

⁷Be asymptotic normalization coefficients from ³He + ⁴He elastic scattering and validation of extrapolation methods

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We present the first direct determination of asymptotic normalization coefficients (ANCs) for the ground and first excited states of ⁷Be from ³He + ⁴He elastic scattering phase shifts. The analysis draws upon the classic datasets of two older low-energy measurements, long considered difficult due to internal inconsistencies and the absence of systematic uncertainties. Employing single-level *R*-matrix fits, the mapping between reduced widths and ANCs, and Bayesian inference, we reconstruct posterior distributions and quantify uncertainties. A key innovation of this study is the application and validation of the extrapolation method developed by our group, specifically designed to extract ANC values in cases where direct low-energy information is lacking. Benchmarking against the latest measurements [Paneru *et al.*, *Phys. Rev. C* **109**, 015802 (2024)], which reported only *R*-matrix fits with fixed ANCs, confirms the predictive power and robustness of our approach. Our findings show that ANCs deduced from older scattering measurements are systematically lower than those inferred from radiative capture, thereby confirming earlier indications of a discrepancy between the two methods. This work demonstrates both the originality and reliability of the extrapolation method as a powerful tool for ANC determination in nuclear astrophysics.

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I. INTRODUCTION

Asymptotic normalization coefficients (ANCs) determine the amplitude of the asymptotic behavior of the overlap function for the bound-state wave functions of nuclei $a = (bc)$, b and c . The residue of the $b + c$ elastic scattering partial-wave amplitude at the bound-state pole $a = (bc)$ is articulated in terms of the ANC for the virtual decay $a \rightarrow b + c$. The general properties of the ANCs are discussed in the review paper [1]. The role of ANCs is especially important in determining the cross sections for peripheral nuclear reactions, in particular, reactions with charged particles at low energies, which are inaccessible for direct measurement due to the large Coulomb barrier. The most important class of such processes are astrophysical nuclear reactions in the cores of stars. ANCs determine the overall normalization of the cross sections and astrophysical *S* factors for direct radiative capture reactions and resonant reactions proceeding through subthreshold resonances occurring at astrophysically relevant energies [2–4]. One of the important astrophysical reactions is the ³He(α , γ)⁷Be radiative capture. This process is the primary reaction for

chains II and III of the $p - p$ chain of hydrogen burning in stars.

The ³He(α , γ)⁷Be process, despite its critical significance, still holds a mystery. The cause is that, even after substantial experimental endeavors, a singular characterization of the low-energy cross section remains unobtainable [5]. The low-energy ³He(α , γ)⁷Be radiative capture represents a peripheral direct transition from a continuum to a loosely bound state. Given this scenario, it's often assumed that the *S* factors are presumed to be an accurate way of ascertaining the ANCs of the final bound states. Yet, the most extensive Bayesian analysis indicated that it did not hold true [6]. The Bayesian method (see Appendix), as opposed to the usual χ^2 minimization, enables a complete analysis of the probability distribution of the model's parameters (in the current scenario, ANCs $C_{3/2}$ and $C_{1/2}$, which correspond to the virtual decays ⁷Be($3/2^-$; 0 MeV) $\rightarrow$ α + ³He and ⁷Be($1/2^-$; 0.429 MeV) $\rightarrow$ α + ³He channels, respectively). As presented in [6], the two-dimensional posterior resulting from the Bayesian analysis of the effective field theory (EFT) predictions [7,8], utilizing exclusively radiative capture data, exhibited a notably wide span for the ⁷Be ANCs (refer to Fig. 12 of [6]). It renders ⁷Be as an excellent laboratory for research.

One of the reasons is that despite contemporary astrophysical *S*₃₄ factors being measured at extremely low energies with an accuracy under 9% [9,10], the *S*₃₄ factor does not exhibit proportionality to the square of the final bound state ⁷Be ANCs $C_{3/2}$ and $C_{1/2}$.

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TABLE I. ANC's $C_{3/2}$ and $C_{1/2}$ values in $\text{fm}^{-1/2}$ for ${}^7\text{Be} \rightarrow \alpha + {}^3\text{He}$. The word “theory” or “extraction” in the last column indicates whether the result for the ANC is a theory prediction or extracted from the data.

$C_{3/2}$	$C_{1/2}$	Method	References
4.60–4.91	3.81–4.05	Analysis of S_{34} factor in a two-body model	[11] (extraction)
4.82 ± 0.18	3.99 ± 0.14	Relation between ERF and ANC	[12] (extraction)
3.55 ± 0.15	2.9 ± 0.1	Variational Monte Carlo method	[13] (theory)
3.7	3.6	R -matrix fit	[14] (extraction)
3.75	3.6	NLO EFT	[7] (extraction)
3.91	3.53	No-core shell model (NCSM)	[15] (theory)
4.57 ± 0.12	3.59 ± 0.07	${}^6\text{Li}({}^3\text{He}, d){}^7\text{Be}$, DWBA	[16] (extraction)

The underlying factor is that the low-energy ${}^3\text{He}(\alpha, \gamma){}^7\text{Be}(3/2^-)$ and ${}^3\text{He}(\alpha, \gamma){}^7\text{Be}(1/2^-)$ radiative captures, associated with the dominant transitions $1/2^+ \rightarrow 3/2^-$ and $1/2^+ \rightarrow 1/2^-$, respectively, cannot be regarded as strictly peripheral processes. The reason is that the initial ${}^3\text{He} + \alpha$ system is in an s wave, so there is no centrifugal barrier to confine the reaction to the asymptotic region in which the ${}^3\text{He} - \alpha$ nuclear interaction can be neglected. In other words, the lack of a centrifugal barrier in the initial channel means that contributions from the domain adjacent to the nuclear surface, where the nuclear interaction remains significant, cannot be overlooked. Omitting this transitional area and focusing merely on the extreme periphery means the complete normalization of the S_{34} factor relies entirely on $C_{3/2}^2$ and $C_{1/2}^2$. The resulted ANC's become too high [11] (see Table I). Also, too high ANC's are obtained using the relationship between the effective-range function at positive energies and the ANC [12].

II. CURRENT STATUS OF ${}^7\text{Be}$ ANC DETERMINATION

Determining the ${}^7\text{Be}$ bound state ANC's through theoretical analysis of low-energy S_{34} factors requires matrix element calculations that are more sophisticated than a simple potential model. The variational Monte Carlo Method saw one of its first sophisticated implementations in Ref. [13], see Table I. The work of Ref. [14] marked a crucial advancement by fitting the measured experimental S_{34} factors using the R -matrix method with classic χ^2 optimization and Monte Carlo-simulated uncertainties. The $C_{3/2}$ and $C_{1/2}$ ANC's were used as part of the fitting parameters. The resulting ANC's are shown in Table I of Ref. [7]. Finally we need to mention important *ab initio* calculations. In the work of Ref. [7], next-to-leading-order (NLO) EFT formalism was used to calculate the amplitude for the radiative capture ${}^3\text{He}(\alpha, \gamma){}^7\text{Be}$. The fitting of experimental low-energy S_{34} factors was achieved utilizing this amplitude. The median values for the ANC's $C_{3/2}$ and $C_{1/2}$, incorporated among the fitting parameters in a Bayesian analysis, can be found in Table I. Finally, among the most advanced theoretical approaches, we highlight *ab initio* no-core shell-model (NCSM) calculations applied to the S_{34} factor [15]. The resulting ${}^7\text{Be}$ ANC's did not meet expectations. A situation was attributed to the poor binding energies of the interacting clusters. However, the manual adjustment of cluster binding energies to align with actual experimental data (phenomenological NCSM) led to the ANC's matching

those from Refs. [14] and [7] (see Table I). One may employ these ANC's as sensible comparative numbers or priors in the Bayesian analysis. The ANC for $C_{3/2}$ determined from the α -particle transfer reaction ${}^6\text{Li}({}^3\text{He}, d){}^7\text{Be}$ was too high [16] (see Table I).

To constrain the ${}^7\text{Be}$ ANC's, beyond radiative capture, focus was also placed on examining low-energy ${}^3\text{He} + {}^4\text{He}$ elastic scattering. However, older measurements of the phase shifts [17,18] had minimal impact on specifying the ${}^7\text{Be}$ ANC's. Differences were noted between the ANC's acquired from scattering measurements and those obtained from the radiative capture data. In particular, it was impossible using the scattering phase shifts from Ref. [18] to reproduce the measured S_{34} factors [6]. Obtained low S_{34} factors from the parameters of elastic scattering data are correlated with low ${}^7\text{Be}$ ANC's. Thus the ANC's obtained through the older scattering measurements differed from those produced by the radiative capture data. It resulted in a different low-energy extrapolation of the S_{34} factor obtained based on the ANC's extracted from the elastic scattering.

