

Quantum correlation dynamics and in-medium $3 \leftrightarrow 3$ collisions of fermions

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It is the aim of this study to quantify the role of in-medium $3 \leftrightarrow 3$ collisions for systems of fermions which initially are out-of equilibrium. The formulation of the three-body dynamics is based on the equations of motion method for identical fermions—also denoted as quantum correlation dynamics—and presented in detail. The on-shell two-body collision integral is briefly reviewed and the on-shell three-two-body collision integral is derived on the basis of the same two-body interaction in leading order. The resulting equations obey particle number as well as energy-momentum conservation. For a quantification of the relative impact of three-body interactions a model study is employed for a homogeneous system in space in a finite box with periodic boundary conditions. Here, spin-isospin symmetric nuclear matter systems are addressed with momentum distributions that are given by shifted Fermi spheres (without overlap) as encountered in the initial phase of nucleus-nucleus collisions after contact. The results for the relaxation times—employing an effective two-body interaction—are compared to Boltzmann-Uehling-Uhlenbeck (BUU) transport calculations in the continuum limit. It is found that the additional three-body interactions reduce the relaxation times up to a factor of 3 at 130 MeV/nucleon. Furthermore, it is shown in BUU transport calculations that an enhanced stopping by $3 \leftrightarrow 3$ collisions shows up in the angular distribution of energetic nucleons (>60 MeV), e.g., in central $^{97}_{45}\text{Rh}$ collisions at 40 MeV/nucleon that lead to the formation of a compound nucleus. The angular distribution of the energetic nucleons changes from a slightly forward peaked angular distribution to a slightly sideways peaked angular distribution which might be controlled experimentally.

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I. INTRODUCTION

The dynamics of quantum many-body systems is of particular interest in the context of the nuclear many-body problem and in heavy-ion collisions, which probe the matter at high nuclear densities in and out-of equilibrium. While at low bombarding energies below about 10 MeV/nucleon the dynamics of the heavy-ion collision can reasonably be described by time-dependent-Hartree-Fock (TDHF) [1,2] the impact of two-body collisions becomes important with increasing bombarding energies since the Pauli-blocking effect weakens for higher energies.

In this energy range extended TDHF approaches have been introduced such as the time-dependent-density-matrix (TDDM) approach [3,4], which accounts for the additional impact of binary elastic nucleon-nucleon collisions that are not blocked by the Pauli principle [5,6]. However, this approach is limited to light nuclear systems and still low bombarding energies due to the limited number of basis states. On the other hand semiclassical transport models have been formulated in the past [7–9], that can be solved within the test particle approximation and also be applied for heavy systems at higher energy. These models are denoted by Boltzmann-Uehling-Uhlenbeck (BUU) or Vlasov-Uehling-

Uhlenbeck (VUU) type approaches [10] and have been employed by many transport groups worldwide for the analysis of heavy-ion collision data in particular for the production of hard photons, pions, or kaons [11].

While at bombarding energies below about 130 MeV/nucleon the production of pions is subthreshold, the role of inelastic nucleon-nucleon collisions (e.g., $NN \rightarrow N\Delta$, $\Delta \rightarrow N\pi$) rises with increasing bombarding energy. Such processes can easily be incorporated in semiclassical transport simulations because the excited states (nucleon resonances or produced mesons) have different quantum numbers than the nucleons and also have low or moderate phase-space densities. The role of inelastic reaction channels becomes important for the stopping and equilibration in heavy-ion collisions since in particular the elastic nucleon-nucleon collisions become forward peaked and decrease in magnitude for higher invariant energies. However, the question remains about the role of higher-order interactions in the energy domain below about 130 MeV/nucleon where the degrees of freedom are just identical fermions (nucleons) and the dominant processes are elastic scattering.

It is the aim of this study to formulate and/or derive a corresponding collision term and to investigate quantitatively the impact of additional $3 \leftrightarrow 3$ interactions while keeping the full antisymmetry of the problem. The derivation of transport theories either conveniently starts from the BBGKY¹ hierarchy

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¹Bogolyubov, Born, Green, Kirkwood, Yvon.

in case of on-shell theories [10] or from the Kadanoff-Baym equations for n -body propagators [12,13], which are the counterparts of the reduced density matrices. Such derivations have been performed in the past in the non-relativistic and relativistic case [11,14–16]² as well as for $2 \leftrightarrow n$ reactions ($n = 3, 4 \dots$) in case of high density systems at relativistic energies [18], where Pauli-blocking or Bose-enhancement factors play a minor role since almost the full ‘zoo’ of hadronic states is involved. As mentioned above a rigid formulation for $3 \leftrightarrow 3$ processes in the medium is still lacking and will be presented here. This is of particular relevance since there is practically no information from experimental data on three-body scattering events especially in the nuclear physics context. Information about three-body scattering is practically only available from chiral perturbation theory (ChPT) or lattice QCD (IQCD) calculations [19].

The outline of this work is as follows. In Sec. II the cluster decomposition of the reduced density matrices ρ_n is briefly recalled as well as the derivation of the equations of motion for two-body and three-body correlation functions on the basis of Ref. [20]. These equations will be expanded in a single-particle basis and the physical interpretation of the various terms be discussed for the two-body and three-body correlation functions. Section III is devoted to the derivation of on-shell collision terms in the two-body and three-body case (in leading order) within a natural basis $|\alpha\rangle$ that diagonalizes the one-body density matrix. Explicit results for the matrix elements of the two-body and three-body correlations will be given while shifting the numerous interaction terms to the Appendix. In Sec. IV a first model study for spin-isospin symmetric nuclear matter systems is presented with momentum distributions that are given by shifted Fermi spheres (without overlap) as encountered in the initial phase of nucleus-nucleus collisions after contact. The explicit time evolution of the quadrupole moment in momentum space will be presented for different bombarding energies and the corresponding relaxation times be extracted. For comparison BUU transport calculations are performed in a finite box in the infinite nuclear matter limit employing the same (simple) two-body interaction. This will allow to quantify the role of the three-body interactions which appear on top of the two-body scatterings.

The results of the model study are primarily academic and the question arises if a higher stopping power by additional $3 \leftrightarrow 3$ collisions could be seen in experiments. To this aim in Sec. V actual BUU transport calculations (with enhanced cross sections) are performed for finite systems at 40 MeV/nucleon—leading to a fusing system with particle emission—and the quadrupole moment in momentum space of very energetic nucleons is calculated that are emitted early in the system. A summary of the results and its implications

will be given in Sec. VI while the full list of interaction terms is presented in the Appendix.

II. EQUATIONS OF MOTION FOR CORRELATION FUNCTIONS

The dynamical description of the nuclear many-body problem in the nonrelativistic limit is based on the von Neumann equation for the A -body density matrix ρ_A [21], i.e., ($\hbar = 1$)

$$i \frac{\partial}{\partial t} \rho_A = [H_A, \rho_A], \quad (1)$$

where H_A denotes the A -body Hamiltonian in the restriction to two-body interactions $v(ij)$,

$$H_A = \sum_{i=1}^A t(i) + \sum_{i<j}^{A-1} v(ij). \quad (2)$$

The quantity $t(i)$ denotes the kinetic energy of particle i , which in case of nucleons is characterized by position $\mathbf{r}_i$ (or momentum $\mathbf{p}_i$), spin σ_i , and isospin τ_i , while $v(ij)$ denotes the two-body interaction. Equation (1) is equivalent to a coupled set of equations for the reduced density matrices of rank n ,

$$\begin{aligned} \rho_n(1 \dots n, 1' \dots n') &= \frac{1}{(A-n)!} \text{Tr}_{(n+1, \dots, A)} \\ &\times \rho_A(1 \dots A, 1' \dots n', n+1 \dots A), \end{aligned} \quad (3)$$

known as the BBGKY hierarchy [12,21], i.e.,

$$\begin{aligned} i \frac{\partial}{\partial t} \rho_n &= \left[\sum_{i=1}^n t(i), \rho_n \right] + \left[\sum_{i<j}^{n-1} v(ij), \rho_n \right] \\ &+ \sum_{i=1}^n [v(i, n+1), \rho_{n+1}] \end{aligned} \quad (4)$$

for $1 \leq n \leq A$ with $\rho_{A+1} = 0$. In these equations the time evolution of ρ_n couples to ρ_{n+1} and truncation schemes are necessary to obtain adequate solutions to the set of equations (4). Different expansion and truncation schemes have been proposed and investigated in the past [22–26] and compared to the exact solution in case of simple few-body or few-level systems [27–30]. Depending on the model systems different approximation schemes are found to be suited in each case.

In order to obtain a suitable truncation scheme for the nuclear many-body problem and in particular for the dynamics of low energy heavy-ion reactions the density matrices ρ_n are rewritten in terms of antisymmetrized products of density matrices of lower order, i.e., [20]

$$\rho_1(11') = \rho(11'), \quad (5)$$

$$\rho_2(12, 1'2') = \mathcal{A}_{1'2'} \rho(11') \rho(22') + c_2(12, 1'2') = \rho_{20}(12, 1'2') + c_2(12, 1'2') = \mathcal{A}_{12} \rho(11') \rho(22') + c_2(12, 1'2'), \quad (6)$$

²For a comparison of various transport approaches and model results at intermediate energies the reader is referred to the review [17].

$$\begin{aligned}
\rho_3(123, 1'2'3') &= \mathcal{S}_{1'}\rho(11')\rho_{20}(23, 2'3') + \mathcal{S}_{1'}\rho(11')c_2(23, 2'3') + \mathcal{S}_{2'}\rho(22')c_2(13, 1'3') \\
&\quad + \mathcal{S}_{3'}\rho(33')c_2(12, 1'2') + c_3(123, 1'2'3') \\
&= \mathcal{S}_1\rho(11')\rho_{20}(23, 2'3') + \mathcal{S}_1\rho(11')c_2(23, 2'3') + \mathcal{S}_2\rho(22')c_2(13, 1'3') \\
&\quad + \mathcal{S}_3\rho(33')c_2(12, 1'2') + c_3(123, 1'2'3')
\end{aligned} \tag{7}$$

with the two- and three-body antisymmetrization operators

$$\mathcal{A}_{ij} = 1 - P_{ij}; \quad \mathcal{A}_{i'j'} = 1 - P_{i'j'}, \tag{8}$$

$$\begin{aligned}
\mathcal{S}_{1'} &= 1 - P_{1'2'} - P_{1'3'}; \quad \mathcal{S}_1 = 1 - P_{12} - P_{13}, \\
\mathcal{S}_{2'} &= 1 - P_{1'2'} - P_{2'3'}; \quad \mathcal{S}_2 = 1 - P_{12} - P_{23}, \\
\mathcal{S}_{3'} &= 1 - P_{1'3'} - P_{2'3'}; \quad \mathcal{S}_3 = 1 - P_{13} - P_{23},
\end{aligned} \tag{9}$$

where P_{ij} denotes the exchange operator between particle i and j . The cluster decomposition for ρ_4 —that is needed for the equation of motion for ρ_3 —is given by

$$\begin{aligned}
\rho_4(1234, 1'2'3'4') &= \mathcal{A}_{1'2'}\mathcal{S}_{3'}\Lambda_{4'}\rho(11')\rho(22')\rho(33')\rho(44') + \Gamma_{1'}\rho_{20}(12, 1'2')c_2(34, 3'4') \\
&\quad + \Gamma_{2'}\rho_{20}(13, 1'3')c_2(24, 2'4') + \Gamma_{3'}\rho_{20}(14, 1'4')c_2(23, 2'3') + \Gamma_{3'}\rho_{20}(23, 2'3')c_2(14, 1'4') \\
&\quad + \Gamma_{2'}\rho_{20}(24, 2'4')c_2(13, 1'3') + \Gamma_{1'}\rho_{20}(34, 3'4')c_2(12, 1'2') + \Gamma_{1'}c_2(12, 1'2')c_2(34, 3'4') \\
&\quad + \Gamma_{2'}c_2(13, 1'3')c_2(24, 2'4') + \Gamma_{3'}c_2(14, 1'4')c_2(23, 2'3') + \Lambda_{1'}\rho(11')c_3(234, 2'3'4') \\
&\quad + \Lambda_{2'}\rho(22')c_3(134, 1'3'4') + \Lambda_{3'}\rho(33')c_3(124, 1'2'4') + \Lambda_{4'}\rho(44')c_3(123, 1'2'3') + c_4(1234, 1'2'3'4')
\end{aligned} \tag{10}$$

with the four-body antisymmetrization operators

$$\begin{aligned}
\Gamma_{1'} &= (1 - P_{1'3'} - P_{1'4'} - P_{2'3'} - P_{2'4'} + P_{1'3'}P_{2'4'}); \quad \Gamma_1 = (1 - P_{13} - P_{14} - P_{23} - P_{24} + P_{13}P_{24}), \\
\Gamma_{2'} &= (1 - P_{1'2'} - P_{1'4'} - P_{2'3'} - P_{3'4'} + P_{1'2'}P_{3'4'}); \quad \Gamma_2 = (1 - P_{12} - P_{14} - P_{23} - P_{34} + P_{12}P_{34}), \\
\Gamma_{3'} &= (1 - P_{1'2'} - P_{1'3'} - P_{2'4'} - P_{3'4'} + P_{1'3'}P_{2'4'}); \quad \Gamma_3 = (1 - P_{12} - P_{13} - P_{24} - P_{34} + P_{13}P_{24}), \\
\Lambda_{1'} &= (1 - P_{1'2'} - P_{1'3'} - P_{1'4'}); \quad \Lambda_1 = (1 - P_{12} - P_{13} - P_{14}); \\
\Lambda_{2'} &= (1 - P_{2'1'} - P_{2'3'} - P_{2'4'}); \quad \Lambda_2 = (1 - P_{21} - P_{23} - P_{24}); \\
\Lambda_{3'} &= (1 - P_{3'1'} - P_{3'2'} - P_{3'4'}); \quad \Lambda_3 = (1 - P_{31} - P_{32} - P_{34}); \\
\Lambda_{4'} &= (1 - P_{4'1'} - P_{4'2'} - P_{4'3'}); \quad \Lambda_4 = (1 - P_{41} - P_{42} - P_{43}).
\end{aligned} \tag{11}$$

One obtains (after some lengthy algebra)

$$\begin{aligned}
i\frac{\partial}{\partial t} \rho(11'; t) &= [t(1) - t(1')]\rho(11'; t) + \text{Tr}_{(2=2')}[v(12)\mathcal{A}_{12} - v(1'2')\mathcal{A}_{1'2'}]\rho(11'; t)\rho(22'; t) \\
&\quad + \text{Tr}_{(2=2')}[v(12) - v(1'2')]\rho(12, 1'2'; t)
\end{aligned} \tag{12}$$

for the time-evolution of the 1-body density $\rho(11'; t)$ and

$$\begin{aligned}
i\frac{\partial}{\partial t} c_2(12, 1'2'; t) &= [t(1) + t(2) - t(1') - t(2')]\rho(12, 1'2'; t) + \text{Tr}_{(3=3')}[v(13)\mathcal{A}_{13} + v(23)\mathcal{A}_{23} - v(1'3')\mathcal{A}_{1'3'} \\
&\quad - v(2'3')\mathcal{A}_{2'3'}]\rho(33'; t)c_2(12, 1'2'; t) + [v(12) - v(1'2')]\rho_{20}(12, 1'2') \\
&\quad - \text{Tr}_{(3=3')}[v(13)\rho(23'; t)\rho_{20}(13, 1'2'; t) - v(1'3')\rho(32'; t)\rho_{20}(12, 1'3'; t) \\
&\quad + v(23)\rho(13'; t)\rho_{20}(32, 1'2'; t) - v(2'3')\rho(31'; t)\rho_{20}(12, 3'2'; t)]
\end{aligned} \tag{13}$$

$$\begin{aligned}
&+ [v(12) - v(1'2')]\rho(12, 1'2'; t) - \text{Tr}_{(3=3')}[v(13)\rho(23'; t)c_2(13, 1'2'; t) - v(1'3')\rho(32'; t)c_2(12, 1'3'; t) \\
&+ v(23)\rho(13'; t)c_2(32, 1'2'; t) - v(2'3')\rho(31'; t)c_2(12, 3'2'; t)]
\end{aligned} \tag{14}$$

$$\begin{aligned}
&+ \text{Tr}_{(3=3')}([v(13)\mathcal{A}_{13}\mathcal{A}_{1'2'} - v(1'3')\mathcal{A}_{1'3'}\mathcal{A}_{12}]\rho(11'; t)c_2(23, 2'3'; t) \\
&+ [v(23)\mathcal{A}_{23}\mathcal{A}_{1'2'} - v(2'3')\mathcal{A}_{2'3'}\mathcal{A}_{12}]\rho(22'; t)c_2(13, 1'3'; t))
\end{aligned} \tag{15}$$

$$+ \text{Tr}_{(3=3')}[v(13) + v(23) - v(1'3') - v(2'3')]c_3(123, 1'2'3'; t) \tag{16}$$

for the time-evolution of the 2-body correlation $c_2(12, 1'2'; t)$.

The equation of motion for the three-body correlation c_3 amounts to

$$i \frac{\partial}{\partial t} c_3(123, 1'2'3'; t) = [t(1) + t(2) + t(3) - t(1') - t(2') - t(3')]c_3(123, 1'2'3'; t) \\ + [v(12) + v(13) + v(23) - v(1'2') - v(1'3') - v(2'3')]c_3(123, 1'2'3'; t) \\ + [(v(12) + v(13))\mathcal{S}_{1'} - (v(1'2') + v(1'3'))\mathcal{S}_1]\rho(11'; t)c_2(23, 2'3'; t) \\ + [(v(12) + v(23))\mathcal{S}_{2'} - (v(1'2') + v(2'3'))\mathcal{S}_2]\rho(22'; t)c_2(13, 1'3'; t) \\ + [(v(13) + v(23))\mathcal{S}_{3'} - (v(1'3') + v(2'3'))\mathcal{S}_3]\rho(33'; t)c_2(12, 1'2'; t) \quad (17)$$

$$- \text{Tr}_{(4=4')} \{ [v(14)\mathcal{S}_{1'} - v(1'4')\mathcal{S}_1](P_{24} + P_{34} + P_{2'4'} + P_{3'4'})\rho(11'; t)\rho(44'; t)c_2(23, 2'3'; t) \\ + [v(24)\mathcal{S}_{2'} - v(2'4')\mathcal{S}_2](P_{1'4'} + P_{3'4'} + P_{14} + P_{34})\rho(22'; t)\rho(44'; t)c_2(13, 1'3'; t) \\ + [v(34)\mathcal{S}_{3'} - v(3'4')\mathcal{S}_3](P_{14} + P_{24} + P_{1'4'} + P_{2'4'})\rho(33'; t)\rho(44'; t)c_2(12, 1'2'; t) \\ + [(v(24) + v(34))\mathcal{S}_{1'} - (v(2'4') + v(3'4'))\mathcal{S}_1]\rho(41'; t)\rho(14'; t)c_2(23, 2'3'; t) \\ + [(v(14) + v(34))\mathcal{S}_{2'} - (v(1'4') + v(3'4'))\mathcal{S}_2]\rho(42'; t)\rho(24'; t)c_2(13, 1'3'; t) \\ + [(v(14) + v(24))\mathcal{S}_{3'} - (v(1'4') + v(2'4'))\mathcal{S}_3]\rho(43'; t)\rho(34'; t)c_2(12, 1'2'; t) \} \quad (18)$$

$$+ \text{Tr}_{(4=4')} \{ [v(14) + v(24) + v(34) - v(1'4') - v(2'4') - v(3'4')] \{ \mathcal{S}_{3'}c_2(12, 3'4'; t)c_2(34, 1'2'; t) \\ + \mathcal{S}_{2'}c_2(13, 2'4'; t)c_2(24, 1'3'; t) + \mathcal{S}_{1'}c_2(14, 2'3'; t)c_2(23, 1'4'; t) \} \\ + [(v(24) + v(34))\mathcal{S}_{1'} - (v(2'4') + v(3'4'))\mathcal{S}_1]c_2(14, 1'4'; t)c_2(23, 2'3'; t) \\ + [(v(14) + v(34))\mathcal{S}_{2'} - (v(1'4') + v(3'4'))\mathcal{S}_2]c_2(13, 1'3'; t)c_2(24, 2'4'; t) \\ + [(v(14) + v(24))\mathcal{S}_{3'} - (v(1'4') + v(2'4'))\mathcal{S}_3]c_2(12, 1'2'; t)c_2(34, 3'4'; t) \} \quad (19)$$

$$+ \text{Tr}_{(4=4')} \{ [v(14) + v(24) + v(34) - v(1'4') - v(2'4') - v(3'4')] \\ \times (1 - P_{14} - P_{24} - P_{34} - P_{1'4'} - P_{2'4'} - P_{3'4'}) \rho(44'; t)c_3(123, 1'2'3'; t) \\ + [v(14)\mathcal{S}_{1'} - v(1'4')\mathcal{S}_1]\rho(11'; t)c_3(234, 2'3'4'; t) + [v(24)\mathcal{S}_{2'} - v(2'4')\mathcal{S}_2] \\ \times \rho(22'; t)c_3(134, 1'3'4'; t) + [v(34)\mathcal{S}_{3'} - v(3'4')\mathcal{S}_3]\rho(33'; t)c_3(124, 1'2'4'; t) \} \quad (20)$$

$$+ \text{Tr}_{(4=4')} [v(14) + v(24) + v(34) - v(1'4') - v(2'4') - v(3'4')]c_4(1234, 1'2'3'4'; t) \quad (21)$$

and no longer depends on interaction terms of order $v\rho\rho\rho$ and $\text{Tr}_4(v\rho\rho\rho\rho)$ which drop out.

Following Refs. [31–34] one introduces the bare mean-field Hamiltonian by

$$h(i) = t(i) + U(i) = t(i) + \text{Tr}_{(n=n')} v(\text{in})\mathcal{A}_{\text{in}}\rho(nn'; t); \\ h(i') = t(i') + U(i') = t(i') + \text{Tr}_{(n=n')} v(i'n')\mathcal{A}_{i'n'}\rho(nn'; t), \quad (22)$$

the particle-particle and hole-hole (pp-hh) blocking operator by

$$Q_{ij}^{\bar{}} = 1 - \text{Tr}_{(n=n')}(P_{\text{in}} + P_{j\text{n}})\rho(nn'; t); \quad Q_{i'j'}^{\bar{}} = 1 - \text{Tr}_{(n=n')}(P_{i'n'} + P_{j'n'})\rho(nn'; t), \quad (23)$$

and the effective (pp-hh) in-medium interaction $V^=(ij)$ via

$$V^=(ij) = Q_{ij}^{\bar{}}v(ij); \quad V^=(i'j') = Q_{i'j'}^{\bar{}}v(i'j') \quad (24)$$

which act on all terms to the right.

