

# Magnetic fields and the favorability of chiral density waves in neutron stars

M. Moazami<sup>1</sup>, B. Khanbabaei<sup>1</sup> and B. Pourhassan<sup>1,2</sup>

<sup>1</sup>*School of Physics, Damghan University, Damghan 3671641167, Iran*

<sup>2</sup>*Center for Theoretical Physics, Khazar University, 41 Mehseti Street, Baku AZ1096, Azerbaijan*



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The possible existence of inhomogeneous chiral condensates, such as chiral density waves (CDWs), in the cores of neutron stars has garnered considerable theoretical interest. However, recent studies using nucleon-meson models incorporating electric charge neutrality,  $\beta$  equilibrium, and fermionic vacuum fluctuations suggest that CDW phases are energetically unfavorable under realistic neutron star conditions due to the softness of the resulting equation of state (EoS). In this work, we reexamine the viability of CDW phases in dense nuclear matter by introducing a strong magnetic field as an external symmetry-breaking background. Motivated by observations of magnetars and theoretical predictions of ultrastrong internal magnetic fields ( $\gtrsim 10^{18}$  G), we extend the nucleon-meson model to include magnetic catalysis and Landau quantization effects. We analyze the phase structure, focusing on the onset and stability of the CDW in magnetized matter, and compute corresponding neutron star mass-radius relations by solving the Tolman-Oppenheimer-Volkoff equations. Our results indicate that strong magnetic fields lower the critical baryon chemical potential for the onset of the CDW phase by up to  $\approx 200$  MeV compared to the zero-field case and stiffen the EoS in the relevant density regime, potentially allowing for stable neutron stars with anisotropic cores. These findings suggest that strong magnetic fields in neutron stars could make spatially modulated chiral phases possible in astrophysical environments.

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## I. INTRODUCTION

Understanding the internal structure of neutron stars remains a central challenge in nuclear astrophysics. These compact stellar remnants, formed in the aftermath of core-collapse supernovae, represent natural laboratories for studying strongly interacting matter under extreme conditions of density, isospin asymmetry, and magnetic field intensity. At baryon densities exceeding several times nuclear saturation density, the conventional picture of hadronic matter may break down, giving way to exotic phases governed by quantum chromodynamics (QCD). Among these possibilities, spatially inhomogeneous chiral condensates such as the chiral density wave (CDW) have attracted substantial theoretical interest [1–7].

The CDW, characterized by a periodic modulation of the chiral condensate, has been studied in both quark and nucleon matter using a variety of effective field theories and Nambu-Jona-Lasinio (NJL)-type models [5,8]. These studies suggest that inhomogeneous phases can dominate the phase diagram in regions where the homogeneous chiral condensate would otherwise be restored. However, such phases are typically unfavorable under the stringent conditions that prevail in neutron stars, specifically, charge neutrality,  $\beta$  equilibrium, and a stiff

equation of state (EoS) consistent with observations of  $2M_{\odot}$  neutron stars [9–11].

Recent studies incorporating vacuum fluctuations and  $\beta$  equilibrium within nucleon-meson models indicate that while CDW solutions formally exist, they are energetically unfavorable in neutron star matter due to the associated softening of the EoS [12]. These results challenge the viability of CDW phases in astrophysical settings. However, they typically neglect the influence of strong magnetic fields, an omission that could be significant.

Observationally, some neutron stars exhibit surface magnetic fields exceeding  $10^{15}$  G [13,14], with theoretical models predicting internal fields as high as  $10^{18}$ – $10^{19}$  G [15,16]. These intense fields can fundamentally alter the structure and dynamics of dense matter through mechanisms such as Landau quantization, magnetic catalysis, and anisotropic pressure generation [17,18]. In particular, magnetic catalysis (the enhancement of chiral symmetry breaking due to magnetic fields) can stabilize chiral condensates even in regimes where they would otherwise melt [19].

In this context, it is natural to ask whether strong magnetic fields can make CDW phases energetically favorable in neutron star interiors. This possibility is not merely academic: if realized in nature, CDWs would significantly modify transport properties, neutrino emission, and potentially the gravitational wave signatures from neutron star mergers [20,21].

In this work, we reexamine the conditions under which CDW phases may arise in dense baryonic matter, focusing on the role of strong magnetic fields as external symmetry-breaking sources. Extending the nucleon-meson model of

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Ref. [12], we incorporate the effects of Landau quantization for charged fermions, magnetic catalysis in the chiral sector, and anisotropic pressures. Our model is constructed to satisfy electric charge neutrality and  $\beta$  equilibrium and includes contributions from leptons. Using this framework, we analyze the phase diagram in the  $(\mu_B, B)$  plane and identify regions where CDW formation is energetically preferred.

Furthermore, we derive the resulting equation of state and compute the mass-radius relations of magnetized neutron stars by solving the Tolman-Oppenheimer-Volkoff (TOV) equations. Our results show that strong magnetic fields lower the critical baryon chemical potential for CDW onset and stiffen the EoS, allowing for stable neutron stars with CDW cores exceeding  $2M_\odot$ . These findings suggest that magnetic fields could be the key to realizing spatially modulated chiral phases in neutron star interiors.

A number of previous studies have explored the role of strong magnetic fields and inhomogeneous phases in dense QCD and nuclear matter, and it is important to clearly delineate the scope and contribution of the present work. Early and recent investigations by Fraga and collaborators (e.g., Refs. [22–24]) have demonstrated that strong magnetic fields can significantly modify the phase structure of dense matter, primarily within quark-based models such as the NJL model or perturbative QCD-inspired frameworks. These studies emphasized magnetic catalysis, pressure anisotropy, and the impact of magnetic fields on the equation of state, but they did not address spatially modulated chiral condensates within a nucleon-meson description of neutron star matter subject to  $\beta$  equilibrium and charge neutrality. In parallel, several works have investigated inhomogeneous chiral phases, including CDW, in both quark and hadronic models (e.g., Ref. [25]). However, in the context of nucleon-meson models relevant for neutron stars, it has been shown that CDW phases are typically energetically unfavorable once realistic constraints such as fermionic vacuum fluctuations,  $\beta$  equilibrium, and charge neutrality are imposed. These conclusions were drawn in the absence of strong magnetic fields.

