

Massive Thirring/sine-Gordon model with nonzero current densityEric Oevermann ^{*}*Institute for Theoretical Physics, Friedrich Schiller University Jena, 07743 Jena, Germany*Thomas D. Cohen [†]*Department of Physics, University of Maryland, College Park, Maryland 20742, USA*

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This paper determines the zero-temperature equation of state for the massive Thirring/sine-Gordon model. This demonstrates recently derived model-independent upper and lower bounds on the zero-temperature equation of state with fixed number density from systems with a nonzero current density. That approach is potentially valuable as Monte Carlo calculations with a current density avoid the sign problem in the Euclidean formulation. An advantage to illustrating these bounds in the massive Thirring/sine-Gordon model is that the relevant calculations with both a number density and a current density can be done using a Bethe ansatz. For this model, optimal bounds constrain the energy density as a function of number density by a factor of two from above and below at high densities for all choices of couplings. The lower bound becomes exact at low densities, while the upper bound approaches the worst constraint of a factor of 4.90.

DOI: [10.1103/physrevc.114.045202](https://doi.org/10.1103/physrevc.114.045202)**I. INTRODUCTION**

Understanding the equation of state (EoS) of QCD in extreme environments with nonzero temperatures or baryon number density or external magnetic fields has been of interest to the community for a significant period and remains of interest [1–20]. In part, this is to understand environments that arise in experiments or astronomical observations. But there is also substantial interest from a purely theoretical perspective, even in regimes that do not appear to be immediately relevant to current or foreseeable experimental measurements. In this context, it is noteworthy that one class of extreme environment has remained largely unstudied. While environments in which the number density of a conserved $U(1)$ current (such as baryon number in QCD) is nonzero have been of wide interest, environments in which the current density is nonzero have not. For the case of QCD with zero baryon density and nonzero baryon current density, one can envision such an environment as corresponding to baryons flowing in one direction and antibaryons flowing in the opposite direction.

Of course, it is not surprising that QCD with a nonzero baryon current density is of less interest than QCD with a nonzero baryon number density: the latter is a familiar situation and is of direct relevance to some experimental situations.

However, while the former situation is much less familiar and does not directly arise in situations of experimental interest, it ought to be of some theoretical interest on its own. Moreover, these properties (at zero number density) should be accessible in lattice simulations without encountering an intractable sign problem.

Recently, a new reason has emerged to study the properties of field theories such as QCD with nonzero current densities, at least at zero temperature: it is the basis of a novel, model-independent approach that exploits relativity [21] and constrains the EoS. In particular, it provides upper and lower bounds for the energy density as a function of number density. This is significant since direct calculations of the EoS are intractable using standard lattice methods due to a notorious sign problem [3,22–27]. In contrast, calculations for systems with a nonzero current density (and zero number density) might evade the sign problem. Thus, this approach provides a possible method to constrain the zero temperature EoS while evading the sign problem.

This is important since the properties of nuclear matter at nonzero density and zero temperature are key issues in both nuclear physics and in astrophysical studies of neutron stars. The sign problem prevents direct calculations. Attempts at analytic continuation from imaginary to real potentials are limited to high temperatures [28]. At zero temperature, these methods become useless for (baryon) chemical potentials exceeding the mass of the lightest baryon-number carrying state due to the Silver Blaze phenomenon [29,30]. While nonperturbative functional approaches [31] might allow in the future reliable and converged calculations with nonzero number density, they are not yet advanced enough to determine the entire QCD phase diagram. In the meantime, one might hope to at least constrain the zero temperature EoS.

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In this paper, we compute the energy density as a function of current [the analog of current density in (1 + 1) dimensions] at zero number density within the massive Thirring model (MTM) or equivalently its dual, the sine-Gordon model (SGM) at various values of the coupling constant of the theory. As noted above, this quantity should be of at least modest theoretical interest in its own right. It is also the key input into an examination of the usefulness of the bounds derived in Ref. [21]. This paper analyzes these bounds by comparing them at finite current calculations with analogous calculations for finite number density. It is the first illustration of these bounds using a nontrivial quantum field theory.

Note that in order to illustrate how well the bounds work, it is necessary to be able to compute properties of systems with nonzero number densities as well as with nonzero current densities. This is not possible for most nontrivial quantum field theories, since computations with nonzero number densities are not viable. However, for MTM/SGM computations for nonzero number densities are viable using Bethe ansatz techniques, these calculations involve nothing more numerically challenging than solving one-dimensional integral equations that were determined by Haldane more than four decades ago [32]. In our calculations for nonzero currents, we generalized Haldane's method to this new situation. While this situation should be tractable by Euclidean space lattice Monte Carlo methods, the Bethe ansatz approach is numerically cleaner, particularly for zero temperature (which would require an extrapolation in a lattice simulation). Thus, the extent of how constraining these bounds are can be easily checked within the MTM/SGM.

We begin by reviewing some of the model's properties at nonzero number density. In Sec. III, we derive an integral equation for the density of states for a theory with a nonzero current density. It allows us to compute the bounds for the MTM, and we explain our results.

