

Interacting boson model studies of collectivity in even-even and odd-even Te isotopes

E. A. Cederlöf^{1,2,*}, W. Teng,³ Nazeef^{1,4}, T. Wan,^{1,5} C. Qi^{1,†} and Y. Zhang³¹Department of Physics, KTH Royal Institute of Technology, SE-10691 Stockholm, Sweden²Department of Physics and Astronomy, Uppsala University, SE-75120 Uppsala, Sweden³Department of Physics, Liaoning Normal University, Dalian 116029, People's Republic of China⁴Department of Physics, Birla Institute of Technology and Sciences Pilani, Hyderabad Campus, India⁵Department of Applied Physics and MIIT Key Laboratory of Semiconductor Microstructure and Quantum Sensing, Nanjing University of Science and Technology, Nanjing 210094, People's Republic of China

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Background: With the recent progress in lifetime measurements, there has been strong experimental and theoretical interest in studying the evolution of the collectivity in the long Te isotopic chain.**Purpose:** The purpose of the present work is to investigate the structure of low-lying excited states in even-even Te nuclei as well as the $\nu h_{11/2}$ band in the odd-mass systems within the interacting boson model framework.**Methods:** A new open-source Python package is developed to evaluate the eigenvalues and eigenfunctions of a Hamiltonian which contains both the standard $\hat{n}_d$ and $\hat{Q} \cdot \hat{Q}$ terms as well as higher-order terms which could be essential for triaxial deformation and mixed symmetry effects.**Results:** Systematic calculations of low-spin yrast and nonyrast states in even-even ^{108–124}Te and odd-even ^{109–125}Te nuclei have been carried out using the newly developed Python package and compared with the neighboring Cd isotopes. From the resulting eigenfunctions, the reduced transition probabilities $B(E2)$ are calculated and compared to adopted values. Due to the vibrational-like structure of the spectra of the Te nuclei, the Hamiltonian is dominated by the vibrational term, which is reflected in the $B(E2)$ values. This alone does not explain the low $B_{4/2}$ ratios observed in some Te nuclei. However, it is demonstrated via the contour plot that the $B_{4/2}$ ratios can be highly sensitive to the higher-order terms of the Hamiltonian within specific parameter domains.**Conclusions:** We demonstrate that inclusion of higher-order interactions can provide a possible explanation of the anomaly observed in certain Te isotopes. The results highlight both the importance of extended IBM Hamiltonians and the utility of pyIBM as a modern computational tool for collective nuclear structure studies.DOI: [10.1103/vmvz-9m19](https://doi.org/10.1103/vmvz-9m19)

I. INTRODUCTION

The interacting boson model (IBM) is a nuclear structure model that can describe very well many phenomena in atomic nuclei using a system of interacting bosons [1] as the building block rather than treating individual protons and neutrons or the nucleus as a whole [2]. It offers in particular a simple but robust algebraic framework for the description of nuclear collective motion.

Within IBM, one assumes nucleons primarily interact through pairs which behave as bosons (s - and d -bosons). This simplification allows for the prediction of various nuclear

properties, such as energy spectra and transition rates, using a simple Hamiltonian with a relatively small set of parameters which can effectively capture the dominant low-energy behavior of collective nuclei. The simplest form, known as IBM-1, does not distinguish proton from neutron. The more refined version, IBM-2, explicitly distinguishes between proton and neutron bosons, which provides potentially a more microscopic perspective with increased degrees of freedom. In addition, the interacting boson-fermion model (IBFM) extends the IBM to describe odd-mass nuclei by including unpaired nucleons [3].

The IBM has become an indispensable tool for describing a wide array of collective behaviors in nuclei, including vibrations around a spherical shape, rotations of deformed nuclei, and transitions between different nuclear shapes, known as shape phase transitions. The model elegantly incorporates the concept of dynamical symmetries, such as $U(5)$, $SU(3)$, and $O(6)$, which correspond to idealized cases of vibrational, rotational, and γ -soft nuclei, respectively. They are often plotted together within the so-called Casten's triangle.

There has recently been significant renewed interest in the IBM from both experimental and theoretical perspectives

*Contact author: ebbace@kth.se

†Contact author: chongq@kth.se

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[4–25]. Recent studies focus in particular on the emergence of anomalous collective motions and the onset of soft and rigid triaxial deformation [26–36]. Central to these discussions is the characterization of nuclear collectivity through the energy ratio $R_{4/2} = E(4_1^+)/E(2_1^+)$ and the reduced transition probability ratio $B_{4/2} = B(E2; 4_1^+ \rightarrow 2_1^+)/B(E2; 2_1^+ \rightarrow 0_1^+)$. Standard collective limits are well-defined: for a harmonic vibrator, $R_{4/2} \approx 2.0$ and $B_{4/2} = 2.0$; for an axially symmetric rotor, $R_{4/2} \approx 3.33$ and $B_{4/2} \approx 1.43$ (Alaga rule, $10/7$). However, recent observations in open-shell nuclei have identified anomalous cases where $B_{4/2} \ll 1$, including quite a few neutron-deficient W-Os-Pt isotopes around $N = 90$ [37–40] and Te-Xe isotopes, beginning with the well-known early observation in ^{114}Te [35,41–44] as well as lighter $N \sim Z$ nuclei ^{48}Cr [45], challenging standard geometric and algebraic descriptions. The anomaly cannot be reproduced by conventional microscopic large-scale shell model or beyond mean field models yet. Within IBM, it was recently demonstrated that this anomaly can be explained by a low-lying mixed-symmetry collective excitation mode using an extended IBM Hamiltonian that maps a triaxial rotor onto the boson space. Despite these theoretical advances and the accumulation of new experimental data, there remains a very limited choice of IBM codes available for theoreticians and experimentalists to efficiently analyze these complex emergent phenomena.

Bearing in mind the challenges and open questions summarized above, in this work we present a systematic study on the structure and $E2$ transition properties of Te as well as the analogous Cd isotopes with a newly developed open-source Python script for the IBM. The framework is designed to be flexible and extensible, allowing the inclusion of generalized IBM Hamiltonians with higher-order interaction terms. This capability enables a more realistic and comprehensive exploration of complex collective phenomena that go beyond the standard IBM formulations.

From a physics perspective, the collective structure of the tellurium (Te) isotopic chain ($Z = 52$, with two valence protons outside the closed $Z = 50$ core) and the neighboring cadmium nuclei ($Z = 48$, characterized by two proton holes) has long been regarded as an ideal testing ground for quantum vibrational motion in atomic nuclei [46]. Owing to their proximity to a major shell closure, these systems are expected to exhibit near-spherical shapes and are commonly associated, within the IBM framework, with the $U(5)$ vibrational symmetry and its evolution toward the γ -soft $O(6)$ limit near midshell. Indeed, the yrast excitation energies of many mid-shell cadmium and tellurium isotopes display level spacings that closely resemble those of a harmonic quadrupole vibrator. However, recent experimental and theoretical investigations have increasingly challenged the vibrational interpretation of these states [47–51]. In particular, electromagnetic transition properties and the evolution of the excited 0_2^+ states (see, Fig. 17 in Ref. [47]) provide growing evidence that the apparent vibrational character of the spectra is not accompanied by the corresponding $U(5)$ systematics in $E2$ transition strengths. Notably, the anomalous $E2$ transition behavior observed in ^{114}Te [41], as mentioned above, as well as more recently in the odd- A nucleus ^{119}Te [52], deviates significantly from the expectations of standard collective models and available

microscopic calculations. These deviations may be indicative of a structural evolution associated with a shape-phase transition and mixture with intruder configurations that is predicted to occur in tellurium isotopes near midshell. A systematic theoretical investigation capable of incorporating extended collective dynamics is required to clarify the nature of collectivity in this mass region.

