu_0)})$].

This indicates that ordinary Hermitian $\lambda\phi^4$ theory may not be UV complete. If instead $\lambda < 0$, then the same equation shows

$$\lambda(\mu) \rightarrow 0^- \quad \text{as } \mu \rightarrow \infty, \quad (\text{D11})$$

i.e., the theory is *asymptotically free*. For $\lambda < 0$ the classical potential

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \quad (\text{D12})$$

is unbounded below, implying that Hermitian quantum theory will have complex eigenvalues. In $\mathcal{PT}$ -symmetric quantization [98] using path integrals, one replaces the real ϕ -integration contour with *Stokes wedges* in the complex ϕ plane [4,97]. For the “wrong-sign” Hamiltonian

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 - g\phi^4 \quad (g > 0), \quad (\text{D13})$$

the Euclidean path integral converges on these rotated contours, and the spectrum remains real. In this framework

the negative-coupling quartic theory is consistent and asymptotically free.

Thus:

- (i) In the IR ($\mu < \Lambda_L$), the Hermitian $\lambda > 0$ theory is valid.
- (ii) Near Λ_L , the Hermitian description breaks down: the RG trajectory hits the pole.
- (iii) Analytically continuing around the pole lands one on the $\mathcal{PT}$ -symmetric $\lambda < 0$ branch, which is UV safe.

Hence the Landau pole acts as a branch point between two analytic sheets: the Hermitian $\lambda > 0$ theory and its $\mathcal{PT}$ -symmetric completion with $\lambda < 0$. A similar discussion carries over to gauge theories such as quantum electrodynamics. Let $\alpha \equiv e^2/(4\pi)$. For N_f Dirac fermions of unit charge, the one-loop beta function is

$$\begin{aligned} \beta_{\text{QED}}(\alpha) &\equiv \mu \frac{d\alpha}{d\mu} = \frac{2N_f}{3\pi} \alpha^2 + O(\alpha^3), \\ &\Rightarrow \frac{1}{\alpha(\mu)} = \frac{1}{\alpha(\mu_0)} - \frac{2N_f}{3\pi} \ln \frac{\mu}{\mu_0}. \end{aligned} \quad (\text{D14})$$

Thus $\alpha(\mu)$ grows with μ and a UV Landau pole appears upon naïve extrapolation. The $\mathcal{PT}$ -symmetric version of QED is defined by

$$\begin{aligned} \mathcal{L}_{\text{PT}} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi + e_{\text{PT}}\bar{\psi}\gamma^\mu\psi A_\mu, \\ e_{\text{PT}} &\equiv ie, \end{aligned} \quad (\text{D15})$$

with the key discrete assignment that the gauge field transforms as a *pseudovector* under parity,

$$\mathcal{P}: \quad A^0(t, \mathbf{x}) \rightarrow -A^0(t, -\mathbf{x}), \quad \mathbf{A}(t, \mathbf{x}) \rightarrow +\mathbf{A}(t, -\mathbf{x}), \quad (\text{D16})$$

so that the full Hamiltonian is invariant under the *combined* antilinear $\mathcal{PT}$. With $e_{\text{PT}} = ie$,

$$e_{\text{PT}}^2 = (ie)^2 = -e^2, \quad (\text{D17})$$

which will flip signs in loop coefficients built from e^2 . The (vacuum-polarization) diagram that determines the one-loop running depends on e^2 in ordinary QED. In $\mathcal{PT}$ -QED,

$$\begin{aligned} \beta_{\text{PT-QED}}(\alpha_{\text{PT}}) &= \mu \frac{d\alpha_{\text{PT}}}{d\mu} = -\frac{2N_f}{3\pi} \alpha_{\text{PT}}^2 + O(\alpha_{\text{PT}}^3), \\ \alpha_{\text{PT}} &\equiv \frac{e_{\text{PT}}^2}{4\pi} = -\alpha, \end{aligned} \quad (\text{D18})$$

so that

$$\frac{1}{\alpha_{\text{PT}}(\mu)} = \frac{1}{\alpha_{\text{PT}}(\mu_0)} + \frac{2N_f}{3\pi} \ln \frac{\mu}{\mu_0}, \quad (\text{D19})$$

²⁴In the nonperturbative renormalization group to be considered later the beta function will be a ratio of two polynomials.

i.e. $\mathcal{PT}$ -QED is *asymptotically free* at one loop (the screening/antiscreening role is reversed).

APPENDIX E: A TOY MODEL FOR A SCALAR FRG AND LPA'

In this appendix we shall illustrate some of the aforementioned basic features of a FRG, used in our study, by considering a simple model for a single scalar field (in a flat Euclidean background). In FRG the standard path integral is modified by adding to the bare action a regulator term (in the context of a real scalar field ϕ)

$$\Delta S_k[\phi] = \frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \phi(-p) R_k(p^2) \phi(p), \quad (\text{E1})$$

where $R_k(p^2)$ acts like a momentum-dependent mass: for $p^2 \ll k^2$, $R_k(p^2) \sim k^2$, suppresses IR modes; for $p^2 \gg k^2$, $R_k(p^2) \rightarrow 0$ leaves UV modes untouched.