II. MASSIVE THIRRING / SINE-GORDON MODEL

One of the most remarkable discoveries involving field theory in the 1970s, was that the MTM—a bosonic theory—and the SGM—a fermionic theory—were dual to each other, i.e., were essentially the same theory cast into very different languages [33,34]. The Lagrangians of the two theories

$$\mathcal{L}_{\text{MTM}} = \bar{\psi} (i \gamma^\mu \partial_\mu - M_0) \psi - \frac{g}{2} (\bar{\psi} \gamma^\mu \psi)^2, \quad (1a)$$

$$\mathcal{L}_{\text{SGM}} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi + \frac{m_0^2}{\beta^2} \cos(\beta \phi), \quad (1b)$$

are clearly radically different. But the predictions of the two theories are identical if one identifies the couplings [33,34] according to

$$4\pi/\beta^2 = 1 + g/\pi > 0, \quad -\pi/2 < g, \quad (2)$$

and analogous quantities appropriately. Note that with this identification the duality relates the weakly coupled SGM to the strongly coupled MTM. Remarkably, the conserved topological current in the SGM plays an identical role as the

$U(1)$ Noether current in the MTM [35],

$$\mathcal{J}_{\text{SGM}}^\mu = \beta/(2\pi) \epsilon^{\mu\nu} \partial_\nu \phi, \quad \mathcal{J}_{\text{MTM}}^\mu = \bar{\psi} \gamma^\mu \psi. \quad (3)$$

The topological winding number in the SGM is identified with particle number $N = \int dx \bar{\psi} \gamma^\mu \psi$ in the MTM.

Quantum mechanically the Hilbert space of the theory is divided into sectors of fixed integer N . The lightest state of $N = 1$ is a topological soliton in the SGM version of the theory and the fundamental fermion of the MTM. Upon renormalization, the bare fermion mass M_0 obtains a multiplicative factor. The renormalized mass M_r can be identified with the soliton mass m_s of the SGM: $M_r = m_s$. Similarly, the fermion-antifermion bound states of the MTM can be identified with the soliton-antisoliton bound states of the SGM. The lowest-lying bound state of mass m_b , also called breather [32], defines an important length scale and can be identified with the elementary boson in the SGM [35,36]. Before addressing finite density, we introduce our parametrization of the couplings via κ [37],

$$\kappa := \frac{\beta^2/(8\pi)}{1 - \beta^2/(8\pi)} = \frac{1}{1 + 2g/\pi}, \quad (4)$$

and distinguish the regimes of weakly (strongly) coupled SGM (MTM) for $\kappa \ll 1$, strongly (weakly) coupled SGM (MTM) for $\kappa \simeq 1$, and repulsive solitons for $\kappa > 1$. In this notation, the breather and soliton mass (in the attractive case) are related as follows [32]:

$$m_b = 2 \sin(\pi/2 \kappa) m_s \stackrel{\kappa \rightarrow 0}{=} \pi \kappa m_s. \quad (5)$$

Fixed number density

To facilitate the relating of our results, we review some of the known physics of the MTM in different density regimes. Bergknoff and Thacker used a Bethe ansatz to study the spectrum and vacuum properties of the model [38,39]. Haldane extended these results to nonzero number densities [32]. Utilizing the Bethe ansatz technique, one can derive an integral equation for the (renormalized) density of states $\rho(\alpha)$ as a function of rapidity α ,

$$\rho(\alpha) = (2\pi)^{-1} \cosh(\alpha) + \int_{-\alpha_F}^{\alpha_F} d\alpha' R(\alpha - \alpha') \rho(\alpha'), \quad (6)$$

where α_F is the Fermi-rapidity. The kernel R is given by

$$R(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dy e^{i\alpha y} \frac{1}{2} \left(1 - \frac{\tanh(\frac{1}{2}\pi y)}{\tanh(\frac{1}{2}\pi \kappa y)} \right), \quad (7)$$

where κ is the effective coupling constant given in Eq. (4). From the density of states, quantities such as the number n and energy density ϵ can be obtained. We go into more detail about this method, deriving our new result for nonzero current densities below in Eq. (11) and Appendix B.

We distinguish between high- and low-density regimes [32]. The crossover is characterized by the length scale set by the breather mass m_b and not m_s , c.f. Eq. (5). Haldane calls this length scale the breather length and it becomes the effective size of bare solitons dressed by virtual breather fluctuations. At low densities, the system behaves like a noninteracting

Fermi gas. We explicitly note the interaction energy per particle [Eq. (5.2) in Ref. [32]]

$$\frac{\epsilon}{n} - m_s = \frac{1}{6} \frac{\pi^2 n^2}{m_s} \quad (8)$$

to compare with the numerical solution of Eq. (6) below. At high densities, where $n/m_b \gg 1$, the model behaves just as the massless Thirring model and [Eq. (5.22) in Ref. [32]]

$$\frac{\epsilon}{n} - m_s = \frac{\pi}{4} \left(1 + \frac{1}{\kappa}\right) n - m_s. \quad (9)$$

A crystalline or mean-field description becomes valid in the limit of $\kappa \ll 1$, the weakly coupled SGM, in the regime of low densities. However, mean-field theory fails to describe the interaction energy at very low densities, requiring the Fermi gas description of Eq. (8), which is explained in detail in Ref. [37]. The regimes are shown in Fig. 1. In the mean-field regime, the interaction energy is described by [Eq. (6.5) in Ref. [32]]

$$\frac{\epsilon}{n} - m_s = 4 e^{-m_b/n} m_s. \quad (10)$$

In a QCD language, the interaction is mediated by a single meson exchange. At low densities, the dominant contribution to the energy density stems simply from the mass of the individual solitons, which is correctly described by mean-field theory. Looking at the interaction energy, which is of more interest from the point of view of nuclear physics, reveals the crossover from a Fermi gas to a low-density mean-field crystalline description.