The equations of motion for ρ , c_2 , and c_3 then reduce to

$$i \frac{\partial}{\partial t} \rho(11'; t) = [h(1) - h(1')]\rho(11'; t) + \text{Tr}_{(2=2')} [v(12) - v(1'2')]c_2(12, 1'2'; t), \quad (25)$$

$$i \frac{\partial}{\partial t} c_2(12, 1'2'; t) = \left[\sum_{i=1}^2 h(i) - \sum_{i'=1'}^{2'} h(i') \right] c_2(12, 1'2'; t) + [V^=(12) - V^=(1'2')]\rho_{20}(12, 1'2'; t) \quad (26)$$

$$+ [V^=(12) - V^=(1'2')]\rho_{20}(12, 1'2'; t) \quad (27)$$

$$+ \text{Tr}_{(3=3')} \{ [v(13)\mathcal{A}_{13}\mathcal{A}_{1'2'} - v(1'3')\mathcal{A}_{1'3'}\mathcal{A}_{12}]\rho(11'; t)c_2(23, 2'3'; t) \\ + [v(23)\mathcal{A}_{23}\mathcal{A}_{1'2'} - v(2'3')\mathcal{A}_{2'3'}\mathcal{A}_{12}]\rho(22'; t)c_2(13, 1'3'; t) \} \quad (28)$$

$$+ \text{Tr}_{(3=3')} [v(13) + v(23) - v(1'3') - v(2'3')]c_3(123, 1'2'3'; t), \quad (29)$$

and

$$i \frac{\partial}{\partial t} c_3(123, 1'2'3'; t) = \left[\sum_{i=1}^3 h(i) - \sum_{i'=1'}^{3'} h(i') \right] c_3(123, 1'2'3'; t) \\ + [V^=(12)\mathcal{S}_{1'} + V^=(13)\mathcal{S}_{1'} - V^=(1'2')\mathcal{S}_1 - V^=(1'3')\mathcal{S}_1] \rho(11'; t) c_2(23, 2'3'; t) \\ + [V^=(12)\mathcal{S}_{2'} + V^=(23)\mathcal{S}_{2'} - V^=(1'2')\mathcal{S}_2 - V^=(2'3')\mathcal{S}_2] \rho(22'; t) c_2(13, 1'3'; t) \\ + [V^=(13)\mathcal{S}_{3'} + V^=(23)\mathcal{S}_{3'} - V^=(1'3')\mathcal{S}_3 - V^=(2'3')\mathcal{S}_3] \rho(33'; t) c_2(12, 1'2'; t) \quad (30)$$

$$- \text{Tr}_{(4=4')} \{ [v(14)\mathcal{S}_{1'}(P_{2'4'} + P_{3'4'}) - v(1'4')\mathcal{S}_1(P_{24} + P_{34})] \rho(44'; t) \rho(11'; t) c_2(23, 2'3'; t) \\ + [v(24)\mathcal{S}_{2'}(P_{1'4'} + P_{3'4'}) - v(2'4')\mathcal{S}_2(P_{14} + P_{34})] \rho(44'; t) \rho(22'; t) c_2(13, 1'3'; t) \\ + [v(34)\mathcal{S}_{3'}(P_{1'4'} + P_{2'4'}) - v(3'4')\mathcal{S}_3(P_{14} + P_{24})] \rho(44'; t) \rho(33'; t) c_2(12, 1'2'; t) \} \quad (31)$$

$$+ \text{Tr}_{(4=4')} \{ [v(14) + v(24) + v(34) - v(1'4') - v(2'4') - v(3'4')] \{ \mathcal{S}_{1'} c_2(14, 2'3'; t) c_2(23, 1'4'; t) \\ + \mathcal{S}_{2'} c_2(13, 2'4'; t) c_2(24, 1'3'; t) + \mathcal{S}_{3'} c_2(12, 3'4'; t) c_2(34, 1'2'; t) \} \\ + [(v(24) + v(34))\mathcal{S}_{1'} - (v(2'4') + v(3'4'))\mathcal{S}_1] c_2(14, 1'4'; t) c_2(23, 2'3'; t) \\ + [(v(14) + v(34))\mathcal{S}_{2'} - (v(1'4') + v(3'4'))\mathcal{S}_2] c_2(13, 1'3'; t) c_2(24, 2'4'; t) \\ + [(v(14) + v(24))\mathcal{S}_{3'} - (v(1'4') + v(2'4'))\mathcal{S}_3] c_2(12, 1'2'; t) c_2(34, 3'4'; t) \} \quad (32)$$

$$+ [V^=(12) + V^=(13) + V^=(23) - V^=(1'2') - V^=(1'3') - V^=(2'3')] c_3(123, 1'2'3'; t) \quad (33)$$

$$+ \text{Tr}_{(n=n')} \{ [v(1n)\mathcal{A}_{1n}\mathcal{S}_{1'} - v(1'n')\mathcal{A}_{1'n'}\mathcal{S}_1] \rho(11'; t) c_3(n23, n'2'3'; t) \\ + [v(2n)\mathcal{A}_{2n}\mathcal{S}_{2'} - v(2'n')\mathcal{A}_{2'n'}\mathcal{S}_2] \rho(22'; t) c_3(1n3, 1'n'3'; t) \\ + [v(3n)\mathcal{A}_{3n}\mathcal{S}_{3'} - v(3'n')\mathcal{A}_{3'n'}\mathcal{S}_3] \rho(33'; t) c_3(12n, 1'2'n'; t) \} \quad (34)$$

$$+ \text{Tr}_{(4=4')} [v(14) + v(24) + v(34) - v(1'4') - v(2'4') - v(3'4')] c_4(1234, 1'2'3'4'; t), \quad (35)$$

where a couple of interaction terms have been rearranged.

These equations suggest to define the additional operators

$$V_2^\perp(i, i') = \text{Tr}_{(n=n')} [v(\text{in})\mathcal{A}_{\text{in}}\mathcal{A}_{1'2'} - v(i'n')\mathcal{A}_{i'n'}\mathcal{A}_{12}] \rho(ii'; t) P_{\text{in}} P_{i'n'} \quad (36)$$

for the two-body correlation case and

$$V_3^\perp(i, i') = \text{Tr}_{(n=n')} [v(\text{in})\mathcal{A}_{\text{in}}\mathcal{S}_{i'} - v(i'n')\mathcal{A}_{i'n'}\mathcal{S}_i] \rho(ii'; t) P_{\text{in}} P_{i'n'}, \\ V_4^\perp(i, i') = \text{Tr}_{(n=n')} [v(\text{in})\mathcal{A}_{\text{in}}\Lambda_{i'} - v(i'n')\mathcal{A}_{i'n'}\Lambda_i] \rho(ii'; t) P_{\text{in}} P_{i'n'} \quad (37)$$

for the three- and four-body correlation case, where all permutation and antisymmetrization operators act to the right. The operators (36) and (37) describe interactions in the various particle-hole (p-h) channels [33,34] (see below).

Introducing, furthermore, adjoint operators via the relations

$$(O(ij)c_n)^\dagger = c_n O(ij)^\dagger = O(i'j')c_n \quad (38)$$

and a commutator via

$$[O, c_n] = O c_n - c_n O^\dagger, \quad (39)$$

the equations of motion can be written in more compact form, i.e.,

$$i \frac{\partial}{\partial t} \rho - [h(1), \rho] = \text{Tr}_{(2=2')} [v(12), c_2]; \quad (40)$$

$$i \frac{\partial}{\partial t} c_2 - \left[\sum_{i=1}^2 h(i), c_2 \right] - [V^=(12), c_2] - \sum_{i=1}^2 V_2^\perp(i, i') c_2 = [V^=(12), \rho_{20}] + \text{Tr}_{(3=3')} \left[\sum_{i=1}^3 v(i3), c_3 \right] \quad (41)$$

and

$$i \frac{\partial}{\partial t} c_3 - \left[\sum_{i=1}^3 h(i), c_3 \right] - \left[\sum_{i<j}^2 V^=(ij), c_3 \right] - \sum_{i=1}^3 V_3^\perp(i, i') c_3 \\ = [\mathcal{S}_3 (V^=(13) + V^=(23)) \mathcal{S}_{3'}, \rho(33'; t) c_2(12, 1'2'; t)] \quad (42)$$

$$- \text{Tr}_{(4=4')} [\mathcal{S}_1 v(14) \mathcal{S}_{1'} \mathcal{A}_{2'3'} P_{24}, \rho(11'; t) \rho(22'; t) c_2(34, 3'4'; t)] \quad (43)$$

$$+ \text{Tr}_{(4=4')} \{ [\mathcal{S}_3(v(14) + v(24) + v(34))\mathcal{S}_{3'}, c_2(12, 3'4'; t)c_2(34, 1'2'; t)] \\ + [\mathcal{S}_3(v(14) + v(24))\mathcal{S}_{3'}, c_2(12, 1'2'; t)c_2(34, 3'4'; t)] \} \quad (44)$$

$$+ \text{Tr}_{(4=4')} \left[\sum_{i=1}^3 v(i4), c_4(1234, 1'2'3'4'; t) \right]. \quad (45)$$

The set of equations (40)–(45) describe the nonperturbative time-evolution of arbitrary Fermion systems interacting via two-body forces and avoid any double counting of interactions.

Various limiting cases of these equations are known [31–34]. When neglecting two-body correlations c_2 in Eq. (40) one recovers the time-dependent Hartree-Fock theory (TDHF) for weakly interacting systems, while $-i\text{Tr}_{(2=2')} [v(12), c_2]$ gives a collision term. In Eq. (41) the terms $[V^=(12), \rho_{20}]$ describe residual nucleon-nucleon scattering in the Born approximation (BA), the additional terms $[V^=(12), c_2]$ lead to time-dependent G -matrix theory (TDGMT), while the contribution with $V_2^\perp(i, i')c_2$ lead to additional 2p-2h interactions that are important, e.g., for the damping of giant resonances [35]. The physical contents of these equations becomes more transparent within an arbitrary single-particle basis although the compact form of Eqs. (40) to (47) gets partly lost.

A. Expansion in a single-particle basis

Following [31–34] it is of advantage to expand ρ , c_2 and c_3 in an arbitrary complete single-particle basis ψ_α , i.e.,

$$\rho(11'; t) = \sum_{\alpha\alpha'} \rho_{\alpha\alpha'}(t) \psi_\alpha(1) \psi_{\alpha'}^*(1'), \quad (46)$$

$$c_2(12, 1'2'; t) = \sum_{\alpha\beta\alpha'\beta'} C_{\alpha\beta\alpha'\beta'}(t) \psi_\alpha(1) \psi_\beta(2) \psi_{\alpha'}^*(1') \psi_{\beta'}^*(2'), \quad (47)$$

$$c_3(123, 1'2'3'; t) = \sum_{\alpha\beta\gamma\alpha'\beta'\gamma'} C_{\alpha\beta\gamma\alpha'\beta'\gamma'}^3(t) \psi_\alpha(1) \psi_\beta(2) \psi_\gamma(3) \psi_{\alpha'}^*(1') \psi_{\beta'}^*(2') \psi_{\gamma'}^*(3'). \quad (48)$$

A straightforward (but lengthy) calculation then gives

$$i \frac{\partial}{\partial t} \rho_{\alpha\alpha'} = \sum_\lambda [h_{\alpha\lambda} \rho_{\lambda\alpha'} - \rho_{\alpha\lambda} h_{\lambda\alpha'}] + \sum_\beta \sum_{\lambda\gamma} \{ \langle \alpha\beta | v | \lambda\gamma \rangle C_{\lambda\gamma\alpha'\beta} - C_{\alpha\beta\lambda\gamma} \langle \lambda\gamma | v | \alpha'\beta \rangle \} \quad (49)$$

for the one-body density matrix while the time evolution of the two-body correlation matrix is given by

$$i \frac{\partial}{\partial t} C_{\alpha\beta\alpha'\beta'} = \sum_\lambda \{ h_{\alpha\lambda} C_{\lambda\beta\alpha'\beta'} + h_{\beta\lambda} C_{\alpha\lambda\alpha'\beta'} - C_{\alpha\beta\lambda\beta'} h_{\lambda\alpha'} - C_{\alpha\beta\alpha'\lambda} h_{\lambda\beta'} \} \\ + \sum_{\lambda\lambda'\gamma\gamma'} \{ Q_{\alpha\beta\lambda'\gamma'}^\equiv \langle \lambda'\gamma' | v | \lambda\gamma \rangle (\rho_{20})_{\lambda\gamma\alpha'\beta'} - (\rho_{20})_{\alpha\beta\lambda'\gamma'} \langle \lambda'\gamma' | v | \lambda\gamma \rangle Q_{\lambda\gamma\alpha'\beta'}^\equiv \} \quad (50)$$

$$+ \sum_{\lambda\lambda'\gamma\gamma'} Q_{\alpha\beta\lambda'\gamma'}^\equiv \langle \lambda'\gamma' | v | \lambda\gamma \rangle C_{\lambda\gamma\alpha'\beta'} - C_{\alpha\beta\lambda'\gamma'} \langle \lambda'\gamma' | v | \lambda\gamma \rangle Q_{\lambda\gamma\alpha'\beta'}^\equiv \quad (51)$$

$$+ \mathcal{A}_{\alpha\beta} \sum_{\lambda\lambda'\gamma\gamma'} \langle \lambda'\gamma' | v | \lambda\gamma \rangle \mathcal{A} [Q_{\alpha\lambda\alpha'\lambda'}^\perp C_{\gamma\beta\gamma'\beta'} + Q_{\beta\lambda\beta'\lambda'}^\perp C_{\alpha\gamma\alpha'\gamma'}] \quad (52)$$

$$+ \sum_{\lambda\lambda'\gamma\gamma'} \langle \lambda'\gamma' | v | \lambda\gamma \rangle \{ \delta_{\alpha\lambda'} C_{\lambda\beta\gamma,\alpha'\beta'\gamma'}^3 + \delta_{\beta\lambda'} C_{\alpha\lambda\gamma,\alpha'\beta'\gamma'}^3 - \delta_{\lambda\alpha'} C_{\alpha\beta\gamma,\lambda'\beta'\gamma'}^3 - \delta_{\lambda\beta'} C_{\alpha\beta\gamma,\alpha'\lambda'\gamma'}^3 \}, \quad (53)$$

where the explicit time dependence of the expansion coefficients has been suppressed. Furthermore, the particle-hole (p-h) blocking operator

$$Q_{\alpha\lambda\alpha'\lambda'}^\perp = \delta_{\alpha\lambda'} \rho_{\lambda\alpha'} - \rho_{\alpha\lambda'} \delta_{\alpha'\lambda} \equiv (1 \cdot \rho - \rho \cdot 1)_{\alpha\lambda\alpha'\lambda'} = -Q^{\perp\dagger} \quad (54)$$

has been introduced, the (pp-hh) blocking operator [cf. Eq. (23)]

$$Q_{\alpha\beta\lambda'\gamma'}^\equiv = \delta_{\alpha\lambda'} \delta_{\beta\gamma'} - \delta_{\alpha\lambda'} \rho_{\beta\gamma'} - \rho_{\alpha\lambda'} \delta_{\beta\gamma'} \equiv 1 \cdot 1 - 1 \cdot \rho - \rho \cdot 1 = Q^{\equiv\dagger}, \quad (55)$$

and the mean-field Hamiltonian [cf. Eq. (22)]

$$h_{\alpha\lambda} = \langle \alpha | t | \lambda \rangle + \sum_{\gamma\gamma'} \langle \alpha\gamma' | v | \lambda\gamma \rangle \mathcal{A} \rho_{\gamma\gamma'}, \quad (56)$$

while the uncorrelated two-body density matrix is given by [cf. Eq. (6)]

$$(\rho_{20})_{\alpha\beta\alpha'\beta'} = \rho_{\alpha\alpha'}\rho_{\beta\beta'} - \rho_{\alpha\beta'}\rho_{\beta\alpha'}. \quad (57)$$

The index $\mathcal{A}$ denotes antisymmetrization of the two-body interaction v , i.e.,

$$\begin{aligned} \langle \alpha' \beta' | v | \alpha \beta \rangle_{\mathcal{A}} &= \langle \alpha' \beta' | v | \alpha \beta \rangle - \langle \alpha' \beta' | v | \beta \alpha \rangle, \\ \langle \alpha' \beta' | v | \alpha \beta \rangle &= \int d1 \int d2 \psi_{\alpha'}^*(1) \psi_{\beta'}^*(2) v(12) \psi_{\alpha}(1) \psi_{\beta}(2). \end{aligned}$$

The time evolution for the expansion coefficient $C_{\alpha\beta\gamma,\alpha'\beta'\gamma'}^3$, furthermore, reads

$$\begin{aligned} i \frac{\partial}{\partial t} C_{\alpha\beta\gamma,\alpha'\beta'\gamma'}^3 &- \sum_{\lambda} \{ h_{\alpha\lambda} C_{\lambda\beta\gamma,\alpha'\beta'\gamma'}^3 + h_{\beta\lambda} C_{\alpha\lambda\gamma,\alpha'\beta'\gamma'}^3 + h_{\gamma\lambda} C_{\alpha\beta\lambda,\alpha'\beta'\gamma'}^3 - C_{\alpha\beta\gamma,\lambda\beta'\gamma'}^3 h_{\lambda\alpha'} - C_{\alpha\beta\gamma,\alpha'\lambda\gamma'}^3 h_{\lambda\beta'} - C_{\alpha\beta\gamma,\alpha'\beta'\lambda}^3 h_{\lambda\gamma'} \} \\ &= \sum_{\lambda\eta} \{ \langle \alpha \beta | Q^{\bar{}} v | \lambda \eta \rangle_{\mathcal{A}} \mathcal{S}_{\alpha'} \rho_{\lambda\alpha'} C_{\eta\gamma\beta'\gamma'} - \mathcal{S}_{\alpha} C_{\beta\gamma\eta\gamma'} \rho_{\alpha\lambda} \langle \lambda \eta | v Q^{\bar{}} | \alpha' \beta' \rangle_{\mathcal{A}} \\ &\quad + \langle \beta \gamma | Q^{\bar{}} v | \lambda \eta \rangle_{\mathcal{A}} \mathcal{S}_{\beta'} \rho_{\lambda\beta'} C_{\alpha\eta\alpha'\gamma'} - \mathcal{S}_{\beta} C_{\alpha\gamma\alpha'\eta} \rho_{\beta\lambda} \langle \lambda \eta | v Q^{\bar{}} | \beta' \gamma' \rangle_{\mathcal{A}} \\ &\quad + \langle \gamma \alpha | Q^{\bar{}} v | \lambda \eta \rangle_{\mathcal{A}} \mathcal{S}_{\gamma'} \rho_{\lambda\gamma'} C_{\eta\beta\alpha'\beta'} - \mathcal{S}_{\gamma} C_{\alpha\beta\eta\beta'} \rho_{\gamma\lambda} \langle \lambda \eta | v Q^{\bar{}} | \gamma' \alpha' \rangle_{\mathcal{A}} \} \end{aligned} \quad (58)$$

$$\begin{aligned} &+ \sum_{\lambda\lambda'\eta\eta'} \{ \mathcal{S}_{\gamma} \rho_{\alpha\lambda'} \rho_{\beta\eta'} \langle \lambda' \eta' | v | \alpha' \eta \rangle_{\mathcal{A}} \delta_{\lambda\alpha'} C_{\eta\gamma\beta'\gamma'} - \mathcal{S}_{\gamma'} C_{\beta\gamma\eta'\gamma'} \delta_{\alpha\lambda'} \langle \alpha \eta' | v | \lambda \eta \rangle_{\mathcal{A}} \rho_{\lambda\alpha'} \rho_{\eta\beta'} \\ &\quad + \mathcal{S}_{\alpha} \rho_{\beta\lambda'} \rho_{\gamma\eta'} \langle \lambda' \eta' | v | \beta' \eta \rangle_{\mathcal{A}} \delta_{\lambda\beta'} C_{\alpha\eta\alpha'\gamma'} - \mathcal{S}_{\alpha'} C_{\alpha\gamma\alpha'\eta'} \delta_{\beta\lambda'} \langle \beta \eta' | v | \lambda \eta \rangle_{\mathcal{A}} \rho_{\lambda\beta'} \rho_{\eta\gamma'} \\ &\quad + \mathcal{S}_{\beta} \rho_{\gamma\lambda'} \rho_{\alpha\eta'} \langle \lambda' \eta' | v | \gamma' \eta \rangle_{\mathcal{A}} \delta_{\lambda\gamma'} C_{\eta\beta\alpha'\beta'} - \mathcal{S}_{\beta'} C_{\alpha\beta\eta'\beta'} \delta_{\gamma\lambda'} \langle \gamma \eta' | v | \lambda \eta \rangle_{\mathcal{A}} \rho_{\lambda\gamma'} \rho_{\eta\alpha'} \} \end{aligned} \quad (59)$$

$$\begin{aligned} &+ \sum_{\lambda\eta} \{ \langle \alpha \beta | Q^{\bar{}} v | \lambda \eta \rangle C_{\lambda\eta\gamma,\alpha'\beta'\gamma'}^3 - C_{\alpha\beta\gamma,\lambda\eta\gamma'}^3 \langle \lambda \eta | v Q^{\bar{}} | \alpha' \beta' \rangle + \langle \alpha \gamma | Q^{\bar{}} v | \lambda \eta \rangle C_{\lambda\beta\eta,\alpha'\beta'\gamma'}^3 \\ &\quad - C_{\alpha\beta\gamma,\lambda\beta'\eta}^3 \langle \lambda \eta | v Q^{\bar{}} | \alpha' \gamma' \rangle + \langle \beta \gamma | Q^{\bar{}} v | \lambda \eta \rangle C_{\alpha\lambda\eta,\alpha'\beta'\gamma'}^3 - C_{\alpha\beta\gamma,\alpha'\lambda\eta}^3 \langle \lambda \eta | v Q^{\bar{}} | \beta' \gamma' \rangle \} \end{aligned} \quad (60)$$

$$+ \sum_{\lambda\lambda'\eta\eta'} \langle \lambda' \eta' | v | \lambda \eta \rangle_{\mathcal{A}} \{ \mathcal{S}_{\alpha} Q_{\alpha\lambda\alpha'\lambda'}^{\perp} C_{\eta\beta\gamma,\eta'\beta'\gamma'}^3 + \mathcal{S}_{\beta} Q_{\beta\lambda\beta'\lambda'}^{\perp} C_{\alpha\eta\gamma,\alpha'\eta'\gamma'}^3 + \mathcal{S}_{\gamma} Q_{\gamma\lambda\gamma'\lambda'}^{\perp} C_{\alpha\beta\eta,\alpha'\beta'\eta'}^3 \} \quad (61)$$

$$\begin{aligned} &+ \sum_{\eta\delta\lambda} \{ \mathcal{S}_{\alpha'} C_{\beta\gamma\alpha'\eta} \langle \alpha \eta | v | \delta \lambda \rangle C_{\delta\lambda\beta'\gamma'} - \mathcal{S}_{\alpha} C_{\beta\gamma\delta\lambda} \langle \delta \lambda | v | \alpha' \eta \rangle C_{\alpha\eta\beta'\gamma'} + \mathcal{S}_{\beta'} C_{\alpha\gamma\beta'\eta} \langle \beta \eta | v | \delta \lambda \rangle C_{\delta\lambda\alpha'\gamma'} \\ &\quad - \mathcal{S}_{\beta} C_{\alpha\gamma\delta\lambda} \langle \delta \lambda | v | \beta' \eta \rangle C_{\beta\eta\alpha'\gamma'} + \mathcal{S}_{\gamma'} C_{\alpha\beta\gamma'\eta} \langle \gamma \eta | v | \delta \lambda \rangle C_{\delta\lambda\alpha'\beta'} - \mathcal{S}_{\gamma} C_{\alpha\beta\delta\lambda} \langle \delta \lambda | v | \gamma' \eta \rangle C_{\gamma\eta\alpha'\beta'} \} \end{aligned} \quad (62)$$

$$\begin{aligned} &+ \sum_{\lambda\lambda'\eta\eta'} \{ \mathcal{S}_{\alpha'} (1 + P_{\gamma\beta} P_{\gamma'\beta'}) C_{\lambda\gamma\beta'\eta'} \langle \alpha \eta' | v | \lambda \eta \rangle C_{\beta\eta\alpha'\gamma'} \delta_{\alpha\lambda'} - \mathcal{S}_{\alpha} (1 + P_{\gamma\beta} P_{\gamma'\beta'}) C_{\alpha\gamma\beta'\eta'} \langle \lambda' \eta' | v | \alpha' \eta \rangle C_{\beta\eta\lambda'\gamma'} \delta_{\lambda\alpha'} \\ &\quad + \mathcal{S}_{\beta'} (1 + P_{\alpha\gamma} P_{\alpha'\gamma'}) C_{\lambda\gamma\alpha'\eta'} \langle \beta \eta' | v | \lambda \eta \rangle C_{\alpha\eta\beta'\gamma'} \delta_{\beta\lambda'} - \mathcal{S}_{\beta} (1 + P_{\alpha\gamma} P_{\alpha'\gamma'}) C_{\beta\gamma\alpha'\eta'} \langle \lambda' \eta' | v | \beta' \eta \rangle C_{\alpha\eta\lambda'\gamma'} \delta_{\lambda\beta'} \\ &\quad + \mathcal{S}_{\gamma'} (1 + P_{\alpha\beta} P_{\alpha'\beta'}) C_{\beta\lambda\gamma'\eta'} \langle \gamma \eta' | v | \lambda \eta \rangle C_{\alpha\eta\gamma'\beta'} \delta_{\gamma\lambda'} - \mathcal{S}_{\gamma} (1 + P_{\alpha\beta} P_{\alpha'\beta'}) C_{\gamma\beta\lambda'\eta'} \langle \lambda' \eta' | v | \gamma' \eta \rangle C_{\alpha\eta\lambda'\beta'} \delta_{\lambda\gamma'} \} \end{aligned} \quad (63)$$

$$\begin{aligned} &+ \sum_{\lambda\lambda'\eta\eta'} \langle \lambda' \eta' | v | \lambda \eta \rangle \{ [\mathcal{S}_{\alpha'} \mathcal{A}_{\beta\gamma} C_{\beta\lambda\beta'\gamma'} \delta_{\gamma\lambda'} - \mathcal{S}_{\alpha} \mathcal{A}_{\beta'\gamma'} C_{\beta\gamma\beta'\lambda'} \delta_{\lambda\gamma'}] C_{\alpha\eta\alpha'\eta'} \\ &\quad + [\mathcal{S}_{\beta'} \mathcal{A}_{\alpha\gamma} C_{\lambda\gamma\alpha'\gamma'} \delta_{\alpha\lambda'} - \mathcal{S}_{\beta} \mathcal{A}_{\alpha'\gamma'} C_{\alpha\gamma\lambda'\gamma'} \delta_{\lambda\alpha'}] C_{\beta\eta\beta'\eta'} + [\mathcal{S}_{\gamma'} \mathcal{A}_{\alpha\beta} C_{\alpha\lambda\alpha'\beta'} \delta_{\beta\lambda'} - \mathcal{S}_{\gamma} \mathcal{A}_{\alpha'\beta'} C_{\alpha\beta\alpha'\lambda'} \delta_{\lambda\beta'}] C_{\gamma\eta\gamma'\eta'} \}, \end{aligned} \quad (64)$$

where the coupling to explicit four-body correlations c_4 has been neglected such that the set of equations is closed. The three-body antisymmetrization operators $\mathcal{S}_{\alpha}$ are defined as in Eq. (9) by replacing $(1 \rightarrow \alpha)$, $(2 \rightarrow \beta)$, $(3 \rightarrow \gamma)$, $(1' \rightarrow \alpha')$, $(2' \rightarrow \beta')$ and $(3' \rightarrow \gamma')$.