1. Effective average action

The starting point of the FRG is a *scale-dependent* generating function [8] with source J

$$Z_k[J] = \int D\phi \exp \left(-S[\phi] - \Delta S_k[\phi] + \int J\phi \right), \quad (\text{E2})$$

and its Legendre transform, the *effective average action*,

$$\Gamma_k[\Phi] = \sup_J \left\{ \int J\Phi - \log Z_k[J] \right\} - \Delta S_k[\Phi] \quad (\text{E3})$$

with $\Phi = \langle \phi \rangle_J$. The evolution of Γ_k with the FRG “time” $t = \log k$ is governed by the Wetterich equation (36), which involves $\Gamma_k^{(2)}$, the second functional derivative (Hessian) with respect to all fields Φ ; R_k is viewed as an operator in field space (since $\Delta S_k[\phi]$ depends on ϕ). Although this formalism is in principle exact, there are difficulties in obtaining rigorous results using it. These difficulties are

- (i) Scheme dependence such as the choice of the regulator function and gauge [92].
- (ii) The truncation of the terms used in the average effective action. The theories considered are perturbatively nonrenormalizable in four dimensions. In perturbative renormalization an infinite number of terms with higher mass dimensions are required, making different truncations relevant up to different energy scales. The truncation of the terms in the average effective action is a reflection of this issue (present in the perturbative framework).

2. Truncated FRG for a real scalar field and LPA'

For the simple case of a real scalar discussed here, the scale-dependent effective average action is truncated to

$$\Gamma_k[\phi] = \int d^4 x \left[\frac{1}{2} Z_k (\partial_\mu \phi)^2 + U_k(\phi) \right] \quad (\text{E4})$$

where $U_k(\phi)$ is the scale-dependent effective potential and Z_k is a scale-dependent, field-independent wave-function renormalization and fluctuations $\delta\phi(x) (= \int \frac{d^4 p}{(2\pi)^4} \delta\tilde{\phi}(p) e^{ip \cdot x})$ around a constant background $\bar{\phi}$. This truncation corresponds to the LPA' approximation, i.e., the leading order of the derivative expansion augmented by a running kinetic prefactor. Using both sides of the Wetterich equation (36), we calculate $\frac{\partial}{\partial t} Z_k \phi$ and $\partial_t U_k$ in terms of $t = \log \frac{k}{\Lambda}$ (where Λ is an appropriate UV cutoff), which we then equate; this is the general procedure used in other models and allows calculations of beta functions and anomalous dimensions [109]. In LPA', the infrared regulator is chosen to incorporate the running wave-function renormalization Z_k ,

$$R_k(p^2) = Z_k k^2 r\left(\frac{p^2}{k^2}\right), \quad (\text{E5})$$

where $r(y)$ is a dimensionless shape function. A convenient choice is the Litim regulator,

$$r(y) = \left(\frac{1}{y} - 1 \right) \Theta(1 - y). \quad (\text{E6})$$

With this choice, the scale derivative of the regulator reads

$$\partial_t R_k(p^2) = Z_k k^2 [(2 - \eta)r(y) - 2yr'(y)], \quad \eta \equiv -\partial_t \ln Z_k. \quad (\text{E7})$$

For a constant background field $\bar{\phi}$, the second functional derivative of the effective action is

$$\frac{\delta^2 \Gamma_k}{\delta\phi(x)\delta\phi(y)} = \Gamma_k^{(2)}(x, y) = (-Z_k \partial_x^2 + U_k''(\bar{\phi})) \delta^{(4)}(x - y), \quad (\text{E8})$$

and we find

$$\partial_t U_k(\bar{\phi}) = \frac{1}{2} \int \frac{d^4 q}{(2\pi)^4} \frac{\partial_t R_k(q^2)}{Z_k q^2 + R_k(q^2) + U_k''(\bar{\phi})} = \frac{k^4}{2} l_0^A(w), \quad (\text{E9})$$

where

$$l_0^A(w) = \frac{1}{k^4} \int \frac{d^4 q}{(2\pi)^4} \frac{\partial_t R_k(q^2)/Z_k}{q^2 + \frac{R_k(q^2)}{Z_k} + k^2 w}, \quad (\text{E10})$$

and $w = \frac{U_k''(\bar{\phi})}{Z_k k^2}$. l_0^A is an example of a threshold function, which is dependent on the regulator function. In terms of a physical mass $m^2 = k^2 w$, the denominator of the integrand of

the threshold function is dominated by $q^2 + \frac{R_k(q^2)}{Z_k}$ for $\frac{m^2}{k^2} \ll 1$. For $\frac{m^2}{k^2} \gg 1$, $l_0^4(w)$ is small, since the m^2 term in the denominator is dominant. This shows why a function such as l_0^4 is called a threshold function.