The most accurate and up-to-date elastic scattering measurements (SONIC experiment) are presented in Refs. [19,20]. In these works, a new measurement of the ${}^4\text{He}({}^3\text{He}, {}^3\text{He}){}^4\text{He}$ reaction has been made at the center-of-mass energies $E_{\text{c.m.}} = 0.38 - 3.13$ MeV. This was the first measurement of the ${}^3\text{He} + {}^4\text{He}$ elastic scattering performed for energies below 500 keV, which also contains numerous references to earlier works on ${}^3\text{He} + {}^4\text{He}$ scattering. Presented measurements of the ${}^3\text{He} + {}^4\text{He}$ elastic scattering cross section are important in order to improve constraints on theoretical models of the ${}^3\text{He}(\alpha, \gamma){}^7\text{Be}$ reaction. The low-energy elastic scattering data for $E_{{}^3\text{He}} E_{{}^4\text{He}} < 1$ MeV with well-defined uncertainties are pivotal determining $S_{34}(0)$ with constrained extrapolation error. Utilizing SONIC phase shifts [19] improved the extrapolated-to-zero-energy S_{34} factors [21]. A drawback is that the $l = 1$ scattering phase shifts, as given in Ref. [19], were computed via R -matrix fitting, bypassing experimental data, and with ANC's remaining fixed: $C_{3/2} = 3.7 \text{ fm}^{-1/2}$ and $C_{1/2} = 3.6 \text{ fm}^{-1/2}$.

Because the radiative capture S_{34} factors alone were insufficient for constraining the ${}^7\text{Be}$ ANC's, a combined Bayesian R -matrix investigation of the ${}^3\text{He}(\alpha, \gamma){}^7\text{Be}$ and ${}^3\text{He} + {}^4\text{He}$ processes was conducted in Ref. [6] to reduce the ANC's' uncertainty. Simultaneous analysis of the available experimental S_{34} factors and elastic scattering cross section measured in Ref. [19] significantly diminished the uncertainties of the

ANCs. However, adding also the elastic scattering data from Ref. [18] decreases the ⁷Be ANCs and the $S_{34}(0)$ factor. It is clearly seen in two-dimensional posterior [6]. The posterior shows that considering only the EFT analysis of the radiative capture [7] causes $C_{3/2}^2$ to cover a considerably wide span of 10.5–17.5 fm, with $C_{1/2}^2$ ranging from 9 to 16 fm. The posterior maximum *a posteriori* (MAP) (highest probability density) is $C_{3/2}^2 = 13.7$ fm and $C_{1/2}^2 = 13$ fm. Simultaneous analysis of the radiative capture and the elastic scattering data from Ref. [19] gives a much narrower interval: 13.7–18.0 fm for $C_{3/2}^2$ and 7.8–10.3 fm for $C_{1/2}^2$ with the maximum of the posterior $C_{3/2}^2 = 15.8$ fm and $C_{1/2}^2 = 9$ fm. Finally, the simultaneous analysis of the radiative capture, and elastic scattering data from Refs. [19] and [18], resulted in the narrowest intervals: 11–13.3 fm for $C_{3/2}^2$ and 7.3–8.8 fm for $C_{1/2}^2$, with the maximum of the posterior $C_{3/2}^2 = 12$ fm and $C_{1/2}^2 = 7.8$ fm. Thus, a significant reduction in the ANCs occurs with the inclusion in the analysis of the ³He + ⁴He elastic cross section (from old measurements [18]), confirming that the calculated ANCs are very responsive to the precision of the elastic scattering data. It is vital to note that the approach taken by the authors of Ref. [6] for calculating the ANCs involved the differential elastic scattering cross section, with its analysis performed via the *R*-matrix representation of the elastic scattering *S* matrix.

A different collection of old low-energy scattering phase shifts was published in Ref. [22]. These phases were then featured in illustrations from Paneru's research [19], without any accompanying examination. Reference [6] did not incorporate Boykin's phases into its *R*-matrix analysis. This was primarily because crucial systematic uncertainty information was missing and phase shifts were only measured from $E_{^3\text{He } ^4\text{He}} = 1.89$ MeV upward. The analysis conducted in Refs. [6,14,19] was performed utilizing the *R*-matrix formalism parametrized in terms of the reduced widths which are related to the ANCs.¹ Still, even a rudimentary, single-pole *R*-matrix formulation comprises two parameters, the reduced width and an unknown channel radius, which contributes to further ambiguity in defining the reduced width. Besides, often for better fit one needs to add a background term, which leads to an additional uncertainty in pinning down the reduced width. In the EFT technique, several fitting parameters exist, and the ANC is one such parameter. The parameter fitting is performed at positive energies, similar to the *R* matrix, but the ANC is related to the residue at the elastic scattering *S*-matrix's pole, located at negative energies [1]. To optimize the fit, it's necessary to encompass a wider energy domain. However, elevated energies result in a decreased impact from the bound-state pole.

Yet, there is an approach developed by us (see Refs. [23,24] and references therein), which allows one to determine the ANC by utilizing the extrapolation of experimental elastic scattering partial-wave amplitudes to the bound-state pole. Because the extrapolation technique is inherently model independent, ANCs, when extracted in this way, can be regarded as

model independent. As we underscored, despite an extensive research regarding ⁷Be, there are some questions remain about its ANCs $C_{3/2}^2$ and $C_{1/2}^2$. In particular, despite the claims that older phase shifts give lower ANCs, what was confirmed indirectly by Bayesian analysis in [6], no explicit numbers were given. The data from Ref. [22] were never analyzed. To validate the extrapolation method, we can digitize the *R*-matrix-generated phases for $l = 1$ in Ref. [19] and treat them as quasi-experimental points. By extrapolating these phases to the bound-state poles, we can compare the obtained ANCs with those used as fixed input parameters. We also present the Bayesian posterior files for Refs. [22] and [18] phases. Given these facts, there is a strong justification for implementing a model-independent extrapolation to ascertain the ANCs from existing measurements. By applying this extrapolation approach, we are able to validate the statements made in Refs. [6,14,21] regarding the discrepancy between old and modern results for both the ANCs and S_{34} factors.

III. SCOPE AND AIMS OF THE PRESENT STUDY

In summarizing the current status of ⁷Be ANC determinations, one observes a persistent discrepancy between the information obtained from older scattering data and those obtained from radiative capture studies. This difference was mitigated through the use of elastic scattering cross sections reported in Ref. [19]. However, the tension between elastic scattering phase shifts [19] and radiative capture analyses was, until recently, supported only by indirect arguments. In fact, no prior publication had determined the ⁷Be ANCs directly from elastic scattering data. For example, Ref. [6] presented posteriors for the ANCs obtained from a combined analysis of radiative capture together with elastic scattering data from Refs. [19] and [18], while Ref. [19] itself reported only the *R*-matrix fits with fixed ANC values.

The primary goal of this work, therefore, is to determine the ANCs from older elastic scattering phase-shift data. Specifically, we analyze the phases published in Refs. [18] and [22]. Although additional scattering data are available in Ref. [17], the energies were too high to be suitable for ANC extraction. Both of the older datasets also have limitations: Ref. [18] published two contradictory phase-shift sets, rendering the $J^\pi = 1/2^-$ phases unsuitable for analysis, while Ref. [22] did not report systematic uncertainties, complicating their use. Our analysis in this paper covered the $3/2^-$ phases from both sets provided by [18], in addition to the $3/2^-$ and $1/2^-$ phases from Ref. [22].

Consistent with the approach detailed in the authors' earlier publications [23,24], which focused on calculating the ANCs pertinent to the $^{16}\text{O} \rightarrow \alpha + ^{12}\text{C}$ virtual decay, the present study applies the analytical continuation in energy of the measured partial-wave amplitudes for elastic ³He⁴He scattering toward the pole associated with the bound state ⁷Be = (³He α). It is important to note, however, that the methods for deriving the ANCs used herein depend on having information about partial phase shifts. Consequently, the findings from the phase-shift decomposition of elastic ³He + ⁴He scattering are adopted from Refs. [18,22], alongside the *R*-matrix-derived phases from Ref. [19].

¹The observable reduced widths can be expressed in terms of the ANCs; see Eq. (22) below [4].

Different analytical continuation procedures are employed, notably, the universal method newly put forth in Ref. [24]. The R -matrix strategy and the two-particle potential model are also applied as secondary methods. This paper's layout is described below. We did not limit our study to the R -matrix and analytical continuation but additionally incorporated Bayesian analysis of the data from Refs. [22] and [18]. It allowed us to estimate the confidence intervals for the determined ANC. Considering that Ref. [19] solely supplied the R -matrix fit with fixed ANCs, we applied these phases to assess the reliability of our extrapolation methodology by extrapolating the R -matrix phases to the bound-state poles and then likening the obtained results to the initial ANCs. Throughout the paper we use a system of units in which $\hbar = c = 1$.