The physical contents of Eqs. (49) to (64) now becomes transparent when adopting a ‘natural’ basis ψ_{α} that diagonalizes the one-body density matrix, i.e., $\rho_{\alpha\alpha'} = n_{\alpha} \delta_{\alpha\alpha'}$, where n_{α} denotes the occupation number of the state ψ_{α} . The term $h_{\alpha\alpha'} n_{\alpha'}$ then describes the conventional one-loop Hartree-Fock propagation whereas $\sum_{\beta} \langle \alpha \beta | v c_2 | \alpha' \beta \rangle$ and $\sum_{\beta} \langle \alpha \beta | c_2 v | \alpha' \beta \rangle$ describe the coupling to more complicated configurations and will be attributed to two-body scatterings.

B. Two-body correlation dynamics

The equation of motion for the two-body correlation function $C_{\alpha\beta\alpha'\beta'}(t)$ provides more precise information on the higher order configurations considered. Whereas the terms with $h_{\alpha\lambda}$ describe the evolution of the state in the time-dependent mean field, the

limit (50) accounts for interactions in the Born approximation, i.e.,

$$\begin{aligned} & \sum_{\lambda\gamma} \{ \langle \alpha\beta | Q^\pm v | \lambda\gamma \rangle (\rho_{20})_{\lambda\gamma\alpha'\beta'} - (\rho_{20})_{\alpha\beta\lambda\gamma} \langle \lambda\gamma | v Q^{\mp\dagger} | \alpha'\beta' \rangle \} \\ &= \langle \alpha\beta | v | \alpha'\beta' \rangle_{\mathcal{A}} [n_{\alpha'} n_{\beta'} (1 - n_{\alpha}) (1 - n_{\beta}) - n_{\alpha} n_{\beta} (1 - n_{\alpha'}) (1 - n_{\beta'})], \end{aligned} \quad (65)$$

where one has used that $Q^\pm$ becomes diagonal in the ‘natural’ basis,

$$Q_{\alpha\beta\lambda\gamma}^\pm = \delta_{\alpha\lambda} \delta_{\beta\gamma} [1 - n_{\alpha} - n_{\beta}]. \quad (66)$$

In the Hartree-Fock (HF) limit ($n_{\alpha} = 0$ or 1) the operator $Q^\pm$ vanishes for ph or hp states and thus acts as a projector on pp and hh states.

The interaction terms in Eq. (51) correspond to a resummation of ladder diagrams [31–33] which becomes obvious when considering the stationary limit of the particular approach including the interaction terms in Eqs. (50) and (51), i.e.,

$$\begin{aligned} & (\omega - (\epsilon_{\alpha} + \epsilon_{\beta})) C_{\alpha\beta\alpha'\beta'} - (\omega - (\epsilon_{\alpha'} + \epsilon_{\beta'})) C_{\alpha\beta\alpha'\beta'} \\ &= \sum_{\lambda\gamma} (1 - n_{\alpha} - n_{\beta}) \langle \alpha\beta | v | \lambda\gamma \rangle (\rho_2)_{\lambda\gamma\alpha'\beta'} - \sum_{\lambda\gamma} (\rho_2)_{\alpha\beta\lambda\gamma} \langle \lambda\gamma | v | \alpha'\beta' \rangle (1 - n_{\alpha'} - n_{\beta'}), \end{aligned} \quad (67)$$

assuming $h_{\alpha\lambda} = \epsilon_{\alpha} \delta_{\alpha\lambda}$, while $(\rho_2)_{\alpha\beta\alpha'\beta'}$ is given by

$$(\rho_2)_{\alpha\beta\alpha'\beta'} = (\rho_{20})_{\alpha\beta\alpha'\beta'} + C_{\alpha\beta\alpha'\beta'}. \quad (68)$$

Equation (67) is fulfilled for $(\epsilon \rightarrow 0^+)$

$$C_{\alpha\beta\alpha'\beta'} = [\omega - (\epsilon_{\alpha} + \epsilon_{\beta}) + i\epsilon]^{-1} (1 - n_{\alpha} - n_{\beta}) \sum_{\lambda\gamma} \langle \alpha\beta | v | \lambda\gamma \rangle (\rho_2)_{\lambda\gamma\alpha'\beta'} \quad (69)$$

and its hermitian conjugate equation. Defining a G matrix via

$$\langle \alpha\beta | v \rho_2 | \alpha'\beta' \rangle = \langle \alpha\beta | G \rho_{20} | \alpha'\beta' \rangle \quad (70)$$

and using Eq. (68) one obtains

$$\sum_{\lambda'\gamma'} \langle \alpha\beta | G | \lambda'\gamma' \rangle (\rho_{20})_{\lambda'\gamma'\alpha'\beta'} = \sum_{\lambda'\gamma'} \langle \alpha\beta | v | \lambda'\gamma' \rangle (\rho_{20})_{\lambda'\gamma'\alpha'\beta'} + \sum_{\lambda\gamma} \langle \alpha\beta | v | \lambda\gamma \rangle \frac{(1 - n_{\alpha} - n_{\beta})}{[\omega - (\epsilon_{\alpha} + \epsilon_{\beta}) + i\epsilon]} \sum_{\lambda'\gamma'} \langle \lambda\gamma | G | \lambda'\gamma' \rangle (\rho_{20})_{\lambda'\gamma'\alpha'\beta'}, \quad (71)$$

which for arbitrary matrix elements of ρ_{20} gives the result

$$\langle \alpha\beta | G | \alpha'\beta' \rangle = \langle \alpha\beta | v | \alpha'\beta' \rangle + \sum_{\lambda\gamma} \langle \alpha\beta | v | \lambda\gamma \rangle \frac{(1 - n_{\alpha} - n_{\beta})}{[\omega - (\epsilon_{\alpha} + \epsilon_{\beta}) + i\epsilon]} \langle \lambda\gamma | G | \alpha'\beta' \rangle, \quad (72)$$

which is the G -matrix equation [36,37]. Thus the terms (51) account for the ‘short-range’ correlations in the traditional sense. Note that in conventional Brueckner theory the matrix element of $Q^\pm$ is approximated by $(1 - n_{\alpha} - n_{\beta} + n_{\alpha} n_{\beta}) = (1 - n_{\alpha})(1 - n_{\beta})$, i.e. hh scattering is neglected.

The terms in Eq. (52) account for ph interaction diagrams (fully antisymmetrized) since also $Q^\perp$ becomes diagonal in the ‘natural’ basis

$$Q_{\alpha\lambda,\alpha'\lambda'}^\perp = \delta_{\alpha\lambda'} \delta_{\lambda\alpha'} [n_{\alpha'} - n_{\alpha}] \quad (73)$$

and excludes pp or hh interactions in the HF limit.

When neglecting the antisymmetrization $\mathcal{A}_{\alpha\beta}$ in Eq. (52) and assuming $Q^\pm v \equiv 0$, i.e., neglecting pp and hh interactions, the two-body correlation matrix becomes separable, i.e.,

$$C_{\alpha\beta,\alpha'\beta'}(t) = f_{\alpha\alpha'}(t) f_{\beta\beta'}(t) \quad (74)$$

and one obtains

$$\begin{aligned} & f_{\beta\beta'} \left\{ i \frac{\partial}{\partial t} f_{\alpha\alpha'} - (\epsilon_{\alpha} - \epsilon_{\alpha'}) f_{\alpha\alpha'} - \sum_{\gamma\gamma'} (n_{\alpha'} - n_{\alpha}) \langle \alpha\gamma' | v | \alpha'\gamma \rangle_{\mathcal{A}} f_{\gamma\gamma'} \right\} \\ &+ f_{\alpha\alpha'} \left\{ i \frac{\partial}{\partial t} f_{\beta\beta'} - (\epsilon_{\beta} - \epsilon_{\beta'}) f_{\beta\beta'} - \sum_{\gamma\gamma'} (n_{\beta'} - n_{\beta}) \langle \beta\gamma' | v | \beta'\gamma \rangle_{\mathcal{A}} f_{\gamma\gamma'} \right\} = 0 \end{aligned} \quad (75)$$

when assuming again $h_{\alpha\lambda} = \epsilon_\alpha \delta_{\alpha\lambda}$. Equation (75) is fulfilled if $f_{\alpha\alpha'}$ and $f_{\beta\beta'}$ each are the solution of the random phase approximation (RPA) equation [33]

$$[\omega - (\epsilon_\alpha - \epsilon_{\alpha'})]f_{\alpha\alpha'}(\omega) = (n_{\alpha'} - n_\alpha) \sum_{\gamma\gamma'} \langle \alpha\gamma' | v | \alpha'\gamma \rangle_{\mathcal{A}} f_{\gamma\gamma'}(\omega) \quad (76)$$

when performing a Fourier transformation in time ($t \rightarrow \omega$). Thus Eq. (74) describes RPA correlations in the specific limit considered and accounts for ‘long-range’ correlations or ‘density fluctuations’ whereas when including all interaction terms in Eqs. (50)–(52) simultaneously one describes ‘long’ and ‘short’ range correlation phenomena on the same footing without double counting of interactions. Note that the theory dynamically accounts for the lowest order parquet diagrams [22].

C. Three-body correlation dynamics

Within the natural basis ($\rho_{\alpha\beta} = \delta_{\alpha\beta} n_\alpha$) the equations for the expansion coefficients read

$$\begin{aligned} i \frac{\partial}{\partial t} C_{\alpha\beta\gamma, \alpha'\beta'\gamma'}^3 - (\epsilon_\alpha + \epsilon_\beta + \epsilon_\gamma - \epsilon_{\alpha'} - \epsilon_{\beta'} - \epsilon_{\gamma'}) C_{\alpha\beta\gamma, \alpha'\beta'\gamma'}^3 \\ = \sum_{\eta} \{ ((1 - n_\alpha - n_\beta) [\langle \alpha\beta | v | \alpha'\eta \rangle_{\mathcal{A}} n_{\alpha'} C_{\eta\gamma\beta'\gamma'} - \langle \alpha\beta | v | \beta'\eta \rangle_{\mathcal{A}} n_{\beta'} C_{\eta\gamma\alpha'\gamma'} - \langle \alpha\beta | v | \gamma'\eta \rangle_{\mathcal{A}} n_{\gamma'} C_{\eta\gamma\beta'\alpha'}] \\ - (C_{\beta\gamma\eta\gamma'} n_\alpha \langle \alpha\eta | v | \alpha'\beta' \rangle_{\mathcal{A}} - C_{\alpha\gamma\eta\gamma'} n_\beta \langle \beta\eta | v | \alpha'\beta' \rangle_{\mathcal{A}} - C_{\beta\alpha\eta\gamma'} n_\gamma \langle \gamma\eta | v | \alpha'\beta' \rangle_{\mathcal{A}}) (1 - n_{\alpha'} - n_{\beta'}) \\ + ((1 - n_\beta - n_\gamma) [\langle \beta\gamma | v | \beta'\eta \rangle_{\mathcal{A}} n_{\beta'} C_{\alpha\eta\alpha'\gamma'} - \langle \beta\gamma | v | \alpha'\eta \rangle_{\mathcal{A}} n_{\alpha'} C_{\alpha\eta\beta'\gamma'} - \langle \beta\gamma | v | \gamma'\eta \rangle_{\mathcal{A}} n_{\gamma'} C_{\alpha\eta\alpha'\beta'}] \\ - ([C_{\alpha\gamma\alpha'\eta} n_\beta \langle \beta\eta | v | \beta'\gamma' \rangle_{\mathcal{A}} - C_{\beta\gamma\alpha'\eta} n_\alpha \langle \alpha\eta | v | \beta'\gamma' \rangle_{\mathcal{A}} - C_{\alpha\beta\alpha'\eta} n_\gamma \langle \gamma\eta | v | \beta'\gamma' \rangle_{\mathcal{A}}]) (1 - n_{\beta'} - n_{\gamma'}) \\ + ((1 - n_\alpha - n_\gamma) [\langle \gamma\alpha | v | \gamma'\eta \rangle_{\mathcal{A}} n_{\gamma'} C_{\eta\beta\alpha'\beta'} - \langle \gamma\alpha | v | \beta'\eta \rangle_{\mathcal{A}} n_{\beta'} C_{\eta\beta\alpha'\gamma'} - \langle \gamma\alpha | v | \alpha'\eta \rangle_{\mathcal{A}} n_{\alpha'} C_{\eta\beta\gamma'\beta'}] \\ - ([C_{\alpha\beta\eta\beta'} n_\gamma \langle \gamma\eta | v | \gamma'\alpha' \rangle_{\mathcal{A}} - C_{\gamma\beta\eta\beta'} n_\alpha \langle \alpha\eta | v | \gamma'\alpha' \rangle_{\mathcal{A}} - C_{\alpha\gamma\eta\beta'} n_\beta \langle \beta\eta | v | \gamma'\alpha' \rangle_{\mathcal{A}}]) (1 - n_{\alpha'} - n_{\gamma'}) \} \\ - \sum_{\eta} \{ (C_{\beta\gamma\eta\gamma'} \langle \alpha\eta | v | \alpha'\beta' \rangle_{\mathcal{A}} n_{\alpha'} n_{\beta'} - C_{\beta\gamma\eta\beta'} \langle \alpha\eta | v | \alpha'\gamma' \rangle_{\mathcal{A}} n_{\alpha'} n_{\gamma'} - C_{\beta\gamma\eta\alpha'} \langle \alpha\eta | v | \gamma'\beta' \rangle_{\mathcal{A}} n_{\gamma'} n_{\beta'}) \\ - (n_\alpha n_\beta \langle \alpha\beta | v | \alpha'\eta \rangle_{\mathcal{A}} C_{\eta\gamma\beta'\gamma'} - n_\alpha n_\gamma \langle \alpha\gamma | v | \alpha'\eta \rangle_{\mathcal{A}} C_{\eta\beta\beta'\gamma'} - n_\gamma n_\beta \langle \gamma\beta | v | \alpha'\eta \rangle_{\mathcal{A}} C_{\eta\alpha\beta'\gamma'}) \\ + (C_{\alpha\gamma\alpha'\eta} \langle \beta\eta | v | \beta'\gamma' \rangle_{\mathcal{A}} n_{\beta'} n_{\gamma'} - C_{\alpha\gamma\beta'\eta} \langle \beta\eta | v | \alpha'\gamma' \rangle_{\mathcal{A}} n_{\alpha'} n_{\gamma'} - C_{\alpha\gamma\gamma'\eta} \langle \beta\eta | v | \beta'\alpha' \rangle_{\mathcal{A}} n_{\beta'} n_{\alpha'}) \\ - (n_\beta n_\gamma \langle \beta\gamma | v | \beta'\eta \rangle_{\mathcal{A}} C_{\alpha\eta\alpha'\gamma'} - n_\alpha n_\gamma \langle \alpha\gamma | v | \beta'\eta \rangle_{\mathcal{A}} C_{\beta\eta\alpha'\gamma'} - n_\beta n_\alpha \langle \beta\alpha | v | \beta'\eta \rangle_{\mathcal{A}} C_{\gamma\eta\alpha'\gamma'}) \\ + (C_{\alpha\beta\eta\beta'} \langle \gamma\eta | v | \gamma'\alpha' \rangle_{\mathcal{A}} n_{\gamma'} n_{\alpha'} - C_{\alpha\beta\eta\alpha'} \langle \gamma\eta | v | \gamma'\beta' \rangle_{\mathcal{A}} n_{\gamma'} n_{\beta'} - C_{\alpha\beta\gamma'\eta} \langle \gamma\eta | v | \beta'\alpha' \rangle_{\mathcal{A}} n_{\beta'} n_{\alpha'}) \\ - (n_\gamma n_\alpha \langle \gamma\alpha | v | \gamma'\eta \rangle_{\mathcal{A}} C_{\eta\beta\alpha'\beta'} - n_\gamma n_\beta \langle \gamma\beta\eta' | v | \gamma'\eta \rangle_{\mathcal{A}} C_{\eta\alpha\alpha'\beta'} - n_\beta n_\alpha \langle \beta\alpha | v | \gamma'\eta \rangle_{\mathcal{A}} C_{\eta\gamma\alpha'\beta'}) \} \quad (77) \end{aligned}$$

$$\begin{aligned} + \sum_{\lambda\eta} \{ (1 - n_\alpha - n_\beta) \langle \alpha\beta | v | \lambda\eta \rangle C_{\lambda\eta\gamma, \alpha'\beta'\gamma'}^3 - C_{\alpha\beta\gamma, \lambda\eta\gamma'}^3 \langle \lambda\eta | v | \alpha'\beta' \rangle (1 - n_{\alpha'} - n_{\beta'}) \\ + (1 - n_\gamma - n_\alpha) \langle \alpha\gamma | v | \lambda\eta \rangle C_{\lambda\beta\eta, \alpha'\beta'\gamma'}^3 - C_{\alpha\beta\gamma, \lambda\beta'\eta}^3 \langle \lambda\eta | v | \alpha'\gamma' \rangle (1 - n_{\alpha'} - n_{\gamma'}) \\ + (1 - n_\gamma - n_\beta) \langle \beta\gamma | v | \lambda\eta \rangle C_{\alpha\lambda\eta, \alpha'\beta'\gamma'}^3 - C_{\alpha\beta\gamma, \alpha'\lambda\eta}^3 \langle \lambda\eta | v | \beta'\gamma' \rangle (1 - n_{\beta'} - n_{\gamma'}) \} \quad (78) \\ + \sum_{\eta\eta'} \{ \langle \alpha'\eta' | v | \alpha\eta \rangle_{\mathcal{A}} (n_\alpha - n_{\alpha'}) C_{\eta\beta\gamma, \eta'\beta'\gamma'}^3 - \langle \alpha'\eta' | v | \beta\eta \rangle_{\mathcal{A}} (n_\beta - n_{\alpha'}) C_{\eta\alpha\gamma, \eta'\beta'\gamma'}^3 - \langle \alpha'\eta' | v | \gamma\eta \rangle_{\mathcal{A}} (n_\gamma - n_{\alpha'}) C_{\eta\beta\alpha, \eta'\beta'\gamma'}^3 \\ + \langle \beta'\eta' | v | \alpha\eta \rangle_{\mathcal{A}} (n_\alpha - n_{\beta'}) C_{\eta\gamma\eta, \alpha'\eta'\gamma'}^3 - \langle \beta'\eta' | v | \beta\eta \rangle_{\mathcal{A}} (n_\beta - n_{\beta'}) C_{\alpha\eta\gamma, \alpha'\eta'\gamma'}^3 - \langle \beta'\eta' | v | \gamma\eta \rangle_{\mathcal{A}} (n_\gamma - n_{\beta'}) C_{\alpha\eta\beta, \alpha'\eta'\gamma'}^3 \\ + \langle \gamma'\eta' | v | \gamma\eta \rangle_{\mathcal{A}} (n_\gamma - n_{\gamma'}) C_{\alpha\beta\eta, \alpha'\beta'\eta'}^3 - \langle \gamma'\eta' | v | \beta\eta \rangle_{\mathcal{A}} (n_\beta - n_{\gamma'}) C_{\alpha\gamma\eta, \alpha'\beta'\eta'}^3 - \langle \gamma'\eta' | v | \alpha\eta \rangle_{\mathcal{A}} (n_\alpha - n_{\gamma'}) C_{\gamma\beta\eta, \alpha'\beta'\eta'}^3 \}, \quad (79) \end{aligned}$$

where terms of order vc_2c_2 have been discarded. The terms of order $v\rho c_3$ describe corrections to the propagator for c_3 in the pp-hh channel as well as in the p-h channel while the terms of order $v\rho c_2$ and $v\rho\rho c_2$ are the leading order contributions for the three-body collision term. Thus—neglecting the terms with $v\rho c_3$ and vc_2c_2 —one obtains in leading order

$$\begin{aligned} \left(i \frac{\partial}{\partial t} - (\epsilon_\alpha + \epsilon_\beta + \epsilon_\gamma - \epsilon_{\alpha'} - \epsilon_{\beta'} - \epsilon_{\gamma'}) \right) C_{\alpha\beta\gamma, \alpha'\beta'\gamma'}^3 \\ = \sum_{\eta} \{ ((n_\alpha n_\beta \bar{n}_{\alpha'} + \bar{n}_\alpha \bar{n}_\beta n_{\alpha'}) \langle \alpha\beta | v | \alpha'\eta \rangle_{\mathcal{A}} C_{\eta\gamma\beta'\gamma'} - (n_\alpha n_\beta \bar{n}_{\beta'} + \bar{n}_\alpha \bar{n}_\beta n_{\beta'}) \langle \alpha\beta | v | \beta'\eta \rangle_{\mathcal{A}} C_{\eta\gamma\alpha'\gamma'} \\ - (n_\alpha n_\beta \bar{n}_{\gamma'} + \bar{n}_\alpha \bar{n}_\beta n_{\gamma'}) \langle \alpha\beta | v | \gamma'\eta \rangle_{\mathcal{A}} C_{\eta\gamma\beta'\alpha'}) - (C_{\beta\gamma\eta\gamma'} \langle \alpha\eta | v | \alpha'\beta' \rangle_{\mathcal{A}} (n_{\alpha'} n_{\beta'} \bar{n}_\alpha + \bar{n}_{\alpha'} \bar{n}_{\beta'} n_\alpha) \end{aligned}$$

$$\begin{aligned}
& -C_{\alpha\gamma\eta\gamma'} \langle \beta\eta|v|\alpha'\beta' \rangle_{\mathcal{A}} (n_{\alpha'} n_{\beta'} \bar{n}_{\beta} + \bar{n}_{\alpha'} \bar{n}_{\beta'} n_{\beta}) - C_{\beta\alpha\eta\gamma'} \langle \gamma\eta|v|\alpha'\beta' \rangle_{\mathcal{A}} (n_{\alpha'} n_{\beta'} \bar{n}_{\gamma} + \bar{n}_{\alpha'} \bar{n}_{\beta'} n_{\gamma}) \\
& + ((n_{\beta} n_{\gamma} \bar{n}_{\beta'} + \bar{n}_{\beta} \bar{n}_{\gamma} n_{\beta'}) \langle \beta\gamma|v|\beta'\eta \rangle_{\mathcal{A}} C_{\alpha\eta\alpha'\gamma'} - (n_{\beta} n_{\gamma} \bar{n}_{\alpha'} + \bar{n}_{\beta} \bar{n}_{\gamma} n_{\alpha'}) \langle \beta\gamma|v|\alpha'\eta \rangle_{\mathcal{A}} C_{\alpha\eta\beta'\gamma'}) \\
& - (n_{\beta} n_{\gamma} \bar{n}_{\gamma'} + \bar{n}_{\beta} \bar{n}_{\gamma} n_{\gamma'}) \langle \beta\gamma|v|\gamma'\eta \rangle_{\mathcal{A}} C_{\alpha\eta\alpha'\beta'} - (C_{\alpha\gamma\alpha'\eta} \langle \beta\eta|v|\beta'\gamma' \rangle_{\mathcal{A}} (n_{\beta'} n_{\gamma'} \bar{n}_{\beta} + \bar{n}_{\beta'} \bar{n}_{\gamma'} n_{\beta}) \\
& - C_{\beta\gamma\alpha'\eta} \langle \alpha\eta|v|\beta'\gamma' \rangle_{\mathcal{A}} (n_{\beta'} n_{\gamma'} \bar{n}_{\alpha} + \bar{n}_{\beta'} \bar{n}_{\gamma'} n_{\alpha}) - C_{\alpha\beta\alpha'\eta} \langle \gamma\eta|v|\beta'\gamma' \rangle_{\mathcal{A}} (n_{\beta'} n_{\gamma'} \bar{n}_{\gamma} + \bar{n}_{\beta'} \bar{n}_{\gamma'} n_{\gamma})) \\
& + (n_{\alpha} n_{\gamma} \bar{n}_{\gamma'} + \bar{n}_{\alpha} \bar{n}_{\gamma} n_{\gamma'}) \langle \gamma\alpha|v|\gamma'\eta \rangle_{\mathcal{A}} C_{\eta\beta\alpha'\beta'} - (n_{\alpha} n_{\gamma} \bar{n}_{\beta'} + \bar{n}_{\alpha} \bar{n}_{\gamma} n_{\beta'}) \langle \gamma\alpha|v|\beta'\eta \rangle_{\mathcal{A}} C_{\eta\beta\alpha'\gamma'} \\
& - (n_{\alpha} n_{\gamma} \bar{n}_{\alpha'} + \bar{n}_{\alpha} \bar{n}_{\gamma} n_{\alpha'}) \langle \gamma\alpha|v|\alpha'\eta \rangle_{\mathcal{A}} C_{\eta\beta\gamma'\beta'} - (C_{\alpha\beta\eta\beta'} \langle \gamma\eta|v|\gamma'\alpha' \rangle_{\mathcal{A}} (n_{\alpha'} n_{\gamma'} \bar{n}_{\gamma} + \bar{n}_{\alpha'} \bar{n}_{\gamma'} n_{\gamma}) \\
& - C_{\gamma\beta\eta\beta'} \langle \alpha\eta|v|\gamma'\alpha' \rangle_{\mathcal{A}} (n_{\alpha'} n_{\gamma'} \bar{n}_{\alpha} + \bar{n}_{\alpha'} \bar{n}_{\gamma'} n_{\alpha}) - C_{\alpha\gamma\eta\beta'} \langle \beta\eta|v|\gamma'\alpha' \rangle_{\mathcal{A}} (n_{\alpha'} n_{\gamma'} \bar{n}_{\beta} + \bar{n}_{\alpha'} \bar{n}_{\gamma'} n_{\beta})) \} \\
& =: \hat{V}_{\alpha\beta\gamma,\alpha'\beta'\gamma'}^3(t)
\end{aligned} \tag{80}$$

using the identity

$$n_{\beta} n_{\gamma} + (1 - n_{\beta} - n_{\gamma}) n_{\beta'} = n_{\beta} n_{\gamma} \bar{n}_{\beta'} + \bar{n}_{\beta} \bar{n}_{\gamma} n_{\beta'} \tag{81}$$

with $\bar{n}_{\beta} = 1 - n_{\beta}$, which clearly demonstrates the Pauli blocking. Note that the right-hand side (r.h.s.) of Eq. (80) is entirely a function of the occupation numbers and the two-body correlations, which will be determined in leading order below.