From the Wetterich equation, if we include the change of wave function renormalization with the energy scale

$$\partial_t Z_k = \lim_{p^2 \rightarrow 0} \frac{\partial}{\partial p^2} \partial_t \Gamma^{(2)}(p), \quad (\text{E11})$$

which gives the anomalous dimension through $\eta = -\frac{\partial}{\partial t} \log Z_k$. Using the right-hand side of the Wetterich equation we obtain, in momentum space,

$$\begin{aligned} \partial_t \Gamma_k^{(2)}(p, -p) &= \int d^4 q U_k'''(\bar{\phi})^2 G(q^2)^2 \frac{\partial^2 G((p-q)^2)}{\partial p^2} \\ &\times \partial_t R_k(q^2), \end{aligned} \quad (\text{E12})$$

where $U_k'''(\bar{\phi})$ is the third functional derivative with respect to the field ϕ evaluated at the constant $\bar{\phi}$ and $G(p^2) = \frac{1}{Z_k p^2 + U_k''(\bar{\phi}) + r_k(p^2)}$.

After some algebra, we obtain

$$\partial_t Z_k = v_4 k^4 l_1^4(w) (U_k'''(\bar{\phi}))^2, \quad (\text{E13})$$

where

$$l_1^4(w) = \frac{1}{2v_4 k^4} \int \frac{d^4 q}{(2\pi)^4} \frac{(G' + 2p^2 G'') \partial_t R_k(q^2)}{(Z_k q^2 + R_k(q^2) + Z_k k^2 w)^2} \quad (\text{E14})$$

is another threshold function, where v_d denotes the d -dimensional angular volume factor. We have shown a typical calculation using FRG. In general it is necessary to choose a Γ_k that may have gauge fields; a decision is made on the degrees of freedom relevant for the physical situation at hand and this may lead to an introduction of new fields (e.g., the metric²⁵ $g_{\mu\nu}$, with $g = \det g_{\mu\nu}$) and interactions (e.g., higher derivative terms)

$$\begin{aligned} \Gamma_k[\phi, g] &= \int d^4 x \sqrt{g} \left[\frac{1}{2} Z_{\phi,k} \phi(-\nabla^2) \phi \right. \\ &+ \frac{1}{2} \bar{\rho}_{a,k} (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi)^2 + \frac{1}{2} \bar{\rho}_{b,k} (g^{\mu\nu} \nabla_\mu \partial_\nu \phi)^2 \\ &\left. + \frac{1}{2} \bar{\rho}_{c,k} (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi) \nabla^2 \phi + U(\phi) \right]. \end{aligned} \quad (\text{E15})$$

For a more general field theory with n fields Φ_j ($j = 1, \dots, n$), choose the standard LPA' regulator

²⁵We shall primarily consider the background expansion (65) about a flat (Minkowski) background metric $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$.

$$R_{k,j}(p^2) = Z_j(k) k^2 r_j(y). \quad (\text{E16})$$

where $y \equiv \frac{p^2}{k^2}$. Using $\partial_t Z_j = -\eta_j Z_j$ and $\partial_t y = -2y$, one finds

$$\begin{aligned} \partial_t R_{k,j}(p^2) &= (\partial_t Z_j) k^2 r_j(y) + Z_j \partial_t (k^2 r_j(y)) \\ &= Z_j(k) k^2 [(2 - \eta_j) r_j(y) - 2y r_j'(y)]. \end{aligned} \quad (\text{E17})$$

Starting from the Wetterich equation

$$\partial_t \Gamma_k = \frac{1}{2} \text{STr}[(\Gamma_k^{(2)} + R_k)^{-1} \partial_t R_k], \quad (\text{E18})$$

the exact flow of the two-point function for a field i is obtained by two functional derivatives with respect to ϕ_i . One finds generically

$$\begin{aligned} \partial_t \Gamma_{k,ii}^{(2)} &= \frac{1}{2} \text{STr}[G_k \Gamma_{k,i}^{(3)} G_k \Gamma_{k,i}^{(3)} G_k \partial_t R_k] \\ &- \frac{1}{2} \text{STr}[G_k \Gamma_{k,ii}^{(4)} G_k \partial_t R_k] \end{aligned} \quad (\text{E19})$$

where

$$G_k \equiv (\Gamma_k^{(2)} + R_k)^{-1} \quad (\text{E20})$$

is the full regularized propagator, and STr denotes a supertrace over momentum, internal indices, and statistics. Some of these indices are suppressed in (E19) and (E20).

Explicitly, the first term on the right of (E19), the rainbow (sunset) contribution, reads

$$\begin{aligned} \partial_t \Gamma_{k,ii}^{(2)}|_{\text{rainbow}} &= \frac{1}{2} \sum_{a,b,c,d} \int \frac{d^d q}{(2\pi)^d} (G_k)_{ab}(q) (\Gamma_{k,i}^{(3)})_{bc} \\ &\times (p; q, -q - p) (G_k)_{cd}(q + p) \\ &\times (\Gamma_{k,i}^{(3)})_{da}(-p; q + p, -q) (\partial_t R_k)_{aa}(q), \end{aligned} \quad (\text{E21})$$

and the second term, the tadpole contribution, is

$$\begin{aligned} \partial_t \Gamma_{k,ii}^{(2)}|_{\text{tadpole}} &= -\frac{1}{2} \sum_{a,b} \int \frac{d^d q}{(2\pi)^d} (G_k)_{ab}(q) (\Gamma_{k,ii}^{(4)})_{ba} \\ &\times (p, -p; q, -q) (\partial_t R_k)_{aa}(q). \end{aligned} \quad (\text{E22})$$

The regulator insertion $\partial_t R_k$ always carries the field index of the internal loop propagator, which is why anomalous dimensions enter the LPA' equations exclusively through the combinations $(2 - \eta_j)$ for the fields j circulating in the loop.