III. FIXED CURRENT

We seek expressions for the energy-momentum tensor $T^{\mu\nu}$, *c.f.* Eq. (14), that minimize the energy density ϵ under the constraint of having a nonzero current j —the analog of current density for this (1+1)-dimensional system. Many of the details for the construction of this state are presented in Appendix A. Ultimately, all quantities of interest can be derived from the density of states $\rho(\alpha)$, considered as a function of rapidity α . It satisfies the integral equation

$$\begin{aligned} \rho(\alpha) = & \frac{\cosh(\alpha)}{2\pi} + \int_{\alpha_1}^{\alpha_2} d\alpha' \int_{-\infty}^{\infty} \frac{dy}{2\pi} \rho(\alpha') \\ & \times \left[e^{iy(\alpha-\alpha')} \frac{\bar{K}(y)}{1+\bar{K}(y)} - e^{iy(-\alpha-\alpha')} \frac{\bar{L}(y)}{1+\bar{K}(y)} \right], \end{aligned} \quad (11)$$

where the kernel functions $\bar{K}$ and $\bar{L}$ and all details of the derivation are provided in Appendix B. The way in which the integration limits $\alpha_1, \alpha_2 \in \mathbb{R}$ must be chosen is explained in Appendix A. After determining $\rho(\alpha)$, we compute

$$j = 2 \int_{\alpha_1}^{\alpha_2} d\alpha n(\alpha) \tanh \alpha, \quad (12a)$$

$$T^{\mu\nu} = \begin{cases} 2 \int_{\alpha_1}^{\alpha_2} d\alpha \frac{n(\alpha) p^\mu p^\nu}{m_s \cosh \alpha}, & \mu = \nu, \\ 0, & \mu \neq \nu, \end{cases} \quad (12b)$$

where the factor two counts antiparticles and particles, and

$$n(\alpha) = m_s \rho(\alpha), \quad p^\mu = m_s (\cosh \alpha, \sinh \alpha)^\mu. \quad (13)$$

Appendix C describes the details of the numerical setup to obtain data pairs $(j, T^{\mu\nu})$. Using Eq. (14) and applying boosts $\beta \in [0, 1]$ yields upper and lower bounds on the EoS for a range of number densities. Repeating the calculation for multiple current densities leads to a family of curves, the envelope of which defines the optimal bounds that are calculable from this approach.

IV. GENERAL BOUNDS ON THE EQUATION OF STATE

As discussed in Sec. I, one of the motivations for studying this model at fixed current is to illustrate the utility of bounds relating the EoS as a function of density to results for fixed current density. These bounds are reviewed in this section.

Assume that the energy-momentum tensor $T_{\mu\nu}$ of a system with a nonzero current density j and zero number density is known. It was shown in Ref. [21] that this allows one to infer upper (u) and lower (l) bounds on the energy density ϵ as a function of number density n via Lorentz boosts with a boost parameter β :

$$\epsilon(n^u) \leq \frac{T_{tt} + \beta^2 T_{xx}}{1 - \beta^2}, \quad (14a)$$

$$\epsilon(n^l) \geq \frac{1 - \beta^2}{1 + \beta^2 \frac{p(n^l)}{\epsilon(n^l)}} T_{tt} \geq \frac{1 - \beta^2}{1 + \beta^2} T_{tt}, \quad (14b)$$

$$\text{with } n^u := \frac{\beta j}{\sqrt{1 - \beta^2}}, \quad n^l := \frac{\sqrt{1 - \beta^2} j}{\beta}. \quad (14c)$$

The lower bound requires knowledge of the ratio $p/\epsilon \leq 1$, which is usually unknown but constrained by causality, and fixed in any given proposed model for the EoS. The bound is obtained by boosting a system with a zero current and nonzero number density,

$$\begin{pmatrix} n^l \\ 0 \end{pmatrix} \xrightarrow{\beta} \begin{pmatrix} n \\ j \end{pmatrix}, \quad T_{tt}(0, j) \leq T_{tt}^\beta(n, j). \quad (15)$$

The upper bound is derived by boosting a system of zero number and nonzero current density,

$$\begin{pmatrix} 0 \\ j \end{pmatrix} \xrightarrow{\beta} \begin{pmatrix} n^u \\ j^u \end{pmatrix}, \quad \epsilon(n^u) \leq T_{tt}^\beta(n^u, j^u). \quad (16)$$

We work exclusively in Minkowski space, calculating the bounds within the MTM. However, a Lagrange multiplier for a finite current density in Minkowski spacetime, when continued to the Euclidean domain, is equivalent to an imaginary chemical potential in Minkowski spacetime, when translated to the Euclidean domain. Therefore, there is no sign problem in practical lattice calculations. This derivation implicitly uses γ_5 positivity, *cf.*, e.g., Ref. [40], which can be demonstrated for the theory in a straightforward way.