III. COLLISION TERMS

In the context of transport theories the two-body and three-body collision terms are of primary interest. In order to obtain transparent results the leading order terms for $C_{\alpha\beta\alpha'\beta'}(t)$ and $C_{\alpha\beta\gamma,\alpha'\beta'\gamma'}^3(t)$ will be considered.

A. Two-body case

In leading order one considers

$$i \frac{\partial}{\partial t} \rho - [h(1), \rho] = \text{Tr}_{(2=2')} [v(12), c_2] \tag{82}$$

and

$$i \frac{\partial}{\partial t} c_2 - \left[\sum_{i=1}^2 h(i), c_2 \right] = [V^=(12), \rho_{20}]. \tag{83}$$

The collision term in Eq. (82) requires to compute

$$I^2(11'; t) := -i \text{Tr}_{(2=2')} [v(12), c_2(12, 1'2'; t)] \tag{84}$$

or in a single-particle basis

$$\begin{aligned}
I_{\alpha\alpha'}^2(t) &= -i \sum_{\beta} \sum_{\lambda\gamma} \{ \langle \alpha\beta|v|\lambda\gamma \rangle C_{\lambda\gamma\alpha'\beta}(t) \\
&\quad - C_{\alpha\beta\lambda\gamma}(t) \langle \lambda\gamma|v|\alpha'\beta \rangle \},
\end{aligned} \tag{85}$$

i.e., the explicit knowledge of the two-particle correlation function in an arbitrary basis $|\alpha\rangle$.

To calculate the two-particle correlation function in leading order one uses a discrete basis in which the single-particle Hamiltonian $h_{\alpha\lambda}(t)$ and in particular $\rho_{\alpha\alpha'}(t)$ is diagonal, i.e.,

$$h_{\alpha\lambda}(t) \approx \epsilon_{\alpha}(t) \delta_{\alpha\lambda}; \quad \rho_{\alpha\alpha'}(t) = n_{\alpha}(t) \delta_{\alpha\alpha'}. \tag{86}$$

In this basis the equation of motion for the expansion coefficients $C_{\alpha\beta\alpha'\beta'}(t)$ then reduces to (omitting the explicit time dependence of all quantities)

$$\begin{aligned}
& \left\{ i \frac{\partial}{\partial t} - [\epsilon_{\alpha} + \epsilon_{\beta} - \epsilon_{\alpha'} - \epsilon_{\beta'}] \right\} C_{\alpha\beta\alpha'\beta'}(t) \\
&= \sum_{\lambda\gamma} \{ \langle \alpha\beta|Q^=v|\lambda\gamma \rangle (\rho_{20})_{\lambda\gamma\alpha'\beta'} \\
&\quad - (\rho_{20})_{\alpha\beta\lambda\gamma} \langle \lambda\gamma|vQ^=|\alpha'\beta' \rangle \} \\
&= \langle \alpha\beta|v|\alpha'\beta' \rangle_{\mathcal{A}} [n_{\alpha'} n_{\beta'} \bar{n}_{\alpha} \bar{n}_{\beta} - n_{\alpha} n_{\beta} \bar{n}_{\alpha'} \bar{n}_{\beta'}] \\
&=: \langle \alpha\beta|V_B(t)|\alpha'\beta' \rangle,
\end{aligned} \tag{87}$$

where one has taken advantage of the fact that in this basis $Q^=$ is diagonal

$$Q_{\alpha\beta\lambda\gamma}^= = \delta_{\alpha\lambda} \delta_{\beta\gamma} [1 - n_{\alpha} - n_{\beta}]. \tag{88}$$

Furthermore, the identity

$$\begin{aligned}
& n_{\alpha'} n_{\beta'} (1 - n_{\alpha} - n_{\beta}) - n_{\alpha} n_{\beta} (1 - n_{\alpha'} - n_{\beta'}) \\
&= n_{\alpha'} n_{\beta'} \bar{n}_{\alpha} \bar{n}_{\beta} - n_{\alpha} n_{\beta} \bar{n}_{\alpha'} \bar{n}_{\beta'}
\end{aligned} \tag{89}$$

has been used with $\bar{n}_{\gamma} = 1 - n_{\gamma}$.

Equation (87) is a differential equation of first order in time that can be integrated directly. With regards to approximations to be carried out later (within the framework of the energy conservation in two-particle collisions), it is assumed in the following that in particular the single-particle energies $\epsilon_{\alpha}(t) \approx \epsilon_{\alpha}$ are weakly varying functions of time. This approximation is particularly well or exactly fulfilled for the electron states in solids as well as for the single-particle states in a sufficiently large normalization volume. For a vanishing homogeneous solution of (87) $C_{\alpha\beta\alpha'\beta'}(t)$ then is given by

$$C_{\alpha\beta\alpha'\beta'}(t) = -i \int_{t_0}^t dt' \exp\{-i[\epsilon_{\alpha} + \epsilon_{\beta} - \epsilon_{\alpha'} - \epsilon_{\beta'}](t - t')\} \langle \alpha\beta|V_B(t')|\alpha'\beta' \rangle, \tag{90}$$

as one can easily verify by insertion in Eq. (87). Due to the function in the occupation numbers the matrix elements $C_{\alpha\beta\alpha\beta}$ vanish identically and do not contribute to the normalization of the 2-body density matrix ρ_2 , i.e.,

$$\sum_{\alpha\beta} (\rho_{\alpha\alpha}\rho_{\beta\beta} - \rho_{\alpha\beta}\rho_{\beta\alpha} + C_{\alpha\beta\alpha\beta}) = \sum_{\alpha\beta} (n_{\alpha}n_{\beta} - n_{\alpha}\delta_{\alpha\beta}) = A(A-1) \quad (91)$$

with A denoting the total particle number.

For the diagonal element of the two-body collision term (85) one obtains with Eq. (90)

$$\begin{aligned} I_{\alpha\alpha}^2(t) &= -i \sum_{\beta} \sum_{\lambda\gamma} \{ \langle \alpha\beta | v | \lambda\gamma \rangle C_{\lambda\gamma\alpha\beta}(t) - C_{\alpha\beta\lambda\gamma}(t) \langle \lambda\gamma | v | \alpha\beta \rangle \} \\ &= - \sum_{\beta} \sum_{\lambda\gamma} \int_{t_0}^t dt' \{ \exp[-i(\epsilon_{\lambda} + \epsilon_{\gamma} - \epsilon_{\alpha} - \epsilon_{\beta})(t-t')] \langle \alpha\beta | v | \lambda\gamma \rangle \langle \lambda\gamma | V_B(t') | \alpha\beta \rangle \\ &\quad - \exp[-i(\epsilon_{\alpha} + \epsilon_{\beta} - \epsilon_{\lambda} - \epsilon_{\gamma})(t-t')] \langle \alpha\beta | V_B(t') | \lambda\gamma \rangle \langle \lambda\gamma | v | \alpha\beta \rangle \} \\ &= \sum_{\beta} \sum_{\lambda\gamma} \int_{t_0}^t dt' 2 \cos[(\epsilon_{\alpha} + \epsilon_{\beta} - \epsilon_{\lambda} - \epsilon_{\gamma})(t-t')] \\ &\quad \times \langle \alpha\beta | v | \lambda\gamma \rangle \langle \lambda\gamma | v | \alpha\beta \rangle_A [n_{\lambda}(t')n_{\gamma}(t')\bar{n}_{\alpha}(t')\bar{n}_{\beta}(t') - n_{\alpha}(t')n_{\beta}(t')\bar{n}_{\lambda}(t')\bar{n}_{\gamma}(t')] \end{aligned} \quad (92)$$

with $V_B(t')$ from Eq. (87). In Eq. (92)

$$\langle \alpha\beta | V_B(t') | \lambda\gamma \rangle \langle \lambda\gamma | v | \alpha\beta \rangle = - \langle \alpha\beta | v | \lambda\gamma \rangle \langle \lambda\gamma | V_B(t') | \alpha\beta \rangle \quad (93)$$

has been used since the function with the occupation numbers changes sign. The collision integral (92) then can be illustrated as shown in Fig. 1, where the open circles denote a factor n for the occupation number while the full circles denote a factor $\bar{n}$.

The further assumption is that the occupation numbers $n_{\alpha}(t') \approx n_{\alpha}(t)$ are approximately constant in time. In this case one can carry out the time integration in (92) for systems of low density or weak residual interaction and gets

$$\int_{t_0}^t dt' \cos[(\epsilon_{\alpha} + \epsilon_{\beta} - \epsilon_{\lambda} - \epsilon_{\gamma})(t-t')] \quad (94)$$

$$\approx \pi \delta(\epsilon_{\alpha} + \epsilon_{\beta} - \epsilon_{\lambda} - \epsilon_{\gamma}), \quad (95)$$

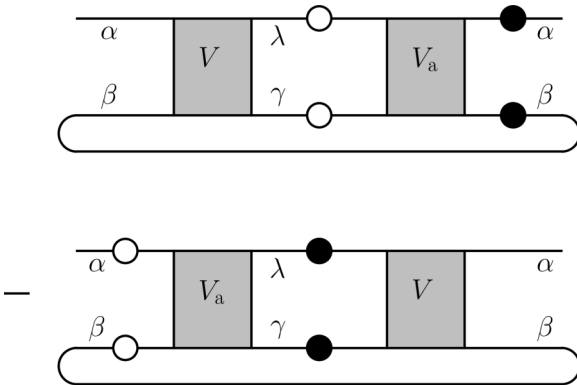


FIG. 1. Illustration of the collision integral (92), where the open circles denote a factor n for the occupation number while the full circles denote a factor $\bar{n}$. The closed line corresponds to a summation over particle 2 (or index β). Here, V_a denotes the antisymmetrized matrix element of the interaction v .

i.e., the energy conservation in the two-body collisions. Formally this result is obtained by taken the limit $t \rightarrow \infty$. Equation (94) implies that the time between two subsequent collisions τ_s is large compared to the actual collision time τ_c , such that the energy uncertainty associated to τ_s , i.e., $\Delta\epsilon \approx \hbar/\tau_s$ will be small. For the diagonal elements of the collision term one then obtains

$$\begin{aligned} I_{\alpha\alpha}^2(t) &\approx 2\pi \sum_{\beta} \sum_{\lambda\gamma} \delta(\epsilon_{\alpha} + \epsilon_{\beta} - \epsilon_{\lambda} - \epsilon_{\gamma}) \\ &\quad \times \langle \alpha\beta | v | \lambda\gamma \rangle \langle \lambda\gamma | v | \alpha\beta \rangle_A [n_{\lambda}n_{\gamma}\bar{n}_{\alpha}\bar{n}_{\beta} - n_{\alpha}n_{\beta}\bar{n}_{\lambda}\bar{n}_{\gamma}](t) \end{aligned} \quad (96)$$

in the basis $|\alpha\rangle$ in which the matrix $\rho_{\alpha\alpha'}$ is diagonal. Note that the result (96) is obtained directly when approximating

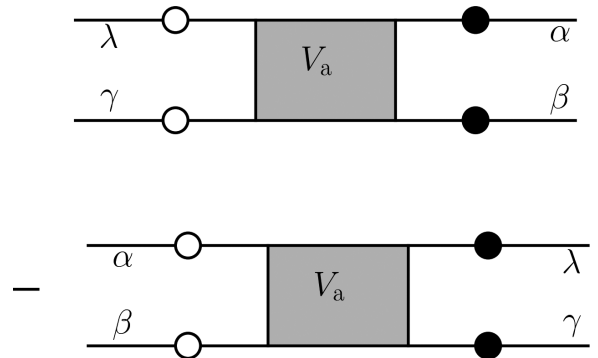


FIG. 2. Illustration of the matrix element $C_{\lambda\gamma\alpha\beta}$ in the on-shell limit, where the open circles denote a factor n for the occupation number while the full circles denote a factor $\bar{n}$. Here, the box with V_a denotes the antisymmetrized matrix element of the interaction v .

$C_{\lambda\gamma\alpha\beta}$ by

$$\begin{aligned} C_{\lambda\gamma\alpha\beta}(t) &= i\pi \delta(\epsilon_\alpha + \epsilon_\beta - \epsilon_\lambda - \epsilon_\gamma) \\ &\times \langle \lambda\gamma | v | \alpha\beta \rangle_{\mathcal{A}} [n_\lambda n_\gamma \bar{n}_\alpha \bar{n}_\beta - n_\alpha n_\beta \bar{n}_\lambda \bar{n}_\gamma](t) \\ &=: -i\pi \langle \lambda\gamma | \tilde{V}_B(t) | \alpha\beta \rangle_{\mathcal{A}}, \end{aligned} \quad (97)$$

i.e., by the on-shell matrix element of the (antisymmetrized) interaction v with its characteristic function in the associated occupation numbers (except for a factor $-i\pi$). Accordingly, $C_{\lambda\gamma\alpha\beta}$ in the on-shell limit can be illustrated by the diagrams shown in Fig. 2 where the open circles denote occupation numbers and the full circles blocking factors.

Equation (96) now is evaluated in the basis of plane waves $|\alpha\rangle \sim \exp\{i\mathbf{p}_\alpha \cdot \mathbf{r}\}$, such that the matrix ρ becomes diagonal in momentum space

$$\rho(\mathbf{p}, \mathbf{p}') = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}') n(\mathbf{p}). \quad (98)$$

In Eq. (98) then $n(\mathbf{p})$ has the physical interpretation of the occupation number of a plane wave with wave number (or momentum) $\mathbf{p}$.

Next, one assumes for simplicity that the matrix elements of the interaction v in Eq. (96) are independent of spin σ_i (and isospin τ_i) and in spatial representation given by

$$\langle \mathbf{r}_1 \mathbf{r}_2 \sigma_1 \sigma_2 \tau_1 \tau_2 | v | \sigma_1' \sigma_2' \tau_1' \tau_2' \mathbf{r}_1' \mathbf{r}_2' \rangle = \delta_{\sigma_1 \sigma_1'} \delta_{\sigma_2 \sigma_2'} \delta_{\tau_1 \tau_1'} \delta_{\tau_2 \tau_2'} \delta^3(\mathbf{r}_1 - \mathbf{r}_1') \delta^3(\mathbf{r}_2 - \mathbf{r}_2') v(\mathbf{r}_1 - \mathbf{r}_2) \quad (99)$$

or in momentum representation by

$$\langle \mathbf{p}_1 \mathbf{p}_2 \sigma_1 \sigma_2 \tau_1 \tau_2 | v | \sigma_1' \sigma_2' \tau_1' \tau_2' \mathbf{p}_1' \mathbf{p}_2' \rangle = \delta_{\sigma_1 \sigma_1'} \delta_{\sigma_2 \sigma_2'} \delta_{\tau_1 \tau_1'} \delta_{\tau_2 \tau_2'} (2\pi)^3 \delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_1' - \mathbf{p}_2') v(\mathbf{p}_2 - \mathbf{p}_2'), \quad (100)$$

which expresses the conservation of momentum in the two-body interaction.

With these approximations one can evaluate Eq. (96) and obtains with $\epsilon(\mathbf{p}) = \mathbf{p}^2/(2m)$ (nonrelativistically),

$$\begin{aligned} I^2(\mathbf{p}_1, \mathbf{p}_1; t) &= (2s+1)(2\tau+1) \int \frac{d^3 p_2}{(2\pi)^3} \frac{d^3 p_3}{(2\pi)^3} \frac{d^3 p_4}{(2\pi)^3} \\ &\times 2\pi \delta\left(\frac{1}{2m}[p_1^2 + p_2^2 - p_3^2 - p_4^2]\right) (2\pi)^3 \delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) v(\mathbf{p}_2 - \mathbf{p}_4) v_{\mathcal{A}}(\mathbf{p}_4 - \mathbf{p}_2) \\ &\times \{n(\mathbf{p}_3; t) n(\mathbf{p}_4; t) \bar{n}(\mathbf{p}_1; t) \bar{n}(\mathbf{p}_2; t) - n(\mathbf{p}_1; t) n(\mathbf{p}_2; t) \bar{n}(\mathbf{p}_3; t) \bar{n}(\mathbf{p}_4; t)\}. \end{aligned} \quad (101)$$

Equation (101) describes scattering processes $\mathbf{p}_1 + \mathbf{p}_2 \rightarrow \mathbf{p}_3 + \mathbf{p}_4$ ('loss' term) and $\mathbf{p}_3 + \mathbf{p}_4 \rightarrow \mathbf{p}_1 + \mathbf{p}_2$ ('gain' term) with the conservation of energy and momentum. Furthermore, the momentum states $\mathbf{p}_3, \mathbf{p}_4$ (for the 'loss' terms) or $\mathbf{p}_1, \mathbf{p}_2$ (for the 'gain' terms) cannot be completely occupied due to the factors $\bar{n}(\mathbf{p}_i; t)$, which contain the Pauli blocking for fermions.

Using the Born approximation for the elastic differential cross section

$$\frac{d\sigma_{el}}{d\Omega} = \frac{m^2}{16\pi^2 \hbar^4} v v_{\mathcal{A}}^\dagger \quad (102)$$

for two-body scattering of particles (of mass m) the collision integral—after integration over the δ -functions—becomes (cf. [10] p. 62)

$$\begin{aligned} I^2(\mathbf{p}_1, \mathbf{p}_1; t) &= (2s+1)(2\tau+1) \int \frac{d^3 p_2}{(2\pi)^3} \int d\Omega v_{\text{rel}} \frac{d\sigma_{el}}{d\Omega}(\mathbf{p}_1 + \mathbf{p}_2, \mathbf{p}_2 - \mathbf{p}_4) \\ &\times \{n(\mathbf{p}_3; t) n(\mathbf{p}_4; t) \bar{n}(\mathbf{p}_1; t) \bar{n}(\mathbf{p}_2; t) - n(\mathbf{p}_1; t) n(\mathbf{p}_2; t) \bar{n}(\mathbf{p}_3; t) \bar{n}(\mathbf{p}_4; t)\} \end{aligned} \quad (103)$$

with the relative velocity

$$v_{\text{rel}} = \frac{|\mathbf{p}_1 - \mathbf{p}_2|}{m}, \quad (104)$$

which is of the familiar Uehling-Uhlenbeck form [6]. Note that in Eq. (103) the four-momenta are constrained by energy and momentum conservation.