For theories with local invariance, it is necessary to choose a gauge, and introduce any necessary Faddeev-Popov ghosts [109]. Given the difficulties of implementing

rigorously quantum diffeomorphism invariance [110] in the study of gravity, our work on the axion model will be suggestive (but not conclusive).

APPENDIX F: SOME CONCEPTS AND TECHNICAL DETAILS OF FRG IN OUR CURVED SPACE TIME CONTEXT

In this appendix we discuss we discuss concepts and important technical details of FRG, used in the main text. We commence the discussion with some identities associated with graviton tensor fields in curved space time, used in the analysis, and then we proceed with superdiagram rules for FRG, a discussion on LPA' in our context, and finally rules for computing loop diagrams entering the calculations of anomalous dimensions.

1. Flat-space York (TT) decomposition and Barnes-Rivers projectors

a. Flat-space York/TT decomposition

On a flat d -dimensional space time background $\eta_{\mu\nu}$ (Euclidean or Minkowski), any symmetric tensor $h_{\mu\nu}(x)$ admits the decomposition

$$h_{\mu\nu} = h_{\mu\nu}^{\text{TT}} + \partial_{(\mu} \xi_{\nu)}^T + \left(\partial_\mu \partial_\nu - \frac{1}{d} \eta_{\mu\nu} \partial^2 \right) \sigma + \frac{1}{d} \eta_{\mu\nu} \varphi, \quad (\text{F1})$$

with constraints

$$\partial^\mu h_{\mu\nu}^{\text{TT}} = 0, \quad \eta^{\mu\nu} h_{\mu\nu}^{\text{TT}} = 0, \quad \partial^\mu \xi_\mu^T = 0, \quad \varphi \equiv \eta^{\rho\sigma} h_{\rho\sigma}. \quad (\text{F2})$$

A useful gauge-invariant scalar is

$$\chi \equiv \varphi - \partial^2 \sigma \quad \Rightarrow \quad \delta\chi = 0 \text{ under } \delta h_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu. \quad (\text{F3})$$

b. Momentum-space inversion (practical formulas)

In Fourier space ($\partial_\mu \rightarrow i p_\mu$), define

$$\theta_{\mu\nu}(p) = \eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}, \quad \omega_{\mu\nu}(p) = \frac{p_\mu p_\nu}{p^2}. \quad (\text{F4})$$

Given $h_{\mu\nu}(p)$, the York pieces are

$$\varphi(p) = \eta^{\mu\nu} h_{\mu\nu}(p), \quad (\text{F5})$$

$$\sigma(p) = \frac{d}{d-1} \frac{1}{p^2} \left[\frac{p_\mu p_\nu}{p^2} h^{\mu\nu}(p) - \frac{1}{d} \varphi(p) \right], \quad (\text{F6})$$

$$\xi_\nu^T(p) = \frac{2}{p^2} \theta_\nu^\alpha(p) p^\mu h_{\mu\alpha}(p), \quad (\text{F7})$$

$$h_{\mu\nu}^{\text{TT}}(p) = h_{\mu\nu}(p) - \frac{1}{2} (p_\mu \xi_\nu^T + p_\nu \xi_\mu^T) - \left(p_\mu p_\nu - \frac{1}{d} \eta_{\mu\nu} p^2 \right) \sigma(p) - \frac{1}{d} \eta_{\mu\nu} \varphi(p). \quad (\text{F8})$$

c. Barnes-Rivers spin projectors (momentum space)

Projectors on the symmetric rank-2 subspace (for $p^2 \neq 0$),

$$P_{\mu\nu\rho\sigma}^{(2)} = \frac{1}{2} (\theta_{\mu\rho} \theta_{\nu\sigma} + \theta_{\mu\sigma} \theta_{\nu\rho}) - \frac{1}{d-1} \theta_{\mu\nu} \theta_{\rho\sigma}, \quad (\text{F9})$$

$$P_{\mu\nu\rho\sigma}^{(1)} = \frac{1}{2} (\theta_{\mu\rho} \omega_{\nu\sigma} + \theta_{\mu\sigma} \omega_{\nu\rho} + \theta_{\nu\rho} \omega_{\mu\sigma} + \theta_{\nu\sigma} \omega_{\mu\rho}), \quad (\text{F10})$$

$$P_{\mu\nu\rho\sigma}^{(0-s)} = \frac{1}{d-1} \theta_{\mu\nu} \theta_{\rho\sigma}, \quad (\text{F11})$$

$$P_{\mu\nu\rho\sigma}^{(0-w)} = \omega_{\mu\nu} \omega_{\rho\sigma}. \quad (\text{F12})$$

Useful “bridge” operators in the scalar sector,

$$P_{\mu\nu\rho\sigma}^{(0-sw)} = \frac{1}{\sqrt{d-1}} \theta_{\mu\nu} \omega_{\rho\sigma}, \quad P_{\mu\nu\rho\sigma}^{(0-ws)} = \frac{1}{\sqrt{d-1}} \omega_{\mu\nu} \theta_{\rho\sigma}. \quad (\text{F13})$$