IV. BASIC FORMALISM

In this section we recapitulate basic formulas which are necessary for the subsequent discussion. Let us begin with a brief description of the formalism of the two-body elastic scattering in the presence of short-range and Coulombic interactions. A more detailed presentation can be found, for example, in Refs. [23,24].

The renormalized Coulomb-nuclear partial-wave scattering amplitude $\tilde{f}_l(E)$ of particles b and c is of the form

$$\tilde{f}_l = \frac{\exp(2i\delta_l) - 1}{2ik} \mathbb{C}_l^{-2}(\eta) \equiv \frac{k^{2l}}{D_l(E)}, \quad (1)$$

where δ_l is the Coulomb-nuclear phase shift, $k = \sqrt{2\mu E}$ is the relative momentum, $\eta = Z_b Z_c e^2 \mu / k$ is the Coulomb parameter for the bc scattering state, $\mu = m_b m_c / (m_b + m_c)$, m_i and Z_{ie} are the mass and the electric charge of particle i , and $\mathbb{C}_l(\eta)$ is the Coulomb penetration factor (or Gamow factor):

$$\mathbb{C}_l(\eta) = \left[\frac{2\pi\eta}{\exp(2\pi\eta) - 1} v_l(\eta) \right]^{1/2}, \quad (2)$$

$$v_l(\eta) = \prod_{n=1}^l (1 + \eta^2/n^2) \quad (l > 0), \quad v_0(\eta) = 1. \quad (3)$$

The denominator $D_l(E)$ in Eq. (1) can be written in the following equivalent forms:

$$D_l(E) = \tilde{\Delta}_l(E) - ik^{2l+1} \mathbb{C}_l^2(\eta), \quad (4)$$

$$D_l(E) = K_l(E) - 2\eta k^{2l+1} h(\eta) v_l(\eta), \quad (5)$$

where $K_l(E)$ is the Coulomb-modified effective-range function (ERF),

$$K_l(E) = k^{2l+1} [\mathbb{C}_l^2(\eta) (\cot \delta_l - i) + 2\eta h(k) v_l(\eta)], \quad (6)$$

$$h(\eta) = \psi(i\eta) + \frac{1}{2i\eta} - \ln(i\eta), \quad (7)$$

$\psi(x)$ is the digamma function, and

$$\tilde{\Delta}_l(E) = k^{2l+1} \mathbb{C}_l^2(\eta) \cot \delta_l. \quad (8)$$

The analytic properties of $\tilde{f}_l(E)$ on the physical sheet of energy E are analogous to those of the partial-wave scattering amplitude for the short-range potential, and $\tilde{f}_l$ can be analytically continued into the negative-energy region.

If the $b + c$ system has in the partial wave l the bound state a with the binding energy $\varepsilon = \kappa^2/2\mu > 0$, then the amplitude $\tilde{f}_l$ has a pole at $E = -\varepsilon$ and the ANC $C_l(a \rightarrow b + c)$ is expressed through the residue of $\tilde{f}_l$ at this pole as follows [25]:

$$\begin{aligned} C_l^2 &= -2\mu \left[\frac{\Gamma(l+1+\eta_b)}{l!} \right]^2 \text{res} \tilde{f}_l(E)|_{E=-\varepsilon} \\ &= -2\mu \left[\frac{\Gamma(l+1+\eta_b)}{l!} \right]^2 (-1)^l \kappa^{2l} \left(\frac{dD_l(E)}{dE} \right)_{E=-\varepsilon}^{-1}, \end{aligned} \quad (9)$$

where $\Gamma(z)$ is the Gamma function and $\eta_b = Z_b Z_c e^2 \mu / \kappa$ is the Coulomb parameter for the bound state a .

The traditional approach to determining the ANC from scattering data is based on the use of the ERF $K_l(E)$ (ERF method). $K_l(E)$ is analytic near $E = 0$. It can be expanded into a Taylor series in E and analytically continued to $E = -\varepsilon$. Then the ANC can be found by using Eqs. (5) and (10). However, the ERF method, although formally rigorous, is practically suitable only for light nuclei due to the presence of a large background of purely Coulombic terms. These terms may far exceed, in the absolute value, the informative part of the ERF containing the phase shifts. It could restrict the practical execution of analytic continuation and compromise its accuracy.

As a result of this, the authors of Ref. [26] recommended extending the function $\tilde{\Delta}_l(E)$ analytically, as described in Eq. (8), rather than $K_l(E)$ (the Δ method). Yet, from a mathematical standpoint, the Δ method is not entirely rigorous or accurate. Moreover, this method is applicable only to systems with sufficiently large electric charges and masses. It was shown in Refs. [27,28] that this method can be employed to obtain reliable information on a bound state if its binding energy and the energy of scattering states used to approximate the $\tilde{\Delta}_l(E)$ function satisfy the condition

$$|E| \leq (Z_b Z_c e^2)^2 \mu / 2 = 1 \text{ Ry}, \quad (11)$$

where 1 Ry is the nuclear Rydberg energy.

In connection with the aforementioned, a new method of analytical continuation, free from the above disadvantages, was proposed in Ref. [24]. We will call this method the subtraction method (or S method). Within this method, the quantity $g_l(E)$ is introduced according to

$$g_l(E) = D_l(E) - D_l^{\text{SW}}(E) \quad (12)$$

$$= \tilde{\Delta}_l(E) - \tilde{\Delta}_l^{\text{SW}}(E) \quad (13)$$

$$= K_l(E) - K_l^{\text{SW}}(E), \quad (14)$$

where $D_l^{\text{SW}}(E)$, $\tilde{\Delta}_l^{\text{SW}}(E)$, and $K_l^{\text{SW}}(E)$ are the functions $D_l(E)$, $\tilde{\Delta}_l(E)$, and $K_l(E)$ for a potential that is the sum of the Coulomb and square-well (SW) potentials. $\tilde{\Delta}_l^{\text{SW}}(E)$ is expressed explicitly in terms of parameters of the SW potential (see Eq. (16) from Ref. [23]). $K_l(E)$ and $K_l^{\text{SW}}(E)$, as distinct from $\tilde{\Delta}_l(E)$, have no essential singularity at $E = 0$. Hence, these functions can be articulated as a series in E in the neighborhood of $E = 0$. Consequently, $g_l(E)$ can also be expressed as a series in E near $E = 0$. We want to underscore that this property is not contingent upon the parameters of the

SW potential. Accordingly, $g_l(E)$ can be approximated by a polynomial at $E > 0$ and continued into the negative-energy region. Note that $g_l(E)$ is real for both positive and negative energies.

The condition of the pole of $\tilde{f}_l(E)$ at $E = -\varepsilon = -\kappa^2/2\mu$ is

$$D_l(-\varepsilon) = g_l(-\varepsilon) + D_l^{\text{SW}}(-\varepsilon) = 0. \quad (15)$$

ANC C_l is determined by the expression (10). The SW potential is chosen to determine the auxiliary function $\tilde{\Delta}_l^{\text{SW}}(E)$, because this potential, combined with the Coulomb potential, allows an analytical solution of the Schrödinger equation for arbitrary values of l . In principle, instead of $\tilde{\Delta}_l^{\text{SW}}(E)$, one can take another analytical expression, for example, an expression obtained within the R -matrix formalism [29], allowing analytical continuation into the domain $E < 0$.

The main results for determining the ANCs, presented in the next section, were obtained using ERF, Δ , and S methods. The values of ANCs $C_{3/2}$ and $C_{1/2}$ found within the R -matrix approach and the two-particle potential model are also given.