V. NUMERICAL RESULTS

Our numerical results for both the EoS at fixed number density and the bounds obtained using solutions at fixed

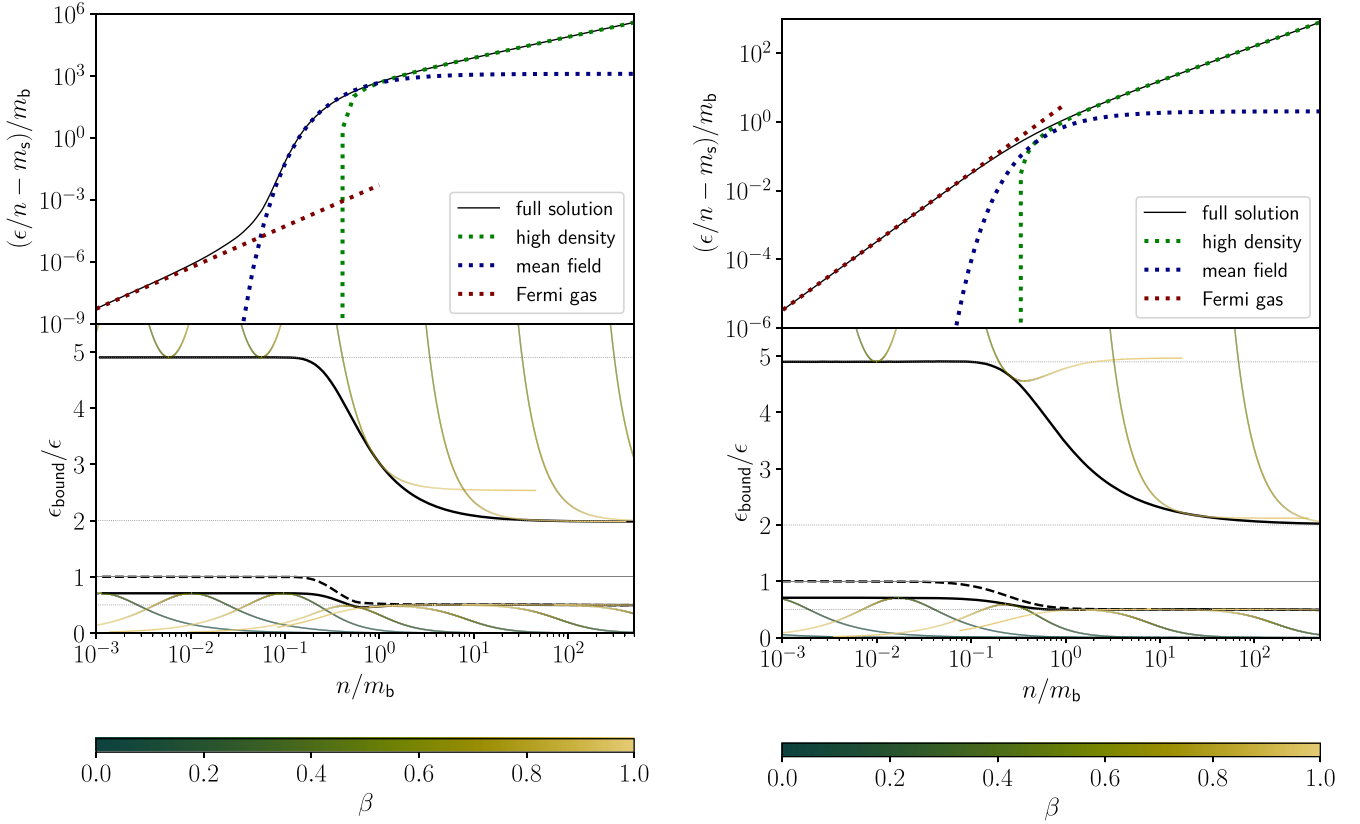


FIG. 1. *Top*: The energy density ϵ , obtained by solving the integral equation (6), enters the interaction energy per soliton as a function of number density n ; the results are all in units of the boson mass m_b . The regimes of high-density [Eq. (9)], low-density mean-field [Eq. (10)] and very-low-density Fermi gas [Eq. (8)] are indicated. *Bottom*: The ratio of upper and lower bounds (with the best lower bound shown in dashed lines) to the exact result. A selection of curves corresponding to different $\beta \in [10^{-6}, 0.9999]$ boosted current densities are plotted with the bound being the envelope of all curves. The gray horizontal lines indicate the limiting values of the ratio at low and high densities.

current are shown in Fig. 1 for both weak and strong SGM coupling. As expected, the energy density as a function of density is well approximated at low densities by a Fermi gas and at high densities by the high density approximation of Eq. (9) for both strong and weak coupling. Away from these two limits, the weak coupling theory is well described by mean-field theory (or equivalently via classical bosonic theory). The results for the upper and lower bounds are of particular interest. The best lower bound uses the value of p/ϵ obtained from the numerical solution to the finite number density integral equation. Since the ratio becomes one at high densities, both lower bounds coincide in this regime.

A. Choice of current and boost for optimal bounds

First, consider the upper bound. A low density n^u is obtained by a small boost of some current or by some boost of a small current. The latter provides a better bound because it keeps the boosted current j^u low, bringing the system closer to the ground state with $\epsilon(n^u)$. A high density results from either a large boost or a large current. In this regime, the former provides a slightly better bound for the same reason as previously. The curves become constant because an even larger boost keeps the ratio $n^u/j^u = \beta$ nearly constant. It must be as large as possible to approach the equality in Eq. (14a).

This explains why at low densities and with smaller boosts, the upper bound is worse than at large densities.