From a numerical perspective, the algorithm developed in this work is designed to describe a wide range of collective modes, including rotational, vibrational, γ -rigid, and γ -soft dynamics, as well as their possible mixing. It provides a practical and computationally efficient framework for identifying the dominant collective character of nuclear bands and for calculating both in-band and inter-band electromagnetic transition properties. The flexibility of the implementation allows for systematic studies across isotopic chains and the exploration of extended IBM Hamiltonians. The code has been applied successfully in our recent analysis of the excitation spectrum and electromagnetic observables of the neutron-deficient nucleus ^{108}Te [53].

II. THEORETICAL FRAMEWORK

Several computational tools have been developed to perform calculations within the IBM framework, including PHINT (Fortran), ArbModel [54], and IBAR [55] (both written in C) for IBM-1, as well as NPBOS (Fortran) for IBM-2. Among these, IBAR is the only code that has been formally published and remains available in a public repository, while others are distributed more informally within the community. Many of the existing IBM codes were developed several decades ago and are implemented in programming languages that are less commonly used by the current generation of nuclear physicists, which can hinder accessibility, extensibility, and long-term maintainability.

At the same time, substantial theoretical effort has been devoted to extending the IBM Hamiltonian beyond its standard form by introducing higher-order interaction terms to describe more complex collective phenomena. Incorporating such extensions into existing codes is often nontrivial and may require significant restructuring of legacy software. Motivated by these considerations, we have developed a new implementation of the IBM based on the Python programming language, designed from the outset to support extended Hamiltonians in a flexible and modular manner. The goal of this work is to provide a transparent, easy-to-use theoretical tool that can be readily employed by both theorists and experimentalists for the rapid analysis and interpretation of complex nuclear-structure data.

To describe both even-even and odd- A nuclei within a unified theoretical framework, we have developed a computational implementation that incorporates both IBM and IBFM. In this approach, odd- A systems are treated as boson-fermion systems, in which an even-even core described within the IBM is coupled to a single unpaired nucleon. For simplicity and to maintain numerical tractability, the odd particle is assumed to be restricted to a selected single- j orbital, an approximation that is commonly adopted in IBFM studies and is adequate

for capturing the dominant features of low-lying spectroscopic observables.

The code is benchmarked against known analytical solutions [56] and the unpublished Mathematica script for IBM-1 [57]. That script was used in our recent calculations presented in Refs. [26–28,52]. That script is composed of several independent components for basis construction, Hamiltonian matrix evaluation and electromagnetic transition calculations, which may be quite hard for a nonexpert to use.

A. IBM and IBFM Hamiltonian

In the following, we give a unified description of the code for the IBM and IBFM with given single- j shell. For the present calculation, we consider an effective Hamiltonian in the form [26–28]

$$H = a_1 \hat{n}_d + a_2 [\hat{Q}^\chi \cdot \hat{Q}^\chi] + a_3 (d^\dagger \times d^\dagger \times d^\dagger)^{(3)} \cdot (\tilde{d} \times \tilde{d} \times \tilde{d})^{(3)} + a_4 (\hat{L} \times \hat{Q}^\chi \times \hat{L})^{(0)} + a_5 \hat{L}^2 + a_6 (\hat{L} \times \hat{Q}^\chi)^{(1)} \cdot (\hat{L} \times \hat{Q}^\chi)^{(1)} + a_F [\hat{Q}^\chi \cdot \hat{q}_F], \quad (1)$$

where the first two terms describe the nuclear vibration and rotation (axial or γ soft). The third and fourth terms are higher-order terms that may lead to triaxial deformation and associated band mixing. Only the LQL term is considered in some previous works for simplicity [27,29]. The angular momentum operator and quadrupole operator are defined by

$$\hat{L}_u = \sqrt{10} (d^\dagger \times \tilde{d})_u^{(1)}, \quad (2)$$

$$\hat{Q}_u^\chi = (d^\dagger \times \tilde{s} + s^\dagger \times \tilde{d})_u^{(2)} + \chi (d^\dagger \times \tilde{d})_u^{(2)}, \quad (3)$$

with the parameter $\chi \in [-\frac{\sqrt{7}}{2}, +\frac{\sqrt{7}}{2}]$. Those operators generate the $SU(3)$ algebra for $\chi = -\frac{\sqrt{7}}{2}$ and the conjugate $SU(3)$ algebra, commonly denoted $\overline{SU}(3)$, for $\chi = +\frac{\sqrt{7}}{2}$, respectively. The corresponding band states in $SU(3)$ and $\overline{SU}(3)$ are not orthonormal, and their overlaps do not generally equal unity. They become a part of the $O(6)$ algebra when $\chi = 0$. The two cases are applied to describe γ -soft rotor ($\chi = 0$) and axial rotor ($\chi = -\frac{\sqrt{7}}{2}$ for prolate and $\chi = \frac{\sqrt{7}}{2}$ for oblate), respectively. The d boson number operator is defined as

$$\hat{n}_d = \sqrt{5} (d^\dagger \times \tilde{d})^{(0)}, \quad (4)$$

with $\tilde{d}_u = (-1)^u d_{-u}$, which is meanwhile the first-order Casimir operator of the $U(5)$ algebra. The widely used consistent- Q Hamiltonian [58] in the IBM just contains the

first two terms of the Hamiltonian as defined above. With the extension to including triaxial dynamics, the present IBM Hamiltonian for the description of even-even nuclei contains the first six terms in Eq. (1).

Apart from the IBM terms, we simply consider a quadrupole type of term describing the interaction between the odd neutron particle (or hole) and the rest of the boson core. This interaction term is defined as in Eq. (1) as $\hat{Q}^\chi \cdot \hat{q}_F$, with

$$\hat{q}_F = (a_j^\dagger \times \tilde{a}_j)^{(2)}. \quad (5)$$

Clearly, the coupling to the odd-particle and the last Hamiltonian term are relevant only for IBFM calculations.

For the IBFM, one should additionally take into account the single-particle term [3]

$$\hat{H}_F = \varepsilon_j \hat{n}_j, \quad (6)$$

where ε_j represents the single-particle energies of the spherical orbit j and

$$\hat{n}_j = -\sqrt{2j+1} (a_j^\dagger \times \tilde{a}_j)^{(0)}, \quad (7)$$

with $\tilde{a}_{j,m_j} = (-1)^{j-m_j} a_{j,-m_j}$ is the fermion number operator. However, this term for a given j only contributes a constant (which can be thus set to zero for convenience).

In IBFM, the exchange interaction may also be expected to play a crucial role in shaping the detailed spectroscopy of odd-mass nuclei, particularly in determining the relative ordering of bands arising from different boson-fermion couplings [59]. In the present work, however, we restrict our attention to the yrast structures dominated by a single- j configuration coupled to an even-even core. Within this limited scope, exchange and monopole interactions are not explicitly included to minimize the number of free parameters, while retaining the essential features of the quadrupole collectivity under investigation. The effects of the monopole interaction can, to some extent, be effectively absorbed into a renormalization of the n_d term of the bosonic Hamiltonian [3].