V. SOURCES OF PHASE SHIFTS

All results detailed in this section stem from the phase-shift analysis of the elastic ^3He - ^4He scattering data as compiled from three specific sources (Refs. [18–20,22]). Given that articles [18,22] are considerably old, their standard may be dubious. Moreover, whereas Ref. [18] encompasses both systematic and statistical uncertainties, Ref. [22] does not account for systematic uncertainties. Additionally, two distinct measurements for $3/2^-$ and $1/2^-$ scattering phase shifts were reported in Ref. [18], which proved to be inconsistent. While the $1/2^-$ phases in variant (A) exceeded the $3/2^-$ ones, variant (B) showed an inverse arrangement. This type of division of phases $1/2^-$ and $3/2^-$ is inconsistent with measurements from Ref. [22] and recent data from Ref. [19]. Additional difficulties become apparent with the consistency of the data for $1/2^-$ phases. The R -matrix formulation failed to fit these phases for variant (B). The study found that these $1/2^-$ phases are also generated by contributions other than solely the first excited $1/2^-$ state in ^7Be . Moreover, the findings from the experimental data suggest that the sign of splitting $1/2^-$ phases and $3/2^-$ in variant (A) is flawed. As a result, we will forgo analyzing the $1/2^-$ phases pertinent to variants (A) and (B). Afterward, we will analyze only the two $3/2^-$ variants. The third publication [19], being the most up-to-date, has improved quality. Detailed measurements of the differential elastic scattering cross section are presented. However, rather than revealing experimental scattering phase shifts, Ref. [19] contains only the R -matrix fit with the fixed ANCs. The R -matrix phases presented in this study can serve solely to validate the extrapolation technique, achieved by contrasting the ANCs obtained via extrapolation with the original ones.

In Figs. 1 and 2 we show the available scattering phase shifts for the spin parities of ^7Be $J^\pi = 3/2^-$ as functions of $E_{^3\text{He}}$, where $E_{^3\text{He}}$ is the laboratory ^3He energy related to the center-of-mass energy E as $E_{^3\text{He}} = (m_\alpha + m_{^3\text{He}})/m_\alpha E = 1.619E$. Note that in Fig. 1 the phases from Ref. [18] are shown for set (A) and in Fig. 2 for set (B). But we note that

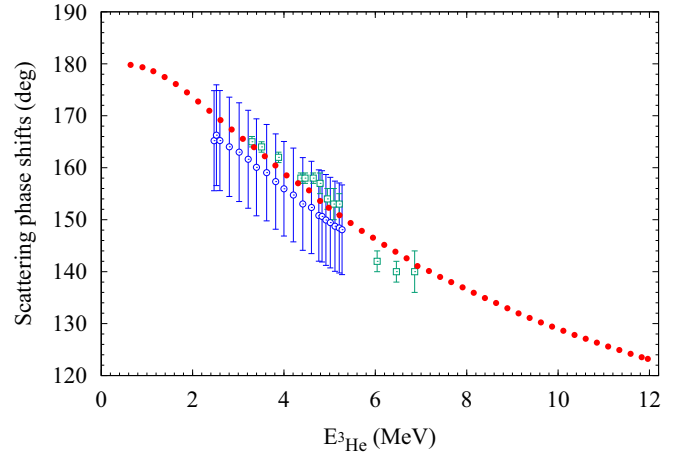


FIG. 1. $J^\pi = 3/2^-$ scattering phase shifts: red dots Ref. [19], blue circles represent set A from Ref. [18], and green squares Ref. [22].

the analysis of the $1/2^-$ scattering phases from Ref. [18] is bypassed owing to the complications detailed therein.

VI. ANALYSIS OF BOYKIN'S PHASES BY VARIOUS METHODS

A. R matrix

In Ref. [22] the single-level R matrices were constructed for ^3He - ^4He scattering in channels $3/2^-$ and $1/2^-$. However, the authors did not explain in sufficient detail how the parameters of the R matrices were obtained, and the given values of these parameters contain very large errors. In this regard, we redetermined the parameters of the R matrices based on the requirement of the best description of the results of phase-shift analysis from Ref. [22].

The simplest single-channel, single-level R matrix is of the form

$$R(E) = \frac{\gamma^2}{E_R - E}, \quad (16)$$

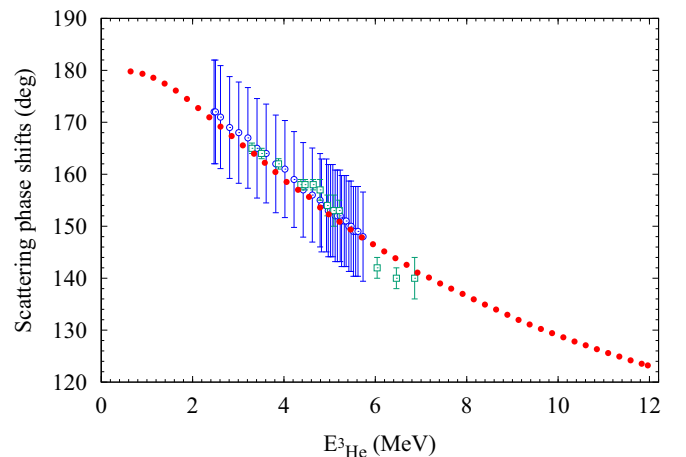


FIG. 2. The same as in Fig. 1, but blue circles represent set B from Ref. [18].

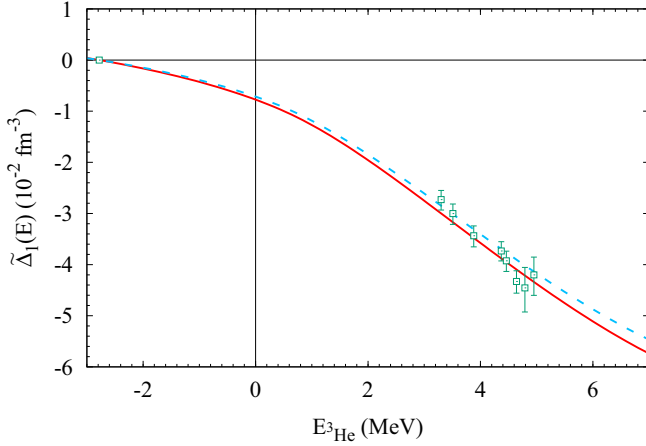


FIG. 3. Function $\tilde{\Delta}_l(E)$, $J^\pi = 3/2^-$, for the optimal R matrix (solid red line) and for the SW potential (dashed blue line).

where γ^2 is the formal reduced width used in the standard R -matrix parametrization. If in a given channel there is a bound state with binding energy ε , then in order for the scattering amplitude to have a pole at $E = -\varepsilon$, it is necessary to set $E_R = -\varepsilon$. In the case of the $\alpha + {}^3\text{He}$ system, $\varepsilon = 1.586$ MeV for channel $3/2^-$.

The R -matrix $R(E)$ function is expressed through the phase shift $\delta(E)$ and the channel radius a [30]. Utilizing the χ^2 criterion on expression (16) for a constant a , we can establish the connection

$$\gamma^2 \sum_{i=1}^n \frac{1}{\epsilon_i^2 (E_R - E_i)^2} = \sum_{i=1}^n \frac{R_i}{\epsilon_i^2 (E_R - E_i)}. \quad (17)$$

Here E_i is the value of energy E for which the phase shift δ_i with the corresponding error is known, $R_i = R(E_i)$, and ϵ_i is the error of R_i determined by the error of δ_i . From Eq. (17) it is possible to express the reduced width γ^2 in terms of the remaining quantities included in Eq. (17), which depend on a . Now, applying the criterion χ^2 with respect to a , it is possible to determine the parameters a and γ^2 that lead to the best description of the results of the phase-shift analysis.

To determine the parameters of the R matrices using the procedure described above, in this work, as in Ref. [22], we used the values of the eight lowest energy phase shifts $\delta_{3/2}(E)$ presented together with errors in Table 2 of Ref. [22]. These eight phase shifts lie in the range $3.30 \text{ MeV} \leq E_{^3\text{He}} \leq 4.95 \text{ MeV}$. Note that these same eight phase shifts are used in all the approaches described below. The calculations are illustrated in Figs. 3 and 4.

As a result, we obtained the following values of parameters of the R matrices:

$$\begin{aligned} J^\pi = \frac{3}{2}^- : a &= 3.068 \text{ fm}, \quad \gamma^2 = 3.008 \text{ MeV}; \\ J^\pi = \frac{1}{2}^- : a &= 3.300 \text{ fm}, \quad \gamma^2 = 2.509 \text{ MeV}. \end{aligned} \quad (18)$$

R matrices with parameters (18) describe the results of the phase-shift analysis quite well (see red lines in Figs. 3 and 4). From Eqs. (21) and (22), along with the R -matrix parameters

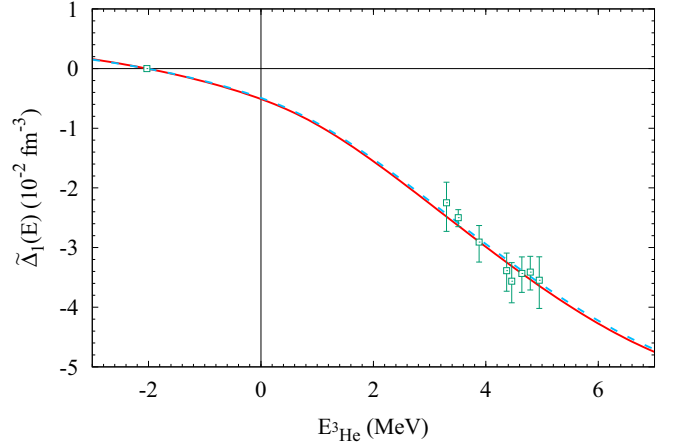


FIG. 4. Function $\tilde{\Delta}_l(E)$, $J^\pi = 1/2^-$, for the optimal R matrix (solid red line) and for the SW potential (dashed blue line).

in Eq. (18), we derive $C_{3/2} = 2.66 \text{ fm}^{-1/2}$ and $C_{1/2} = 2.54 \text{ fm}^{-1/2}$.