B. The three-body case

In leading order one now considers the following set of equations for $\rho(t)$, $c_2(t)$, and $c_3(t)$:

$$i \frac{\partial}{\partial t} \rho - [h(1), \rho] = \text{Tr}_{(2=2')} [v(12), c_2]; \quad (105)$$

$$i \frac{\partial}{\partial t} c_2 - \left[\sum_{i=1}^2 h(i), c_2 \right] = [V^=(12), \rho_{20}] + \text{Tr}_{(3=3')} [v(13) + v(23), c_3] \quad (106)$$

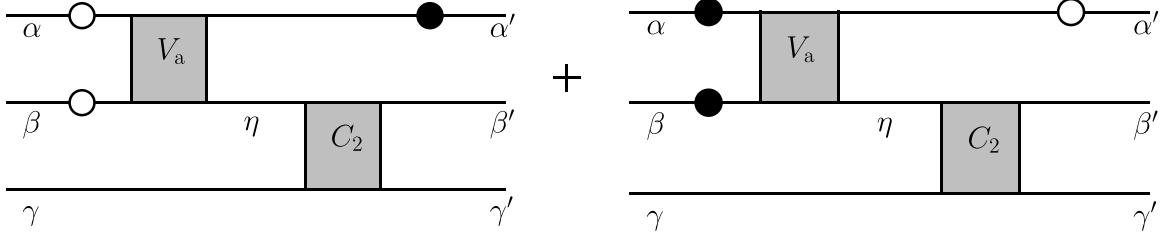


FIG. 3. Illustration of a term in the three-body correlation $\tilde{C}_{\alpha\beta\gamma,\alpha'\beta'\gamma'}^3(t)$ where the initial and final lines are constrained by energy conservation.

and

$$i \frac{\partial}{\partial t} c_3 - \left[\sum_{i=1}^3 h(i), c_3 \right] = [S_3(V^=(13) + V^=(23))S_{3'}, \rho(33'; t)c_2(12, 1'2'; t)] \\ - \text{Tr}_{(4=4')} [S_1 v(14)S_{1'} \mathcal{A}_{2'3'} P_{24}, \rho(11'; t)\rho(22'; t)c_2(34, 3'4'; t)]. \quad (107)$$

Here, the left-hand side (l.h.s.) describes the propagation of all particles (initial and final) in the time-dependent mean field, whereas the r.h.s. presents the individual inhomogeneous terms of lower order, which represent the source terms. In analogy to the previous section one obtains for $c_3(t)$ in the natural basis in the limit (107)

$$C_{\alpha\beta\gamma,\alpha'\beta'\gamma'}^3(t) = -i \int_{t_0}^t dt' \exp(-i[\epsilon_\alpha + \epsilon_\beta + \epsilon_\gamma - \epsilon_{\alpha'} - \epsilon_{\beta'} - \epsilon_{\gamma'}](t - t')) \\ \times \sum_{\eta} \{ (n_\alpha n_\beta \bar{n}_{\alpha'} + \bar{n}_\alpha \bar{n}_\beta n_{\alpha'}) \langle \alpha\beta | v | \alpha'\eta \rangle_{\mathcal{A}} C_{\eta\gamma\beta'\gamma'} - (n_\alpha n_\beta \bar{n}_{\beta'} + \bar{n}_\alpha \bar{n}_\beta n_{\beta'}) \langle \alpha\beta | v | \beta'\eta \rangle_{\mathcal{A}} C_{\eta\gamma\alpha'\gamma'} \\ - (n_\alpha n_\beta \bar{n}_{\gamma'} + \bar{n}_\alpha \bar{n}_\beta n_{\gamma'}) \langle \alpha\beta | v | \gamma'\eta \rangle_{\mathcal{A}} C_{\eta\gamma\beta'\alpha'} - (C_{\beta\gamma\eta\gamma'} \langle \alpha\eta | v | \alpha'\beta' \rangle_{\mathcal{A}} (n_{\alpha'} n_{\beta'} \bar{n}_\alpha + \bar{n}_{\alpha'} \bar{n}_{\beta'} n_\alpha) \\ - C_{\alpha\gamma\eta\gamma'} \langle \beta\eta | v | \alpha'\beta' \rangle_{\mathcal{A}} (n_{\alpha'} n_{\beta'} \bar{n}_\beta + \bar{n}_{\alpha'} \bar{n}_{\beta'} n_\beta) - C_{\beta\alpha\eta\gamma'} \langle \gamma\eta | v | \alpha'\beta' \rangle_{\mathcal{A}} (n_{\alpha'} n_{\beta'} \bar{n}_\gamma + \bar{n}_{\alpha'} \bar{n}_{\beta'} n_\gamma) \\ + ((n_\beta n_\gamma \bar{n}_{\beta'} + \bar{n}_\beta \bar{n}_\gamma n_{\beta'}) \langle \beta\gamma | v | \beta'\eta \rangle_{\mathcal{A}} C_{\alpha\eta\alpha'\gamma'} - (n_\beta n_\gamma \bar{n}_{\alpha'} + \bar{n}_\beta \bar{n}_\gamma n_{\alpha'}) \langle \beta\gamma | v | \alpha'\eta \rangle_{\mathcal{A}} C_{\alpha\eta\beta'\gamma'} \\ - (n_\beta n_\gamma \bar{n}_{\gamma'} + \bar{n}_\beta \bar{n}_\gamma n_{\gamma'}) \langle \beta\gamma | v | \gamma'\eta \rangle_{\mathcal{A}} C_{\alpha\eta\alpha'\beta'} - (C_{\alpha\gamma\alpha'\eta} \langle \beta\eta | v | \beta'\gamma' \rangle_{\mathcal{A}} (n_{\beta'} n_{\gamma'} \bar{n}_\beta + \bar{n}_{\beta'} \bar{n}_{\gamma'} n_\beta) \\ - C_{\beta\gamma\alpha'\eta} \langle \alpha\eta | v | \beta'\gamma' \rangle_{\mathcal{A}} (n_{\beta'} n_{\gamma'} \bar{n}_\alpha + \bar{n}_{\beta'} \bar{n}_{\gamma'} n_\alpha) - C_{\alpha\beta\alpha'\eta} \langle \gamma\eta | v | \beta'\gamma' \rangle_{\mathcal{A}} (n_{\beta'} n_{\gamma'} \bar{n}_\gamma + \bar{n}_{\beta'} \bar{n}_{\gamma'} n_\gamma) \\ + (n_\alpha n_\gamma \bar{n}_{\gamma'} + \bar{n}_\alpha \bar{n}_\gamma n_{\gamma'}) \langle \gamma\alpha | v | \gamma'\eta \rangle_{\mathcal{A}} C_{\eta\beta\alpha'\beta'} - (n_\alpha n_\gamma \bar{n}_{\beta'} + \bar{n}_\alpha \bar{n}_\gamma n_{\beta'}) \langle \gamma\alpha | v | \beta'\eta \rangle_{\mathcal{A}} C_{\eta\beta\alpha'\gamma'} \\ - (n_\alpha n_\gamma \bar{n}_{\alpha'} + \bar{n}_\alpha \bar{n}_\gamma n_{\alpha'}) \langle \gamma\alpha | v | \alpha'\eta \rangle_{\mathcal{A}} C_{\eta\beta\gamma'\beta'} - (C_{\alpha\beta\eta\beta'} \langle \gamma\eta | v | \gamma'\alpha' \rangle_{\mathcal{A}} (n_{\alpha'} n_{\gamma'} \bar{n}_\gamma + \bar{n}_{\alpha'} \bar{n}_{\gamma'} n_\gamma) \\ - C_{\gamma\beta\eta\beta'} \langle \alpha\eta | v | \gamma'\alpha' \rangle_{\mathcal{A}} (n_{\alpha'} n_{\gamma'} \bar{n}_\alpha + \bar{n}_{\alpha'} \bar{n}_{\gamma'} n_\alpha) - C_{\alpha\gamma\eta\beta'} \langle \beta\eta | v | \gamma'\alpha' \rangle_{\mathcal{A}} (n_{\alpha'} n_{\gamma'} \bar{n}_\beta + \bar{n}_{\alpha'} \bar{n}_{\gamma'} n_\beta) \} \}, \quad (108)$$

where the explicit dependence of all terms on t' has been suppressed. It is of advantage to rewrite Eq. (A1) in more compact form using, e.g.,

$$\langle \alpha\beta | \hat{V} | \alpha'\eta \rangle_{\mathcal{A}} := (n_\alpha n_\beta \bar{n}_{\alpha'} + \bar{n}_\alpha \bar{n}_\beta n_{\alpha'}) \langle \alpha\beta | v | \alpha'\eta \rangle_{\mathcal{A}}, \\ \langle \beta\eta | \hat{V}^\dagger | \gamma'\alpha' \rangle_{\mathcal{A}} := \langle \beta\eta | v | \gamma'\alpha' \rangle_{\mathcal{A}} (n_{\alpha'} n_{\gamma'} \bar{n}_\beta + \bar{n}_{\alpha'} \bar{n}_{\gamma'} n_\beta), \quad (109)$$

which explicitly depend on time t' via the occupation numbers involved. This leads to

$$C_{\alpha\beta\gamma,\alpha'\beta'\gamma'}^3(t) = -i \int_{t_0}^t dt' \exp(-i[\epsilon_\alpha + \epsilon_\beta + \epsilon_\gamma - \epsilon_{\alpha'} - \epsilon_{\beta'} - \epsilon_{\gamma'}](t - t')) \\ \times \sum_{\eta} \{ (1 - P_{\alpha'\beta'} - P_{\alpha'\gamma'}) \langle \alpha\beta | \hat{V} | \alpha'\eta \rangle_{\mathcal{A}} C_{\eta\gamma\beta'\gamma'} - (1 - P_{\alpha\beta} - P_{\alpha\gamma}) C_{\beta\gamma\eta\gamma'} \langle \alpha\eta | \hat{V}^\dagger | \alpha'\beta' \rangle_{\mathcal{A}} \\ + (1 - P_{\alpha'\beta'} - P_{\beta'\gamma'}) \langle \beta\gamma | \hat{V} | \beta'\eta \rangle_{\mathcal{A}} C_{\alpha\eta\alpha'\gamma'} - (1 - P_{\alpha\beta} - P_{\beta\gamma}) C_{\alpha\gamma\alpha'\eta} \langle \beta\eta | \hat{V}^\dagger | \beta'\gamma' \rangle_{\mathcal{A}} \\ + (1 - P_{\alpha'\gamma'} - P_{\beta'\gamma'}) \langle \gamma\alpha | \hat{V} | \gamma'\eta \rangle_{\mathcal{A}} C_{\eta\beta\alpha'\beta'} - (1 - P_{\alpha\gamma} - P_{\beta\gamma}) C_{\alpha\beta\eta\beta'} \langle \gamma\eta | \hat{V}^\dagger | \gamma'\alpha' \rangle_{\mathcal{A}} \} \quad (110)$$

in more compact form with $\hat{V}(t')$ and $C(t')$. Note that $\sum_{\eta}$ does not involve an intermediate propagator. An illustration of a single term in $C_{\alpha\beta\gamma,\alpha'\beta'\gamma'}$ is presented in Fig. 3.

Now one uses the result (90) for the two-body correlations, i.e.,

$$C_{\alpha\beta\alpha'\beta'}(t') = -i \int_{t_0}^{t'} dt'' \exp\{-i[\epsilon_\alpha + \epsilon_\beta - \epsilon_{\alpha'} - \epsilon_{\beta'}](t' - t'')\} \langle \alpha\beta | V_B(t'') | \alpha'\beta' \rangle \quad (111)$$

and inserts in Eq. (110). Since this is straight forward (but lengthy) an explicit representation is discarded.

The next step is to insert $C_{\alpha\beta\gamma,\alpha'\beta'\gamma'}(t')$ Eq. (110) in

$$\begin{aligned} \langle \alpha\beta | V_C^3 | \alpha'\beta' \rangle(t') := & \sum_{\gamma=\gamma'} \sum_{\lambda\mu} \{ \langle \alpha\gamma | v | \lambda\mu \rangle C_{\lambda\beta\mu,\alpha'\beta'\gamma'}^3(t') + \langle \beta\gamma | v | \lambda\mu \rangle C_{\alpha\lambda\mu,\alpha'\beta'\gamma'}^3(t') \} \\ & - \sum_{\gamma=\gamma'} \sum_{\lambda\mu} \{ C_{\alpha\beta\gamma,\lambda\beta'\mu}^{3\dagger}(t') \langle \lambda\mu | v | \alpha'\gamma' \rangle + C_{\alpha\beta\gamma,\alpha'\lambda\mu}^{3\dagger}(t') \langle \lambda\mu | v | \beta'\gamma' \rangle \}, \end{aligned} \quad (112)$$

which is the additional inhomogeneous term in Eq. (106) ($\text{Tr}_{(3=3')}[(v(13) + v(23)), c_3]$).

One obtains the extended two-body correlation $\hat{C}_{\alpha\beta\alpha'\beta'}(t)$ with the additional inhomogeneous term (112) by a further time integration as

$$\hat{C}_{\alpha\beta\alpha'\beta'}(t) = -i \int_{t_0}^t dt' \exp\{-i[\epsilon_\alpha + \epsilon_\beta - \epsilon_{\alpha'} - \epsilon_{\beta'}](t - t')\} \langle \alpha\beta | V_B(t') + V_C^3(t') | \alpha'\beta' \rangle, \quad (113)$$

which now includes the effects from 3-body correlations via Eq. (112).

For the diagonal element of the collision term one proceeds as in Sec. III A and obtains

$$\begin{aligned} \hat{I}_{\alpha\alpha}(t) = & -i \sum_{\beta} \sum_{\delta v} \{ \langle \alpha\beta | v | \delta v \rangle \hat{C}_{\delta v\alpha\beta}(t) - \hat{C}_{\alpha\beta\delta v}(t) \langle \delta v | v | \alpha\beta \rangle \} \\ = & I_{\alpha\alpha}^2(t) - \sum_{\beta} \sum_{\delta v} \int_{t_0}^t dt' \{ \exp\{-i[\epsilon_\delta + \epsilon_v - \epsilon_\alpha - \epsilon_\beta](t - t')\} \langle \alpha\beta | v | \delta v \rangle \langle \delta v | V_C^3(t') | \alpha\beta \rangle \\ & - \exp\{-i[\epsilon_\alpha + \epsilon_\beta - \epsilon_\delta - \epsilon_v](t - t')\} \langle \alpha\beta | V_C^{3\dagger}(t') | \delta v \rangle \langle \delta v | v | \alpha\beta \rangle \}, \end{aligned} \quad (114)$$

which includes the two-body contribution $I_{\alpha\alpha}^2(t)$ from Sec. III A and the additional three-body contributions by $V_C^3(t)$. The other two time integrals—appearing in the matrix elements of the two-body correlations and three-body correlations—are treated in the same fashion and approximately give additional factors $-i\pi\delta(\text{s.p. energies})$. Although one might question this approximation, which is invalid for short time intervals, this approximation leads to energy conservation in each individual interaction matrix element.

One then gets for the first two contributions in the three-body collision part

$$\begin{aligned} \tilde{I}_{\alpha\alpha}^3(t) \approx & -\pi^3 \sum_{\beta} \sum_{\delta v} \delta(\epsilon_\alpha + \epsilon_\beta - \epsilon_\delta - \epsilon_v) \langle \alpha\beta | v | \delta v \rangle \sum_{\gamma=\gamma'} \sum_{\lambda\mu} \sum_{\eta} \{ \delta(\epsilon_\lambda + \epsilon_v + \epsilon_\mu - \epsilon_\alpha - \epsilon_\beta - \epsilon_\gamma) \\ & \times \delta(\epsilon_\eta + \epsilon_\mu - \epsilon_\beta - \epsilon_\gamma) \langle \delta\gamma | v | \lambda\mu \rangle \langle \lambda v | \hat{V}(t) | \alpha\eta \rangle_{\mathcal{A}} \langle \eta\mu | V_B | \beta\gamma' \rangle \\ & + \delta(\epsilon_\delta + \epsilon_\lambda + \epsilon_\mu - \epsilon_\alpha - \epsilon_\beta - \epsilon_\gamma) \delta(\epsilon_\eta + \epsilon_\mu - \epsilon_\beta - \epsilon_\gamma) \langle v\gamma | v | \lambda\mu \rangle \langle \delta\lambda | \hat{V}(t) | \alpha\eta \rangle_{\mathcal{A}} \langle \eta\mu | V_B | \beta\gamma' \rangle \}, \end{aligned} \quad (115)$$

$$\begin{aligned} = & -\pi^3 \sum_{\beta} \sum_{\delta v} \delta(\epsilon_\alpha + \epsilon_\beta - \epsilon_\delta - \epsilon_v) \langle \alpha\beta | v | \delta v \rangle \sum_{\gamma=\gamma'} \sum_{\lambda\mu} \sum_{\eta} \{ \delta(\epsilon_\lambda + \epsilon_v - \epsilon_\alpha - \epsilon_\eta) \\ & \times \delta(\epsilon_\eta + \epsilon_\mu - \epsilon_\beta - \epsilon_\gamma) \langle \delta\gamma | v | \lambda\mu \rangle \langle \lambda v | \hat{V}(t) | \alpha\eta \rangle_{\mathcal{A}} \langle \eta\mu | V_B | \beta\gamma' \rangle \\ & + \delta(\epsilon_\delta + \epsilon_\lambda - \epsilon_\alpha - \epsilon_\eta) \delta(\epsilon_\eta + \epsilon_\mu - \epsilon_\beta - \epsilon_\gamma) \langle v\gamma | v | \lambda\mu \rangle \langle \delta\lambda | \hat{V}(t) | \alpha\eta \rangle_{\mathcal{A}} \langle \eta\mu | V_B | \beta\gamma' \rangle \}. \end{aligned} \quad (116)$$

The three δ functions in energy ensure that all four matrix elements of the interaction are taken on-shell. Defining

$$\begin{aligned} \langle \alpha\beta | \tilde{v} | \mu v \rangle & := \delta(\epsilon_\alpha + \epsilon_\beta - \epsilon_\mu - \epsilon_v) \langle \alpha\beta | v | \mu v \rangle, \\ \langle \alpha\beta | \tilde{V}(t) | \mu v \rangle_{\mathcal{A}} & := \delta(\epsilon_\alpha + \epsilon_\beta - \epsilon_\mu - \epsilon_v) \langle \alpha\beta | \hat{V}(t) | \mu v \rangle_{\mathcal{A}}, \\ \langle \alpha\beta | \tilde{V}_B(t) | \mu v \rangle_{\mathcal{A}} & := \delta(\epsilon_\alpha + \epsilon_\beta - \epsilon_\mu - \epsilon_v) \langle \alpha\beta | V_B(t) | \mu v \rangle_{\mathcal{A}}, \end{aligned} \quad (117)$$

one can rewrite Eq. (116) in more compact form as

$$\tilde{I}_{\alpha\alpha}^3(t) \approx -\pi^3 \sum_{\beta} \sum_{\delta v} \langle \alpha\beta | \tilde{v} | \delta v \rangle \sum_{\gamma} \sum_{\lambda\mu} \sum_{\eta} \{ \langle \delta\gamma | v | \lambda\mu \rangle \langle \lambda v | \tilde{V}(t) | \alpha\eta \rangle_{\mathcal{A}} \langle \eta\mu | \tilde{V}_B | \beta\gamma \rangle + \langle v\gamma | v | \lambda\mu \rangle \langle \delta\lambda | \tilde{V}(t) | \alpha\eta \rangle_{\mathcal{A}} \langle \eta\mu | \tilde{V}_B | \beta\gamma \rangle \}. \quad (118)$$

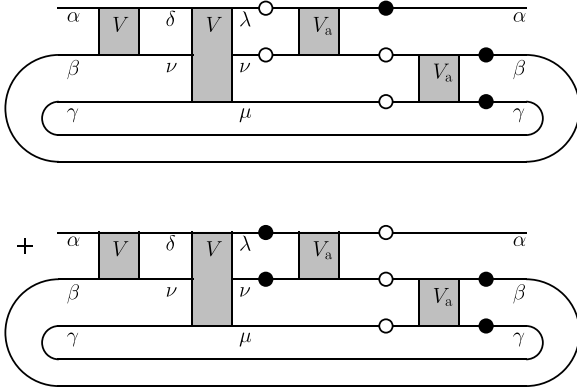


FIG. 4. Illustration of the interaction terms in Eq. (118) for the ‘loss terms’, i.e., including only the occupation factors $-n_\beta n_\gamma \bar{n}_\eta \bar{n}_\mu$ in the matrix element $\langle \eta\mu | \tilde{V}_B | \beta\gamma' \rangle$. The open circles denote occupation numbers while the full circles stand for blocking factors. The closed lines corresponds to a summation over particle 2 (or index β) and particle 3 (or index γ).

Here, $\sum_\beta \equiv \text{Tr}_2$ denotes the summation over particle 2 while $\sum_\gamma \equiv \text{Tr}_3$ denotes the summation over an additional particle 3. Figure 4 shows an illustration of the interaction terms in Eq. (118) for the ‘loss terms’, i.e., including only the occupation factors $-n_\beta n_\gamma \bar{n}_\eta \bar{n}_\mu$ in the matrix element $\langle \eta\mu | \tilde{V}_B | \beta\gamma' \rangle$. The open circles denote occupation numbers while the full circles stand for blocking factors. The full list of interaction terms is given in the Appendix.

IV. EQUILIBRATION IN A BOX WITH PERIODIC BOUNDARY CONDITIONS

In this section a numerical analysis of the relative impact of three-body contributions is addressed in model systems that are related to shifted Fermi spheres in a finite or infinite box with periodic boundary conditions. Such model systems reflect the initial overlap phase of heavy-ion collisions at low or intermediate energies where elastic collisions and Pauli blocking dominate the relaxation processes.

In case of periodic boundary conditions for a homogeneous system in a finite volume of size $V = a^3$ the single-particle eigenstates in coordinate space (for nucleons) are

$$\begin{aligned} \langle \mathbf{r} | n_x n_y n_z, n_s, n_\tau \rangle \\ = \langle \mathbf{r} | \alpha \rangle = \frac{1}{a^{3/2}} \exp \left(i \frac{2\pi}{a} [n_x x + n_y y + n_z z] \right) \chi(n_s) \tau(n_\tau), \end{aligned} \quad (119)$$

where $\chi(n_s)$ denote the two orthogonal eigenstates for the spin and $\tau(n_\tau)$ for isospin, respectively. By construction these states are orthonormal in all quantum numbers. The momentum components of particles are given by ($i = \pm 1, \pm 2, \pm 3, \dots$)

$$p_i(n_i) = \frac{2\pi \hbar}{a} n_i \quad (120)$$

and the single-particle energies by

$$\epsilon(n_x, n_y, n_z) = \frac{2\hbar^2 \pi^2}{M_N a^2} (n_x^2 + n_y^2 + n_z^2) =: \bar{\omega}(n_x^2 + n_y^2 + n_z^2) \quad (121)$$

when assuming no contributions from spin and isospin projections. Here, M_N stands for the nucleon mass. Within this basis the conservation of energy and momentum in two-body transitions can be strictly fulfilled and easily controlled via integer numbers n_i for momentum or n_i^2 for energy. Furthermore, the matrix elements of a local $\delta^3(\mathbf{r}_1 - \mathbf{r}_2)$ -type interaction, i.e.,

$$v(\mathbf{r}_1 - \mathbf{r}_2) = V_0 \delta^3(\mathbf{r}_1 - \mathbf{r}_2) \quad (122)$$

with interaction strength V_0 are, assuming the interactions again to be diagonal in spin and isospin,

$$\begin{aligned} \langle n_x^3, n_y^3, n_z^3; n_x^4, n_y^4, n_z^4 | v | n_x^1, n_y^1, n_z^1; n_x^2, n_y^2, n_z^2 \rangle \\ = \frac{V_0}{a^6} \int_{-a/2}^{a/2} dx \int_{-a/2}^{a/2} dy \int_{-a/2}^{a/2} dz \\ \times \exp \left(-i \frac{2\pi}{a} (n_x^1 + n_x^2 - n_x^3 - n_x^4) x \right) \\ \times \exp \left(-i \frac{2\pi}{a} (n_y^1 + n_y^2 - n_y^3 - n_y^4) y \right) \\ \times \exp \left(-i \frac{2\pi}{a} (n_z^1 + n_z^2 - n_z^3 - n_z^4) z \right) \\ = \frac{V_0}{a^3} \delta(n_x^1 + n_x^2 - n_x^3 - n_x^4) \delta(n_y^1 + n_y^2 - n_y^3 - n_y^4) \\ \times \delta(n_z^1 + n_z^2 - n_z^3 - n_z^4), \end{aligned} \quad (123)$$

which implies momentum conservation in the transitions $1 + 2 \leftrightarrow 3 + 4$.

A. The two-body case

In case of a diagonal density matrix $\rho_{\alpha\alpha'}(t) = \delta_{\alpha\alpha'} n_\alpha(t)$ the change in the occupation numbers $n_\alpha(t)$ is given by the two-body collision term $I_{\alpha\alpha}^2(t)$,

$$\begin{aligned} \frac{d}{dt} n_\alpha(t) = I_{\alpha\alpha}^2(t) = -\frac{i}{\hbar} \sum_\beta \sum_{\lambda\gamma} [\langle \alpha\beta | v | \lambda\gamma \rangle C_{\lambda\gamma\alpha\beta}(t) \\ - C_{\alpha\beta\lambda\gamma}(t) \langle \lambda\gamma | v | \alpha\beta \rangle]. \end{aligned} \quad (124)$$

When inserting the on-shell solution for $C_2(t)$ Eq. (97) one ends up with

$$\begin{aligned} I_{\alpha\alpha}^2(t) = \frac{2\pi}{\hbar} \sum_\beta \sum_{\lambda\gamma} \delta(\epsilon_\alpha + \epsilon_\beta - \epsilon_\lambda - \epsilon_\gamma) \\ \times \langle \alpha\beta | v | \lambda\gamma \rangle \langle \lambda\gamma | v | \alpha\beta \rangle_{\mathcal{A}} \\ \times [n_\lambda(t) n_\gamma(t) \bar{n}_\alpha(t) \bar{n}_\beta(t) - n_\alpha(t) n_\beta(t) \bar{n}_\lambda(t) \bar{n}_\gamma(t)]. \end{aligned} \quad (125)$$

These expressions can be worked out further in case of periodic boundary conditions by using Eqs. (121) and (123)

giving (in the on-shell case)

$$I_{\alpha\alpha}^2(t) = \frac{2\pi}{\hbar\bar{\omega}} \frac{3}{4} \frac{V_0^2}{a^6} \sum_{\beta} \sum_{\lambda\gamma} \delta(n_x^1 + n_x^2 - n_x^3 - n_x^4) \times \delta(n_y^1 + n_y^2 - n_y^3 - n_y^4) \delta(n_z^1 + n_z^2 - n_z^3 - n_z^4) \times \delta(m_{\alpha}^2 + m_{\beta}^2 - m_{\lambda}^2 - m_{\gamma}^2) [n_{\lambda}(t)n_{\gamma}(t)\bar{n}_{\alpha}(t)\bar{n}_{\beta}(t) - n_{\alpha}(t)n_{\beta}(t)\bar{n}_{\lambda}(t)\bar{n}_{\gamma}(t)] \quad (126)$$

using the notation

$$m_{\alpha}^2 = (n_x^1)^2 + (n_y^1)^2 + (n_z^1)^2 \quad (127)$$

which (except for a factor $\bar{\omega}$) gives the discrete single-particle energies.