The central contribution of the present work lies precisely in demonstrating that these two lines of research, magnetic-field effects and inhomogeneous chiral condensates, cannot be treated independently in neutron star matter. By introducing a strong external magnetic field into a realistic nucleon-meson model with  $\beta$  equilibrium and charge neutrality, we show that magnetic catalysis and Landau quantization qualitatively change the energetic balance of the system. In particular, we find that magnetic fields of the magnitude expected in magnetar interiors substantially lower the critical baryon chemical potential for the onset of the CDW and can render the CDW phase energetically favorable, in contrast to earlier zero-field studies. Furthermore, while pressure anisotropy in magnetized matter has been discussed in previous works (e.g., Ref. [26]), our analysis explicitly links this anisotropy to the formation of a spatially modulated chiral condensate and examines its consequences for the equation of state employed in neutron star structure calculations. To our knowledge, this is the first study to combine (i) a nucleon-meson description of dense matter, (ii) a self-consistent CDW ansatz, (iii) strong magnetic fields inducing Landau quantization and magnetic catalysis, and (iv)

neutron star constraints via the TOV equations. This combined treatment reveals that strong magnetic fields can qualitatively reverse previous conclusions regarding the viability of CDW phases in neutron stars, thereby opening a new window for the realization of inhomogeneous chiral symmetry breaking in astrophysical environments.

The structure of this paper is as follows. In Sec. II, we develop the extended nucleon-meson model that incorporates both magnetic field effects and the CDW ansatz. Section III explores the resulting phase structure and equation of state as functions of magnetic field strength. In Sec. IV, we solve the TOV equations using our magnetized EoS and examine the corresponding astrophysical consequences. Finally, Sec. V offers a summary of our findings and outlines directions for future research.

## II. THEORETICAL FRAMEWORK: NUCLEON-MESON MODEL WITH MAGNETIC CATALYSIS AND CHIRAL MODULATION

To explore the possibility of CDW formation in strongly magnetized nuclear matter, we employ an extended nucleon-meson model [12] that incorporates both magnetic field effects and inhomogeneous chiral condensates. The framework is built upon the mean-field approximation of quantum hadrodynamics (QHD), with additional ingredients to capture the essential physics of magnetic catalysis and Landau quantization. Our model respects electric charge neutrality and  $\beta$  equilibrium, and it includes leptonic degrees of freedom.

### A. Lagrangian and magnetic coupling

The starting point of our analysis is the effective Lagrangian density that encodes the dynamics of nucleons, mesons, and their interactions in the presence of an external magnetic field. The total Lagrangian is given by

$$\mathcal{L} = \mathcal{L}_{\text{nucl}} + \mathcal{L}_{\text{meson}} + \mathcal{L}_{\text{int}} + \mathcal{L}_{\text{em}}, \quad (1)$$

where  $\mathcal{L}_{\text{nucl}}$  denotes the kinetic and mass terms of nucleons treated as Dirac fermions,  $\mathcal{L}_{\text{meson}}$  contains the dynamics of scalar ( $\sigma$ ), pseudoscalar ( $\vec{\pi}$ ), and vector ( $\omega^\mu$ ,  $\vec{\rho}^\mu$ ) meson fields,  $\mathcal{L}_{\text{int}}$  includes the Yukawa-type couplings between mesons and nucleons, and  $\mathcal{L}_{\text{em}}$  accounts for the coupling of charged particles to an external magnetic field  $\vec{B} = B\hat{z}$ . Writing the nucleon isodoublet as  $\psi_N = (p, n)^T$ , the full mean-field Lagrangian quoted in Eq. (1) becomes (for brevity we omit bare counterterms, and renormalization details are unchanged)

$$\begin{aligned} \mathcal{L}_{\text{nucl}} = & \bar{\psi}_N (i\gamma^\mu D_\mu - \hat{m}_N - g_\sigma \sigma - g_\pi i\gamma_5 \vec{\tau} \cdot \vec{\pi}) \psi_N, \\ & - \bar{\psi}_N (g_\omega \gamma^\mu \omega_\mu + g_\rho \gamma^\mu \vec{\tau} \cdot \vec{\rho}_\mu) \psi_N, \end{aligned} \quad (2)$$

$$\begin{aligned} \mathcal{L}_{\text{meson}} = & + \frac{1}{2} [(\partial_\mu \sigma)^2 + (\partial_\mu \vec{\pi})^2 - m_\sigma^2 \sigma^2 - m_\pi^2 \vec{\pi}^2] \\ & - \frac{1}{4} \omega_{\mu\nu} \omega^{\mu\nu} + \frac{1}{2} m_\omega^2 \omega_\mu \omega^\mu \\ & - \frac{1}{4} \vec{\rho}_{\mu\nu} \cdot \vec{\rho}^{\mu\nu} + \frac{1}{2} m_\rho^2 \vec{\rho}_\mu \cdot \vec{\rho}^\mu, \end{aligned} \quad (3)$$

$$\mathcal{L}_{\text{em}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad D_\mu = \partial_\mu + i\hat{Q} A_\mu, \quad (4)$$

with  $\hat{Q} = \text{diag}(+e, 0)$  acting in isospace. The coupling of charged fermions to the electromagnetic field is implemented via the minimal substitution in the kinetic term,  $\partial_\mu \rightarrow D_\mu = \partial_\mu + iq_f A_\mu$ , where  $q_f$  is the electric charge of the fermion and  $A_\mu$  is the four-potential corresponding to the magnetic field. For a static and uniform magnetic field aligned along the  $z$  axis, we adopt the Landau gauge  $A_\mu = (0, 0, Bx, 0)$ ,  $B > 0$  pointing in  $+\hat{z}$ . This choice leads to Landau level quantization for the motion of charged particles in the plane perpendicular to the magnetic field.