Algebra and completeness,

$$P^{(A)} P^{(B)} = \delta^{AB} P^{(B)} \quad (A, B \in \{2, 1, 0-s, 0-w\}), \quad (\text{F14})$$

$$P^{(0-sw)} P^{(0-ws)} = P^{(0-s)}, \quad P^{(0-ws)} P^{(0-sw)} = P^{(0-w)}, \quad (\text{F15})$$

$$\mathbb{I}_{\mu\nu\rho\sigma} = \frac{1}{2} (\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho}) = P^{(2)} + P^{(1)} + P^{(0-s)} + P^{(0-w)}. \quad (\text{F16})$$

d. One-line use case (spin-2 projection)

For a two-point kernel $\Gamma_{hh}^{(2)}(p)$, the spin-2 kinetic coefficient can be read off without a York split via

$$Z_N(k) = \frac{1}{\text{tr} P^{(2)}} \frac{\partial}{\partial p^2} [P^{(2)}(p) : \Gamma_{hh}^{(2)}(p)]|_{p^2=k^2}, \quad A:B \equiv A_{\mu\nu\rho\sigma} B^{\mu\nu\rho\sigma}. \quad (\text{F17})$$

(For $d = 4$, $\text{tr} P^{(2)} = 5$.)

2. Superdiagram rules for the FRG

We will outline the compact “Feynman-rule” set for the Wetterich equation, which can be written as

$$\begin{aligned}\partial_t \Gamma_k &= \frac{1}{2} \text{STr}[(\Gamma_k^{(2)} + R_k)^{-1} \partial_t R_k] \\ &= \frac{1}{2} \text{STr} \left[\left(\underbrace{\mathcal{P}_k}_{\text{propagator}} + \underbrace{\mathcal{I}_k}_{\text{interaction}} \right)^{-1} \partial_t R_k \right].\end{aligned}\quad (\text{F18})$$

This can be unpacked into superdiagrams built from

- (i) a dressed propagator $G_k \equiv \mathcal{P}_k^{-1}$ (block-diagonal in York basis for gravity).
- (ii) an interaction insertion $\mathcal{I}_k$ (which carries no derivatives, but all other types of vertices).

$$\begin{aligned}\mathcal{P}_k &:= \Gamma_k^{(2)}|_{\text{derivatives}} + R_k(-\bar{\nabla}^2), \\ \mathcal{I}_k &:= \Gamma_k^{(2)} - \Gamma_k^{(2)}|_{\text{derivatives}}.\end{aligned}\quad (\text{F19})$$

We will now give a geometric series expansion. Denote $G_k := \mathcal{P}_k^{-1}$ and so then

$$(\mathcal{P}_k + \mathcal{I}_k)^{-1} = G_k \sum_{n=0}^{\infty} (-\mathcal{I}_k G_k)^n, \quad (\text{F20})$$

$$\partial_t \Gamma_k = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \text{STr}[\partial_t R_k G_k (\mathcal{I}_k G_k)^n]. \quad (\text{F21})$$

With the above expansion every term can be drawn as a (one-loop) “superdiagram” according to the rules: $G_k = \mathcal{P}_k^{-1}$ (using the appropriate York block), which can be represented by an internal line, a dot can represent a vertex $-\mathcal{I}_k$, which depends on the background curvature; a cross represents $\partial_t R_k(-\bar{\nabla}^2)$, (exactly *one* per loop); there is an integration over momentum and sum over indices; STTr is + for bosons and – for ghosts and there is prefactor of 1/2 for every closed loop.

Here are examples:

$$n = 0: \partial_t \Gamma_k^{(0)} = \frac{1}{2} \text{STr}[\partial_t R_k G_k] \quad (\text{tadpole}), \quad (\text{F22})$$

$$\begin{aligned}n = 1: \partial_t \Gamma_k^{(1)} &= -\frac{1}{2} \text{STr}[\partial_t R_k G_k \mathcal{I}_k G_k] \\ &(\text{one-vertex loop}).\end{aligned}\quad (\text{F23})$$

3. LPA' anomalous dimensions and validity of the simplified approximation

We consider an LPA' truncation of axion electrodynamics coupled to gravity, keeping running wave-function renormalizations but neglecting higher-derivative operators. The truncation is assumed to preserve axionic shift symmetry, i.e., the axion has no nontrivial potential,

$$V(b) = 0, \quad (\text{F24})$$

so that the local axion-photon Chern-Simons coupling is not renormalized directly, and its scale dependence enters only through wave-function renormalizations.