Since the loss-term in Eq. (126) is proportional to n_{α} one can determine the on-shell collision width for the state α by

$$\Gamma_{\alpha} = \frac{2\pi}{\hbar\bar{\omega}} \frac{3}{4} \frac{V_0^2}{a^6} \sum_{\beta} \sum_{\lambda\gamma} \delta(n_x^1 + n_x^2 - n_x^3 - n_x^4) \times \delta(n_y^1 + n_y^2 - n_y^3 - n_y^4) \delta(n_z^1 + n_z^2 - n_z^3 - n_z^4) \times \delta(m_{\alpha}^2 + m_{\beta}^2 - m_{\lambda}^2 - m_{\gamma}^2) n_{\beta}\bar{n}_{\lambda}\bar{n}_{\gamma}. \quad (128)$$

In the continuum limit $a \rightarrow \infty$ the collision width of a particle with momentum $\mathbf{p}_{\alpha}$ reads for isotropic scattering with a constant and isotropic cross section

$$\sigma_{el} = \frac{M_N^2}{4\pi\hbar^4} \frac{3}{4} V_0^2 \quad (129)$$

(≈ 32.4 mb for $V_0 = \pm 300$ MeV fm³),

$$\Gamma(\mathbf{p}_{\alpha}) = \frac{\hbar c}{(2\pi\hbar)^3} d \int d^3 p_{\beta} \int d\Omega v_{\text{rel}}(\mathbf{p}_{\alpha} - \mathbf{p}_{\beta}) \times \frac{\sigma_{el}}{4\pi} n(\mathbf{p}_{\beta})(1 - n(\mathbf{p}_3))(1 - n(\mathbf{p}_4)), \quad (130)$$

where the final states $\mathbf{p}_3$ and $\mathbf{p}_4$ are fixed by energy and momentum conservation except for an angle $\Omega = (\cos(\vartheta), \phi)$ in the individual center of mass which has to be integrated over. The relative velocity—in the nonrelativistic limit—is given by

$$v_{\text{rel}}(\mathbf{p}_{\alpha} - \mathbf{p}_{\beta}) = \frac{|\mathbf{p}_{\alpha} - \mathbf{p}_{\beta}|}{M_N}, \quad (131)$$

while the factor $d = 4$ in Eq. (130) stems from summation over spin and isospin. The factor $3/4$ accounts for the fact that the antisymmetrized matrix element of the interaction vanishes in case of a δ force for nucleons with identical spin and isospin. Thus, in practice one has an effective degeneracy factor $d = 3$ for the scattering of non-identical particles.

The numerical calculations are carried out in boxes of side-length $a = 15$ – 18 fm (depending on energy) employing the lowest 1016 basis states. This basis was found to be sufficient since by increasing the number of basis states by a factor of 2 the results obtained for the quadrupole moment in momentum space

$$Q_2(t) = \sum_{\alpha} (2p_{\alpha,z}^2 - p_{\alpha,x}^2 - p_{\alpha,y}^2) n_{\alpha}(t) \quad (132)$$

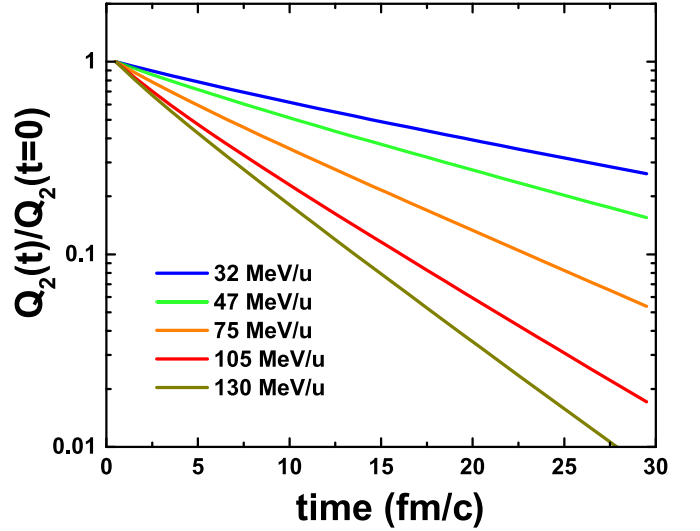


FIG. 5. Numerical results for the time evolution of the quadrupole moment (132)—normalized by $Q_2(t=0)$ —for collisions at 32, 47, 75, 105, and 130 MeV/nucleon (MeV/u).

do not change within the linewidth. It is noted in passing that the conservation of energy, momentum and total particle number is fulfilled within the numerical accuracy ($\approx 10^{-6}$).

The actual numerical results for $Q_2(t)$ are displayed in Fig. 5 for a couple of bombarding energies from 32 MeV/nucleon to 130 MeV/nucleon showing a pronounced exponential decay which can be well approximated by

$$Q_2(t) = Q_2(t=0) \exp(-t/\tau) \quad (133)$$

in all cases. The extracted results for the relaxation time τ are displayed in Fig. 6 for different bombarding energies by the

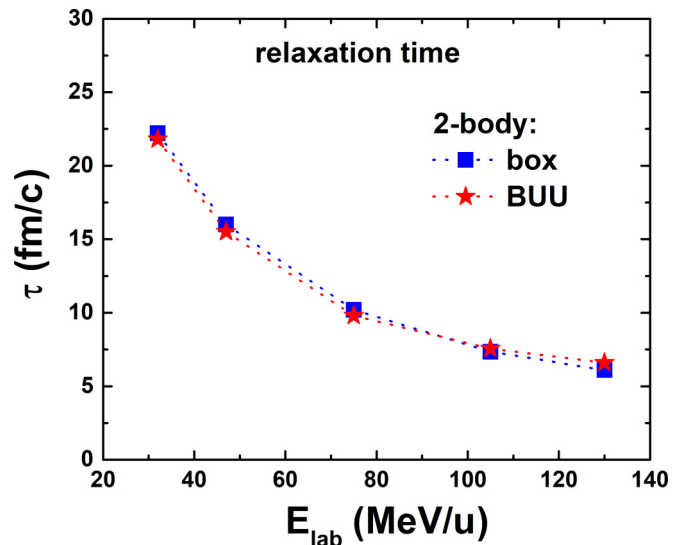


FIG. 6. The relaxation times τ extracted from the approximation (133) are shown by blue boxes for different bombarding energies. The corresponding results from BUU calculations are displayed by red stars.

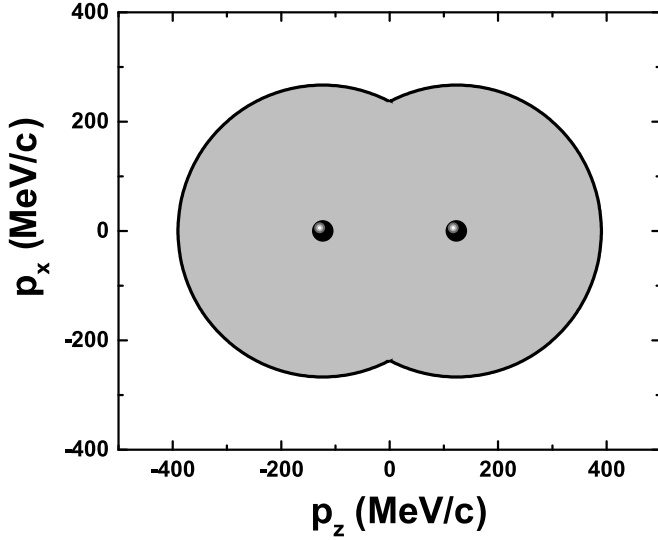


FIG. 7. The initial occupation number as a function of p_x and p_z in case of a bombarding energy of 130 MeV/nucleon in the continuum limit. The shaded area here indicates an occupation number of 1 while the outside region in momentum is unoccupied. The Fermi momentum is taken as $p_F = 268$ MeV/c which gives a nucleon density of ≈ 0.335 fm $^{-3}$.

blue boxes and found to drop rapidly with energy due to a weakening of the Pauli blocking in collisions.

How does this compare to the continuum limit within standard BUU transport calculations? To this end the BUU transport code from Ref. [11] is used in a finite box (of similar size) with an isotropic cross section $\sigma_{el} = 32.4$ mb, which corresponds to the interaction strength $V_0 = \pm 300$ MeV fm ^{3}

according to Eq. (129). The initial distribution in occupation number is illustrated in Fig. 7 as a function of p_x and p_z in case of a bombarding energy of 130 MeV/nucleon. The shaded area here indicates an occupation number of 1 while the outside region in momentum is unoccupied. The initial conditions for the BUU calculations are chosen by Monte Carlo (for 1000 parallel ensembles) according to Fig. 7 in momentum space and distributed homogeneously in a box of volume (17 fm) ^{3} . The actual densities in the calculations are 0.280 fm $^{-3}$, 0.298 fm $^{-3}$, 0.318 fm $^{-3}$, 0.335 fm $^{-3}$, and 0.34 fm $^{-3}$ for 32, 47, 75, 105, and 130 MeV/u, respectively. The densities for the discrete basis amount to 0.274 fm $^{-3}$, 0.294 fm $^{-3}$, 0.32 fm $^{-3}$, 0.33 fm $^{-3}$, and 0.335 fm $^{-3}$ for 32, 47, 75, 105, and 130 MeV/u, respectively, while the side-length of the boxes varies from 15–18 fm.

The results of the BUU calculations for the relaxation time τ are shown in Fig. 6 by the red stars, which are found to agree within a few percent with the discrete finite box calculations (blue boxes). One cannot expect identical results since the continuum limit implies slightly different initial conditions and slightly different initial densities as quantified above. Nevertheless, the good agreement shows that the on-shell correlation dynamics and standard 2-body collision transport simulations well compare with each other.

B. The three-body case

Let us now turn to the three-body collision integral. The three δ functions in energy ensure that all four matrix elements of the interaction are taken on-shell. Defining

$$\langle \alpha\beta | \hat{v} | \mu\nu \rangle := \delta(\epsilon_\alpha + \epsilon_\beta - \epsilon_\mu - \epsilon_\nu) \langle \alpha\beta | v | \mu\nu \rangle \quad (134)$$

one can rewrite the first terms in $\tilde{I}_{\alpha\alpha}^3(t)$ as

$$\begin{aligned} \tilde{I}_{\alpha\alpha}^3(t) \approx & -\pi^3 \sum_{\beta} \sum_{\delta\nu} \langle \delta\nu | v | \alpha\beta \rangle \sum_{\gamma} \sum_{\lambda\mu} \sum_{\eta} \{ \langle \alpha\gamma | \hat{v} | \lambda\mu \rangle [n_\lambda n_\beta \bar{n}_\delta + \bar{n}_\lambda \bar{n}_\beta n_\delta] \langle \lambda\beta | \hat{v} | \delta\eta \rangle_{\mathcal{A}} \langle \eta\mu | \hat{v} | \nu\gamma \rangle_{\mathcal{A}} [n_\nu n_\gamma \bar{n}_\eta \bar{n}_\mu - n_\eta n_\mu \bar{n}_\nu \bar{n}_\gamma] \\ & + \langle \beta\gamma | \hat{v} | \lambda\mu \rangle [n_\alpha n_\lambda \bar{n}_\delta + \bar{n}_\alpha \bar{n}_\lambda n_\delta] \langle \alpha\lambda | \hat{v} | \delta\eta \rangle_{\mathcal{A}} \langle \eta\mu | \hat{v} | \nu\gamma \rangle_{\mathcal{A}} [n_\nu n_\gamma \bar{n}_\eta \bar{n}_\mu - n_\eta n_\mu \bar{n}_\nu \bar{n}_\gamma] \\ & - [n_\alpha n_\beta \bar{n}_\lambda + \bar{n}_\alpha \bar{n}_\beta n_\lambda] \langle \alpha\beta | \hat{v} | \lambda\eta \rangle_{\mathcal{A}} \langle \eta\gamma | \hat{v} | \nu\mu \rangle_{\mathcal{A}} [n_\nu n_\mu \bar{n}_\eta \bar{n}_\gamma - n_\eta n_\gamma \bar{n}_\nu \bar{n}_\mu] \langle \lambda\mu | \hat{v} | \delta\gamma \rangle \\ & - [n_\alpha n_\beta \bar{n}_\delta + \bar{n}_\alpha \bar{n}_\beta n_\delta] \langle \alpha\beta | \hat{v} | \delta\eta \rangle_{\mathcal{A}} \langle \eta\gamma | \hat{v} | \lambda\mu \rangle_{\mathcal{A}} [n_\lambda n_\mu \bar{n}_\eta \bar{n}_\gamma - n_\eta n_\gamma \bar{n}_\lambda \bar{n}_\mu] \langle \lambda\mu | \hat{v} | \nu\gamma \rangle \} \dots \end{aligned} \quad (135)$$

Here, $\sum_{\beta} \equiv \text{Tr}_2$ denotes the summation over particle 2 while $\sum_{\gamma} \equiv \text{Tr}_3$ denotes the summation over an additional particle 3.

The calculations for the time evolution of the occupation numbers (A9) including the three-body collisions—listed in the Appendix—are carried in the same basis (of 1016 or 1568 states) as for the two-body case, where the possible final states ($\lambda\mu$) are precalculated for all initial states ($\alpha\beta$) and stored in a file. Due to the large number of interactions (see Appendix) and internal loops the required CPU time increases by a factor $\approx 2 \times 10^5$ compared to the two-body case. Again the conservation of energy, momentum and particle number is fulfilled within the numerical accuracy. A typical result for the relaxation of the quadrupole moment is shown in Fig. 8 for 32 MeV/nucleon (lower blue line), where the results are

compared to the two-body case (upper red line). This result remains practically unchanged when enhancing the number of basis states from 1016 to 1568 (black dotted line), which also increases the CPU time by approximately another order of magnitude.

Extracting again a relaxation time τ by an exponential fit one obtains the results displayed in Fig. 9 for different bombarding energies which shows that by including the three-body interactions the relaxation time τ decreases by about a factor 1.5 at 32 MeV/nucleon. This decrease becomes larger (up to a factor of about ≈ 3) with increasing bombarding energy since the systems becomes more dense—due to the enhanced number of occupied states—and transverse momentum states are more easily accessed by three-body scattering than in case of Pauli-blocked two-body scatterings.

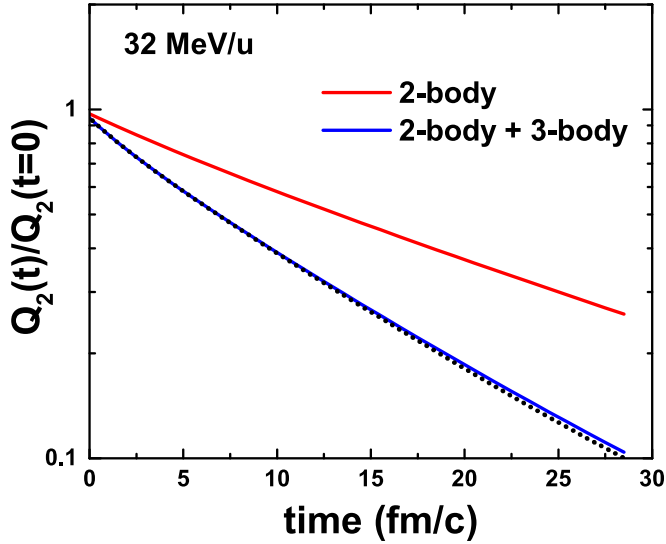


FIG. 8. Comparison of the results for the quadrupole moment $Q_2(t)$ —normalized by $Q_2(t=0)$ —for the two-body case (upper red line) with the result including the additional three-body interactions (lower blue line) for 32 MeV/nucleon (MeV/u). The result for the two-body + three-body case within an enhanced basis of states is shown by the black dotted line and almost identical to the blue line.

These results indicate that standard transport calculations—only including two-body scattering—might underestimate the relaxation in low energy heavy-ion collisions.

Some comments to these results are in order: First of all the effects of three-body collisions might be overestimated in the model study due to the assumption of isotropic collisions and energy independent cross sections. In reality the individual two-body collisions get more forward peaked with increasing energy, which leads to a stronger blocking for initial times. Moreover, the size of the elastic cross sec-

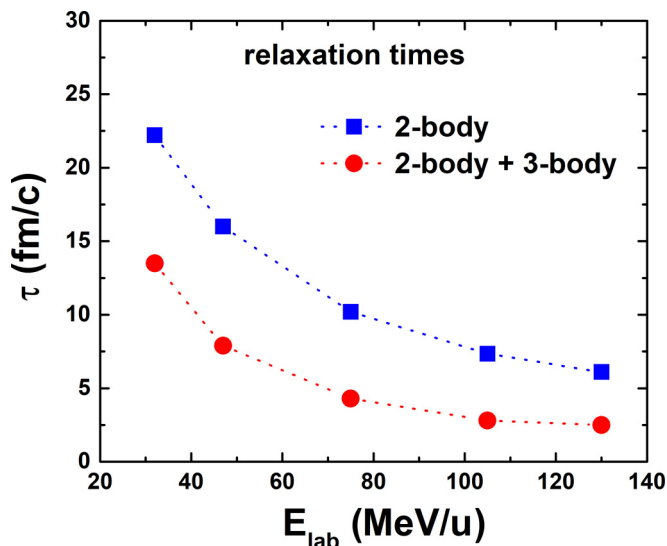


FIG. 9. Comparison of the results for the relaxation time τ for the two-body case (blue boxes) with the result including the additional three-body interactions (red dots) for different bombarding energies.

tion decreases with collision energy. Furthermore, the leading order approximation for the three-body interactions might be questioned. Higher order terms ($\sim \text{Tr}(vc_2c_2)$)—leading to an additional ‘interaction box’ (cf. the illustration in Fig. 4 for the collision term)—might change the present results and also reduce the net interaction cross section. Unfortunately, concrete studies will increase the CPU time by further orders of magnitude if one tries to quantify the influence of these higher order terms. Accordingly, this question remains open until increasing CPU/GPU power or new algorithms are available.

V. BUU CALCULATIONS FOR HIGH ENERGY PARTICLE EMISSION IN FUSION REACTIONS

The results presented in the previous section are primarily academic and the question arises if a higher stopping power by elastic three-body collisions might leave traces in experimental observables. At first sight one might look at the proton rapidity distributions, which should become narrower with increasing cross section, but this will require a systematic study for various target masses as well as a scan in bombarding energy. An alternative is the emission of energetic nucleons which preferentially occurs before kinetic equilibrium is achieved.

To this end a standard BUU code (described in more detail in [11]) is used with a mean field of Skyrme type and a finite range Yukawa interaction for nucleons. Coulomb forces are included as well as the excitation and decay of Δ resonances. The elastic scattering cross section σ_{el} for binary nucleon-nucleon collisions is provided by the Cugnon parametrization [38] and Pauli blocking for the final states by calculating the ‘local’ occupation in phase-space by an average over all ensembles. The simulations are performed within the parallel ensemble method by propagating simultaneously ≈ 2000 ensembles. No coalescence of nucleons for the formation of light clusters is included.

In this BUU model the quadrupole moment in momentum space of the emitted nucleons is calculated for different cross sections, i.e., multiplying the actual elastic cross section σ_{el} by a factor 5/3 or 2 in fusion reactions of central collisions. Here, the bombarding energy should not be too high because the compound system in reality will break into pieces. A suitable system is found to be central collisions of, e.g., $^{97}\text{Rh} + ^{97}\text{Rh}$ at 40 MeV/nucleon, which leads to a fusion process with the emission of particles. This is demonstrated in Fig. 10 where the nucleon density distribution $\rho(x, y=0, z; t)$ is shown (in units of fm^{-3}) at times of 5, 40, 80, 120 fm/c employing the default elastic cross section σ_{el} from [38] while x, z are given in fm. The first contour plot at $t = 10$ fm/c displays the two nuclei just before contact while in the second plot for $t = 40$ fm/c a maximum overlap of the impinging nuclei is achieved. They form a fusing system as seen from the density distributions at 80 fm/c and 120 fm/c while throughout the reaction nucleons are emitted. The emitted nucleons can clearly be separated from the fusing remnant by a cut in coordinate space for $r > 10$ fm.

Only very energetic nucleons are emitted early and provide information about the dynamics (or stopping power) at early

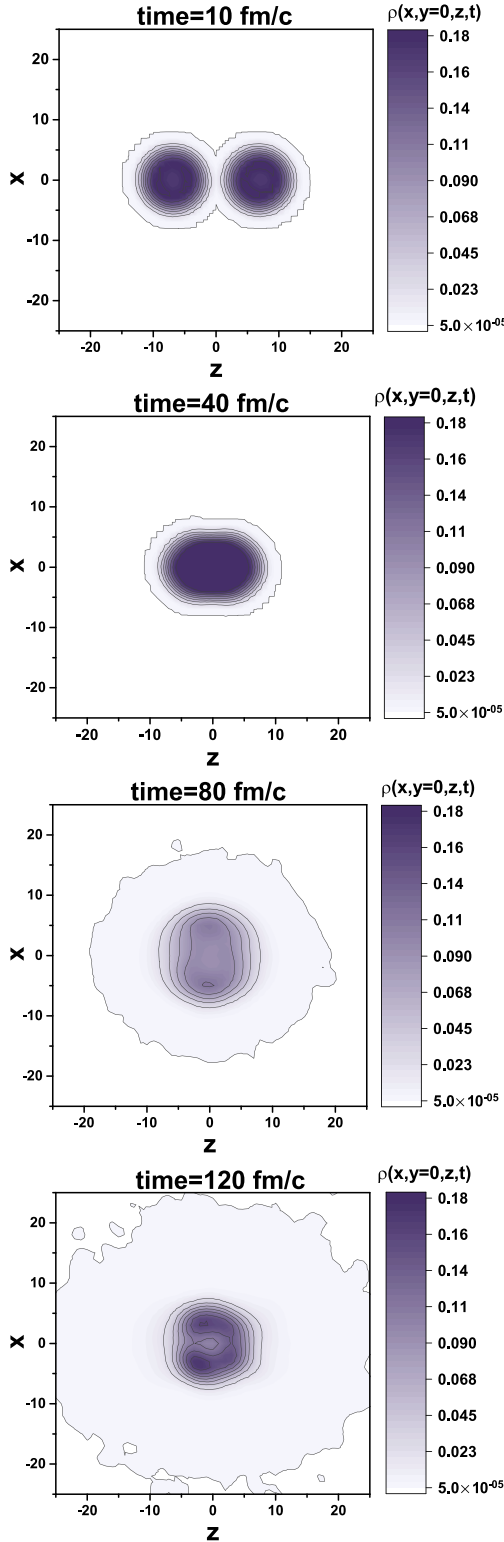


FIG. 10. The nucleon density distribution $\rho(x, y = 0, z; t)$ for central collisions of $^{97}\text{Rh} + ^{97}\text{Rh}$ at 40 MeV/nucleon for times of 10, 40, 80, and 120 fm/c. The density contours—specified on the r.h.s.—are given in units of fm^{-3} .

times. To this end high statistics BUU calculations have been performed and the quadrupole moment in momentum space has been calculated for nucleons outside of the compound

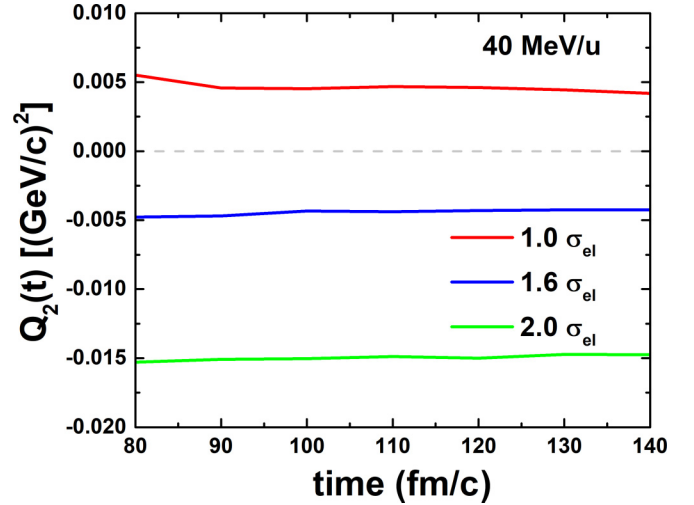


FIG. 11. Comparison of the results for the quadrupole moment in momentum space of energetic nucleons (>60 MeV) in central collisions of $^{97}\text{Rh} + ^{97}\text{Rh}$ at 40 MeV/nucleon (MeV/u) for different elastic nucleon-nucleon cross sections of $1.0 \sigma_{el}$, $1.6 \sigma_{el}$ and $2.0 \sigma_{el}$, where σ_{el} denotes the elastic Cugnon cross section from [38].

nucleus (>10 fm) for different cutoff kinetic energies E_c for the nucleons in steps of 10 MeV. It is found that only very energetic nucleons above 60 MeV of kinetic energy provide a quadrupole moment that is approximately constant in time after 80 fm/c and can be used as an observable for the stopping of energetic nucleons.