B. Two-body square-well potential model

In Sec. VIB, the ANC $C_{3/2}$ is found by solving the two-body Schrödinger equation with the potential in the form of the sum of the Coulomb potential and the square-well (SW) potential. Based on the insights of the shell model, we require that the chosen potential leads to two bound states, the lower of which is forbidden. The radius R and depth V_0 of the SW potential are selected by the χ^2 method from the condition of the best description of the experimental phase shifts and the correct reproduction of the binding energy $\varepsilon_{3/2}$ and $\varepsilon_{1/2}$ of the ground ($3/2^-$) and excited ($1/2^-$) states of ${}^7\text{Be}$. Using this method, the following parameters were found:

$$\begin{aligned} J^\pi = \frac{3}{2}^- : R &= 3.27 \text{ fm}, \quad V_0 = 52.31 \text{ MeV}; \\ J^\pi = \frac{1}{2}^- : R &= 3.50 \text{ fm}, \quad V_0 = 45.70 \text{ MeV}. \end{aligned} \quad (19)$$

These parameters in turn lead to $C_{3/2} = 2.73 \text{ fm}^{-1/2}$ and $C_{1/2} = 2.60 \text{ fm}^{-1/2}$.

The SW potential may appear to be a very simple choice; however, at the low energies considered here, the detailed form of the potential is not very significant. Note that the SW potential with parameters given in Eq. (19) describes the experimental data quite well (see blue lines in Figs. 3 and 4). This same potential will be used below within the S method.

C. ERF method

In this approach, as well as in the Δ - and S-methods, the functions we consider are determined by the experimental data. These are approximated in the physical region $E > 0$ by the expression

$$\sum_{n=0}^N c_n P_n(E), \quad (20)$$

where P_n are Chebyshev polynomials of degree n . The maximal degree of the polynomial N and the coefficients c_n are

fitted to the experimental data using the χ^2 criterion as well as the F criterion (see the monograph [31]). F statistics compare the χ^2 per degree of freedom for two different fits. Hence, it can be used to reject the null hypothesis (with a certain confidence), meaning that improvement in that fit metric is solely due to a statistical fluctuation.

In the ERF method, the approximated function is ERF $K_l(E)$ (6). The results of the analytical continuation of $K_l(E)$ turn out to be very sensitive to the phase-shift values at the lowest energies. Figure 3 shows that the experimental phase shift $\delta_{3/2}(E_{\text{He}})$ at the lowest energy $E_{\text{He}} = 3.30$ MeV noticeably deviates from the smooth energy dependence. Applying the ERF method using all eight phase shifts from Ref. [22] is characterized by poor convergence with increasing maximal degree N of the approximating polynomial and leads to an unrealistic result $C_{3/2} = 9.88 \text{ fm}^{-1/2}$. In this regard, the ERF method was applied to the set of phase shifts in which, at $E_{\text{He}} = 3.30$ MeV, the phase shift $\delta_{3/2}$ from [22] was replaced by $\delta_{3/2}^*$, determined by the R matrix constructed above. This procedure leads to $C_{3/2} = 2.96 \text{ fm}^{-1/2}$. A largely similar situation is observed in channel $1/2^-$ (see Fig. 4). Replacing $\delta_{1/2}(E_{\text{He}} = 3.30 \text{ MeV})$ from [22] with the value following from the R -matrix approach, we obtain $C_{1/2} = 2.87 \text{ fm}^{-1/2}$.

D. Δ method

For the $\alpha + {}^3\text{He}$ system $1 \text{ Ry} = 0.68 \text{ MeV}$, while the interval used to approximate the experimental data is $2.04 \text{ MeV} \leq E \leq 3.06 \text{ MeV}$. Therefore, the condition (11) for the applicability of the Δ method is noticeably violated. Nevertheless, we decided to find out what results the formal application of this method would lead to. In the Δ method, as in Refs. [23,24], the function $\ln(A - \tilde{\Delta}_l(E))$ is approximated and analytically continued. Using the logarithmic function makes it possible to soften the strong dependence of $\tilde{\Delta}_l(E)$ near $E = 0$ and improve the quality of approximation of the considered function by polynomials. The constant $A > 0$ is added to make $A - \tilde{\Delta}_l(-\varepsilon)$ positive. The value of A is chosen so that within the approximating energy range the condition $A \ll |\tilde{\Delta}_l(E)|$ holds, and the approximated function is as close to a straight line as possible so that it could be approximated by a polynomial of a low degree N . Under these conditions, as shown in [24], the calculation results are practically insensitive to changes in A .

Applying the χ^2 and F criteria to a polynomial approximation of the function $\ln(A - \tilde{\Delta}_l(E))$ followed by analytical continuation to the point $E = -\varepsilon$ results in $C_{3/2} = 2.36 \text{ fm}^{-1/2}$ and $C_{1/2} = 2.17 \text{ fm}^{-1/2}$.

E. S method

Within the S method, quantity $g_l(E)$ in the form (13) is approximated and continued into the region $E < 0$; the ANC values are found using Eqs. (4) and (10). Formally, the S method can be used for any parameters of the SW potential that determine the function $\tilde{\Delta}_l^{\text{SW}}(E)$. We chose the same parameters that, within the framework of the two-particle model, lead to the best description of the experimental data. With this choice, the approximated quantity $g_l(E)$ makes a small

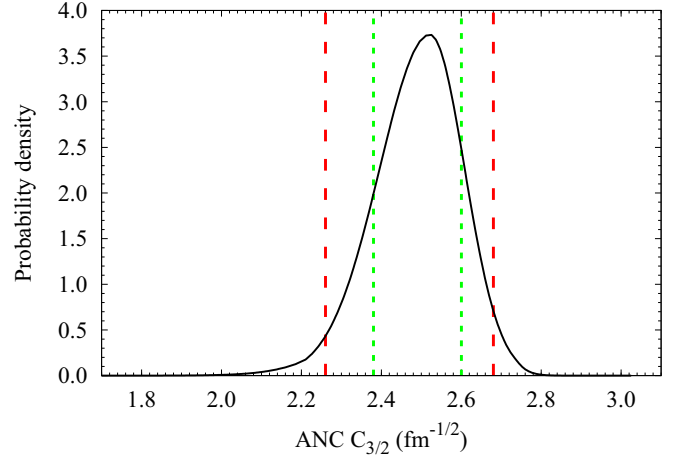
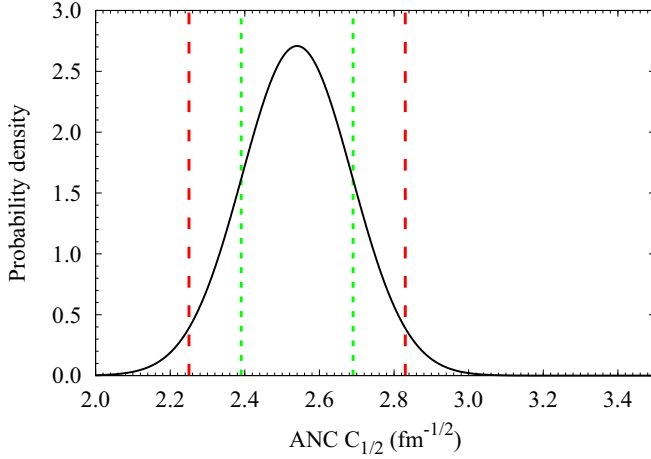


FIG. 5. Black solid line is the posterior from Bayesian analysis of the $3/2^-$ scattering phase shifts from Ref. [22]. The probability density is normalized to unity on the prior interval of the ANC. Green dotted and red dashed lines are 68% and 95% CIs of the ANC $C_{3/2}$, respectively.

contribution to the full function $\tilde{\Delta}_l(E)$, and possible inaccuracies in its calculation should have little effect on the results. The polynomial approximation of the function $g_l(E)$ constructed in this way leads to $C_{3/2} = 2.73 \text{ fm}^{-1/2}$ and $C_{1/2} = 2.61 \text{ fm}^{-1/2}$. Note that the results for $C_{3/2}$ and $C_{1/2}$ obtained with direct use of the SW potential and within the S method using the SW potential with the same parameters are practically the same. This is because, as can be seen from Figs. 3 and 4, the SW potential, despite its simplicity, describes the experimental data very well. As a result, the difference function $g_l(E)$ turns out to be very small.