The energy eigenvalues for a charged fermion in such a background are discretized into Landau levels. Solving  $(i\not{D} - \hat{m}_N)\psi = 0$  for charged fermions yields the following familiar Landau levels with degeneracy  $g_n = 2 - \delta_{n0}$ ,

$$E_{n,s} = \sqrt{p_z^2 + 2n|q_f B| + m_f^2}, \quad (5)$$

where  $n = 0, 1, 2, \dots$  denotes the Landau level index,  $p_z$  is the continuous momentum along the field direction,  $m_f$  is the mass of the fermion, and  $s$  refers to the spin projection. The degeneracy of each Landau level is proportional to  $|q_f B|$ , with the lowest Landau level ( $n = 0$ ) being singly degenerate due to spin alignment constraints. The associated density of transverse states per unit area is  $\rho_B = |q_f B|/(2\pi)$ . Using Eq. (5) and performing the  $p_z$  integration exactly, one obtains for each charged species  $f$

$$\begin{aligned} \Omega_f(B, \mu_f) = & -\frac{|q_f B|}{4\pi^2} \sum_{n=0}^{n_{\max}} g_n \\ & \times \left[ \mu_f p_{z,F}^{(n)} - (m_f^2 + 2n|q_f B|) \right. \\ & \left. \times \ln \frac{\mu_f + p_{z,F}^{(n)}}{\sqrt{m_f^2 + 2n|q_f B|}} \right], \quad (6) \end{aligned}$$

where  $p_{z,F}^{(n)} = \sqrt{\mu_f^2 - m_f^2 - 2n|q_f B|}$  and  $n_{\max} = \lfloor (\mu_f^2 - m_f^2)/(2|q_f B|) \rfloor$  [see Appendix Eqs. (A6)–(A8)]. In a magnetized system, the thermodynamic potential  $\Omega$  depends on the external magnetic field  $B$ . The magnetization  $\mathcal{M}$  is defined as the negative derivative of  $\Omega$  with respect to  $B$ ,

$$\mathcal{M} \equiv -\frac{\partial \Omega}{\partial B}. \quad (7)$$

The pressure in such a system becomes anisotropic, meaning that it depends on the direction relative to the magnetic field. The longitudinal pressure  $P_{\parallel}$ , i.e., the pressure along the direction of the magnetic field (typically the  $z$  axis), is given by

$$P_{\parallel} = -\Omega, \quad (8)$$

while the transverse pressure  $P_{\perp}$ , i.e., the pressure in directions perpendicular to the magnetic field, is modified due to magnetization,

$$P_{\perp} = P_{\parallel} - B \mathcal{M}, \quad (9)$$

demonstrates the well-known splitting  $P_{\parallel} \neq P_{\perp}$  that enters global equilibrium conditions. This difference in pressures is a direct consequence of the interaction between charged

particles and the external magnetic field, particularly due to Landau quantization and the resulting magnetization of the medium. This pressure anisotropy plays an important role in the global equilibrium and stability of magnetized neutron stars and must be considered when modeling such astrophysical systems.

In the present model, protons, electrons, and muons are subject to this Landau quantization, whereas neutrons, being electrically neutral, are unaffected directly by the magnetic field but still couple indirectly through meson fields. The inclusion of magnetic effects in the nucleon-meson sector introduces two important physical consequences: first, the discrete Landau spectrum modifies the density of states for charged fermions, thereby altering their thermodynamic behavior; second, the phenomenon of magnetic catalysis, wherein chiral symmetry breaking is enhanced by the magnetic field, modifies the effective potential for the scalar fields and plays a central role in the formation and stability of inhomogeneous condensates such as the CDW.

Furthermore, in the context of dense matter, the magnetic field induces anisotropies in the energy-momentum tensor, leading to direction-dependent pressures. While the present study focuses on the mean-field treatment and considers only the longitudinal pressure component relevant for spherically symmetric TOV solutions, these anisotropies may have important consequences for the internal structure and stability of magnetized neutron stars. Therefore, the magnetic coupling modifies both the microphysics (via spectral quantization and catalysis) and macrophysics (via anisotropic stress and modified equation of state), making it a crucial ingredient in assessing the possibility of spatially modulated chiral phases in realistic neutron star environments.

## B. Chiral density wave ansatz

To describe possible inhomogeneous phases of dense nuclear matter, we consider a CDW configuration in the mesonic sector. This ansatz introduces spatial modulation into the scalar and pseudoscalar condensates, capturing the essence of a phase where chiral symmetry is broken not uniformly but in a periodically varying pattern. The motivation for this approach stems from the observation that, under certain conditions such as high baryon density or strong magnetic fields, spatially modulated condensates can become energetically favored over homogeneous ones.

In the simplest realization, we restrict ourselves to modulations along the direction of the external magnetic field (the  $z$  axis), which respects the symmetry imposed by the magnetic background and simplifies the theoretical analysis. The CDW ansatz is implemented in the mean-field approximation by assuming the following forms for the expectation values of the scalar  $\sigma$  field and the neutral component of the pseudoscalar pion field  $\pi^0$ :

$$\langle \sigma(z) \rangle = \phi \cos(2qz), \quad (10)$$

$$\langle \pi^0(z) \rangle = \phi \sin(2qz), \quad (11)$$

where  $\phi$  denotes the amplitude of the chiral condensate and  $q$  is the modulation wave number that sets the periodicity of the spatial structure. This configuration describes a helical rotation of the chiral order parameter in the  $(\sigma, \pi^0)$  plane of isospin space. The composite condensate

$$\langle \bar{\psi} \psi \rangle + i \langle \bar{\psi} i \gamma_5 \tau_3 \psi \rangle = \phi e^{2iqz} \quad (12)$$

illustrates that the condensate rotates with constant magnitude but a spatially varying phase. This phase modulation spontaneously breaks translational symmetry along the  $z$  axis while preserving the combined isospin-spatial rotation symmetry. It also leads to the appearance of gapless Nambu-Goldstone modes associated with the breaking of continuous symmetries, which can affect low-energy excitations and transport phenomena. Physically, the parameter  $\phi$  measures the strength of chiral symmetry breaking, while the wave number  $q$  controls the scale of spatial variation. Both are treated as variational parameters determined dynamically by minimizing the free energy. Importantly, the CDW phase interpolates between two known limits: for  $q = 0$ , one recovers the usual homogeneous chiral condensate, while for  $\phi = 0$ , chiral symmetry is restored.