B. Basis

We use $|N(\lambda, \mu) \tilde{\chi} L M_L\rangle$ to denote the $SU(3)$ orthogonal basis for a system with the total boson number N , spin value L and projection M_L , with (λ, μ) characterizing the $SU(3)$ irreducible representation and $\tilde{\chi}$ being used to denote the multiplicity of L in the reduction of $SU(3) \supset SO(3)$. The irreducible representations (IRREPs) (λ, μ) are given by

$$\begin{aligned} (\lambda, \mu) = & (2N, 0) \oplus (2N-4, 2) \oplus \cdots \oplus (2, N-1) \quad \text{or} \quad (0, N) \\ & \oplus (2N-6, 0) \oplus (2N-10, 2) \oplus \cdots \oplus (2, N-4) \quad \text{or} \quad (0, N-3) \\ & \oplus (2N-12, 0) \oplus (2N-16, 2) \oplus \cdots \oplus (2, N-7) \quad \text{or} \quad (0, N-6) \\ & \cdots \\ & \oplus (4, 0) \oplus (0, 2) \quad \text{for} \quad N(\bmod 3) = 2 \\ & \oplus (2, 0) \quad \text{for} \quad N(\bmod 3) = 1 \\ & \oplus (0, 0) \quad \text{for} \quad N(\bmod 3) = 0. \end{aligned} \quad (8)$$

If (λ, μ) are known, the allowed angular momentum quantum number can be extracted from the following rules:

$$\begin{aligned} K &= \min[\lambda, \mu], \quad \min[\lambda, \mu] - 2, \quad \min[\lambda, \mu] - 4, \dots, 0 \quad \text{or} \quad 1 \\ L &= \max[\lambda, \mu], \quad \max[\lambda, \mu] - 2, \dots, 0 \quad \text{or} \quad 1 \quad \text{for} \quad K = 0, \\ L &= K, \quad K + 1, \quad K + 2, \quad \dots, \quad K + \max[\lambda, \mu], \quad \text{for} \quad K > 0. \end{aligned} \quad (9)$$

For the IBFM, the eigenstates can be expanded in terms of the weak-coupling basis vectors $|N(\lambda, \mu)\tilde{\chi}(Lj)JM_J\rangle$ as

$$|\xi, JM_J\rangle = \sum_{(\lambda, \mu)\tilde{\chi}L} C_{(\lambda, \mu)\tilde{\chi}L}^{\xi, JM_J} |N(\lambda, \mu)\tilde{\chi}(Lj)JM_J\rangle, \quad (10)$$

with

$$\begin{aligned} &|N(\lambda, \mu)\tilde{\chi}(Lj)JM_J\rangle \\ &= \sum_{M_L, m_j} \langle LM_L, jm_j | JM_J \rangle |N(\lambda, \mu)\tilde{\chi}LM_L\rangle |jm_j\rangle, \end{aligned} \quad (11)$$

where ξ denotes an additional quantum number, $C_{(\lambda, \mu)\tilde{\chi}L}^{\xi, JM_J}$ is the expansion coefficient, L , j , and J represent the angular momentum for the boson core, odd particle and the entire system, respectively, with their third components denoted by M_L , m_j , and M_J . The SU(3) coupling basis can be characterized by the group chain

$$\left| \begin{array}{c} \text{U}(6) \supset \text{SU}(3) \supset \text{SO}(3) \otimes \text{SU}_j(2) \supset \text{SU}_J(2) \supset \text{SO}_J(2) \\ N \quad (\lambda, \mu) \quad \tilde{\chi} \quad L \quad j \quad J \quad M_J \end{array} \right\}. \quad (12)$$

Note that the complete group symmetry for the fermion part with single j should replace $\text{SU}_j(2)$ with

$$\text{U}(2j+1) \supset \text{SU}(2j+1) \supset \text{SP}(2j+1) \supset \text{SU}_j(2), \quad (13)$$

particularly for the multifermion situation because the n^2 operators $(a^\dagger \times \tilde{a})_q^k$ with $k = 0, 1, \dots, n = 2j$ can generate the maximal group symmetry $\text{U}(2j+1)$. If only one fermion is considered, as in the current scenario, the nontrivial sub-symmetry is only $\text{SU}_j(2)$.

C. Hamiltonian matrix element evaluation

With the basis vector constructed above, one can establish the Hamiltonian matrix. For example, the LQL term in the IBM that was studied extensively recently can be evaluated as

$$\begin{aligned} &\langle \alpha' L' M'_L | (\hat{L} \times \hat{Q}^x \times \hat{L})^{(0)} | \alpha L M_L \rangle \\ &= \delta_{L'L} \delta_{M'_L M_L} \sum_{\alpha_a L_a \alpha_b L_b} \left\{ \begin{array}{c} 1 \ 2 \ 1 \\ L_a L' \ L_b \end{array} \right\} \langle \alpha' L' | \hat{L} | \alpha_b L_b \rangle \\ &\quad \times \langle \alpha_b L_b | \hat{Q}^x | \alpha_a L_a \rangle \langle \alpha_a L_a | \hat{L} | \alpha L \rangle \\ &= \delta_{L'L} \delta_{M'_L M_L} L(L+1) \left\{ \begin{array}{c} 1 \ 2 \ 1 \\ L \ L \ L \end{array} \right\} \langle \alpha' L' | \hat{Q}^x | \alpha L \rangle. \end{aligned} \quad (14)$$

Their full expansion into single-boson matrix elements would be quite lengthy and complex. Therefore we will not give the expression of those elements for simplicity. To exemplify the derivation of the Hamiltonian matrix element in the IBFM, we use as an example a quadrupole type of core-particle coupling

term, which is defined as

$$\hat{M}_c = (d^\dagger \times \tilde{d})^2 \cdot (a^\dagger \times \tilde{a})^2. \quad (15)$$

Using the Wigner-Eckart theorem, the matrix element can be derived as

$$\begin{aligned} &\langle \alpha' (L' j) J' M'_J | \hat{M}_c | \alpha (L j) J M_J \rangle \\ &= \delta_{J' J} \delta_{M'_J M_J} (-1)^{L+j+J} \langle \alpha' L' | (d^\dagger \times \tilde{d})^2 | \alpha L \rangle \\ &\quad \times \left\{ \begin{array}{c} L' \ j \ J \\ j \ L \ 2 \end{array} \right\} \langle j | (a^\dagger \times \tilde{a})^2 | j \rangle \\ &= -5 \delta_{J' J} \delta_{M'_J M_J} (-1)^{L'+j+J} \sum_{\alpha'' L''} \left\{ \begin{array}{c} 2 \ 2 \ 2 \\ L L' L'' \end{array} \right\} \\ &\quad \times \left\{ \begin{array}{c} L' \ j \ J \\ j \ L \ 2 \end{array} \right\} \langle \alpha' L' | d^\dagger | \alpha'' L'' \rangle \langle \alpha'' L'' | \tilde{d} | \alpha L \rangle, \end{aligned} \quad (16)$$

where the abbreviation $\alpha \equiv N(\lambda, \mu)\tilde{\chi}$ is used. Clearly, the final results will be determined by the reduced elements of the boson operator under the SU(3) basis. The d -boson operators can be further expressed as the SU(3) irreducible tensors, $\hat{T}_{\tilde{\chi}, LM_L}^{(\lambda, \mu)}$. In particular, it is expressed as

$$d_u^\dagger = A_{1,2u}^{(2,0)}, \quad \tilde{d}_u = B_{1,2u}^{(0,2)}. \quad (17)$$

The double-barred reduced matrix elements contained in Eq. (16) can be further expanded as

$$\begin{aligned} &\langle \alpha' L' | d^\dagger | \alpha'' L'' \rangle \\ &= \sqrt{2L''+1} \langle [N](\lambda', \mu') || A^{(2,0)} || [N-1](\lambda'', \mu'') \rangle \\ &\quad \times \langle (\lambda', \mu') \tilde{\chi}', L'; (2, 0) 1, 2 | (\lambda'', \mu'') \tilde{\chi}'', L'' \rangle \end{aligned} \quad (18)$$

and

$$\begin{aligned} &\langle \alpha'' L'' | \tilde{d} | \alpha L \rangle \\ &= \sqrt{2L''+1} \langle [N-1](\lambda'', \mu'') || B^{(0,2)} || [N](\lambda, \mu) \rangle \\ &\quad \times \langle (\lambda'', \mu'') \tilde{\chi}'', L''; (0, 2) 1, 2 | (\lambda, \mu) \tilde{\chi}, L \rangle, \end{aligned} \quad (19)$$

The last term is the Clebsch-Gordan (CG) coefficient.