F. Bayesian analysis

We finalize our analysis of Boykin phases by utilizing Bayesian analysis and generating a posterior distribution, which allows us to determine the MAP value (see Appendix A). To compute the conditional probability density $P(C|D)$, in line with Eq. (A1), we opt for a constant (uniform) prior $P(C) = \text{const}$, confining the ANCs in the interval $0 < C < 3.7 \text{ fm}^{-1/2}$. Outside this interval, $P(C) = 0$. This means that $P(C)$ is simply a constant across the allowed ANC range, so the posterior $P(C|D)$ is directly proportional to the likelihood $P(D|C)$ inside that range. The MAP estimate from the Bayesian posterior analysis, obtained using a flat prior over $0 < C_{3/2} < 3.7 \text{ fm}^{-1/2}$ and $0 < C_{1/2} < 3.7 \text{ fm}^{-1/2}$ (thus, we assume equal probability of all the ANCs from the adopted searching interval) and experimental scattering phase-shift data for $3/2^-$ and $1/2^-$ [22] together with the R -matrix fit, yields the ANC $C_{3/2}(\text{MAP}) \approx 2.525 \text{ fm}^{-1/2}$. A 68% credibility interval (CI) is $2.38 - 2.60 \text{ fm}^{-1/2}$ and 95% CI is $2.26 - 2.68 \text{ fm}^{-1/2}$. For $1/2^-$ the ANC $C_{1/2}(\text{MAP}) \approx 2.54 \text{ fm}^{-1/2}$. A 68% CI is $2.39 - 2.69 \text{ fm}^{-1/2}$ and 95% CI is $2.25 - 2.83 \text{ fm}^{-1/2}$. The calculated posteriors are shown in Figs. 5 and 6. To make these posteriors realistic, we assumed the same aggregate uncertainty as in Ref. [18], 5.8%. Given the CI intervals, the ANCs $C_{3/2}(\text{MAP})$ and $C_{1/2}(\text{MAP})$

FIG. 6. Notations are the same as in Fig. 5 but for $C_{1/2}$.

established by the Bayesian analysis align remarkably well with ANC's acquired through different methodologies, including extrapolation.

VII. ANALYSIS OF BARNARD SET (A) SCATTERING PHASE SHIFTS

Reference [18] provides two different sets of the phase shifts. Figure 6 [18] presents one instance, labeled set (A). The second set, illustrated in Fig. 4 [18], is referred to as set (B).

A. *R*-matrix fitting

Although the authors admitted that the fit of set (A) was not satisfactory, we attempted to determine the ANC from the Barnard $3/2^-$ phases using a single-level *R* matrix (and variations). It should be noted that the phase shifts for $J^\pi = 3/2^-$ in set (A) are found to be lower than those for $J^\pi = 1/2^-$, contradicting set (B). For 26 energy points, including the bound state, we tried to fit the experimental data using a single-level *R* matrix. The best fit using standard $\min\chi^2$ resulted in the following parameters: channel radius $R_{\text{ch}} = 3.634$ fm, the formal reduced width $\gamma^2 = 1.465$ MeV. The observable reduced width is given by

$$(\gamma^{\text{obs}})^2 = \frac{\gamma^2}{1 + \gamma^2 \frac{d\hat{S}(E)}{dE} \Big|_{E=-\epsilon}}, \quad (21)$$

where $\hat{S}(E)$ is the shift function. For the one-channel case, the ANC is expressed in terms of the observable reduced width as [4]

$$C^2 = (\gamma^{\text{obs}})^2 \frac{2\mu a}{W_{-\eta_b, l+1/2}^2(2\kappa a)}, \quad (22)$$

where $W_{-\eta_b, l+1/2}(2\kappa a)$ is the Whittaker function describing the radial behavior of the tail of the overlap function of nuclei ^7Be , ^3He , and ^4He , η_b is the Coulomb parameter of the bound state, and κ is the bound-state wave number. From Eqs. (21) and (22), ANC $C_{3/2}$ is determined to be $3.10 \pm 0.18 \text{ fm}^{-1/2}$.

The *R*-matrix fitting to the phases $3/2^-$ set (A) from Ref. [18] is omitted due to its inadequacy.

The aggregate uncertainties, which account for systematic and statistical errors in each instance, are 5.83%, according to Ref. [18]. It is apparent that the one-level *R*-matrix fit is insufficient, notwithstanding the fact that the *R* matrix remains within the designated uncertainty range. Note that adding one more level as a background term did not improve the fitting.

B. Extrapolation analysis

We also attempted to determine the ANC by an extrapolation procedure using the Δ method. Based on Eq. (9), the ANC's calculation relies on the derivative $(\frac{dD_l(E)}{dE})^{-1}$. Given that all our extrapolation curves exhibited a negative sign, the ANC was thus imaginary. Hence, our extrapolation failed to determine the ANC.

Bayesian analysis

The final attempt to determine the ANC was the Bayesian analysis with the goal to determine the MAP value of the ANC from posterior distribution. It also failed. Data-only analyses preferred $C_{3/2} \approx 3.7 \text{ fm}^{-1/2}$, inconsistent with prior expectations ($0 < C_{3/2} < 3.7 \text{ fm}^{-1/2}$), analysis from Ref. [6], and the above-mentioned single-level fits near $C_{3/2} \approx 3.1 \text{ fm}^{-1/2}$. The χ^2 landscape is flat and non-Gaussian in $C_{3/2}$, indicating model-data tension. This result indicates that set (A) may contain intrinsic issues, in line with the findings of Ref. [18].

As a final result, we merely have the ANC value $C_{3/2} = 3.10 \pm 0.18 \text{ fm}^{-1/2}$, obtained through a deficient *R*-matrix fit.

VIII. ANALYSIS OF BARNARD SET (B) SCATTERING PHASE SHIFTS

A. *R*-matrix fitting

Here we discuss the ANC's obtained from the analysis of the elastic scattering phase shifts $3/2^-$ set (B) [18]. The superiority of set (B) over set (A), as posited by Ref. [18]'s authors, is affirmed by our findings. First off, the $3/2^-$ phases are higher than the $1/2^-$ phases, a finding that agrees with Ref. [22]. Employing the single-level *R* matrix for 32 energy points (inclusive of the bound state) and a channel radius of $R_{\text{ch}} = 3.8$ fm, sourced from Ref. [18], we observed that the most accurate match to the experimental phases necessitated a formal reduced width of $\gamma^2 = 0.38$ MeV, which is identical to the $\gamma^2 = 0.38$ MeV found in Ref. [18] (see Fig. 7). From our fitting we obtain the ANC of $C_{3/2} = 1.97 \pm 0.11 \text{ fm}^{-1/2}$, markedly below the value associated with set (A).

1. Extrapolation analysis

We also attempted to determine the ANC $C_{3/2}$ using the $\text{Re}\Delta(E)$ method, see Fig. 8. In this case the sign of the derivative $(\frac{dD_l(E)}{dE})^{-1}$ was correct when keeping two terms or three terms in the Chebyshev polynomial extrapolation of the function used for extrapolation. We find $C_{3/2} = 2.15 \pm 0.13 \text{ fm}^{-1/2}$ for $n = 2$ and $C_{3/2} = 2.08 \pm 0.12 \text{ fm}^{-1/2}$ for $n = 3$. These values are very close to the ANC obtained from the *R*-matrix fitting.

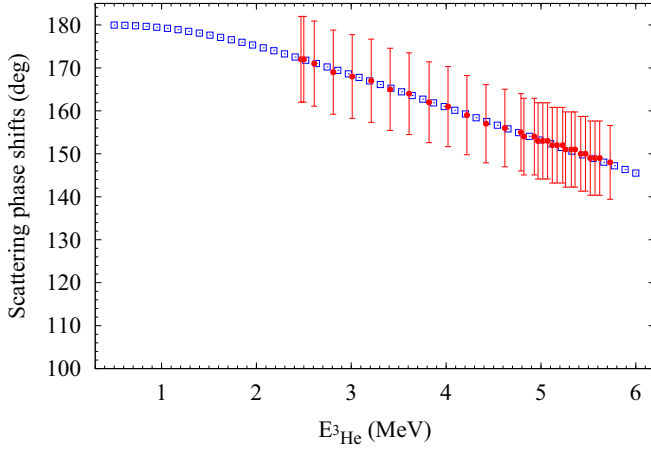


FIG. 7. $3/2^-$ phase shifts from set (B) [18]. Red dots with uncertainties are experimental data [18]. Blue squares are our R -matrix fit.