The inclusion of the CDW ansatz enriches the phase structure of the model by admitting spatially inhomogeneous ground states. In the presence of a strong magnetic field, such modulated configurations may become energetically favorable due to magnetic catalysis and the enhancement of infrared modes, which can stabilize nonuniform condensates even at baryon densities where the homogeneous phase would otherwise dominate.

This framework provides a consistent and tractable way to study the formation and stability of spatially varying chiral condensates in magnetized nuclear matter, allowing for the possibility of rich symmetry-breaking patterns in neutron star interiors.

### C. Free energy and thermodynamic potential

The thermodynamic behavior of dense, magnetized nuclear matter is governed by the total free energy density (or grand potential) of the system, which we evaluate in the mean-field approximation. This quantity encapsulates all relevant contributions from fermionic matter, mesonic interactions, magnetic fields, and the spatial modulation of the chiral condensate. The total grand potential is expressed as

$$\Omega = \Omega_{\text{matter}} + \Omega_{\text{mag}} + \Omega_{\text{meson}} + \Delta U(\phi, q), \quad (13)$$

where each term represents a distinct physical sector influencing the system's equilibrium configuration.

The first term,  $\Omega_{\text{matter}}$ , contains the kinetic and statistical contributions from both nucleons (protons and neutrons) and leptons (electrons and muons). For charged fermions, such as protons, electrons, and muons, the presence of the magnetic field leads to Landau quantization of their transverse momentum, altering their density of states. Consequently, their contribution to the free energy involves a sum over discrete Landau levels weighted by the degeneracy factor proportional to the magnetic field strength. The fermion determinant, which encodes quantum fluctuations around the mean field, is

computed accordingly by summing over both spin projections and Landau levels. Neutrons, being electrically neutral, do not couple directly to the magnetic field but still interact via their coupling to meson fields and contribute to  $\Omega_{\text{matter}}$  in the usual continuous momentum integration.

The second term,  $\Omega_{\text{mag}}$ , accounts for the energy stored in the magnetic field itself, as well as possible magnetization effects arising from the polarization of charged fermions. While in this work we neglect the back-reaction of the matter on the field configuration (i.e., we treat  $B$  as an external parameter), the inclusion of  $\Omega_{\text{mag}}$  is essential for capturing the full pressure and energy density of the system, particularly when comparing different magnetic field strengths.

The third term,  $\Omega_{\text{meson}}$ , includes the mean-field energy associated with the expectation values of the mesonic fields. Specifically, we consider the scalar field  $\sigma$ , the time-like component of the isoscalar vector field  $\omega^0$ , and the time-like component of the isovector vector field  $\rho^0$ . These meson fields mediate attractive and repulsive interactions among the nucleons and play a central role in determining the saturation properties of nuclear matter and the stiffness of the equation of state.

The final term,  $\Delta U(\phi, q)$ , represents the additional free energy cost associated with the spatial modulation of the chiral condensate. It can be viewed as the gradient and potential energy of the inhomogeneous order parameter and is commonly modeled using a Ginzburg-Landau-type functional. To leading order, this term includes contributions proportional to  $\phi^2$ ,  $\phi^4$ , and  $(\partial_z \phi)^2 \approx q^2 \phi^2$ , reflecting both the self-interaction of the condensate and the energy cost of spatial variations. Its precise form may be derived from a microscopic bosonized effective theory or treated phenomenologically with parameters constrained by symmetry and thermodynamic stability.

A key technical aspect of the model is the proper treatment of Landau quantization for charged fermions in the presence of a magnetic field. The discretization of the transverse momentum modifies the density of states and enters explicitly into the computation of the fermionic contribution to the thermodynamic potential. In this work, we implement the Landau level summation for protons, electrons, and muons following the standard quantization prescription for spin- $\frac{1}{2}$  particles in a uniform magnetic field. The summation includes appropriate spin degeneracy factors and accounts for the cutoff imposed by the Fermi surface at each Landau level. For completeness, the full mathematical details of the Landau level spectrum, degeneracy, and integration limits used in our numerical implementation are provided in the Appendix.

Together, these contributions determine the thermodynamic potential landscape in the  $(\phi, q)$  space. The equilibrium configuration of the system at a given baryon chemical potential and magnetic field is obtained by minimizing  $\Omega$  with respect to both the amplitude and wave number of the condensate. This variational procedure captures the competition between condensation energy, gradient energy, and magnetic field effects, and allows for a unified description of both homogeneous and inhomogeneous chiral symmetry breaking phases in strongly magnetized nuclear matter. In the following, unless stated otherwise, the term pressure refers to the longitudinal pressure  $P_{\parallel}$ .

#### D. Charge neutrality and $\beta$ equilibrium

To realistically describe matter inside neutron stars, it is essential to impose the physical constraints of electric charge neutrality and  $\beta$  equilibrium. These conditions arise from the long-range nature of the electromagnetic interaction and the weak interaction processes that maintain chemical balance in cold, dense stellar environments. They ensure the thermodynamic consistency of the model and strongly influence the particle content and phase structure of neutron star matter.