The numerical results for the quadrupole moment in momentum space of the emitted nucleons (>60 MeV) are presented in Fig. 11 as a function of time for different interaction strength, i.e., multiplication factors of the elastic cross section σ_{el} in the BUU code. Inelastic reaction channels are found to play no role at the energy of 40 MeV/nucleon. The actual numbers in Fig. 11 show that the quadrupole moment of the energetic nucleons is approximately stable in time and positive for the default interaction strength (upper line), which implies a preferential forward emission of these nucleons. With increasing cross section the quadrupole moment turns negative, which implies a preferential emission in sideward direction. The expected factor at 40 MeV/nucleon is about 5/3 according to Fig. 9 and indicates that the emission of these energetic nucleons should be slightly enhanced in sideward direction contrary to the expectation in the default two-body interaction scenario. Although such experimental studies might suffer from low statistics these investigations could shed further light on the stopping power of elastic nucleon-nucleon collisions and the role of three-body collision processes.

Some comments to these results are in order: One might argue that actual transport simulations for heavy-ion collisions, that are based on the two-body level using an effective in-medium cross section for $2 \leftrightarrow 2$ scattering, describe actual data on rapidity distributions or collective flow quite well and that there is no need for additional $3 \leftrightarrow 3$ interactions. In principle, these effective in-medium cross sections might be derived from G -matrix calculations [cf. Eq. (72)], but actually

are ‘fitted’ to experimental data for ‘stopping’ or ‘collective flow’ within the two-body approach alone. Even in case of microscopic G -matrix calculations the situation is not unique since these calculations start from a bare two-body interaction, which is iterated for the T matrix that finally describes the scattering phase-shifts. However, different two-body interactions may give the same on-shell T matrix but differ in the off-shell matrix elements of the interaction that appear in the G -matrix equation (72). Thus there is no unique answer from theory for the ‘right’ in-medium two-body cross section so far. On the other hand, systematic experimental studies e.g. on pre-equilibrium particle emission as a function of the system size (or mass of projectile and target)—in comparison to two-body transport calculations—might indicate the necessity to incorporate $3 \leftrightarrow 3$ interactions.

VI. SUMMARY

In this study the formulation of $3 \leftrightarrow 3$ interactions has been based on the equations of motion method for identical fermions which has also been denoted as quantum correlation dynamics. All explicit interactions terms for the two-body and three-body correlations have been derived and/or presented in full detail and discussed in a single-particle basis $|\alpha\rangle$. In this framework the on-shell two-body collision integral has been reviewed and the on-shell three-body collision integral has been derived in the on-shell limit on the basis of the same two-body interaction in leading order. Here, no explicit three-body interaction has been accounted for. The resulting equations obey particle number as well as energy-momentum conservation and avoid a double-counting of interactions and maintain full antisymmetry with respect to particle exchange.

For a quantification of the relative impact of three-body interactions a model study for a homogeneous system in space in a finite box with periodic boundary conditions has been performed. The systems investigated are spin-isospin symmetric nuclear matter configurations with initial occupation distributions that are given by shifted Fermi spheres (without overlap, cf. Fig. 7) as encountered in the initial phase of nucleus-nucleus collisions after contact. The results for the relaxation times—employing an effective two-body interaction—have been compared to Boltzmann-Uehling-Uhlenbeck (BUU) transport calculations in the continuum limit for the same bombarding energies and agree on the level of a few percent. It has been found that the additional three-body interactions reduce the relaxation times up to a factor of ≈ 3 —depending on bombarding energy—which indicates that the equilibration in low-energy nucleus-nucleus collisions by elastic nucleon-nucleon scattering might have been underestimated so far.

In principle, one expects that the influence of three-body reactions should become more important with increasing bombarding energy in actual heavy-ion collisions due to compression effects, but in this case the inelastic channels between hadrons of different quantum numbers (resonances, mesons, etc.) become dominant as noted in the Introduction. This particular situation is not in the range of the model calculations addressed here because they only describe elastic interactions of identical fermions including exact Pauli blocking. Thus the model cases considered had to be restricted to lower energies

where the exact implementation of Pauli blocking is important. As shown in Fig. 6 the blocking effect decreases for higher bombarding energies and the proper implementation of Pauli blocking becomes less important such that classical estimates might be used. The classical collision criterion involves the critical impact parameter $b = \sqrt{\sigma/\pi}$, which in the present model case gives $b \approx 1$ fm. This has to be compared to the average distance between nucleons at density ρ given by $d = \rho^{-1/3}$, i.e., amounts to $d \approx 1.45$ fm at twice nuclear matter density. Note that the model studies performed here refer to shifted Fermi spheres (cf. Fig. 7) and give a nucleon density which at most gives twice nuclear matter density, i.e., in case the shifted Fermi spheres just touch each other in momentum space or are fully separated. On the other hand screening effects by, e.g., by mesons reduce the probability for three-body fermion scattering such that the geometrical estimates at higher density have to be taken with care. A fully quantum mechanically solution to this problem is not available so far.

Furthermore, it has been shown within BUU transport calculations that the enhanced stopping by $3 \leftrightarrow 3$ collisions—modeled by a corresponding increase in the elastic two-body cross section—shows up in the angular distribution of energetic nucleons (>60 MeV of kinetic energy) in, e.g., central $^{97}_{45}\text{Rh}$ collisions at 40 MeV/nucleon that lead to the formation of a compound nucleus, i.e., in a partial fusion reaction. The angular distribution of the energetic nucleons changes from a slightly forward peaked angular distribution to a slightly side-wards peaked angular distribution which could be controlled experimentally. However, a single experiment will not provide a final answer on the role of elastic three-body reactions because the ‘correct’ in-medium two-body cross section cannot be determined sufficiently accurate by G -matrix calculations. Accordingly, systematic experimental studies e.g. on pre-equilibrium particle emission will be necessary—in comparison to two-body transport calculations—for different sizes of projectile and target to unravel the role of elastic three-body nucleon interactions.

In principle such three-body interactions can be implemented in transport codes employing the in-cell method [39], which is based on matrix elements and a probabilistic interpretation of the collision integral, and added to the two-body matrix element squared as in case of the calculations in the discrete basis performed in Sec. IV. Such concepts have been employed, e.g., in Refs. [18,40] for $2 \leftrightarrow 3$ channels.

One should note, furthermore, that the energy-independent isotropic cross section employed in the model calculations might overestimate the effect of $3 \leftrightarrow 3$ reactions and future studies are required with more realistic interactions. Additionally, the role of higher order interaction terms has to be clarified.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX

In order to transparently derive all terms in the three-body part of the collision integral the result for $c_3(t)$ in the natural basis in the limit (107) is recalled:

$$\begin{aligned}
C_{\alpha\beta\gamma,\alpha'\beta'\gamma'}^3(t) = & -i \int_{t_0}^t dt' \exp(-i[\epsilon_\alpha + \epsilon_\beta + \epsilon_\gamma - \epsilon_{\alpha'} - \epsilon_{\beta'} - \epsilon_{\gamma'}](t - t')) \\
& \times \sum_{\eta} \{ (n_\alpha n_\beta \bar{n}_{\alpha'} + \bar{n}_\alpha \bar{n}_\beta n_{\alpha'}) \langle \alpha\beta | v | \alpha' \eta \rangle_{\mathcal{A}} C_{\eta\gamma\beta'\gamma'} - (n_\alpha n_\beta \bar{n}_{\beta'} + \bar{n}_\alpha \bar{n}_\beta n_{\beta'}) \langle \alpha\beta | v | \beta' \eta \rangle_{\mathcal{A}} C_{\eta\gamma\alpha'\gamma'} \\
& - (n_\alpha n_\beta \bar{n}_{\gamma'} + \bar{n}_\alpha \bar{n}_\beta n_{\gamma'}) \langle \alpha\beta | v | \gamma' \eta \rangle_{\mathcal{A}} C_{\eta\gamma\beta'\alpha'} - (C_{\beta\gamma\eta\gamma'} \langle \alpha\eta | v | \alpha' \beta' \rangle_{\mathcal{A}} (n_{\alpha'} n_\beta \bar{n}_\alpha + \bar{n}_{\alpha'} \bar{n}_\beta n_\alpha) \\
& - C_{\alpha\gamma\eta\gamma'} \langle \beta\eta | v | \alpha' \beta' \rangle_{\mathcal{A}} (n_{\alpha'} n_\beta \bar{n}_\beta + \bar{n}_{\alpha'} \bar{n}_\beta n_\beta) - C_{\beta\alpha\eta\gamma'} \langle \gamma\eta | v | \alpha' \beta' \rangle_{\mathcal{A}} (n_{\alpha'} n_\beta \bar{n}_{\gamma'} + \bar{n}_{\alpha'} \bar{n}_\beta n_{\gamma'})) \\
& + ((n_\beta n_\gamma \bar{n}_{\beta'} + \bar{n}_\beta \bar{n}_\gamma n_{\beta'}) \langle \beta\gamma | v | \beta' \eta \rangle_{\mathcal{A}} C_{\alpha\eta\alpha'\gamma'} - (n_\beta n_\gamma \bar{n}_{\alpha'} + \bar{n}_\beta \bar{n}_\gamma n_{\alpha'}) \langle \beta\gamma | v | \alpha' \eta \rangle_{\mathcal{A}} C_{\alpha\eta\beta'\gamma'} \\
& - (n_\beta n_\gamma \bar{n}_{\gamma'} + \bar{n}_\beta \bar{n}_\gamma n_{\gamma'}) \langle \beta\gamma | v | \gamma' \eta \rangle_{\mathcal{A}} C_{\alpha\eta\alpha'\beta'} - (C_{\alpha\gamma\alpha'\eta} \langle \beta\eta | v | \beta' \gamma' \rangle_{\mathcal{A}} (n_{\beta'} n_{\gamma'} \bar{n}_\beta + \bar{n}_{\beta'} \bar{n}_{\gamma'} n_\beta) \\
& - C_{\beta\gamma\alpha'\eta} \langle \alpha\eta | v | \beta' \gamma' \rangle_{\mathcal{A}} (n_{\beta'} n_{\gamma'} \bar{n}_\alpha + \bar{n}_{\beta'} \bar{n}_{\gamma'} n_\alpha) - C_{\alpha\beta\alpha'\eta} \langle \gamma\eta | v | \beta' \gamma' \rangle_{\mathcal{A}} (n_{\beta'} n_{\gamma'} \bar{n}_{\gamma'} + \bar{n}_{\beta'} \bar{n}_{\gamma'} n_{\gamma'})) \\
& + (n_\alpha n_\gamma \bar{n}_{\gamma'} + \bar{n}_\alpha \bar{n}_\gamma n_{\gamma'}) \langle \gamma\alpha | v | \gamma' \eta \rangle_{\mathcal{A}} C_{\eta\beta\alpha'\beta'} - (n_\alpha n_\gamma \bar{n}_{\beta'} + \bar{n}_\alpha \bar{n}_\gamma n_{\beta'}) \langle \gamma\alpha | v | \beta' \eta \rangle_{\mathcal{A}} C_{\eta\beta\alpha'\gamma'} \\
& - (n_\alpha n_\gamma \bar{n}_{\alpha'} + \bar{n}_\alpha \bar{n}_\gamma n_{\alpha'}) \langle \gamma\alpha | v | \alpha' \eta \rangle_{\mathcal{A}} C_{\eta\gamma\beta'\beta'} - (C_{\alpha\beta\eta\beta'} \langle \gamma\eta | v | \gamma' \alpha' \rangle_{\mathcal{A}} (n_{\alpha'} n_{\gamma'} \bar{n}_{\gamma'} + \bar{n}_{\alpha'} \bar{n}_{\gamma'} n_{\gamma'}) \\
& - C_{\gamma\beta\eta\beta'} \langle \alpha\eta | v | \gamma' \alpha' \rangle_{\mathcal{A}} (n_{\alpha'} n_{\gamma'} \bar{n}_\alpha + \bar{n}_{\alpha'} \bar{n}_{\gamma'} n_\alpha) - C_{\alpha\gamma\eta\beta'} \langle \beta\eta | v | \gamma' \alpha' \rangle_{\mathcal{A}} (n_{\alpha'} n_{\gamma'} \bar{n}_\beta + \bar{n}_{\alpha'} \bar{n}_{\gamma'} n_\beta) \} \}, \quad (A1)
\end{aligned}$$

where the explicit dependence of all terms on t' has been suppressed. In the on-shell limit this gives

$$\begin{aligned}
C_{\alpha\beta\gamma,\alpha'\beta'\gamma'}^3(t) \approx & -\pi^2 \delta(\epsilon_\alpha + \epsilon_\beta + \epsilon_\gamma - \epsilon_{\alpha'} - \epsilon_{\beta'} - \epsilon_{\gamma'}) \\
& \times \sum_{\eta} \{ (n_\alpha n_\beta \bar{n}_{\alpha'} + \bar{n}_\alpha \bar{n}_\beta n_{\alpha'}) \langle \alpha\beta | v | \alpha' \eta \rangle_{\mathcal{A}} \langle \eta\gamma | \tilde{v} | \beta' \gamma' \rangle_{\mathcal{A}} (n_{\beta'} n_{\gamma'} \bar{n}_\eta \bar{n}_\gamma - \bar{n}_{\beta'} \bar{n}_{\gamma'} n_\eta n_\gamma) \\
& - (n_\alpha n_\beta \bar{n}_{\beta'} + \bar{n}_\alpha \bar{n}_\beta n_{\beta'}) \langle \alpha\beta | v | \beta' \eta \rangle_{\mathcal{A}} \langle \eta\gamma | \tilde{v} | \alpha' \gamma' \rangle_{\mathcal{A}} (n_{\alpha'} n_{\gamma'} \bar{n}_\eta \bar{n}_\gamma - \bar{n}_{\alpha'} \bar{n}_{\gamma'} n_\eta n_\gamma) \\
& + (n_\alpha n_\beta \bar{n}_{\gamma'} + \bar{n}_\alpha \bar{n}_\beta n_{\gamma'}) \langle \alpha\beta | v | \gamma' \eta \rangle_{\mathcal{A}} \langle \eta\gamma | \tilde{v} | \alpha' \beta' \rangle_{\mathcal{A}} (n_{\alpha'} n_{\beta'} \bar{n}_\eta \bar{n}_\gamma - \bar{n}_{\alpha'} \bar{n}_{\beta'} n_\eta n_\gamma) \\
& - \langle \beta\gamma | \tilde{v} | \eta\gamma' \rangle_{\mathcal{A}} (n_{\gamma'} n_\eta \bar{n}_\beta \bar{n}_\gamma - \bar{n}_{\gamma'} \bar{n}_\eta n_\beta n_\gamma) \langle \alpha\eta | v | \alpha' \beta' \rangle_{\mathcal{A}} (n_{\alpha'} n_\beta \bar{n}_\alpha + \bar{n}_{\alpha'} \bar{n}_\beta n_\alpha) \\
& + \langle \alpha\gamma | \tilde{v} | \eta\gamma' \rangle_{\mathcal{A}} (n_{\gamma'} n_\eta \bar{n}_\alpha \bar{n}_\gamma - \bar{n}_{\gamma'} \bar{n}_\eta n_\alpha n_\gamma) \langle \beta\eta | v | \alpha' \beta' \rangle_{\mathcal{A}} (n_{\alpha'} n_\beta \bar{n}_\beta + \bar{n}_{\alpha'} \bar{n}_\beta n_\beta) \\
& - \langle \alpha\beta | \tilde{v} | \eta\gamma' \rangle_{\mathcal{A}} (n_{\gamma'} n_\eta \bar{n}_\beta \bar{n}_\alpha - \bar{n}_{\gamma'} \bar{n}_\eta n_\beta n_\alpha) \langle \gamma\eta | v | \alpha' \beta' \rangle_{\mathcal{A}} (n_{\alpha'} n_\beta \bar{n}_{\gamma'} + \bar{n}_{\alpha'} \bar{n}_\beta n_{\gamma'}) \\
& + (n_\beta n_\gamma \bar{n}_{\beta'} + \bar{n}_\beta \bar{n}_\gamma n_{\beta'}) \langle \beta\gamma | v | \beta' \eta \rangle_{\mathcal{A}} \langle \alpha\eta | \tilde{v} | \alpha' \gamma' \rangle_{\mathcal{A}} (n_{\alpha'} n_{\gamma'} \bar{n}_\eta \bar{n}_\alpha - \bar{n}_{\alpha'} \bar{n}_{\gamma'} n_\eta n_\alpha) \\
& - (n_\beta n_\gamma \bar{n}_{\alpha'} + \bar{n}_\beta \bar{n}_\gamma n_{\alpha'}) \langle \beta\gamma | v | \alpha' \eta \rangle_{\mathcal{A}} \langle \alpha\eta | \tilde{v} | \beta' \gamma' \rangle_{\mathcal{A}} (n_{\beta'} n_{\gamma'} \bar{n}_\eta \bar{n}_\alpha - \bar{n}_{\beta'} \bar{n}_{\gamma'} n_\eta n_\alpha) \\
& - (n_\beta n_\gamma \bar{n}_{\gamma'} + \bar{n}_\beta \bar{n}_\gamma n_{\gamma'}) \langle \beta\gamma | v | \gamma' \eta \rangle_{\mathcal{A}} \langle \alpha\eta | \tilde{v} | \alpha' \beta' \rangle_{\mathcal{A}} (n_{\alpha'} n_\beta \bar{n}_\eta \bar{n}_\alpha - \bar{n}_{\alpha'} \bar{n}_\beta n_\eta n_\alpha) \\
& - \langle \alpha\gamma | \tilde{v} | \alpha' \eta \rangle_{\mathcal{A}} (n_{\alpha'} n_\eta \bar{n}_\gamma \bar{n}_\alpha - \bar{n}_{\alpha'} \bar{n}_\eta n_\gamma n_\alpha) \langle \beta\eta | v | \beta' \gamma' \rangle_{\mathcal{A}} (n_{\beta'} n_{\gamma'} \bar{n}_\beta + \bar{n}_{\beta'} \bar{n}_{\gamma'} n_\beta) \\
& + \langle \beta\gamma | \tilde{v} | \alpha' \eta \rangle_{\mathcal{A}} (n_{\alpha'} n_\eta \bar{n}_\beta \bar{n}_\gamma - \bar{n}_{\alpha'} \bar{n}_\eta n_\beta n_\gamma) \langle \alpha\eta | v | \beta' \gamma' \rangle_{\mathcal{A}} (n_{\beta'} n_{\gamma'} \bar{n}_\alpha + \bar{n}_{\beta'} \bar{n}_{\gamma'} n_\alpha) \\
& + \langle \alpha\beta | \tilde{v} | \alpha' \eta \rangle_{\mathcal{A}} (n_{\alpha'} n_\eta \bar{n}_\beta \bar{n}_\alpha - \bar{n}_{\alpha'} \bar{n}_\eta n_\beta n_\gamma) \langle \gamma\eta | v | \beta' \gamma' \rangle_{\mathcal{A}} (n_{\beta'} n_{\gamma'} \bar{n}_{\gamma'} + \bar{n}_{\beta'} \bar{n}_{\gamma'} n_{\gamma'}) \\
& + n_\alpha n_\gamma \bar{n}_{\gamma'} + \bar{n}_\alpha \bar{n}_\gamma n_{\gamma'} \langle \gamma\alpha | v | \gamma' \eta \rangle_{\mathcal{A}} \langle \eta\beta | \tilde{v} | \alpha' \beta' \rangle_{\mathcal{A}} (n_{\alpha'} n_\beta \bar{n}_\eta \bar{n}_\beta - \bar{n}_{\alpha'} \bar{n}_\beta n_\eta n_\beta) \\
& - (n_\alpha n_\gamma \bar{n}_{\beta'} + \bar{n}_\alpha \bar{n}_\gamma n_{\beta'}) \langle \gamma\alpha | v | \beta' \eta \rangle_{\mathcal{A}} \langle \eta\beta | \tilde{v} | \alpha' \gamma' \rangle_{\mathcal{A}} (n_{\alpha'} n_{\gamma'} \bar{n}_\eta \bar{n}_\beta - \bar{n}_{\alpha'} \bar{n}_{\gamma'} n_\eta n_\beta)
\end{aligned}$$

$$\begin{aligned}
& - (n_\alpha n_\gamma \bar{n}_{\alpha'} + \bar{n}_\alpha \bar{n}_\gamma n_{\alpha'}) \langle \gamma \alpha | v | \alpha' \eta \rangle_{\mathcal{A}} \langle \eta \beta | \tilde{v} | \gamma' \beta' \rangle_{\mathcal{A}} (n_{\gamma'} n_\beta \bar{n}_\eta \bar{n}_\beta - \bar{n}_{\gamma'} \bar{n}_\beta n_\eta n_\beta) \\
& - \langle \alpha \beta | \tilde{v} | \eta \beta' \rangle_{\mathcal{A}} (n_{\beta'} n_\eta \bar{n}_\beta \bar{n}_\alpha - \bar{n}_{\beta'} \bar{n}_\eta n_\alpha n_\beta) \langle \gamma \eta | v | \gamma' \alpha' \rangle_{\mathcal{A}} (n_{\alpha'} n_{\gamma'} \bar{n}_\gamma + \bar{n}_{\alpha'} \bar{n}_{\gamma'} n_\gamma) \\
& + \langle \gamma \beta | \tilde{v} | \eta \beta' \rangle_{\mathcal{A}} (n_{\beta'} n_\eta \bar{n}_\beta \bar{n}_\gamma - \bar{n}_{\beta'} \bar{n}_\eta n_\gamma n_\beta) \langle \alpha \eta | v | \gamma' \alpha' \rangle_{\mathcal{A}} (n_{\alpha'} n_{\gamma'} \bar{n}_\alpha + \bar{n}_{\alpha'} \bar{n}_{\gamma'} n_\alpha) \\
& + \langle \alpha \gamma | \tilde{v} | \eta \beta' \rangle_{\mathcal{A}} (n_{\beta'} n_\eta \bar{n}_\gamma \bar{n}_\alpha - \bar{n}_{\beta'} \bar{n}_\eta n_\gamma n_\alpha) \langle \beta \eta | v | \gamma' \alpha' \rangle_{\mathcal{A}} (n_{\alpha'} n_{\gamma'} \bar{n}_\beta + \bar{n}_{\alpha'} \bar{n}_{\gamma'} n_\beta) \} \quad (\text{A2})
\end{aligned}$$

using

$$\langle \alpha \gamma | \tilde{v} | \eta \lambda \rangle_{\mathcal{A}} = \delta(\epsilon_\alpha + \epsilon_\gamma - \epsilon_\eta - \epsilon_\lambda) \langle \alpha \gamma | v | \eta \lambda \rangle_{\mathcal{A}},$$

which is dimensionless.