2. Bayesian analysis

Finally, we applied Bayesian analysis (see Appendix) to generate Bayesian posterior $P(C_{3/2}|D)$ and to determine the MAP value of the ANC. The MAP estimate from the Bayesian posterior analysis, obtained using a flat prior over $0 < C_{3/2} < 3.7 \text{ fm}^{-1/2}$ (thus, we assume equal probability of all the ANCs from the adopted searching interval) and experimental scattering phase-shift data for Barnard's $3/2^-$ variant (B) [18] together with the R -matrix fit, yields $C_{3/2}(\text{MAP}) \approx 1.98 \text{ fm}^{-1/2}$. The calculated posterior is shown in Fig. 9. This result is in excellent agreement with the ANC obtained independently by the R -matrix fitting and extrapolation of the elastic scattering phase shift to the bound-state pole. The 68% and 95% CI intervals are $1.84 - 2.12 \text{ fm}^{-1/2}$ and $1.7 - 2.26 \text{ fm}^{-1/2}$, respectively.

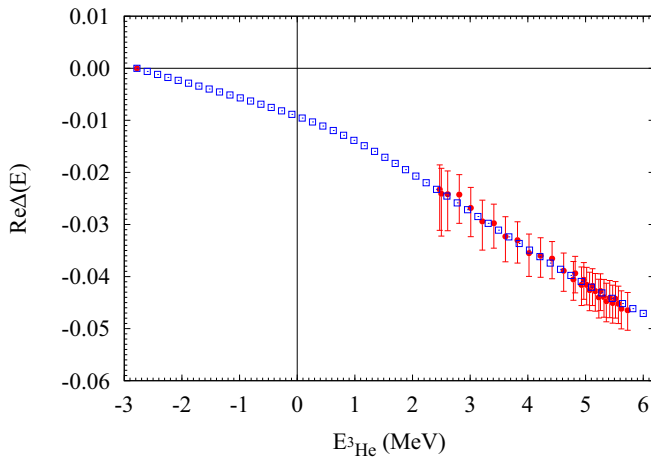


FIG. 8. Red dots with uncertainties is experimental $\text{Re}\Delta(E)$ function for $J^\pi = 3/2^-$ set (B) [18]. Blue squares represent our extrapolation.

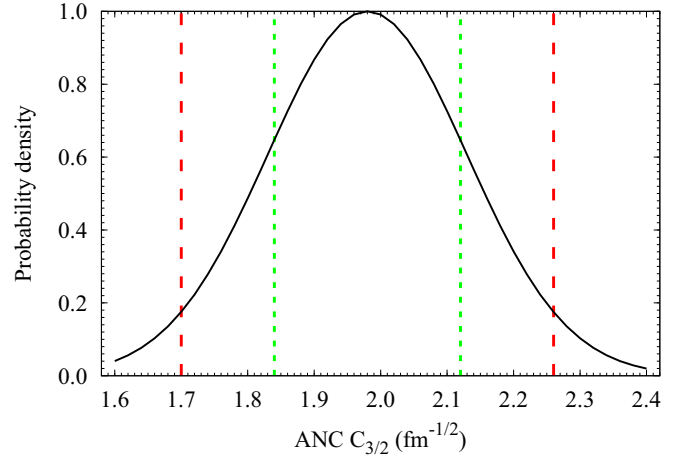


FIG. 9. Black solid line is the posterior from Bayesian analysis of the $3/2^-$ scattering phase shifts from [18] variant (B). The probability density is normalized to unity on the prior interval of the ANC. Green dotted and red dashed lines are 68% and 95% credible intervals of the ANC $C_{3/2}$, respectively.

B. Testing the validity of the extrapolation approach through analysis of Paneru phases

The up-to-date measurements of the cross section for $^3\text{He} - ^4\text{He}$ elastic scattering are outlined in Ref. [19]. Even though a phase-shift analysis was also executed, the R -matrix fit to the experimental data was the only aspect demonstrated, instead of the experimental phase shifts. The ANCs were maintained at $C_{3/2} = 3.7$ and $C_{1/2} = 3.6 \text{ fm}^{-1/2}$ throughout this fitting. It allows for the validation of our extrapolation technique by extrapolating the R -matrix phases, with pre-defined ANCs, from Ref. [19] to the bound-state poles of ^7Be 's ground and first excited states, and subsequently, cross-referencing the ANCs thus produced with the initial ANCs. To extrapolate we fitted the R -matrix Paneru phases using a

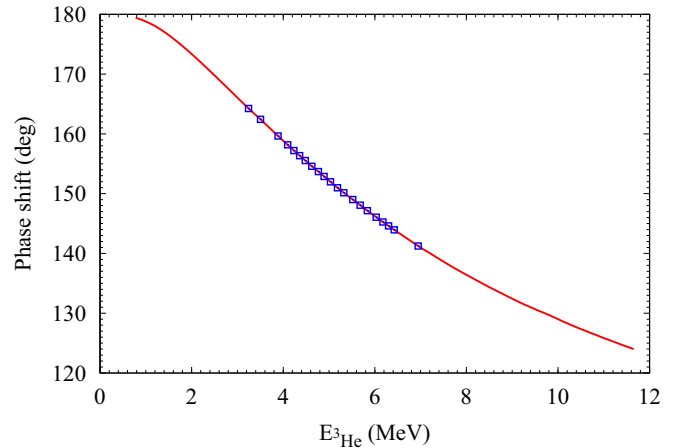
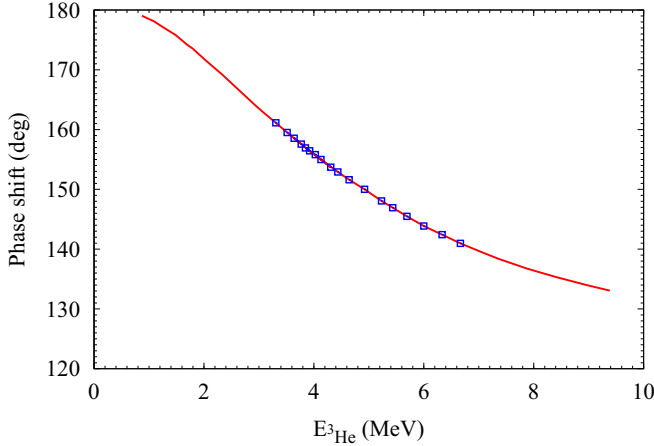


FIG. 10. The $3/2^-$ scattering phase shift from Ref. [19] is displayed as the solid red line. The blue squares are obtained utilizing the R matrix given by Eq. (23).

FIG. 11. Notations are the same as in Fig. 10 but for $J^\pi = 1/2^-$.

two-pole R matrix with the channel radius $R_{\text{ch}} = 4.2$ fm:

$$R_{\text{fit}} = \frac{\gamma_J}{-\varepsilon_J - E} + \frac{\gamma_{J(bg)}}{E_{bg} - E}, \quad (23)$$

where for $J^\pi = 3/2^-$ $\gamma_{3/2} = 0.93 \text{ MeV}^{1/2}$, $\varepsilon_{3/2} = 1.586 \text{ MeV}$, $\gamma_{3/2(bg)} = 2.09 \text{ MeV}^{1/2}$, and $E_{bg} = 24.08 \text{ MeV}$, and ε_J is the ${}^3\text{He} - {}^4\text{He}$ binding energy of the bound state with spin J . Clearly, the second term of the R matrix represents the background term. The E_{He} energy range of 3.3–6.7 MeV was selected for the extrapolation, closely matching the interval explored in the Ref. [22] experiment. Varying numbers of digitized points from this range were then employed for the extrapolation. For $J^\pi = 1/2^-$ we got $\gamma_{1/2} = 1.15 \text{ MeV}^{1/2}$, $\varepsilon_{1/2} = 1.1569 \text{ MeV}$, $\gamma_{1/2(bg)} = 3.60 \text{ MeV}^{1/2}$, and $E_{bg} = 45.70 \text{ MeV}$. The scattering phase shifts from Ref. [19] are displayed in Figs. 10 and 11. Extrapolation utilized the blue squares, which were treated as quasi-experimental phase shifts. The results of the extrapolation are given in Table II.