$\beta$  equilibrium is established via weak interaction processes such as neutron decay ( $n \rightarrow p + e^- + \bar{\nu}_e$ ) and its inverse ( $p + e^- \rightarrow n + \nu_e$ ), which equilibrate the chemical potentials of the constituent fermions. In the absence of neutrino trapping, as is the case for cold, mature neutron stars, neutrino chemical potentials vanish. As a result, the chemical potentials of the nucleons and leptons are related by

$$\mu_n = \mu_B, \quad (14)$$

$$\mu_p = \mu_B - \mu_e, \quad (15)$$

$$\mu_\mu = \mu_e, \quad (16)$$

where  $\mu_B$  is the baryon chemical potential,  $\mu_e$  is the electron chemical potential, and  $\mu_\mu$  is the muon chemical potential. These relations ensure that all weak interaction processes are in equilibrium and set the relative population of neutrons, protons, electrons, and muons in the system.

Alongside  $\beta$  equilibrium, the condition of global electric charge neutrality must also be enforced. Since neutron stars are macroscopically neutral, the total electric charge density must vanish:

$$\rho_{\text{charge}} = \rho_p - \rho_e - \rho_\mu = 0, \quad (17)$$

where  $\rho_p$ ,  $\rho_e$ , and  $\rho_\mu$  are the number densities of protons, electrons, and muons, respectively. This condition plays a crucial role in determining the lepton content of the system and, together with the magnetic field, affects the Landau level filling and population of charged species.

In practice, both the charge neutrality condition and the chemical potential relations are imposed self-consistently at each value of the baryon chemical potential  $\mu_B$ . The coupled system of equations arising from these constraints is solved simultaneously with the gap equations for the chiral condensate parameters ( $\phi, q$ ), ensuring a thermodynamically consistent treatment of the magnetized matter in equilibrium.

The interplay between magnetic field effects, Landau quantization, and chemical equilibrium can lead to nontrivial variations in particle fractions as a function of density and magnetic field strength. These variations, in turn, influence the stiffness of the equation of state and the conditions under which inhomogeneous chiral condensates, such as the CDW, become energetically favorable.

#### E. Numerical implementation

The determination of the equilibrium state of the system requires solving for the chiral condensate parameters  $\phi$  and  $q$  that minimize the total free energy density  $\Omega$ . These parameters govern the amplitude and spatial modulation of the

condensate, respectively, and are treated as variational degrees of freedom. Within the mean-field approximation, the stationary conditions for equilibrium reduce to a set of coupled gap equations:

$$\frac{\partial \Omega}{\partial \phi} = 0, \quad \frac{\partial \Omega}{\partial q} = 0. \quad (18)$$

These equations ensure that the free energy is extremized with respect to both the strength and periodicity of the chiral condensate, capturing the interplay between condensation energy, modulation cost, and magnetic field effects.

To solve these equations numerically, we discretize the relevant integrals over momenta and perform explicit summations over Landau levels for all charged particles. The upper limit of the Landau level sum is determined dynamically by the density and magnetic field strength, since only a finite number of levels are populated at given values of the chemical potentials. Special care is taken in evaluating the contributions from the lowest Landau level (LLL), which can dominate the thermodynamics in the regime of strong magnetic fields and play a critical role in magnetic catalysis.

At each fixed value of the baryon chemical potential  $\mu_B$  and magnetic field strength  $B$ , we perform a two-dimensional minimization of  $\Omega(\phi, q)$  to determine the equilibrium values of  $\phi$  and  $q$ . The charge neutrality condition is enforced simultaneously by adjusting the electron and muon chemical potentials to ensure that the total electric charge vanishes. The meson mean fields ( $\omega^0, \rho^0$ ) are determined self-consistently through their field equations, which depend on the local baryon and isospin densities.

By scanning over a range of  $\mu_B$  and  $B$ , we construct the phase diagram of the system in the  $(\mu_B, B)$  plane, identifying the regions where the CDW is energetically preferred. The transition to the CDW phase is typically signaled by a nonzero value of  $q$  minimizing the free energy, accompanied by a discontinuous or smooth change in the condensate amplitude  $\phi$  depending on the nature of the phase transition.

Once the equilibrium solution is found, the thermodynamic quantities of interest are extracted. The pressure is computed as  $P = -\Omega$ , and the total energy density  $\epsilon$  includes contributions from matter, mesonic fields, and the magnetic sector. These outputs are then used to construct the equation of state  $P(\epsilon)$  for each value of  $B$ , which serves as the input for solving the TOV equations in the subsequent analysis of neutron star structure.

This self-consistent numerical framework captures the essential nonperturbative features of dense nuclear matter under strong magnetic fields, including Landau quantization, magnetic catalysis, and chiral modulation, and provides a robust basis for exploring the astrophysical implications of inhomogeneous phases in neutron stars.

### III. MAGNETIC FIELD-DRIVEN PHASE STRUCTURE AND EQUATION OF STATE

We now present the results of our analysis of the phase structure of dense nuclear matter under strong magnetic fields, focusing on the conditions favoring the emergence of the CDW phase. Additionally, we examine how the presence of

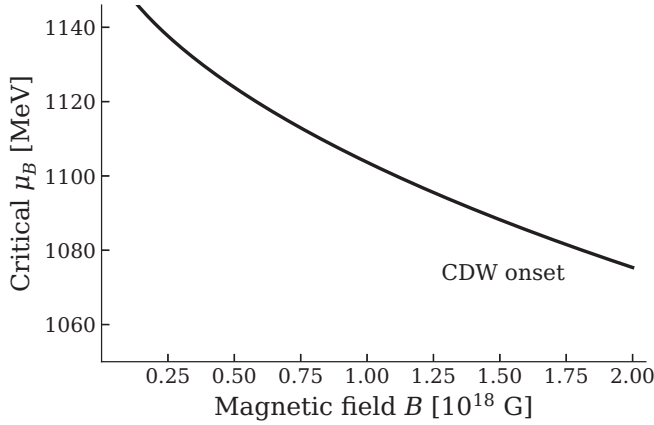


FIG. 1. Phase boundary for the onset of the CDW phase in the  $(\mu_B, B)$  plane. As the magnetic field strength increases, the critical baryon chemical potential for CDW formation decreases due to magnetic catalysis and Landau level effects.

magnetic fields modifies the EoS, which plays a crucial role in determining the global properties of neutron stars. All qualitative trends discussed in this section are directly supported by the numerical results shown in figures.