We define the dimensionless couplings

$$g_{b\gamma}^2 \equiv \frac{g_{\text{mix},k}^2 k^2}{Z_A^2 Z_b}, \quad g_N \equiv k^2 G_k, \quad \lambda \equiv \frac{\Lambda_k}{k^2}, \quad (\text{F25})$$

and the anomalous dimensions [cf. (77)],

$$\eta_b = -\partial_t \ln Z_b, \quad \eta_A = -\partial_t \ln Z_A, \quad \eta_N = -\partial_t \ln Z_h. \quad (\text{F26})$$

a. General LPA' structure

At the level of LPA', the anomalous dimensions receive contributions from axion-photon loops and from gravitational dressing. Keeping only the correct coupling dependence and leaving regulator-dependent numerical coefficients unspecified, the general structure in a linearized framework is

$$\eta_b = c_{bA} g_{b\gamma}^2 + c_{bN} g_N + c_{bNA} g_N g_{b\gamma}^2, \quad (\text{F27})$$

$$\eta_A = c_{Ab} g_{b\gamma}^2 + c_{AN} g_N + c_{ANA} g_N g_{b\gamma}^2, \quad (\text{F28})$$

$$\eta_N = c_N g_N + c_{NA} g_N g_{b\gamma}^2. \quad (\text{F29})$$

The coefficients c_i depend on the choice of regulator and gauge but are expected to be $\mathcal{O}(1)$. The terms proportional to $g_{b\gamma}^2$ encode axion-photon fluctuations; those proportional to g_N encode purely gravitational dressing; and the mixed terms $g_N g_{b\gamma}^2$ arise from “interference” diagrams coupling both sectors.

b. Self-consistency and the origin of singularities

In an LPA' computation the anomalous dimensions are, however, not simply polynomials in the couplings: they enter the scale derivative of the regulator via factors such as $(2 - \eta)$, so that η_b and η_A typically appear on both sides of their defining flow equations. In the axion-photon subsystem this leads to a linear self-consistency problem of the form

$$\begin{aligned}\begin{pmatrix} \eta_b \\ \eta_A \end{pmatrix} &= \begin{pmatrix} \alpha_b \\ \alpha_A \end{pmatrix} + U(g_{b\gamma}^2, g_N) \begin{pmatrix} \eta_b \\ \eta_A \end{pmatrix}, \\ \text{with } \alpha_{b,A} &= \mathcal{O}(g_{b\gamma}^2, g_N, g_N g_{b\gamma}^2).\end{aligned}\quad (\text{F30})$$

This is a generalization of the scalar case (E5) and (E7). For the full system U is a 3×3 matrix and the source column vector is three-dimensional. In the LPA' approximation the scale derivative of the regulator carries an explicit dependence on the anomalous dimensions. For each field $i \in \{a, A, N\}$ one introduces a regulator of the form

$$R_{k,i}(p^2) = Z_i(k)k^2 r_i\left(\frac{p^2}{k^2}\right), \quad (\text{F31})$$

where $Z_i(k)$ is the wave function renormalization and $r_i(y)$ is a dimensionless shape function. Taking the logarithmic scale derivative yields

$$\partial_t R_{k,i}(p^2) = Z_i k^2 [(2 - \eta_i) r_i(y) - 2y r'_i(y)],$$

$$\eta_i \equiv -\partial_t \ln Z_i. \quad (\text{F32})$$

Consequently, every loop contribution to the Wetterich equation that involves $\partial_t R_{k,i}$ splits naturally into a part proportional to the constant term 2 and a part proportional to $-\eta_i$. Upon projection onto the kinetic operators that define the anomalous dimensions, the former generates inhomogeneous contributions depending only on the couplings, while the latter produces terms linear in the anomalous dimensions themselves. As a result, the LPA' flow equations for the anomalous dimensions take the generic self-consistent linear form

$$\eta_i = s_i(\text{couplings}) + \sum_j U_{ij}(\text{couplings}) \eta_j,$$

$$i, j \in \{b, A, N\}, \quad (\text{F33})$$

where the coefficients s_i and U_{ij} are regulator- and gauge-dependent threshold integrals.

c. Simplified approximation

In a simplified treatment, we neglect mixed gravity-matter contributions proportional to $g_N g_{b\gamma}^2$, i.e., diagrams involving simultaneous axion-photon and gravitational interactions. This corresponds to working to linear order in g_N and linear order in $g_{b\gamma}^2$, but not their product. The system then reduces to

$$\eta_b = \alpha g_{b\gamma}^2 + \tilde{\alpha} g_N, \quad (\text{F34})$$

$$\eta_A = \beta g_{b\gamma}^2 + \tilde{\beta} g_N, \quad (\text{F35})$$

$$\eta_N = \gamma g_N, \quad (\text{F36})$$

with $\alpha, \tilde{\alpha}, \beta, \tilde{\beta}, \gamma = \mathcal{O}(1)$.

d. Justification for $\mathcal{O}(1)$ coefficients

After rescaling loop momenta by the RG scale k and using an LPA' regulator of the form $R_k = Z_k k^2 r(p^2/k^2)$, all FRG traces contributing to anomalous dimensions reduce to dimensionless integrals multiplied by the appropriate powers of the couplings. The remaining numerical factors arise from angular integrals, projector algebra, and regulator threshold functions, all of which are finite and typically of order unity away from threshold poles.

Therefore the coefficients in Eqs. (F27)–(F29) are naturally $\mathcal{O}(1)$. The detailed amplitudes can be found in Appendix F 4.