We note that derived via extrapolation with $N = 14$ and $n = 6$ $C_{3/2} = 3.68 \text{ fm}^{-1/2}$ practically coincides with the input ANC $C_{3/2} = 3.7 \text{ fm}^{-1/2}$ [19]. Concerning $C_{1/2}$ for $N = 19$ and $n = 4$, a 3.60% difference was determined between the value derived from extrapolation and the original input. This slight discrepancy could stem from the accuracy of digitization, R -matrix parametrization, and the energy interval selected for the extrapolation.

TABLE II. The $C_{3/2}$ and $C_{1/2}$ ANCs obtained by extrapolating the digitized R -matrix phase shifts from Ref. [19]. N is the number of the data points used for extrapolation, and n is the order of the Chebyshev polynomial used for extrapolation.

J	N	n	ANC [$\text{fm}^{-1/2}$]
3/2	9	5	3.65
3/2	14	6	3.68
1/2	9	5	3.77
1/2	14	5	3.77
1/2	19	4	3.73

IX. DISCUSSIONS

Elastic scattering phase shifts from Refs. [18,22], and [19] were investigated. Table III displays the total outcomes. In the rows representing the Bayesian results alongside the ANCs are shown in parentheses 68% CI. The evidence from our study suggests that reviewing existing elastic scattering phase-shift data unequivocally demonstrates that the resulting ANCs $C_{3/2}$ and $C_{1/2}$ are substantially reduced in comparison to values derived from radiative capture data, specifically, $C_{3/2} = 3.7 \text{ fm}^{-1/2}$ and $C_{1/2} = 3.6 \text{ fm}^{-1/2}$. This strengthens the conclusions of Refs. [21] and [6] regarding the conflict between radiative capture and elastic scattering data. Furthermore, the ANC $C_{3/2}$ derived using the data from Ref. [22] consistently exceeds the values found in set (B) [18]. A higher $C_{3/2}$ value, obtained from an inferior R -matrix fit of set (A) [18], is not considered due to the failure of all other methods to determine from set (A) this particular ANC.

X. SUMMARY

In this work we revisited the elastic scattering phase-shift data for the ${}^3\text{He} + {}^4\text{He}$ system in order to determine the ANCs of ${}^7\text{Be}$. Our results demonstrate that the extracted ANCs, $C_{3/2}$ and $C_{1/2}$, are consistently smaller than those obtained from radiative capture measurements ($C_{3/2} = 3.7 \text{ fm}^{-1/2}$ and $C_{1/2} = 3.6 \text{ fm}^{-1/2}$), thereby supporting earlier conclusions by Refs. [21] and [6] about the disagreement between elastic scattering and radiative capture data. Moreover, we find that the ANCs $C_{3/2}$ determined from Ref. [22] are systematically higher than those obtained from set (B) of Ref. [18], whereas the values derived from set (A) of Ref. [18] are excluded due to the unreliability of the corresponding R -matrix fits.

A central component of this study is the application and testing of the extrapolation method, originally developed by us, which provides a model-independent means of determining ANCs. By cross-validating this method against the fixed-ANC R -matrix fits of Ref. [19], we demonstrated its robustness and predictive power.

In addition, we employed Bayesian analysis to extract maximum *a posteriori* (MAP) values and credible intervals for the ANCs from the phase-shift data. The Bayesian results are consistent with the extrapolation method, confirming the reliability of both approaches. The Bayesian framework also quantified the statistical confidence in the extracted ANCs, which had not been available in earlier analyses.

Taken together, our findings highlight that the extrapolation method is a powerful and reliable tool for extracting ANCs from scattering data and that Bayesian analysis provides a complementary approach for quantifying uncertainties and validating results.

As a final comment, we note that the asymptotic normalization coefficients used in this work follow the standard two-body cluster and R -matrix definition. According to this definition, the radial overlap function asymptotically behaves like $u_l(r) \rightarrow C_l W_{-\eta, l+1/2}(2\kappa r)$ as $r \rightarrow \infty$, so that ANC C_l has the dimension of $\text{fm}^{-1/2}$. This dimensional convention is universally adopted in modern nuclear astrophysics (see, e.g., Refs. [2–4,6,14,19,25]). It differs from the dimensionless

TABLE III. Determined ANCs $C_{3/2}$ and $C_{1/2}$ values for $^7\text{Be} \rightarrow \alpha + ^3\text{He}$.

$C_{3/2}$ (fm $^{-1/2}$)	$C_{1/2}$ (fm $^{-1/2}$)	Method	Source
2.67 ± 0.16	2.54 ± 0.21	R -matrix fitting	[22]
2.73 ± 0.14	2.60 ± 0.19	Square-well potential	[22]
2.96 ± 0.24	2.87 ± 0.26	ERF method	[22]
2.73 ± 0.15	2.61 ± 0.19	S method	[22]
2.53 (2.38-2.60)	2.54 (2.39-2.69)	Bayesian analysis	[22]
3.10 ± 0.18		R -matrix fitting	[18] (A)
1.97 ± 0.11		R -matrix fitting	[18] (B)
2.08 ± 0.12		Δ method	[18] (B)
1.98 (1.84-2.12)		Bayesian analysis	[18] (B)
3.68	3.77	Δ method	[19]

normalization employed in early few-body formulations (see, e.g., Ref. [32]). The difference between these two definitions is purely conventional and involves only a multiplicative constant that ensures dimensional consistency.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX: BAYESIAN ANALYSIS FORMALISM

An important advantage of the Bayesian framework over the standard χ^2 minimization is that Bayesian inference yields the full probability distribution for the parameter of interest rather than a single best-fit value with an assumed Gaussian uncertainty. While the χ^2 approach provides an optimal estimate only under the assumption of linearity and normally distributed residuals, the Bayesian method incorporates both the likelihood from the experimental data and any prior physical knowledge about the parameter, producing a posterior probability density $P(C|D)$ that directly quantifies the credibility of different ANC values. This allows straightforward determination of credible intervals (e.g., 68% or 95%), which remain valid even when the likelihood is non-Gaussian. Moreover, the Bayesian approach naturally accommodates model uncertainties, parameter bounds, and correlations, which are difficult to handle consistently within the classical χ^2 method. The posterior distribution also reveals the presence of multiple maxima or long tails, features that are often hidden in a single χ^2 minimum. As a result, Bayesian inference provides a more complete and transparent statistical characterization of the ANC consistent with both the experimental data and prior physical constraints. The Bayesian analysis is based on Bayes'

theorem,

$$P(C | D) = \frac{P(D | C), P(C)}{P(D)}, \quad (\text{A1})$$

where C is the model parameter of interest. In our case, C denotes the ANC for the bound state of ^7Be . D represents the set of observations and the theoretical framework used to interpret them. Here D includes the experimental scattering phase shifts together with their reported uncertainties and the theoretical R -matrix phase shifts $\delta(E; C)$ calculated for each trial C . $P(C | D)$ is the posterior probability density function (posterior PDF). It quantifies the probability that the ANC takes a value C given the experimental data and the theoretical model D . $P(D | C)$ is the likelihood function, which measures the probability of obtaining the observed data D for a given value of C . In this analysis, $P(D | C)$ is obtained from a χ^2 comparison between the experimental and theoretical phase shifts over measured energies, incorporating the reported experimental uncertainties. $P(C)$ is the prior probability distribution for the ANC. It reflects our state of knowledge before incorporating the current dataset. In the present analysis, we adopt a flat (uniform) prior in the physically reasonable range corresponding to no preferential bias toward any particular ANC value in that interval. $P(D)$ is the evidence or marginal likelihood, acting as a normalization factor ensuring that the posterior integrates to unity. The evidence $P(D)$ in Eq. (A1) is given by

$$P(D) = \int_{C_{\min}}^{C_{\max}} dC P(D | C) P(C). \quad (\text{A2})$$

Here $C_{\min}$ and $C_{\max}$ define the lower and upper limits of the ANC interval specified by the prior $P(C)$. This integral sums the likelihood over all possible ANC values allowed by the prior, thereby ensuring that the posterior $P(C|D)$ is properly normalized. While $P(D)$ is essential for normalization, it does not influence the relative variation of the posterior with respect to C . The shape of $P(C|D)$ is determined entirely by the likelihood $P(D|C)$ and the prior $P(C)$. For each candidate ANC C we compute the predicted R -matrix phase shifts and evaluate how likely it is that these could have produced the measured phases D , assuming Gaussian experimental errors.

It is done via

$$\chi^2(C) = \sum_i \frac{[\delta_{\text{exp}}(E_i) - \delta_{th}(E_i, C)]^2}{\sigma_i^2} \quad (\text{A3})$$

and $P(D|C) \approx e^{-\chi^2(C)/2}$. σ_i is the standard deviation in point E_i . The MAP is the maximum *a posteriori*, that is, the maximum of the conditional likelihood $P(C|D)$ for a given posterior.

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