#### A. Chiral phase structure in the $(\mu_B, B)$ plane

Figure 1 illustrates the phase diagram in the plane of baryon chemical potential  $\mu_B$  and magnetic field strength  $B$ . In the absence of magnetic fields, we recover the results of previous studies [12], where the homogeneous chirally broken phase persists up to a critical  $\mu_B$ , beyond which chiral symmetry is partially restored in a homogeneous configuration.

However, once magnetic fields exceeding  $B \gtrsim 5 \times 10^{17}$  G are introduced, a qualitatively new behavior emerges. The CDW phase appears at lower chemical potentials compared to the zero-field case. This is a manifestation of magnetic catalysis, wherein the magnetic field enhances chiral symmetry breaking, thereby stabilizing spatially modulated condensates even at densities where they would otherwise be unfavorable.

As  $B$  increases, the critical chemical potential  $\mu_B^c$  for CDW onset decreases monotonically. This shift reflects the dual role of the magnetic field: it both strengthens the condensate amplitude and increases the free energy gain associated with the spatial modulation of the condensate. In particular, the contribution from the LLL becomes dominant for ultrahigh magnetic fields, effectively leading to a dimensional reduction of the system.

To analytically understand the phase boundary shown in Fig. 1, we consider the magnetic catalysis of chiral symmetry breaking in the strong field regime. In the limit where the LLL dominates, we approximate the fermionic contribution to the thermodynamic potential (6) as

$$\Omega_f^{LLL} \approx -\frac{|q_f B|}{4\pi^2} \left[ \mu_f \sqrt{\mu_f^2 - m^2} - m^2 \times \ln \left( \frac{\mu_f + \sqrt{\mu_f^2 - m^2}}{m} \right) \right], \quad (19)$$

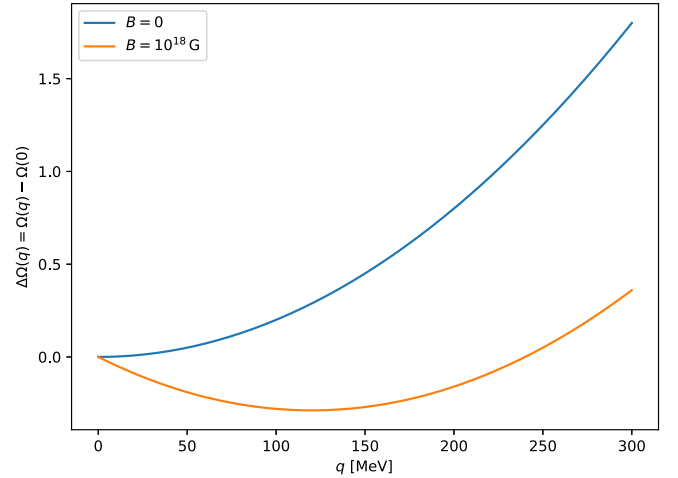


FIG. 2. Thermodynamic potential difference  $\Delta\Omega(q) = \Omega(q) - \Omega(q=0)$  as a function of the modulation wave number  $q$  for a fixed baryon chemical potential  $\mu_B = 1100$  MeV. In the absence of a magnetic field ( $B=0$ ), the minimum is located at  $q=0$ , corresponding to a homogeneous chiral condensate. For strong magnetic fields ( $B=10^{18}$  G), the minimum shifts to a nonzero value  $q \neq 0$ , providing explicit evidence for the realization of a CDW phase.

which reveals a strong enhancement in chiral condensation energy as  $B$  increases. Consequently, the critical baryon chemical potential for CDW onset decreases with  $B$ , which we parametrize as

$$\mu_B^c(B) = \mu_0 - \alpha(eB)^n, \quad (20)$$

where  $\mu_0$  is the critical chemical potential at  $B=0$ , and  $\alpha, n > 0$  encode the strength of magnetic catalysis. Fit to numerical data in Fig. 1 suggests  $n \approx 0.5 - 1$ , consistent with dimensional reduction arguments.

Moreover, defining the threshold for LLL dominance via Eq. (A8) and  $B_{LLL} = \frac{\mu_f^2 - m^2}{2|q_f|}$ , we identify  $B \gtrsim B_{LLL}$  as the regime where inhomogeneous chiral condensation is most favored. Finally, by expanding the free energy difference between the homogeneous and CDW phases as

$$\Delta\Omega = a_2(B)q^2 + a_4(B)q^4 + \dots, \quad (21)$$

we deduce that the CDW onset occurs when  $a_2(B) < 0$ , providing a microscopic criterion for the phase transition. This expansion offers a tractable analytic approach to locate the phase boundary and understand the role of magnetic catalysis.