D. EM transition

The $E2$ transition operator is chosen in its standard form as

$$\hat{T}^{E2} = e \hat{Q}^x, \quad (20)$$

where e denotes the effective charge and χ is taken to be the same parameter as in the Hamiltonian. Using this operator, the reduced transition probability between initial and final states can be evaluated as

$$B(E2; L_i \rightarrow L_f) = \frac{|\langle \alpha_f L_f | \hat{T}^{E2} | \alpha_i L_i \rangle|^2}{2L_i + 1}. \quad (21)$$

As discussed previously, a particularly sensitive observable for characterizing nuclear collectivity is the $B_{4/2}$ ratio, defined as

$$B_{4/2} = \frac{B(E2; 4_1^+ \rightarrow 2_1^+)}{B(E2; 2_1^+ \rightarrow 0_1^+)}. \quad (22)$$

This ratio provides a clear benchmark for distinguishing different collective regimes: in the vibrational $U(5)$ limit one expects $B_{4/2} \approx 2$, whereas in the rotational $SU(3)$ limit the predicted value is $B_{4/2} \approx 10/7 \simeq 1.43$. Deviations from these canonical values, and in particular values below unity, are indicative of anomalous collective behavior, which may arise from triaxial deformation or other complex structural effects. Thus, the $B_{4/2}$ ratio serves as a sensitive probe of the underlying nuclear dynamics and provides a useful measure for comparing calculations with experimental data.

E. Numerical details

We present the Python code in the following packages:

- (1) A stand-alone package with all CG coefficients up to $N=10$ stored in a library. One can do calculations with user-defined Hamiltonian parameters, optimize the parameters by providing a list of energy levels, or study the sensitivity of the solution to the choice of parameters by performing a two-dimensional (2D) contour plot.
- (2) A supplementary code package coupled to an executable file of the Fortran code SU3CGVCS [60] which generates the necessary CG coefficients. The package first tries to read from the CG library, if the CG coefficients are not found, it calls the Fortran CG code to generate the coefficients and updates the library to higher N .

In the standard package three programs are available. The ENERGY_BVALUES program does calculations of the energy levels and reduced transition probabilities of the selected nucleus. The user provides the boson number and a list of parameters a_1 – a_6 , a_F for the Hamiltonian defined in Eq. (1). To omit any of the terms in the Hamiltonian, set the corresponding parameter to zero. If the Hamiltonian parameters need to be determined the program FITTING_PARAMETERS can be used. This program optimizes the parameters by minimizing the RMSE between the provided experimental energy levels and the corresponding calculated energy levels through a genetic algorithm. It is possible to set any number of parameters to fix values or free ranges. Finally, if more information is needed on the variation of the solution with different choices of parameters, the program CONTOUR produces a 2D contour plot of a selected feature of the solution as a function of two of the parameters. More information on the three programs is available in the attached README file. They are also available on GitHub [61] and Zenodo [62].

III. RESULTS

Systematic theoretical studies of the tellurium isotopic chain have been carried out using a variety of approaches,

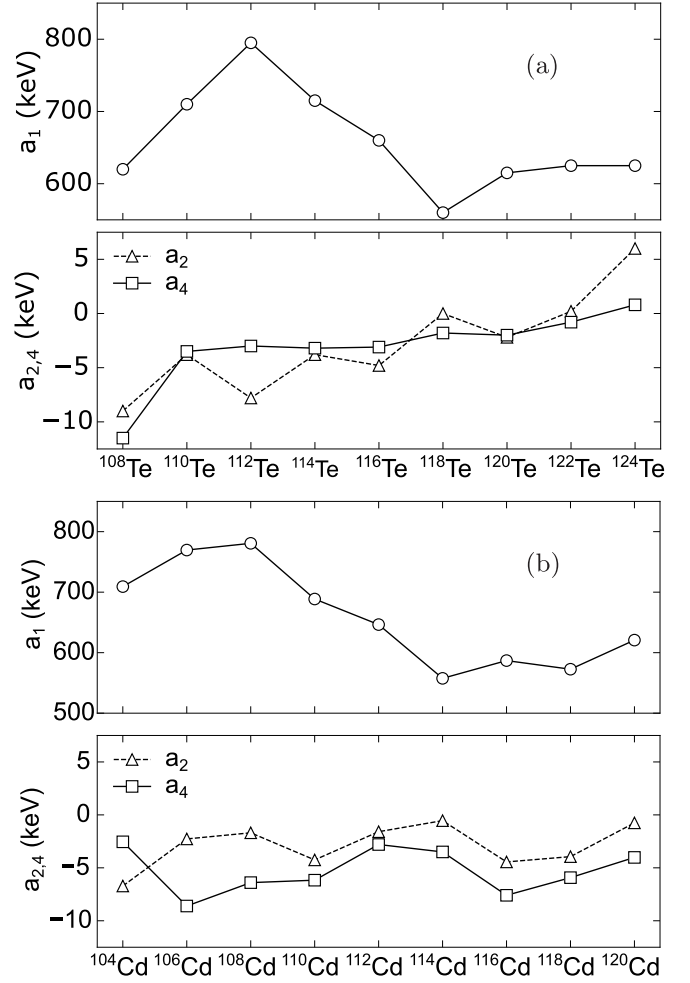


FIG. 1. Slow isotopic variation of the IBM parameters adopted for the description of the even-even Te (upper) and Cd (lower) isotopes.

including IBM-1 calculations [63,64], self-consistent mean-field methods [65], Hartree-Fock-Bogoliubov calculations combined with the generator coordinate method [66], as well as large-scale shell-model calculations [42] and effective field theory studies [67]. In particular, Ref. [64] employed a simplified IBM-1 Hamiltonian containing four interaction terms to investigate vibration-related features in the Te isotopes and to assess the extent to which their low-lying spectra can be interpreted within a vibrational framework. On the experimental side, there have been significant recent efforts in measuring the E2 transition properties of Te isotopes [35,52,68–75].

As a simple illustrative case to demonstrate the capabilities of the code developed in this work, we begin by considering an IBM-1 Hamiltonian that includes only the a_1 , a_2 , and a_4 interaction terms. Given the simplicity of the IBM-1 framework, it is not expected that a single parameter set with strictly fixed values can quantitatively reproduce the structural evolution along the entire tellurium or cadmium isotopic chain. Instead, we allow for a modest, smooth variation of the Hamiltonian parameters from isotope to isotope, as shown in Fig. 1,

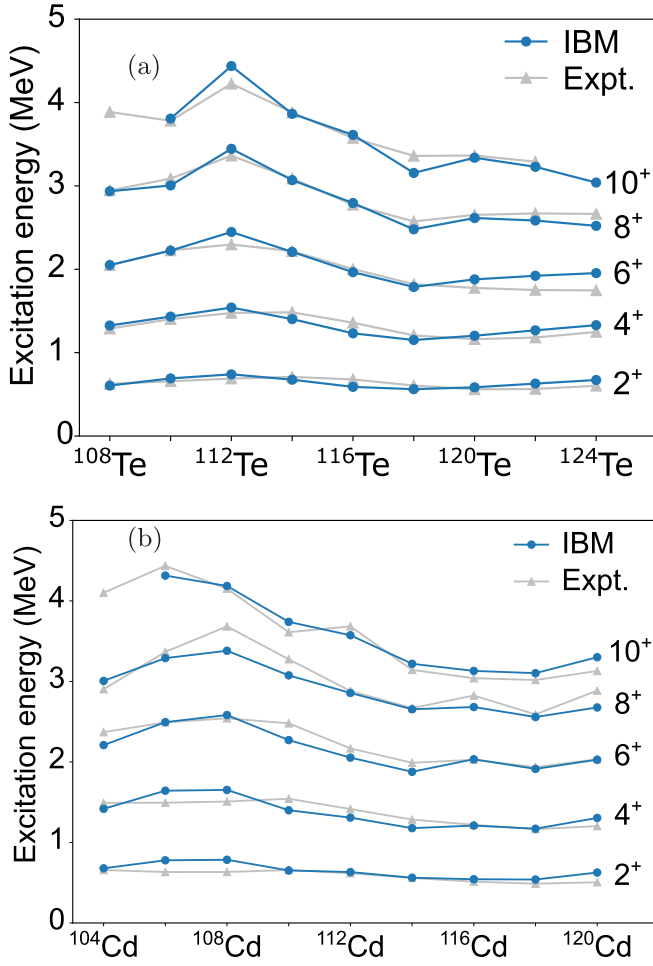


FIG. 2. Experimental [76] and calculated ground-state bands of even-even Te (upper) and Cd (lower) isotopes.

reflecting the gradual changes in underlying nuclear structure across the chain.