Next, turn to the lower bounds before addressing the limiting values of $\epsilon_{\text{bound}}/\epsilon$. A low density n^l can lead to a moderate current from a large boost or to a small current from a moderate boost. The latter provides a better bound because it keeps the density $n = n^l/\sqrt{1 - \beta^2}$ low in the boosted system, making the equality in Eq. (14a) closer. A high density can lead to some moderate current from a small boost or to a large current from a moderate boost. The latter provides a better bound.

B. Low densities

At very low densities, where the Fermi gas description is valid, the ratio for the optimal upper bound approaches a value of about 4.90. This can be understood in the following way: The system consists of nearly noninteracting particles. The energy density is given by $\epsilon(n^u) = m_s n^u$, while the pressure vanishes. At zero number and nonzero current density j we compute

$$T_{tt}(j) = 2m_s j, \quad T_{xx}(j) = m_s j, \quad (17)$$

using Eqs. (11), (12b), and (13), $\rho(\alpha) \simeq \cosh(\alpha)/(2\pi)$, and expanding around the minimum $\alpha_{\min}$, cf. Eq. (A4). By

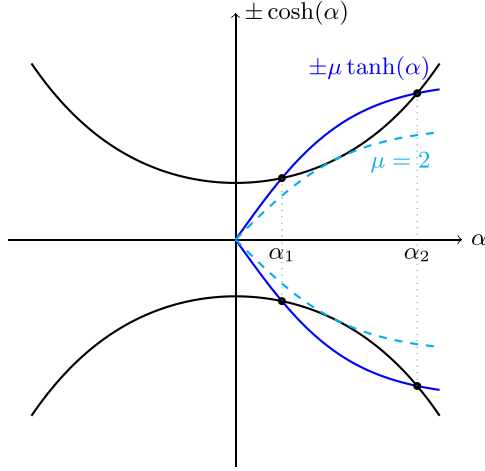


FIG. 2. States of rapidities $\alpha_1 \leq \alpha \leq \alpha_2$ are populated in the nonzero current ground state. All quantities are in units of m_s ; the energy per state of rapidity α is drawn in black, its current contribution times the Lagrange multiplier μ in blue.

Eqs. (14a) and (14c), the ratio

$$\frac{(T_{tt} + \beta^2 T_{xx})/(1 - \beta^2)}{\epsilon(n^u)} = \frac{(2 + \beta^2)/(1 - \beta^2)}{\beta} \quad (18)$$

has a minimum of $2\sqrt{6} \approx 4.90$ at $\beta = \sqrt{2/5} \approx 0.63$.

The best lower bound approaches the actual energy density. This happens for a boost where $n^1 = j$ with

$$\beta = 1/\sqrt{2} \approx 0.7 \Leftrightarrow \beta \gamma \equiv \beta/\sqrt{1 - \beta^2} = 1, \quad (19)$$

and can be explained as follows: The energy density is given by $\epsilon(n^1) = m_s n^1$. In the boosted frame, we find $T_{tt} = \gamma^2 \epsilon(n^1)$. Moreover, the energy density for the zero number but nonzero current density was computed to be $\epsilon(j) = 2m_s j$. The inequality Eq. (14b) is derived from $\epsilon(j) \leq T_{tt}$, cf. Eq. (15). We therefore conclude

$$2m_s \beta \gamma n^1 \leq \gamma^2 m_s n^1. \quad (20)$$

The equality holds for the condition in Eq. (19).

C. High densities

The upper and lower bounds constrain the energy density by a factor of two at high densities. This can be understood in the following way: The energy density grows quadratically with the number density, cf. Eq. (9). Besides, the energy density as a function of current density is the same function as for number density in this regime,

$$T_{tt}(j) = \epsilon(n)|_{n=j}. \quad (21)$$

The ways of filling up levels lead to the same function:

- (i) number density: filling all particle states from minus to plus the Fermi rapidity being very large;
- (ii) current density: filling up anti- and particle states from zero to a large rapidity values, cf. Fig. 2.

For the upper bound one finds from Eqs. (14a) and (14c)

$$\epsilon(n^u) \leq \frac{T_{tt} + \beta^2 T_{xx}}{\beta^2} \left(\frac{n^u}{j}\right)^2 \xrightarrow{\beta \rightarrow 1} 2T_{tt} \left(\frac{n^u}{j}\right)^2, \quad (22)$$

where $T_{tt} = T_{xx}$ holds at high densities. This relation and Eqs. (14b) and (14c) are utilized for the lower bound to express

$$\epsilon(n^l) \geq \frac{\beta^2 T_{tt}}{1 + \beta^2} \left(\frac{n^l}{j}\right)^2 \xrightarrow{\beta \rightarrow 1} \frac{T_{tt}}{2} \left(\frac{n^l}{j}\right)^2. \quad (23)$$

Note that this understanding of the high-density limits is valid for the (1 + 1)-dimensional model and would need to be modified for higher dimensions.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this paper. Requests for further information or data should be sent to the authors.