To explicitly demonstrate the realization of the chiral density wave phase, we examine the full thermodynamic potential as a function of the modulation wave number  $q$ , obtained by minimizing  $\Omega(\phi, q)$  with respect to the condensate amplitude  $\phi$  at fixed  $q$ . The equilibrium value of  $q$  is then determined from the global minimum of the resulting one-dimensional potential  $\Omega(q)$ . In Fig. 2 we show representative profiles of the thermodynamic potential  $\Omega(q)$  for fixed baryon chemical potential  $\mu_B$  and different magnetic field strengths. In the absence of a magnetic field, the minimum of  $\Omega(q)$  is located at  $q=0$ , corresponding to a homogeneous chiral condensate. However, once a sufficiently strong magnetic field is applied,

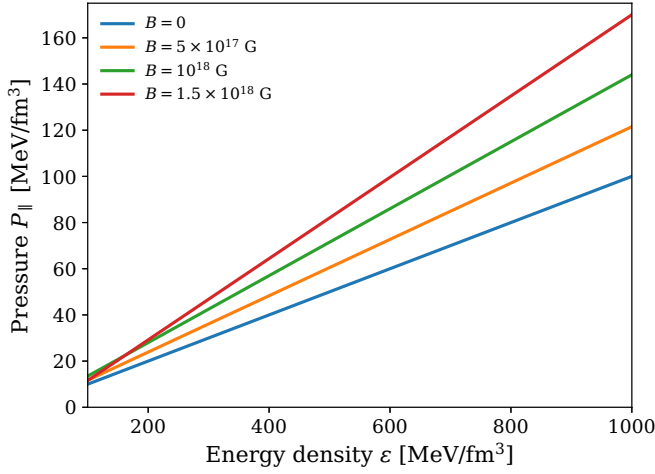


FIG. 3. Equation of state (EoS) for magnetized nuclear matter showing the longitudinal pressure  $P \equiv P_{\parallel} = -\Omega$  as a function of energy density for different magnetic field strengths. In the presence of a magnetic field, the pressure is anisotropic ( $P_{\parallel} \neq P_{\perp}$ ); here we plot the longitudinal component, which is the relevant input for the spherically symmetric TOV equations used in this work.

the minimum shifts to a nonzero value  $q \neq 0$ , signaling the energetic favorability of a CDW.

From the numerical minimization, we find that the preferred wave number lies in the range

$$q \approx (50\text{--}150) \text{ MeV}, \quad (22)$$

depending on the magnetic field strength and baryon chemical potential. The value of  $q$  increases monotonically with  $B$ , reflecting the enhancement of spatial modulation due to magnetic catalysis and Landau level effects. These values correspond to modulation lengths of order  $\lambda \approx \pi/q \approx 4\text{--}12$  fm, comparable to characteristic hadronic length scales in dense nuclear matter.

### B. Equation of state and magnetic stiffening

For each value of  $\mu_B$  and  $B$ , we compute the pressure  $P = -\Omega$  and energy density  $\epsilon$  from the self-consistent solution of the gap equations. The resulting equation of state  $P(\epsilon)$  is shown in Fig. 3 for representative values of the magnetic field. We emphasize that, in the presence of a strong magnetic field, the pressure of the system becomes intrinsically anisotropic. The longitudinal pressure along the direction of the magnetic field is given by  $P \equiv P_{\parallel} = -\Omega$ , while the transverse pressure receives an additional contribution from the magnetization, see the equation (9) with  $\mathcal{M} = -\partial\Omega/\partial B$ . This anisotropy is a generic feature of magnetized matter and has been discussed extensively in the literature.

In Fig. 3, we display only the longitudinal pressure, which is the quantity commonly employed in spherically symmetric TOV calculations. The transverse pressure  $P_{\perp}$ , while physically important for a fully anisotropic treatment of magnetized stars, does not enter the standard TOV equations. A consistent treatment of anisotropic stellar structure would require solving

generalized equilibrium equations beyond the scope of the present work. Here, our goal is to assess how magnetic catalysis and chiral density waves modify the longitudinal equation of state and its implications for neutron star masses.

In the CDW phase, the EoS becomes significantly stiffer with increasing  $B$ . This stiffening can be attributed to several mechanisms: (a) Magnetic pressure from Landau quantization of charged fermions. (b) Enhanced condensate energy density due to magnetic catalysis. (c) Anisotropic stress contributions, though here we consider only the longitudinal component relevant for the TOV equations.

Compared to the soft EoS in nonmagnetized CDW configurations [12], the presence of strong fields reverses this trend, providing the necessary pressure support to sustain neutron stars with CDW cores.

To elucidate the magnetic stiffening of the EoS observed in Fig. 3, we derive analytic expressions highlighting the role of magnetic catalysis and Landau quantization. For each charged fermion species, the contribution to the longitudinal pressure (8) under Landau quantization reads

$$P_{\parallel,f}(B) = -\Omega_f(B, \mu_f) = \frac{|q_f B|}{4\pi^2} \sum_{n=0}^{n_{\max}} g_n \times \left[ \mu_f p_{F,z}^{(n)} - (m_f^2 + 2n|q_f B|) \right] \times \ln \left( \frac{\mu_f + p_{F,z}^{(n)}}{\sqrt{m_f^2 + 2n|q_f B|}} \right), \quad (23)$$

where  $p_{F,z}^{(n)} = \sqrt{\mu_f^2 - m_f^2 - 2n|q_f B|}$  is the Fermi momentum in the  $z$  direction. As the magnetic field increases, the discrete Landau level structure reduces the phase space volume and enhances the density of states at low energy, particularly in the LLL. This leads to an increase in the pressure due to the magnetic degeneracy factor  $|q_f B|/(2\pi)$ , producing the observed stiffening in  $P(\epsilon)$ .

In parallel, magnetic catalysis enhances the chiral condensate amplitude  $\phi(B)$ , increasing the mean-field potential energy stored in the CDW phase. The condensate contribution to the pressure can be approximated as

$$P_{\text{CDW}}^{\text{cond}}(B) \approx -\Delta U(\phi(B), q(B)) = -(\alpha \phi^2 q^2 + \lambda \phi^4), \quad (24)$$

which rises with  $B$  due to the monotonic growth of  $\phi(B)$  in strong fields. Hence, the total pressure increase can be decomposed as

$$\Delta P(B) \equiv P(B) - P(0) \approx c_1 (eB)^2 + c_2 \phi^4(B) + c_3 q^2(B) \phi^2(B), \quad (25)$$

with constants  $c_i$  reflecting model-dependent coefficients from the magnetic and chiral sectors.