Experimental and calculated ground-state bands up to spin $I = 10$ for the even-even Te and Cd isotopes are presented in Fig. 2. The calculated spectra reproduce the observed level energies with good accuracy across the isotopic chains, in particular capturing the near-equidistant spacing characteristic of vibrational excitations in midshell tellurium nuclei. This agreement demonstrates that the adopted Hamiltonian provides an adequate description of the dominant collective dynamics governing these systems and is well suited to account for the vibrational-like nature of the low-lying states. Deviations between calculation and experiment at higher spins remain small and follow systematic trends, indicating that the essential features of the collective motion are consistently described within the present framework.

Calculations have also been performed for odd-A tellurium nuclei. In these cases, only the boson-fermion coupling strength a_F is adjusted, while all remaining parameters are retained from those of the corresponding even-even core. The odd nucleon is assumed to occupy the $h_{11/2}$ orbital, which is expected to dominate the low-lying structure in this mass region and provides a reasonable single- j approximation for

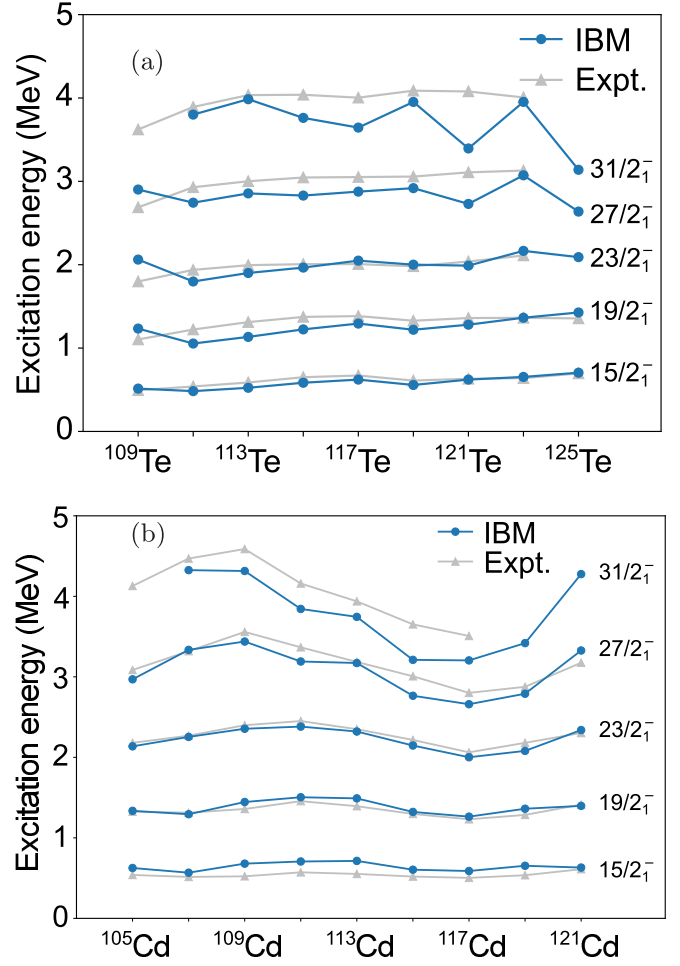


FIG. 3. Experimental and calculated $h_{11/2}$ bands of even-odd Te (upper) and Cd (lower) isotopes.

the states considered. Within this framework, the experimental excitation energies of the band built on the $h_{11/2}$ bandhead, comprising the $(11/2^-, 15/2^-, \dots)$ spin sequence, are reproduced with good accuracy by the IBFM calculations, as illustrated in Fig. 3. The side band formed by $(13/2^-, 17/2^-)$ can be reasonably reproduced. Possible contributions from additional orbitals and multi- j configurations, which may become relevant for higher-lying states, are not included in the present calculations and will be the subject of future work.

In addition, there are several low-lying yrast and yrare states in odd-A tellurium and cadmium isotopes, such as $9/2^-$, $7/2^-$, and $5/2^-$ in those odd-A isotopes. Those are not well reproduced by the present IBFM calculations which tend to overestimate their excitation energy. These discrepancies arise because the current model assumes a single- j orbital for the odd nucleon, which captures the dominant low-lying band built on the $h_{11/2}$ orbital but cannot account for states originating from additional orbitals or multi- j configurations or from the coupling to core excitation or particle-vibration coupled states. As a result, low-spin yrast and yrare states, as well as certain $13/2^-$ states, are beyond the scope of the present single- j description.

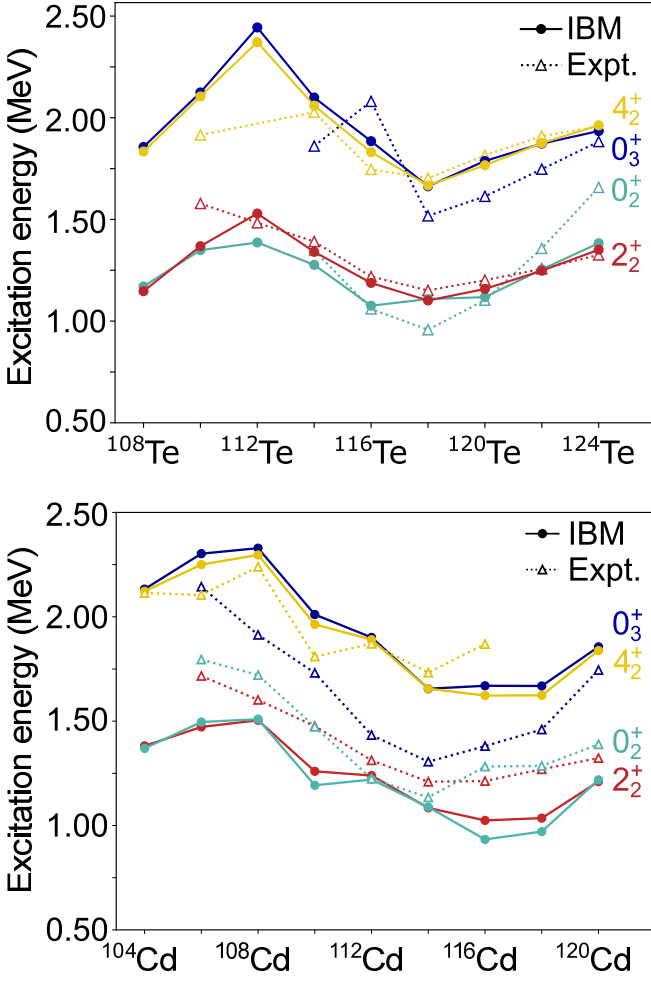


FIG. 4. Experimental [76] and calculated evolution of the excited $0_{2,3}^+$, 2_2^+ and 4_2^+ states in even-mass Te (Upper) and Cd (Lower) isotopes.

The evolution of yrare and higher-lying states in even-even and odd-A tellurium and cadmium nuclei is illustrated in Figs. 4 and 5, respectively. For the even-even isotopes, we show the $0_{2,3}^+$, 2_2^+ , and 4_2^+ states, while for the odd-A isotopes the corresponding band members $11/2_{2,3}^-$, $15/2_2^-$, and $19/2_2^-$ are presented.