Equation (A2) can be rewritten as

$$\begin{aligned}
C_{\alpha\beta\gamma,\alpha'\beta'\gamma'}^3(t) & \approx -\pi^2 \sum_{\eta} \{ \langle \alpha \beta | \tilde{v} | \alpha' \eta \rangle_{\mathcal{A}} \langle \eta \gamma | \tilde{v} | \beta' \gamma' \rangle_{\mathcal{A}} N(\alpha\beta\alpha'\beta'\gamma'\eta\gamma) \\
& - \langle \alpha \beta | \tilde{v} | \beta' \eta \rangle_{\mathcal{A}} \langle \eta \gamma | \tilde{v} | \alpha' \gamma' \rangle_{\mathcal{A}} N(\alpha\beta\beta'\alpha'\gamma'\eta\gamma) + \langle \alpha \beta | \tilde{v} | \gamma' \eta \rangle_{\mathcal{A}} \langle \eta \gamma | \tilde{v} | \alpha' \beta' \rangle_{\mathcal{A}} N(\alpha\beta\gamma'\alpha'\beta'\eta\gamma) \\
& + \langle \beta \gamma | \tilde{v} | \beta' \eta \rangle_{\mathcal{A}} \langle \alpha \eta | \tilde{v} | \alpha' \gamma' \rangle_{\mathcal{A}} N(\beta\gamma\beta'\alpha'\gamma'\eta\alpha) - \langle \beta \gamma | \tilde{v} | \alpha' \eta \rangle_{\mathcal{A}} \langle \alpha \eta | \tilde{v} | \beta' \gamma' \rangle_{\mathcal{A}} N(\beta\gamma\alpha'\beta'\gamma'\eta\alpha) \\
& - \langle \beta \gamma | \tilde{v} | \gamma' \eta \rangle_{\mathcal{A}} \langle \alpha \eta | \tilde{v} | \alpha' \beta' \rangle_{\mathcal{A}} N(\beta\gamma\gamma'\alpha'\beta'\eta\alpha) + \langle \gamma \alpha | \tilde{v} | \gamma' \eta \rangle_{\mathcal{A}} \langle \eta \beta | \tilde{v} | \alpha' \beta' \rangle_{\mathcal{A}} N(\alpha\gamma\gamma'\alpha'\beta'\eta\beta) \\
& - \langle \gamma \alpha | \tilde{v} | \beta' \eta \rangle_{\mathcal{A}} \langle \eta \beta | \tilde{v} | \alpha' \gamma' \rangle_{\mathcal{A}} N(\alpha\gamma\beta'\alpha'\gamma'\eta\beta) - \langle \gamma \alpha | \tilde{v} | \alpha' \eta \rangle_{\mathcal{A}} \langle \eta \beta | \tilde{v} | \gamma' \beta' \rangle_{\mathcal{A}} N(\alpha\gamma\alpha'\beta'\gamma'\eta\beta) \} \quad (\text{A3})
\end{aligned}$$

with a function N including the time-dependent occupation numbers and blocking factors

$$N(\alpha\gamma\alpha'\beta'\gamma'\eta\beta) = [n_\alpha n_\gamma \bar{n}_{\alpha'} + \bar{n}_\alpha \bar{n}_\gamma n_{\alpha'}] [n_{\beta'} n_{\gamma'} \bar{n}_\eta \bar{n}_\beta - \bar{n}_{\beta'} \bar{n}_{\gamma'} n_\eta n_\beta]. \quad (\text{A4})$$

Thus the matrix elements of C^3 are entirely described by antisymmetrized on-shell matrix elements of the interaction v and associated occupation numbers. Due to the antisymmetry of the matrix elements one can rewrite Eq. (A3) as

$$\begin{aligned}
C_{\alpha\beta\gamma,\alpha'\beta'\gamma'}^3(t) & \approx -\pi^2 \sum_{\eta} \{ \langle \alpha \beta | \tilde{v} | \alpha' \eta \rangle_{\mathcal{A}} \langle \eta \gamma | \tilde{v} | \beta' \gamma' \rangle_{\mathcal{A}} N(\alpha\beta\alpha'\beta'\gamma'\eta\gamma) \\
& - \langle \alpha \beta | \tilde{v} | \beta' \eta \rangle_{\mathcal{A}} \langle \eta \gamma | \tilde{v} | \alpha' \gamma' \rangle_{\mathcal{A}} N(\alpha\beta\beta'\alpha'\gamma'\eta\gamma) + \langle \alpha \beta | \tilde{v} | \gamma' \eta \rangle_{\mathcal{A}} \langle \eta \gamma | \tilde{v} | \alpha' \beta' \rangle_{\mathcal{A}} N(\alpha\beta\gamma'\alpha'\beta'\eta\gamma) \\
& - \langle \beta \gamma | \tilde{v} | \beta' \eta \rangle_{\mathcal{A}} \langle \eta \alpha | \tilde{v} | \alpha' \gamma' \rangle_{\mathcal{A}} N(\beta\gamma\beta'\alpha'\gamma'\eta\alpha) + \langle \beta \gamma | \tilde{v} | \alpha' \eta \rangle_{\mathcal{A}} \langle \eta \alpha | \tilde{v} | \beta' \gamma' \rangle_{\mathcal{A}} N(\beta\gamma\alpha'\beta'\gamma'\eta\alpha) \\
& + \langle \beta \gamma | \tilde{v} | \gamma' \eta \rangle_{\mathcal{A}} \langle \eta \alpha | \tilde{v} | \alpha' \beta' \rangle_{\mathcal{A}} N(\beta\gamma\gamma'\alpha'\beta'\eta\alpha) - \langle \alpha \gamma | \tilde{v} | \gamma' \eta \rangle_{\mathcal{A}} \langle \eta \beta | \tilde{v} | \alpha' \beta' \rangle_{\mathcal{A}} N(\alpha\gamma\gamma'\alpha'\beta'\eta\beta) \\
& + \langle \alpha \gamma | \tilde{v} | \beta' \eta \rangle_{\mathcal{A}} \langle \eta \beta | \tilde{v} | \alpha' \gamma' \rangle_{\mathcal{A}} N(\alpha\gamma\beta'\alpha'\gamma'\eta\beta) - \langle \alpha \gamma | \tilde{v} | \alpha' \eta \rangle_{\mathcal{A}} \langle \eta \beta | \tilde{v} | \gamma' \beta' \rangle_{\mathcal{A}} N(\alpha\gamma\alpha'\beta'\gamma'\eta\beta) \}. \quad (\text{A5})
\end{aligned}$$

Furthermore, inserting the result (A5) in

$$\begin{aligned}
\langle \alpha \beta | V_C^3 | \alpha' \beta' \rangle(t') & := \sum_{\gamma=\gamma'} \sum_{\lambda\mu} \{ \langle \alpha \gamma | v | \lambda \mu \rangle C_{\lambda\beta\mu,\alpha'\beta'\gamma'}^3(t') + \langle \beta \gamma | v | \lambda \mu \rangle C_{\alpha\lambda\mu,\alpha'\beta'\gamma'}^3(t') \} \\
& - \sum_{\gamma=\gamma'} \sum_{\lambda\mu} \{ C_{\alpha\beta\gamma,\lambda\beta'\mu}^3(t') \langle \lambda \mu | v | \alpha' \gamma' \rangle + C_{\alpha\beta\gamma,\alpha'\lambda\mu}^3(t') \langle \lambda \mu | v | \beta' \gamma' \rangle \} \quad (\text{A6})
\end{aligned}$$

gives

$$\begin{aligned}
\langle \alpha \beta | V_C^3(t) | \alpha' \beta' \rangle & = -\pi^2 \sum_{\gamma=\gamma'} \sum_{\lambda\mu} \sum_{\eta} \langle \alpha \gamma | v | \lambda \mu \rangle \{ \langle \lambda \beta | \tilde{v} | \alpha' \eta \rangle_{\mathcal{A}} \langle \eta \mu | \tilde{v} | \beta' \gamma' \rangle_{\mathcal{A}} N(\lambda\beta\alpha'\beta'\gamma'\eta\mu) \\
& - \langle \lambda \beta | \tilde{v} | \beta' \eta \rangle_{\mathcal{A}} \langle \eta \mu | \tilde{v} | \alpha' \gamma' \rangle_{\mathcal{A}} N(\lambda\beta\beta'\alpha'\gamma'\eta\mu) + \langle \lambda \beta | \tilde{v} | \gamma' \eta \rangle_{\mathcal{A}} \langle \eta \mu | \tilde{v} | \alpha' \beta' \rangle_{\mathcal{A}} N(\lambda\beta\gamma'\alpha'\beta'\eta\mu) \\
& - \langle \beta \mu | \tilde{v} | \beta' \eta \rangle_{\mathcal{A}} \langle \eta \lambda | \tilde{v} | \alpha' \gamma' \rangle_{\mathcal{A}} N(\beta\mu\beta'\alpha'\gamma'\eta\lambda) + \langle \beta \mu | \tilde{v} | \alpha' \eta \rangle_{\mathcal{A}} \langle \eta \lambda | \tilde{v} | \beta' \gamma' \rangle_{\mathcal{A}} N(\beta\mu\alpha'\beta'\gamma'\eta\lambda) \\
& + \langle \beta \mu | \tilde{v} | \gamma' \eta \rangle_{\mathcal{A}} \langle \eta \lambda | \tilde{v} | \alpha' \beta' \rangle_{\mathcal{A}} N(\beta\mu\gamma'\alpha'\beta'\eta\lambda) - \langle \lambda \mu | \tilde{v} | \gamma' \eta \rangle_{\mathcal{A}} \langle \eta \beta | \tilde{v} | \alpha' \beta' \rangle_{\mathcal{A}} N(\lambda\mu\gamma'\alpha'\beta'\eta\beta) \\
& + \langle \lambda \mu | \tilde{v} | \beta' \eta \rangle_{\mathcal{A}} \langle \eta \beta | \tilde{v} | \alpha' \gamma' \rangle_{\mathcal{A}} N(\lambda\mu\beta'\alpha'\gamma'\eta\beta) - \langle \lambda \mu | \tilde{v} | \alpha' \eta \rangle_{\mathcal{A}} \langle \eta \beta | \tilde{v} | \gamma' \beta' \rangle_{\mathcal{A}} N(\lambda\mu\alpha'\beta'\gamma'\eta\beta) \} \\
& - \pi^2 \sum_{\gamma=\gamma'} \sum_{\lambda\mu} \sum_{\eta} \langle \beta \gamma | v | \lambda \mu \rangle \{ \langle \alpha \lambda | \tilde{v} | \alpha' \eta \rangle_{\mathcal{A}} \langle \eta \mu | \tilde{v} | \beta' \gamma' \rangle_{\mathcal{A}} N(\alpha\lambda\alpha'\beta'\gamma'\eta\mu) \\
& - \langle \alpha \lambda | \tilde{v} | \beta' \eta \rangle_{\mathcal{A}} \langle \eta \mu | \tilde{v} | \alpha' \gamma' \rangle_{\mathcal{A}} N(\alpha\lambda\beta'\alpha'\gamma'\eta\mu) + \langle \alpha \lambda | \tilde{v} | \gamma' \eta \rangle_{\mathcal{A}} \langle \eta \mu | \tilde{v} | \alpha' \beta' \rangle_{\mathcal{A}} N(\alpha\lambda\gamma'\alpha'\beta'\eta\mu) \\
& - \langle \lambda \mu | \tilde{v} | \beta' \eta \rangle_{\mathcal{A}} \langle \eta \alpha | \tilde{v} | \alpha' \gamma' \rangle_{\mathcal{A}} N(\lambda\mu\beta'\alpha'\gamma'\eta\alpha) + \langle \lambda \mu | \tilde{v} | \alpha' \eta \rangle_{\mathcal{A}} \langle \eta \alpha | \tilde{v} | \beta' \gamma' \rangle_{\mathcal{A}} N(\lambda\mu\alpha'\beta'\gamma'\eta\alpha) \\
& + \langle \lambda \mu | \tilde{v} | \gamma' \eta \rangle_{\mathcal{A}} \langle \eta \alpha | \tilde{v} | \alpha' \beta' \rangle_{\mathcal{A}} N(\lambda\mu\gamma'\alpha'\beta'\eta\alpha) - \langle \alpha \mu | \tilde{v} | \gamma' \eta \rangle_{\mathcal{A}} \langle \eta \lambda | \tilde{v} | \alpha' \beta' \rangle_{\mathcal{A}} N(\alpha\mu\gamma'\alpha'\beta'\eta\lambda) \}
\end{aligned}$$

$$\begin{aligned}
& + \langle \alpha\mu | \tilde{v} | \beta'\eta \rangle_{\mathcal{A}} \langle \eta\lambda | \tilde{v} | \alpha'\gamma' \rangle_{\mathcal{A}} N(\alpha\mu\beta'\alpha'\gamma'\eta\lambda) - \langle \alpha\mu | \tilde{v} | \alpha'\eta \rangle_{\mathcal{A}} \langle \eta\lambda | \tilde{v} | \beta'\gamma' \rangle_{\mathcal{A}} N(\alpha\mu\alpha'\beta'\gamma'\eta\lambda) \} \\
& + \pi^2 \sum_{\gamma=\gamma'} \sum_{\lambda\mu} \sum_{\eta} \{ \langle \alpha\beta | \tilde{v} | \lambda\eta \rangle_{\mathcal{A}} \langle \eta\gamma | \tilde{v} | \beta'\mu \rangle_{\mathcal{A}} N(\beta'\mu\gamma\lambda\eta\alpha\beta) \\
& - \langle \alpha\beta | \tilde{v} | \beta'\eta \rangle_{\mathcal{A}} \langle \eta\gamma | \tilde{v} | \lambda\mu \rangle_{\mathcal{A}} N(\lambda\mu\gamma\beta'\eta\alpha\beta) + \langle \alpha\beta | \tilde{v} | \mu\eta \rangle_{\mathcal{A}} \langle \eta\gamma | \tilde{v} | \lambda\beta' \rangle_{\mathcal{A}} N(\lambda\beta'\gamma\mu\eta\alpha\beta) \\
& - \langle \beta\gamma | \tilde{v} | \beta'\eta \rangle_{\mathcal{A}} \langle \eta\alpha | \tilde{v} | \lambda\mu \rangle_{\mathcal{A}} N(\lambda\mu\alpha\beta'\eta\beta\gamma) + \langle \beta\gamma | \tilde{v} | \lambda\eta \rangle_{\mathcal{A}} \langle \eta\alpha | \tilde{v} | \beta'\mu \rangle_{\mathcal{A}} N(\beta'\mu\alpha\lambda\eta\beta\gamma) \\
& + \langle \beta\gamma | \tilde{v} | \mu\eta \rangle_{\mathcal{A}} \langle \eta\alpha | \tilde{v} | \lambda\beta' \rangle_{\mathcal{A}} N(\lambda\beta'\alpha\mu\eta\beta\gamma) - \langle \alpha\gamma | \tilde{v} | \mu\eta \rangle_{\mathcal{A}} \langle \eta\beta | \tilde{v} | \lambda\beta' \rangle_{\mathcal{A}} N(\lambda\beta'\beta\mu\eta\alpha\gamma) \\
& + \langle \alpha\gamma | \tilde{v} | \beta'\eta \rangle_{\mathcal{A}} \langle \eta\beta | \tilde{v} | \lambda\mu \rangle_{\mathcal{A}} N(\lambda\mu\beta\beta'\eta\alpha\gamma) - \langle \alpha\gamma | \tilde{v} | \lambda\eta \rangle_{\mathcal{A}} \langle \eta\beta | \tilde{v} | \beta'\mu \rangle_{\mathcal{A}} N(\beta'\mu\beta\lambda\eta\alpha\gamma) \} \langle \lambda\mu | v | \alpha'\gamma' \rangle \\
& + \pi^2 \sum_{\gamma=\gamma'} \sum_{\lambda\mu} \sum_{\eta} \{ \langle \alpha\beta | \tilde{v} | \alpha'\eta \rangle_{\mathcal{A}} \langle \eta\gamma | \tilde{v} | \lambda\mu \rangle_{\mathcal{A}} N(\lambda\mu\gamma\alpha'\eta\alpha\beta) \\
& - \langle \alpha\beta | \tilde{v} | \lambda\eta \rangle_{\mathcal{A}} \langle \eta\gamma | \tilde{v} | \alpha'\mu \rangle_{\mathcal{A}} N(\alpha'\mu\gamma\lambda\eta\alpha\beta) + \langle \alpha\beta | \tilde{v} | \mu\eta \rangle_{\mathcal{A}} \langle \eta\gamma | \tilde{v} | \alpha'\lambda \rangle_{\mathcal{A}} N(\alpha'\lambda\gamma\mu\eta\alpha\beta) \\
& - \langle \beta\gamma | \tilde{v} | \lambda\eta \rangle_{\mathcal{A}} \langle \eta\alpha | \tilde{v} | \alpha'\mu \rangle_{\mathcal{A}} N(\alpha'\mu\alpha\lambda\eta\beta\gamma) + \langle \beta\gamma | \tilde{v} | \alpha'\eta \rangle_{\mathcal{A}} \langle \eta\alpha | \tilde{v} | \lambda\mu \rangle_{\mathcal{A}} N(\lambda\mu\alpha\alpha'\eta\beta\gamma) \\
& + \langle \beta\gamma | \tilde{v} | \mu\eta \rangle_{\mathcal{A}} \langle \eta\alpha | \tilde{v} | \alpha'\lambda \rangle_{\mathcal{A}} N(\alpha'\lambda\alpha\mu\eta\beta\gamma) - \langle \alpha\gamma | \tilde{v} | \mu\eta \rangle_{\mathcal{A}} \langle \eta\beta | \tilde{v} | \alpha'\lambda \rangle_{\mathcal{A}} N(\alpha'\lambda\beta\mu\eta\alpha\gamma) \\
& + \langle \alpha\gamma | \tilde{v} | \lambda\eta \rangle_{\mathcal{A}} \langle \eta\beta | \tilde{v} | \alpha'\mu \rangle_{\mathcal{A}} N(\alpha'\mu\beta\lambda\eta\alpha\gamma) - \langle \alpha\gamma | \tilde{v} | \alpha'\eta \rangle_{\mathcal{A}} \langle \eta\beta | \tilde{v} | \lambda\mu \rangle_{\mathcal{A}} N(\lambda\mu\beta\alpha'\eta\alpha\gamma) \} \langle \lambda\mu | v | \beta'\gamma' \rangle.
\end{aligned} \tag{A7}$$

It is recalled that Eq. (A7) is the additional inhomogeneous term in the equation for $\dot{c}_2$ in Eq. (106), i.e.,

$$\text{Tr}_{(3=3')} [v(13) + v(23), c_3],$$

which has to be integrated in time. In the on-shell approximation this gives a factor $-i\pi\delta(\epsilon_\alpha + \epsilon_\beta - \epsilon_\lambda - \epsilon_\mu)$, which leads to an additional factor of $-i\pi$ and a replacement

$$\langle \alpha\beta | v | \lambda\mu \rangle \delta(\epsilon_\alpha + \epsilon_\beta - \epsilon_\lambda - \epsilon_\mu) = \langle \alpha\beta | \tilde{v} | \lambda\mu \rangle.$$

For the three-body part of the collision integral $I_{\alpha\alpha}^3$ one, furthermore, has to evaluate the commutator with the interaction v and sum over the states β , i.e.,

$$I_{\alpha\alpha}^3(t) = -\pi \sum_{\beta} \sum_{\delta\nu} \{ \langle \alpha\beta | v | \delta\nu \rangle \langle \delta\nu | \tilde{V}_C^3(t) | \alpha\beta \rangle - \langle \alpha\beta | \tilde{V}_C^{3\dagger}(t) | \delta\nu \rangle \langle \delta\nu | v | \alpha\beta \rangle \}, \tag{A8}$$

where in $\tilde{V}_C^3(t)$ all terms in V_C^3 with v have to be replaced by $\tilde{v}$ as argued above (on-shell limit). This provides all interaction terms in a transparent way for numerical computations. All terms involve an interaction v , the product of 3 on-shell interactions $\tilde{v}$ and the function of the occupation numbers N (A4). All matrix elements of the interaction are constrained by energy-momentum conservation, which reduces the sum over the intermediate states drastically and provides total energy-momentum conservation.

The equation for the time evolution of occupation numbers finally is given by

$$\frac{\partial}{\partial t} n_\alpha(t) = I_{\alpha\alpha}^2(t) + I_{\alpha\alpha}^3(t) =: \hat{I}_{\alpha\alpha}(t) \tag{A9}$$

with $I_{\alpha\alpha}^2(t)$ given by (125) and $I_{\alpha\alpha}^3(t)$ given by Eq. (A8).

Furthermore, summing $\hat{I}_{\alpha\alpha}(t)$ additionally over α obtains

$$\sum_{\alpha} \frac{\partial}{\partial t} n_\alpha(t) = \sum_{\alpha} \hat{I}_{\alpha\alpha}(t) = 0, \tag{A10}$$

which implies particle number conservation and provides a further control for numerical realizations.

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- [1] P. Bonche, S. Koonin, and J. W. Negele, *Phys. Rev. C* **13**, 1226 (1976).
 - [2] J. W. Negele, *Rev. Mod. Phys.* **54**, 913 (1982).
 - [3] M. Tohyama and M. Gong, *Z. Phys. A* **332**, 269 (1989).
 - [4] M. Gong and M. Tohyama, *Z. Phys. A* **335**, 153 (1990).
 - [5] L. W. Nordheim, *Proc. Roy. Soc. A (London)* **119**, 689 (1928).
 - [6] E. A. Uehling and G. Uhlenbeck, *Phys. Rev.* **43**, 552 (1933).
 - [7] G. F. Bertsch, H. Kruse, and S. Das Gupta, *Phys. Rev. C* **29**, 673 (1984); **33**, 1107(E) (1986).
 - [8] J. Aichelin and G. Bertsch, *Phys. Rev. C* **31**, 1730 (1985).
 - [9] G. F. Bertsch and S. Das Gupta, *Phys. Rep.* **160**, 189 (1988).
 - [10] W. Cassing, *Transport Theories for Strongly-Interacting Systems*, Lecture Notes in Physics, Vol. 989 (Springer, Cham, 2021), pp. 1–248.
 - [11] W. Cassing, V. Metag, U. Mosel, and K. Niita, *Phys. Rep.* **188**, 363 (1990).
 - [12] L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (W. A. Benjamin, Inc., New York, 1962).

- [13] P. Danielewicz, *Ann. Phys.* **152**, 305 (1984).
- [14] C. M. Ko, Q. Li, and R.-C. Wang, *Phys. Rev. Lett.* **59**, 1084 (1987).
- [15] B. Blättel, V. Koch, W. Cassing, and U. Mosel, *Phys. Rev. C* **38**, 1767 (1988).
- [16] W. Botermans and R. Malfliet, *Phys. Rep.* **198**, 115 (1990).
- [17] H. Wolter *et al.* (TMEP Collaboration), *Prog. Part. Nucl. Phys.* **125**, 103962 (2022).
- [18] W. Cassing, *Nucl. Phys. A* **700**, 618 (2002).
- [19] S. M. Dawid, Z. T. Draper, A. D. Hanlon, B. Hörz, C. Morningstar, F. Romero-López, S. R. Sharpe, and S. Skinner, [arXiv:2502.14348](https://arxiv.org/abs/2502.14348).
- [20] S. J. Wang and W. Cassing, *Ann. Phys.* **159**, 328 (1985).
- [21] R. Balescu, *Equilibrium and Non-Equilibrium Statistical Mechanics* (John Wiley & Sons, Chichester, New York, Sydney, Toronto, 1975).
- [22] R. Krieg, Ph.D. thesis, University of Erlangen, 1991 (unpublished).
- [23] T. Ghorega, R. Krieg, P. G. Reinhard, and C. Toepffer, *Nucl. Phys. A* **560**, 166 (1993).
- [24] M. Tohyama and P. Schuck, *Eur. Phys. J. A* **45**, 257 (2010).
- [25] M. Tohyama and P. Schuck, *Eur. Phys. J. A* **50**, 77 (2014).
- [26] P. Schuck, D. S. Delion, J. Dukelsky, M. Jemai, E. Litvinova, G. Roepke, and M. Tohyama, *Phys. Rep.* **929**, 1 (2021).
- [27] E. Litvinova and P. Schuck, *Phys. Rev. C* **100**, 064320 (2019); **107**, 029903(E) (2023).
- [28] M. Tohyama and P. Schuck, *Eur. Phys. J. A* **55**, 74 (2019).
- [29] M. Tohyama and P. Schuck, *Eur. Phys. J. A* **53**, 186 (2017).
- [30] P. Schuck and M. Tohyama, *Phys. Rev. B* **93**, 165117 (2016).
- [31] W. Cassing and S. Wang, *Z. Phys. A* **328**, 423 (1987).
- [32] W. Cassing, K. Niita, and S. Wang, *Z. Phys. A* **331**, 439 (1988).
- [33] W. Cassing and A. Pfitzner, *Z. Phys. A* **337**, 175 (1990).
- [34] W. Cassing and A. Pfitzner, *Z. Phys. A* **342**, 161 (1992).
- [35] F. V. De Blasio, W. Cassing, M. Tohyama, P. F. Bortignon, and R. A. Broglia, *Phys. Rev. Lett.* **68**, 1663 (1992).
- [36] B. D. Day, *Rev. Mod. Phys.* **50**, 495 (1978).
- [37] K. A. Brueckner, *Phys. Rev.* **97**, 1353 (1955).
- [38] J. Cugnon, T. Mizutani, and J. Vandermeulen, *Nucl. Phys. A* **352**, 505 (1981).
- [39] A. Lang, H. Babovsky, W. Cassing, U. Mosel, H.-G. Reusch, and K. Weber, *J. Comput. Phys.* **106**, 391 (1993).
- [40] O. Linnyk, E. L. Bratkovskaya, and W. Cassing, *Prog. Part. Nucl. Phys.* **87**, 50 (2016).