APPENDIX A: NONZERO CURRENT DENSITY AND ZERO NUMBER DENSITY GROUND-STATE OCCUPATION

In this Appendix, we outline how to populate the available states such that we get the ground state at nonzero current density. The vacuum state of zero number and current density would correspond to filling all “negative energy” states in the language of the Dirac sea. A nonzero density at zero temperature is obtained by filling (or unfilling) all states up to the Fermi rapidity. Here, a nonzero current but zero number density is achieved by simultaneously populating positive and negative energy states of equal spatial momentum such that their velocities are opposite, i.e.,

$$\alpha \leftrightarrow i\pi - \alpha, \quad \frac{p^\mu}{m_s} = \begin{pmatrix} \cosh \alpha \\ \sinh \alpha \end{pmatrix} \leftrightarrow \begin{pmatrix} -\cosh \alpha \\ \sinh \alpha \end{pmatrix}. \quad (A1)$$

Furthermore, we must minimize

$$\epsilon - \mu j, \quad (A2)$$

where μ is the Lagrange multiplier fixing a nonzero current density. This amounts to filling the states around the minimum of $\epsilon(\alpha)/j(\alpha)$ up to μ , which is illustrated in Fig. 2. We find that the ratio

$$\frac{\epsilon(\alpha)}{j(\alpha)} = m_s \frac{\cosh(\alpha)}{\tanh(\alpha)} \quad (A3)$$

has a minimum of $2m_s$ at $\alpha_{\min}$,

$$\alpha_{\min} = \ln(\sqrt{2} + 1). \quad (A4)$$

Hence, we require $\mu \geq 2m_s$. The recipe is as follows:

- (i) Fix a value for $\mu/m_s \geq 2$;
- (ii) Find the rapidities such that $\epsilon(\alpha_{1,2})/j(\alpha_{1,2}) = \mu$;
- (iii) Solve the integral equation Eq. (11) for $\rho(\alpha)$;
- (iv) Calculate $T^{\mu\nu}$ and j by Eq. (12).

We note that we assume a positive current density $j > 0$ and positive rapidities $\alpha_{1,2} > 0$ without loss of generality.

APPENDIX B: BETHE ANSATZ AND INTEGRAL EQUATION FOR DENSITY OF STATES AT FINITE CURRENT DENSITY

In the following, we present the derivation of the integral equation (11). It is based on a Bethe ansatz that Bergknoff and Thacker used to study the vacuum properties of the model [38,39]. The general idea of the Bethe ansatz for a relativistic, fermionic system is to construct its ground state by starting from a nonphysical reference state—conveniently, the empty Dirac sea. Putting the theory in a box, one fills up the state levels successively, taking into account scattering with all previously added particles. For local scatterings in one dimension, a phase shift variable captures all scattering effects. This construction results in an integral equation for the density of states $\rho(\alpha)$ in the infinite volume limit. The density of states appears on both sides of the equation because filling a high-lying state influences all lower-lying levels as well. Filling the entire Dirac sea corresponds to the vacuum theory. Haldane unfills the sea again to derive the finite density result [32]. We follow these ideas, but we must (un)fill the levels beyond the vacuum as described in Appendix A.

First, we again point out the map from Eq. (A1), which converts a particle into a hole (and vice versa) with the same spatial momentum. Therefore, we assume that the rapidity variable β is complex with an imaginary part of 0 or $\pm i\pi$. The local interaction between two particles/holes of the MTM can be characterized by their phase shift,

$$\phi(\xi) = -i \ln \left(-\frac{\sinh \left[\frac{1}{2} (\xi - 2\mu) \right]}{\sinh \left[\frac{1}{2} (\xi + 2\mu) \right]} \right), \quad \xi \equiv \Delta\beta, \quad (\text{B1})$$

where μ is related to the coupling constant as follows

$$\cot \mu = -\frac{g}{2}. \quad (\text{B2})$$

We list the derivative of the phase shift $K(\xi) \equiv \phi'(\xi)$, $\xi \in \mathbb{R}$, and its Fourier transform [Eq. (2.82) in Ref. [39]]

$$K(\xi) = \frac{\sin 2\mu}{\cosh \xi - \cos 2\mu}, \quad (\text{B3a})$$

$$\bar{K}(y) \equiv \frac{\tilde{K}(y)}{2\pi} = \frac{\sinh[(\pi - 2\mu)y]}{\sinh \pi y}, \quad (\text{B3b})$$

as well its counterpart $L(\xi) \equiv K(\xi \pm i\pi)$ for later use,

$$L(\xi) = \frac{\sin 2\mu}{-\cosh \xi - \cos 2\mu}, \quad (\text{B4})$$

$$\bar{L}(y) \equiv \frac{\tilde{L}(y)}{2\pi} = \begin{cases} \frac{\sinh(-2\mu y)}{\sinh(\pi y)}, & \mu < \frac{\pi}{2}, \\ \frac{\sinh[2(\pi - \mu)y]}{\sinh(\pi y)}, & \frac{\pi}{2} < \mu < \pi. \end{cases} \quad (\text{B5})$$

Remark. The kernel $L(\xi)$ also acquires a $\delta(\xi)$ term because the phase shift $\phi(\xi)$ has a jump of 2π at $\text{Re } \xi = 0$ for $\text{Im } \xi = \pi$ [Eq. (2.91b) in Ref. [39]]. We drop this term because the limits of integration do not include zero in the finite current density case.