The 0_2^+ states in the even-even $^{114-126}\text{Te}$ isotopes have often been considered as intruder bandheads, associated with enhanced $E2$ transitions to the one-phonon 2_1^+ states [47,77]. The 0_2^+ states in $^{110-116}\text{Cd}$ were also suggested to be intruder states [51]. Indeed, the evolution pattern of the 0_2^+ excitation energy, which exhibits a pronounced parabolic dependence along the isotopic chain, with a minimum around the mid-shell ^{118}Te [47,48], has been widely associated with the presence of shape coexistence and the emergence of intruder configurations. Similar systematics of the 0_2^+ states have also been reported in Sn and Cd isotopic chains as well. Experimental data for the transitions of those states, however, remain very limited. In our IBM calculations, the 0_2^+ states are reasonably well reproduced across the tellurium isotopic chain, although the excitation energy of the 0_2^+ state

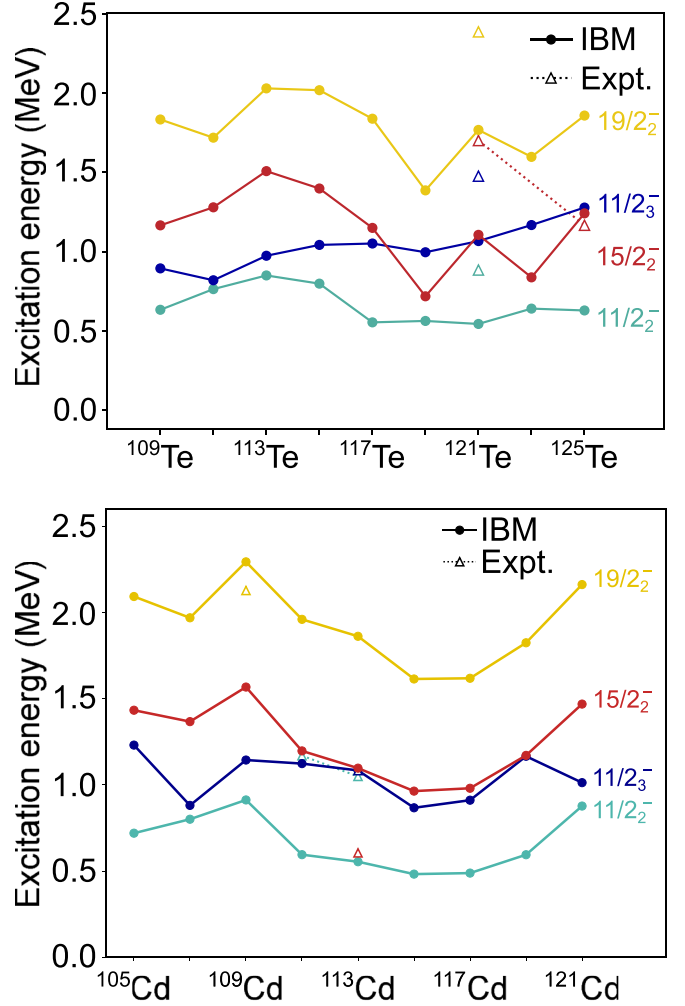


FIG. 5. Calculated evolution of the $11/2_{2,3}^-$, $15/2_2^-$ and $19/2_2^-$ states in odd-mass Te (Upper) and Cd (Lower) isotopes.

in ^{124}Te is noticeably underestimated. It should be noted that the standard IBM-1 framework, which includes only normal valence-space bosons, is generally not well suited for describing intruder configurations that involve particle-hole excitations across closed shells. Consequently, the present agreement should be interpreted with caution and ideally validated against additional experimental data. Nevertheless, the calculations provide a qualitative description of the evolution of the 0_2^+ states, and the reasonable reproduction of their energies suggests that the dominant collective features can still be captured within the IBM framework.

In agreement with experimental trends and recent theoretical studies [65], the IBM calculations predict that the excitation energy of the 0_2^+ state exhibits a pronounced minimum near $N = 66$. This evolution reflects changes in the underlying IBM parameters and the associated collective behavior, pointing to a gradual modification of the nuclear shape. In particular, the results indicate an enhanced γ softness in ^{114}Te , suggesting that triaxial degrees of freedom may play an increasingly important role in this region. These findings imply that a proper understanding of the yrare structures

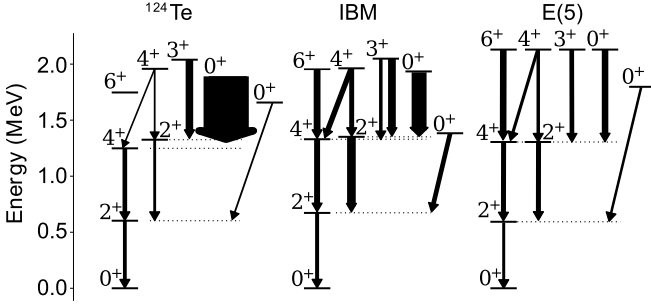


FIG. 8. Experimental level energies [76] and $B(E2)$ values [81] of ^{124}Te compared to results from IBM calculations, performed in this work, and E(5) critical-point symmetry calculations [78,79]. The widths of the arrows represent the $B(E2)$ value, normalized to the $B(E2; 2_1^+ \rightarrow 0_1^+)$ value.

relative to expectations from vibrational or axially symmetric collective models. Recent studies have suggested that this behavior may be linked to the onset of triaxial deformation or enhanced γ softness, which alters the structure of low-lying states and redistributes quadrupole collectivity among them.

To explore this possibility, an extended Hamiltonian including the additional a_3 term is employed, labeled IBM_b in Table I. The inclusion of this term leads to a dramatic improvement in the description of the $B_{4/2}$ ratio for ^{114}Te , bringing the calculated value much closer to the experimental result. This improvement supports interpretations that associate the $B_{4/2}$ anomaly with triaxial correlations not captured by simpler vibrational Hamiltonians.

In contrast, for ^{118}Te the experimental $B(E2)$ ratios are closer to the vibrational limit, and the simpler IBM_a Hamiltonian already provides an adequate description. This difference underscores the rapid structural evolution within the Te isotopic chain and emphasizes the sensitivity of $B(E2)$ ratios to subtle shape changes. Overall, the results presented in Table I demonstrate that while Te isotopes exhibit predominantly vibrational characteristics, certain nuclei—most notably ^{114}Te —require a more nuanced description incorporating triaxial degrees of freedom.

The triaxial deformation in this work is realized through an algebraic formulation of the triaxial rotor within IBM by utilizing the SU(3) dynamical symmetry [26,27], which is not achievable in the standard consistent- Q Hamiltonian. The approach constructs a Hamiltonian divided into static and dynamic components. The static part is responsible for generating a stable collective potential and is built from the second-order and third-order Casimir operators of SU(3). The dynamic part of the Hamiltonian incorporates the higher-order interactions between angular momentum and quadrupole operators. The cubic interaction term $(d^\dagger \times d^\dagger \times d^\dagger)^{(3)} \cdot (\tilde{d} \times \tilde{d} \times \tilde{d})^{(3)}$ in particular can induce triaxial deformation in its classical potential function and induce a stable triaxial minimum at the mean-field level [28,83,84]. In addition, unlike traditional geometric models, this realization accounts for finite-size effects inherent in a system with a limited number of bosons N .

TABLE I. Experimental $B(E2)$ ratios of $^{114,118}\text{Te}$ compared to two different IBM calculations. IBM_a is calculated using the Hamiltonian in Eq. (1) with the a_1, a_2, a_4 terms, while IBM_b is performed with the a_1, a_2, a_3, a_4 terms.