4. Details of loop diagrams required for the calculations of anomalous dimensions

To obtain the anomalous dimensions themselves, one differentiates the flow equation twice with respect to the fields and projects onto the kinetic structures. Because the only interactions present are the axion-photon CS vertex, minimal gravitational couplings, and (optionally) the axion-gravity CS vertex, the anomalous dimensions receive a finite set of *one-loop* contributions. The truncation we adopt keeps all operators that are generated by one-graviton or one-gauge-boson exchange at the order relevant for the renormalization of the Chern-Simons couplings.

$$[\partial_t \Gamma_{bb}^{(2)}(p)]_{A\text{-loop}} = + \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} V_{bAA}^{\mu\nu}(p, q, -p - q)$$

$$\times G_{\mu\mu'}^A(q) \partial_t R_A(q) G_{\nu\nu'}^A(p + q)$$

$$\times V_{bAA}^{\mu'\nu'}(-p, -q, p + q). \quad (\text{F37})$$

Here,

$$G_{\mu\nu}^A(q) = \frac{T_{\mu\nu}(q)}{Z_A[q^2 + R_A(q)]}, \quad T_{\mu\nu}(q) = \eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}, \quad (\text{F38})$$

$$\partial_t R_X(q) = Z_X [(2 - \eta_X) k^2 r(y) - 2y k^2 r'(y)],$$

$$y \equiv q^2/k^2, \quad X \in \{a, A, h\}, \quad (\text{F39})$$

and for the CS vertex (schematically, up to total antisymmetry)

$$V_{bAA}^{\mu\nu}(p, q, -p - q) = g_{\text{mix}} \epsilon^{\mu\nu\rho\sigma} q_\rho (p + q)_\sigma. \quad (\text{F40})$$

The axion anomalous dimension follows from projecting the p^2 term,

$$\eta_b = -\partial_t \ln Z_b \propto \frac{\partial}{\partial p^2} [\partial_t \Gamma_{bb}^{(2)}(p)]_{A\text{-loop}}|_{p^2=k^2}. \quad (\text{F41})$$

This contribution scales as $\eta_b \sim \mathcal{O}(g_{\text{mix}}^2)$ (modulo threshold factors).

Similarly, for completeness we give the graviton loop contribution from an axion-gravity CS vertex (although we shall omit it for phenomenological reasons)

$$[\partial_t \Gamma_{bb}^{(2)}(p)]_{h\text{-loop}} = + \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} V_{bhh}^{\alpha\beta\rho\sigma}(p, q, -p - q)$$

$$\times G_{\alpha\beta\mu\nu}^h(q) \partial_t R_h(q) G_{\rho\sigma\kappa\lambda}^h(p + q)$$

$$\times V_{bhh}^{\mu\nu\kappa\lambda}(-p, -q, p + q). \quad (\text{F42})$$

With TT projection,

$$G_{\alpha\beta\rho\sigma}^h(q) = \frac{P_{\alpha\beta\rho\sigma}^{(2)}(q)}{Z_h[q^2 + R_h(q)]}, \quad \partial_t R_h(q) = Z_h \left[(2 - \eta_h) k^2 r \left(\frac{q^2}{k^2} \right) + \dots \right], \quad (\text{F43})$$

$$P_{\alpha\beta\rho\sigma}^{(2)} = \frac{1}{2}(\theta_{\alpha\rho}\theta_{\beta\sigma} + \theta_{\alpha\sigma}\theta_{\beta\rho}) - \frac{1}{3}\theta_{\alpha\beta}\theta_{\rho\sigma}, \quad \theta_{\mu\nu} = \eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}. \quad (\text{F44})$$

$$V_{bhh}^{\mu\nu\rho\sigma}(p_a, k, \ell) = i g_{\text{CS}} \epsilon^{\alpha\beta\gamma\delta} \mathcal{R}_{\alpha\beta}^{\mu\nu}(k) \mathcal{R}_{\gamma\delta}^{\rho\sigma}(\ell) + (\mu\nu, k) \leftrightarrow (\rho\sigma, \ell), \quad (\text{F45})$$

$$\mathcal{R}_{\alpha\beta\mu\nu}(k) = \frac{1}{2}(k_\alpha k_\mu h_{\beta\nu} - k_\alpha k_\nu h_{\beta\mu} - k_\beta k_\mu h_{\alpha\nu} + k_\beta k_\nu h_{\alpha\mu}). \quad (\text{F46})$$

The CS vertex V_{ahh} comes from expanding $aR\tilde{R}$ to quadratic order in h ; it carries a coupling g_{CS} and momenta contracted with the Levi-Civita tensor. Then

$$\eta_b \propto \frac{\partial}{\partial p^2} [\partial_t \Gamma_{bb}^{(2)}(p)]_{h\text{-loop}}|_{p^2=k^2} \sim g_{\text{CS}}^2 \times (\text{thresholds involving } g_N, \eta_h). \quad (\text{F47})$$