Now follows the main derivation. In order to count states, we place the theory in a box of length L with periodic boundary conditions. This gives rise to a set of equations to be satisfied by the state $|\beta_1, \dots, \beta_n\rangle$ to be an admissible state

$$M_0 L \sinh(\beta_i) = -2\pi n_i - \sum_{j \neq i}^n \phi(\beta_i - \beta_j). \quad (\text{B6})$$

The integers n_i label the branches of the logarithm of the Bethe ansatz equations [Eq. (3.3) in Ref. [38]]. According to the Pauli principle, different states must be labeled by different integers, and the ground state is obtained by choosing $n_{i+1} - n_i = 1$ [Eq. (2.26) in Ref. [39]]. Equation Eq. (B6) holds for $\text{Im}(\beta_i) = \pi$. Next, we take the difference between neighboring states and introduce the density of states $\rho(\beta)$ as

$$\rho(\beta) = \lim_{\Delta\beta \rightarrow 0, (LM_0) \rightarrow \infty} \frac{1}{L M_0 \Delta\beta}, \quad (\text{B7})$$

resulting in

$$\cosh(\beta) = -2\pi \rho(\beta) - \int_{i\pi - \infty}^{i\pi + \infty} d\beta' K(\beta - \beta') \rho(\beta'), \quad (\text{B8})$$

with the derivative of the phase shift $K(\beta) = \phi'(\beta)$ given in Eq. (B3), and the limits of integration corresponding to filling up the entire Dirac sea [Eq. (2.62a) in Ref. [39]]. This is the integral equation for the vacuum density of states which we denote as $\tilde{\rho}_0(\beta)$. The nonzero current density of states is labeled by $\tilde{\rho}(\beta)$, and the difference by $\Delta\tilde{\rho}(\beta) := \tilde{\rho}(\beta) - \tilde{\rho}_0(\beta)$. We remark again that β has an imaginary part of π . The variable α is used for real rapidities in the following. Emptying states from the Dirac sea and adding states of the same momentum above the Dirac sea leads to a finite current density,

$$\begin{aligned} 2\pi \Delta\tilde{\rho}(\beta) = & - \int_{i\pi - \infty}^{i\pi + \infty} d\beta' K(\beta - \beta') \Delta\tilde{\rho}(\beta') \\ & + \int_{i\pi + \alpha_1^h}^{i\pi + \alpha_2^h} d\beta' K(\beta - \beta') \tilde{\rho}(\beta') \\ & - \int_{\alpha_1^p}^{\alpha_2^p} d\alpha' K(\beta - \alpha') \tilde{\rho}(\alpha'). \end{aligned} \quad (\text{B9})$$

We specify the limits of integration below. It is useful to shift the variable $\alpha := \beta - i\pi$, $\rho(\alpha) := \tilde{\rho}(i\pi + \alpha)$, etc.,

$$\begin{aligned} 2\pi \Delta\rho(\alpha) = & - \int_{-\infty}^{\infty} d\alpha' K(\alpha - \alpha') \Delta\rho(\alpha') \\ & + \int_{\alpha_1^h}^{\alpha_2^h} d\alpha' K(\alpha - \alpha') \rho(\alpha') \\ & - \int_{\alpha_1^p}^{\alpha_2^p} d\alpha' K(\alpha - \alpha' + i\pi) \rho(\alpha' - i\pi). \end{aligned} \quad (\text{B10})$$

With this step, $\rho(\alpha)$ ($\alpha \in \mathbb{R}$) still denotes the density of states for negative energy states. If the third term was absent it would be the density of states for negative energy states with negative energy states unfilled just as in Haldane's work. If the second term was absent it would be the density of states for negative energy states with positive energy states filled instead. One would then have to analytically continue the result to arguments $\alpha + i\pi$, hence positive energy states to obtain the equivalent equation.

After applying a Fourier transform to Eq. (B10), bringing the first term of the right-hand side to the left-hand side, solving for $\Delta\rho(\alpha)$ leads to the following solution after Fourier back transform:

$$\rho(\alpha) = \rho_0(\alpha) + \rho_h(\alpha) + \rho_p(\alpha), \quad (\text{B11})$$

with the vacuum solution $\rho_0(\alpha)$ in Eq. (B19) and

$$\rho_h(\alpha) = \int_{\alpha_1^h}^{\alpha_2^h} d\alpha' \int_{-\infty}^{\infty} \frac{dy}{2\pi} e^{iy(\alpha-\alpha')} \frac{\bar{K}(y)}{1 + \bar{K}(y)} \rho(\alpha'), \quad (\text{B12a})$$

$$\rho_p(\alpha) = \int_{\alpha_1^p}^{\alpha_2^p} d\alpha' \int_{-\infty}^{\infty} \frac{dy}{2\pi} e^{iy(\alpha-\alpha')} \frac{-\bar{L}(y)}{1 + \bar{K}(y)} \rho(\alpha' - i\pi). \quad (\text{B12b})$$

Before continuing, we recover the integral equation for nonzero number density by choosing $-\alpha_1^h = \alpha_F = \alpha_2^h$ and $0 = \alpha_1^p = \alpha_2^p$, $\alpha_F > 0$ the Fermi rapidity, [Eq. (3.9) in Ref. [32]]

$$\rho(\alpha) = \rho_0(\alpha) + \int_{-\alpha_F}^{\alpha_F} d\alpha' \rho(\alpha') \times \int_{-\infty}^{\infty} \frac{dy}{2\pi} e^{iy(\alpha-\alpha')} \frac{\bar{K}(y)}{1 + \bar{K}(y)}. \quad (\text{B13})$$