$B(E2)$ ratio	^{114}Te			^{118}Te	
	Expt. [41]	IBM _a	IBM _b	Expt. [68,72,82]	IBM _a
$\frac{B(4_1^+ \rightarrow 2_1^+)}{B(2_1^+ \rightarrow 0_1^+)}$	0.85(11)	1.83	0.53	1.86(10)	1.74
$\frac{B(6_1^+ \rightarrow 4_1^+)}{B(4_1^+ \rightarrow 2_1^+)}$	$1.49^{+0.37}_{-0.26}$	1.27	1.25	$1.12^{+0.20}_{-0.15}$	1.27

By extending the Hamiltonian to include higher-order interaction terms, it becomes possible to reproduce values of the $B_{4/2}$ ratio below unity, as observed experimentally in ^{114}Te . In particular, the inclusion of the a_3 term plays a crucial role in modifying the structure of the low-lying states and reducing the collectivity of the 4_1^+ level relative to that of the 2_1^+ state.

An illustrative example of this effect is shown in Fig. 9, where a two-dimensional contour plot of the $B_{4/2}$ ratios in $^{114-120}\text{Te}$ is presented as a function of the a_4 and a_3 parameters. The figure clearly demonstrates that, within a well-defined region of parameter space, the calculated $B_{4/2}$ ratio falls below unity, as observed in the nucleus ^{114}Te , highlighting the strong sensitivity of this observable to the higher-order interaction terms. This behavior contrasts with calculations employing only lower-order terms, for which such low values cannot be attained.

A second calculation of the $B(E2)$ ratios that includes the a_3 term is listed in Table I and labeled IBM_b. In this calculation, the value of the a_3 parameter was specifically adjusted to reproduce the anomalously low experimental $B_{4/2}$ ratio of ^{114}Te , while maintaining a reasonable description of the remaining transition strengths. The success of this extended Hamiltonian further supports interpretations that associate the $B_{4/2}$ anomaly with correlations beyond simple vibrational dynamics, such as triaxiality or enhanced γ softness, which are effectively captured through the inclusion of higher-order terms.

We have performed analogous calculations for the neighboring odd-A Te isotopes, the results of which are summarized in Fig. 10. In these nuclei, particular attention is given to the negative-parity $11/2^-$ band, as is shown for several isotopes in Fig. 3. This band provides a natural testing ground for examining collective behavior in odd-mass systems and for exploring the influence of the coupling between the unpaired fermion and the even-even core. For the odd-A Te nuclei, an analog of the $B_{4/2}$ ratio is introduced to quantify the relative collectivity of successive $\Delta J = 2$ transitions within the $11/2^-$ band. It is defined as

$$B_{4/2} \equiv \frac{B(E2; 19/2^- \rightarrow 15/2^-)}{B(E2; 15/2^- \rightarrow 11/2^-)}, \quad (23)$$

which directly mirrors the definition used in even-even nuclei while accounting for the presence of the odd fermion.

In Fig. 10, this $B_{4/2}$ ratio is plotted as a function of the a_3 parameter and the a_F parameter, the latter governing

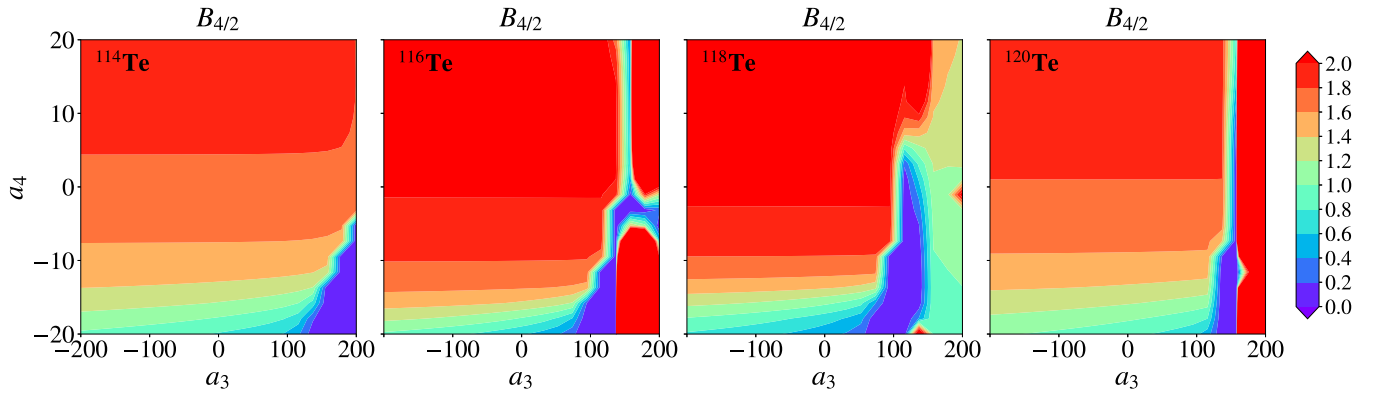


FIG. 9. Two-dimensional contour plot of the $B_{4/2}$ ratio in $^{114-120}\text{Te}$ as a function of the a_4 and a_3 parameters.

the strength of the coupling between the odd nucleon and the collective degrees of freedom of the even-even core. This two-dimensional representation highlights the interplay between intrinsic core correlations and particle-core coupling effects. As in the even-even case, the calculated $B_{4/2}$ ratio exhibits a pronounced sensitivity to these higher-order terms. In particular, modest variations in either a_3 or a_F can lead to substantial changes in the transition strengths, emphasizing that electromagnetic observables in odd- A nuclei provide stringent constraints on the detailed form of the Hamiltonian.

Overall, the results demonstrate that the anomalous behavior of the $B_{4/2}$ ratio is not limited to even-even systems but persists in neighboring odd- A isotopes, as was indeed observed in ^{119}Te [52], where it is further modulated by the fermion-core interaction. This underscores the importance of incorporating higher-order terms and explicit particle-core coupling when modeling collective excitations in transitional odd-mass nuclei.

In Figs. 11 and 12, the $E_{4/2}$ ratios are also displayed for both even-even and even-odd Te isotopes. Over most of the explored parameter space, the $E_{4/2}$ ratio remains relatively stable and exhibits only weak sensitivity to variations in the interaction strengths. Significant deviations from this behavior occur only at the edges of the parameter domain, where the ratio collapses toward values close to unity or below. This robustness of the $E_{4/2}$ ratio, in contrast to the pronounced sensitivity observed for the $B_{4/2}$ ratio, underscores that the

anomalous $B(E2)$ behavior arises primarily from changes in wave-function structure rather than from drastic modifications of the overall energy spectrum.

The excitation energy of the 2_1^+ state and the corresponding transition strength $B(E2; 2_1^+ \rightarrow 0_1^+)$ remain relatively stable over the entire parameter domain investigated above. Similarly, in the odd-mass Te isotopes, the excitation energy of the $15/2^-$ state and the $B(E2; 15/2^- \rightarrow 11/2^-)$ transition strength exhibit only weak dependence on the interaction parameters across the same domain. This overall stability indicates that the higher-order terms primarily affect specific collective correlations in the higher states while leaving the gross features of the low-lying excitation spectra largely unchanged. These correlations, however, can significantly affect the structure of the 4_1^+ state and, consequently, the $4_1^+ \rightarrow 2_1^+$ E2 transition strength, as illustrated in Fig. 13.