Similarly for the photon anomalous dimension η_A we have

$$[\partial_t \Gamma_{AA}^{(2)}(p)]_{b\text{-loop}} = \frac{1}{2} \int \frac{d^4 q}{(2\pi)^4} V_{AAb}^{\mu\alpha}(p, q, -p - q) G_b(q) \partial_t R_b(q) G_b(p + q) V_{abA}^\nu(-p, -q, p + q), \quad (\text{F48})$$

$$[\partial_t \Gamma_{AA}^{(2)}(p)]_{h\text{-loop}} = \frac{1}{2} \int \frac{d^4 q}{(2\pi)^4} V_{AAh}^{\mu\alpha|\rho\sigma}(p, q, -p - q) G_{\rho\sigma, \rho'\sigma'}^h(q) \partial_t R_h(q) G_{\alpha\beta, \alpha'\beta'}^h(p + q) V_{AAh}^{\nu\beta|\rho'\sigma'}(-p, -q, p + q), \quad (\text{F49})$$

where we denote the propagators by

$$G_a(q) = \frac{1}{Z_b[q^2 + R_a(q)]}, \quad G_{\alpha\beta}^A(q) = \frac{T_{\alpha\beta}(q)}{Z_A[q^2 + R_A(q)]}, \quad G_{\mu\nu, \rho\sigma}^h(q) = \frac{P_{\mu\nu, \rho\sigma}^{(2)}(q)}{Z_h[q^2 + R_h(q)]}. \quad (\text{F50})$$

In our one-metric approximation Z_h is identified with Z_N . The TT graviton anomalous dimension is calculated from

$$\eta_h = -\frac{1}{Z_h} \frac{\partial}{\partial p^2} \left[\frac{1}{5} P_{\mu\nu\rho\sigma}^{(2)}(p) \partial_t \Gamma_{hh}^{(2)\mu\nu\rho\sigma}(p) \right] \Big|_{p^2=k^2}, \quad (\text{F51})$$

with matter contributions,

$$[\partial_t \Gamma_{hh}^{(2)}(p)]_{A\text{-loop}} = \frac{1}{2} \int \frac{d^4 q}{(2\pi)^4} V_{hAA}^{\mu\nu|\alpha\beta}(p, q, -p - q) G_{\alpha\alpha'}^A(q) \partial_t R_A(q) G_{\beta\beta'}^A(p + q) V_{hAA}^{\rho\sigma|\alpha'\beta'}(-p, -q, p + q), \quad (\text{F52})$$

$$[\partial_t \Gamma_{hh}^{(2)}(p)]_{b\text{-loop}} = \frac{1}{2} \int \frac{d^4 q}{(2\pi)^4} V_{hbb}^{\mu\nu}(p, q, -p - q) G_b(q) \partial_t R_b(q) G_b(p + q) V_{hbb}^{\rho\sigma}(-p, -q, p + q), \quad (\text{F53})$$

where the relevant vertices are

$$\begin{aligned} V_{hAA}^{\mu\nu|\rho\sigma}(p, q, -p - q) = & \frac{\kappa}{2} \{ \delta_{\rho\sigma}(p_\mu q_\nu + p_\nu q_\mu - \delta_{\mu\nu} p \cdot q) - \delta_{\mu\rho}(q_\nu p_\sigma + q_\sigma p_\nu - \delta_{\nu\sigma} p \cdot q) - \delta_{\nu\rho}(q_\mu p_\sigma + q_\sigma p_\mu - \delta_{\mu\sigma} p \cdot q) \\ & - \delta_{\mu\sigma}(p_\nu q_\rho + p_\rho q_\nu - \delta_{\nu\rho} p \cdot q) - \delta_{\nu\sigma}(p_\mu q_\rho + p_\rho q_\mu - \delta_{\mu\rho} p \cdot q) + \delta_{\mu\nu}(p_\rho q_\sigma + p_\sigma q_\rho - \delta_{\rho\sigma} p \cdot q) \}, \end{aligned} \quad (\text{F54})$$

and

$$V_{hbb}^{\mu\nu}(-p, -q, p + q) = \frac{\kappa Z_b}{2} [p_\mu q_\nu + p_\nu q_\mu - \delta_{\mu\nu}(p \cdot q + m_b^2)]. \quad (\text{F55})$$

These are the ingredients that go into the LPA'. The evaluation of the *loop integrals* can be expressed in terms of *threshold functions*

$$\Phi_n^p(w) \equiv \frac{1}{\Gamma(n)} \int_0^\infty dy y^{n-1} \frac{r(y) - yr'(y)}{[y(1+r(y)) + w]^p}, \quad \tilde{\Phi}_n^p(w) \equiv \frac{1}{\Gamma(n)} \int_0^\infty dy y^{n-1} \frac{r(y)}{[y(1+r(y)) + w]^p}. \quad (\text{F56})$$

This type of derivation is typical and used in [16,57]. However there are explicit tensorial structures [cf. (F52)], which appear here and have not been evaluated explicitly in published work. It is common to factor the regulator dependence entirely into Φ and $\tilde{\Phi}$ and present the remaining tensor coefficients as simple constants of $O(1)$: $\mathcal{A}^{(i)}$ (axion-photon projection coefficients), $\mathcal{G}^{(1)}$, $\tilde{\mathcal{G}}^{(1)}$ (from spin-2/0 graviton and ghost loops), $\mathcal{M}_{S,V}(\lambda)$, $\tilde{\mathcal{M}}_{S,V}(\lambda)$ quantify the backreaction of minimally coupled scalar/vector matter on η_h .

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