The result for a finite current density is a straightforward extension of the above by choosing different limits,

$$\alpha_1^h = \alpha_1, \quad \alpha_2^h = \alpha_2, \quad (\text{B14a})$$

$$\alpha_1^p = -\alpha_1, \quad \alpha_2^p = -\alpha_2. \quad (\text{B14b})$$

Equation (B11) becomes (after $\alpha' \rightarrow -\alpha'$, $y \rightarrow -y$)

$$\rho(\alpha) = \rho_0(\alpha) + \int_{\alpha_1}^{\alpha_2} d\alpha' \int_{-\infty}^{\infty} \frac{dy}{2\pi} \frac{e^{iy(\alpha-\alpha')} \bar{K}(y)}{1 + \bar{K}(y)} \rho(\alpha') + \int_{\alpha_1}^{\alpha_2} d\alpha' \int_{-\infty}^{\infty} \frac{dy}{2\pi} \frac{e^{iy(-\alpha-\alpha')} \bar{L}(y)}{1 + \bar{K}(y)} \rho(-\alpha' - i\pi). \quad (\text{B15})$$

We find by analytic continuation that $\rho(\alpha)$ and $\bar{\rho}(\alpha) := -\rho(-\alpha - i\pi)$ satisfy the same integral equation. The continuation requires some care. Simply replacing α with a complex variable is problematic, since the integral would not necessarily converge. We also need to replace $\bar{K} \leftrightarrow \bar{L}$ in the numerators of Eq. (B15). This can be seen from carefully inspecting the Fourier transformations going back to the

definition of $\rho_p(\alpha)$,

$$\rho_p(\alpha) = - \int_{-\infty}^{\infty} \frac{dy}{2\pi} \frac{e^{iy\alpha}}{1 + \bar{K}(y)} \int_{-\infty}^{\infty} d\tilde{\alpha} e^{-iy\tilde{\alpha}} \times \int_{\alpha_1^p}^{\alpha_2^p} d\alpha' K(\tilde{\alpha} - \alpha' + i\pi) \rho(\alpha' - i\pi). \quad (\text{B16})$$

Equation Eq. (B15) can thus be written as

$$\rho(\alpha) = \rho_0(\alpha) + \int_{\alpha_1}^{\alpha_2} d\alpha' \int_{-\infty}^{\infty} \frac{dy}{2\pi} \frac{e^{-iy\alpha'}}{1 + \bar{K}(y)} \times [e^{iy\alpha} \bar{K}(y) - e^{-iy\alpha} \bar{L}(y)] \rho(\alpha'). \quad (\text{B17})$$

This is our central result. We express it in a more convenient form in the remainder of this Appendix. The phase shift variable μ can be expressed in terms of the variable κ introduced in Eq. (4) as follows (Eq. (4.18) in Ref. [38])

$$\mu = \frac{\pi}{1 + \kappa}. \quad (\text{B18})$$

The vacuum solution $\rho_0(\alpha)$ is given by

$$\rho_0(\alpha) = \rho_0(0) \cosh\left(\frac{1 + \kappa}{2} \alpha\right), \quad (\text{B19})$$

where $\rho_0(0)$ is a diverging renormalization constant [Eq. (3.11) in Ref. [32] and Eq. (4.2') in Ref. [38]]. Renormalizing the solution ρ and rescaling $\alpha(1 + \kappa)/2 \rightarrow \alpha$, $y \rightarrow y(1 + \kappa)/2$ leads to the final form of the integral equation Eq. (11) with kernels

$$\bar{K}(y) = \frac{\sinh[\pi/2(\kappa - 1)y]}{\sinh[\pi/2(\kappa + 1)y]}, \quad (\text{B20a})$$

$$\bar{L}(y) = \begin{cases} \frac{\sinh[-\pi y]}{\sinh[\pi/2(\kappa + 1)y]}, & 1 < \kappa < \infty, \\ \frac{\sinh[\pi \kappa y]}{\sinh[\pi/2(\kappa + 1)y]}, & 0 < \kappa < 1. \end{cases} \quad (\text{B20b})$$

We can rewrite

$$\frac{\bar{K}(y)}{1 + \bar{K}(y)} = \frac{1}{2} \left(1 - \frac{\tanh(\pi/2 y)}{\tanh(\pi/2 \kappa y)} \right), \quad (\text{B21a})$$

$$\frac{\bar{L}(y)}{1 + \bar{K}(y)} = \begin{cases} -\frac{1 - e^{-\pi y}}{1 - e^{-\pi \kappa y}} e^{-\pi/2(\kappa - 1)y}, & 1 < \kappa < \infty, \\ \frac{1 + e^{-\pi \kappa y}}{1 + e^{-\pi y}} e^{-\pi/2(1 - \kappa)y}, & 0 < \kappa < 1. \end{cases} \quad (\text{B21b})$$

APPENDIX C: NUMERICAL SETUP

The integral equations needed to be solved are second-kind Fredholm equations. We use the composite Newton-Cotes trapezoidal method to approximate the integral. The number of grid points used is included in our uploaded data sets. As the chemical potential / Lagrange multiplier increases, the integral becomes more important because the backflow effects become increasingly significant. This requires more grid points than at low densities. Our implementation follows Ref. [41].

The kernels of the integral equations contain integrals that we compute using the scientific python library's integrate package and quad function. Importantly, we control the oscillatory behavior of the integrand using a weight of cos.

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