IV. CONCLUSION

In this work, we have developed and benchmarked a new open-source Python package, pyIBM, capable of performing systematic IBM and IBFM calculations. The package incorporates not only the standard vibrational and quadrupole-quadrupole terms but also higher-order interactions that are essential for exploring triaxiality, mixed-symmetry states, and anomalous collective behaviors. pyIBM provides a user-

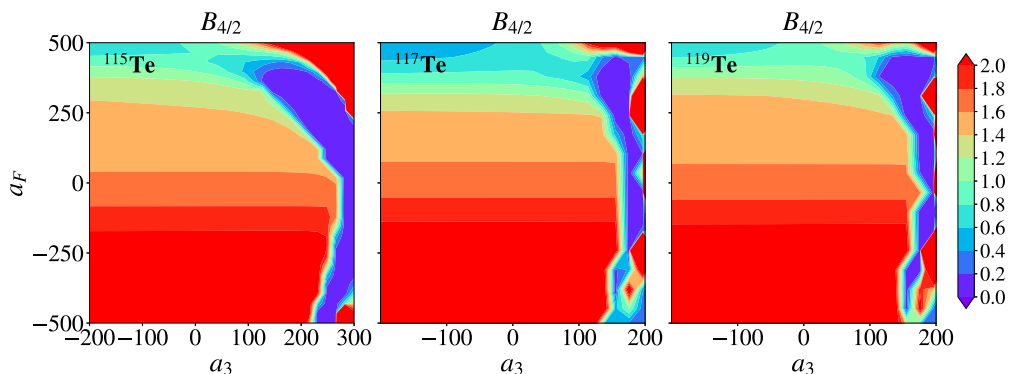


FIG. 10. Two-dimensional contour plot of the $B_{4/2}$ ratio in $^{115-119}\text{Te}$ as a function of the a_F and a_3 parameters.

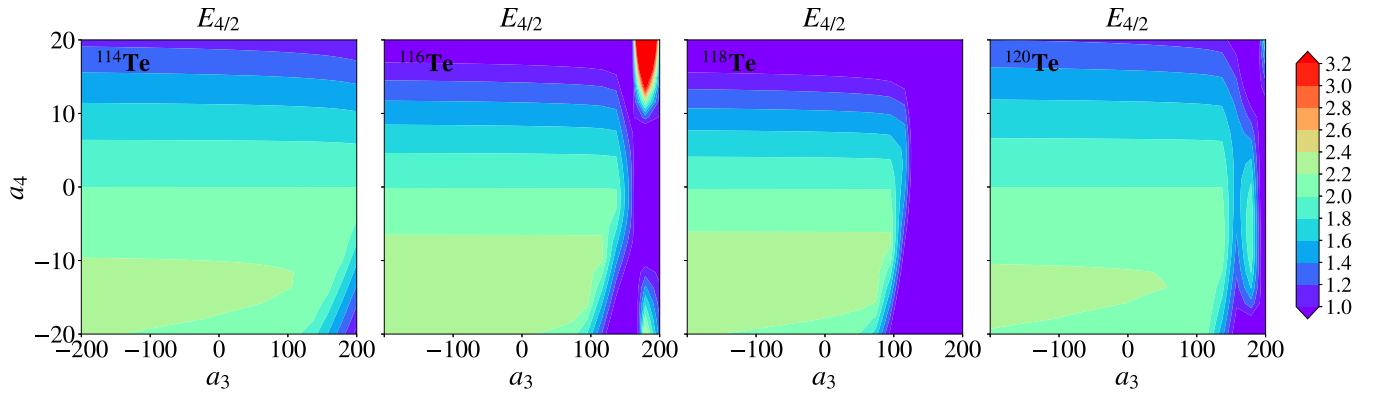


FIG. 11. Two-dimensional contour plot of the $E_{4/2}$ ratio in $^{114-120}\text{Te}$ as a function of the a_4 and a_3 parameters.

friendly platform for IBM/IBFM calculations, complementing older Fortran/C codes.

As an example, the code is applied to study the structure of even-even $^{108-124}\text{Te}$ and odd-even $^{109-125}\text{Te}$ nuclei through energy levels and reduced transition probabilities and compared to those of Cd isotopes. We also investigate the evolution of excited 0^+ states and yrare bands, revealing patterns consistent with recent experimental findings and providing new insight into the delicate interplay between vibrational and γ -soft modes. The Hamiltonian is dominated by the vibrational term, which is reflected in the predicted level schemes and $B(E2)$ values. The 2D contour plot function is introduced to demonstrate the effect of the higher-order terms on various observables like the $B_{4/2}$ ratio.

A particular focus was given to the $B_{4/2}$ anomaly in ^{114}Te , observed 20 years ago. Similar deviations have also been noted recently in odd-mass isotopes ^{119}Te . As one may expect, our results confirm that the standard vibrational Hamiltonian alone cannot reproduce this feature. However, the contour plot demonstrates that $B_{4/2}$ ratios below unity can naturally arise

within the IBM calculations, reflecting enhanced γ softness and the potential role of nonaxial fluctuations. These results further support the view that Te isotopes provide a valuable testing ground for studying collective excitations, the possible emergence of shape evolution.

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DATA AVAILABILITY

The PYIBM code developed for the numerical implementation of this work, together with future updates, is openly available on GitHub [61] and Zenodo [62].

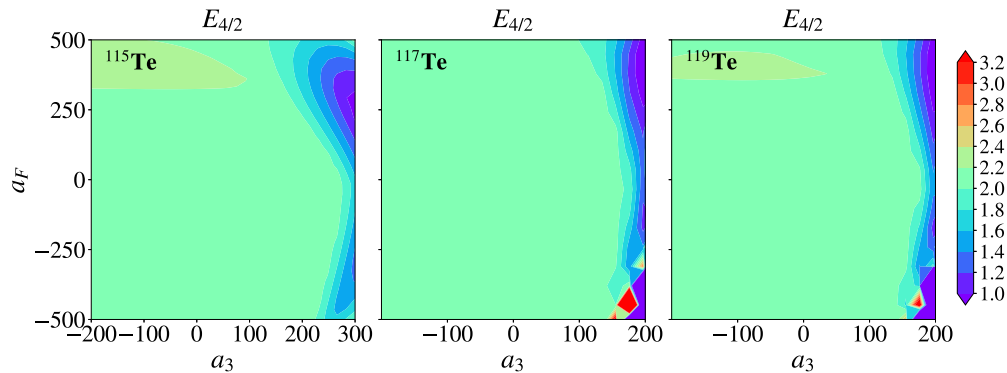


FIG. 12. Two-dimensional contour plot of the $E_{4/2}$ ratio in ^{115}Te as a function of the a_F and a_3 parameters.

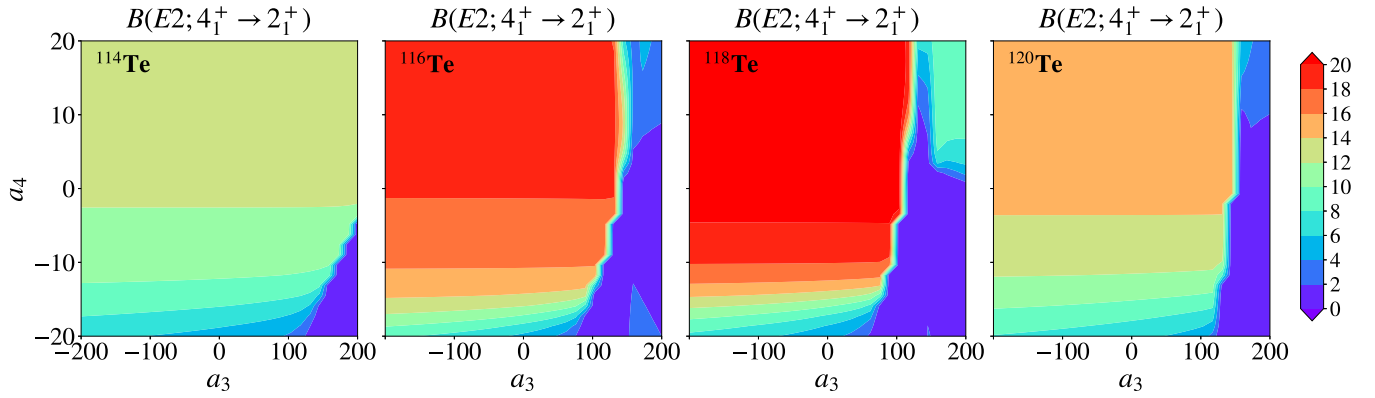


FIG. 13. Two-dimensional contour plot of the $B(E2; 4_1^+ \rightarrow 2_1^+)$ values in $^{114-120}\text{Te}$ as a function of the a_4 and a_3 parameters.

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