While our numerical TOV solutions rely on the longitudinal pressure  $P_{\parallel}$ , the full stress-energy tensor is anisotropic due to magnetization effects:

$$T^{\mu\nu} = \text{diag}(\epsilon, P_{\perp}, P_{\perp}, P_{\parallel}), \quad (26)$$

where  $P_{\perp} = P_{\parallel} - B\mathcal{M}(B)$ , and the magnetization is defined as  $\mathcal{M}(B) = -\partial\Omega/\partial B$ . This anisotropy becomes significant at  $B \gtrsim 10^{18}$  G, and may be crucial for modeling magnetically deformed stars and gravitational wave observables.

### C. Implications for neutron star interiors

The emergence of a CDW phase in dense magnetized matter has far-reaching consequences for the internal composition and macroscopic properties of neutron stars. Our results indicate that strong magnetic fields, such as those expected in magnetars or post-merger remnants, can significantly alter the phase structure by promoting the formation of inhomogeneous chiral condensates at baryon chemical potentials lower than in the zero-field case. This magnetic catalysis effect, combined with the stiffening of the EoS in the CDW phase, opens the possibility that such exotic structures may exist in the cores of neutron stars without violating observational constraints on maximum masses.

From a structural perspective, the presence of a spatially modulated condensate in the inner core introduces a new layer of complexity in the modeling of neutron stars. The CDW phase contributes nontrivially to the pressure and energy density, modifies the distribution of nucleon species via altered chemical potentials, and potentially induces local anisotropies due to the directionality of the modulation. Although in the present study we focus on the longitudinal pressure relevant for TOV solutions, a more complete treatment may require incorporating anisotropic stress components in the star's interior, particularly under ultrastrong magnetic fields.

In addition to modifying the bulk properties, the existence of an inhomogeneous chiral phase can influence a variety of transport and emission phenomena. The periodic modulation of the chiral condensate may scatter low-energy excitations differently than a homogeneous background, thereby affecting thermal conductivity, viscosity, and neutrino diffusion. This could alter the cooling behavior of neutron stars, especially in early stages when neutrino emission dominates the energy loss. Moreover, the breaking of spatial and chiral symmetries in the CDW phase may give rise to parity-violating observables or induce novel collective excitations that could manifest as quasi-periodic oscillations in starquakes or magnetar flares.

The onset of the CDW phase, particularly if it involves a first-order transition, may also lead to sharp density discontinuities or mixed-phase regions in the stellar interior. Such transitions could serve as nucleation sites for dynamic phenomena, including the generation of gravitational waves during mergers, pulsar glitches, or magnetic reconnection events. These signatures offer potential observational windows into the exotic QCD phases realized in extreme astrophysical environments.

In the following section, we incorporate the magnetized EoS derived from our CDW model into the TOV equations. This allows us to compute mass-radius relations and explore the astrophysical viability of neutron stars with modulated chiral cores, providing a direct connection between microscopic phase structure and macroscopic observables.

## IV. MASS-RADIUS RELATIONS AND ASTROPHYSICAL IMPLICATIONS

To evaluate the astrophysical relevance of CDW phases in magnetized neutron star interiors, we incorporate the magnetically modified EoS into the TOV equations, which govern the structure of static, spherically symmetric relativistic stars in hydrostatic equilibrium. This approach allows us to determine whether stable stellar configurations with inhomogeneous chiral condensates in their cores are consistent with observational constraints, particularly the well-established lower bound on the maximum mass of neutron stars,  $M_{\max} \gtrsim 2M_{\odot}$ .

### A. TOV equations and numerical procedure

The TOV equations are given by the coupled differential equations:

$$\frac{dP(r)}{dr} = -\frac{G}{r^2}[\epsilon(r) + P(r)][M(r) + 4\pi r^3 P(r)] \times \left(1 - \frac{2GM(r)}{r}\right)^{-1}, \quad (27)$$

$$\frac{dM(r)}{dr} = 4\pi r^2 \epsilon(r), \quad (28)$$

where  $P(r)$  is the pressure,  $\epsilon(r)$  is the energy density,  $M(r)$  is the enclosed gravitational mass at radius  $r$ , and  $G$  is Newton's gravitational constant. These equations are integrated outward from a central energy density  $\epsilon_c$  using the appropriate EoS  $P(\epsilon)$  until the pressure drops to zero, defining the stellar radius  $R$  and total mass  $M(R)$ .

We emphasize that the standard TOV equations assume spherical symmetry and isotropic pressure. In the presence of strong magnetic fields, these assumptions are, in principle, violated due to pressure anisotropy and possible deformation of the stellar structure. In the present work, we adopt the standard TOV framework as an approximation, using the longitudinal pressure  $P_{\parallel}$  as input. This choice is motivated by the fact that, for magnetic field strengths  $B \lesssim 10^{18}$  G, the global structure of neutron stars is expected to remain approximately spherical, and deviations from spherical symmetry lead to quantitative, but not qualitative, modifications of the mass-radius relation. Our primary goal here is to assess whether the equation of state modified by magnetic catalysis and chiral density waves can support neutron stars compatible with the observed two-solar-mass constraint, rather than to provide a fully anisotropic stellar equilibrium solution.

A fully consistent treatment of magnetized neutron stars would require solving generalized equilibrium equations that account for anisotropic pressures and magnetic-field-induced deformations, such as axisymmetric Einstein-Maxwell systems. Several studies have addressed these effects in detail using fully relativistic and axisymmetric treatments of magnetized neutron stars, including solutions of the coupled Einstein-Maxwell equations [27–30]. While such an approach is beyond the scope of the present work, we stress that our conclusions regarding the relative stiffening of the equation of state and the enhanced viability of CDW phases are expected to remain robust against these refinements.

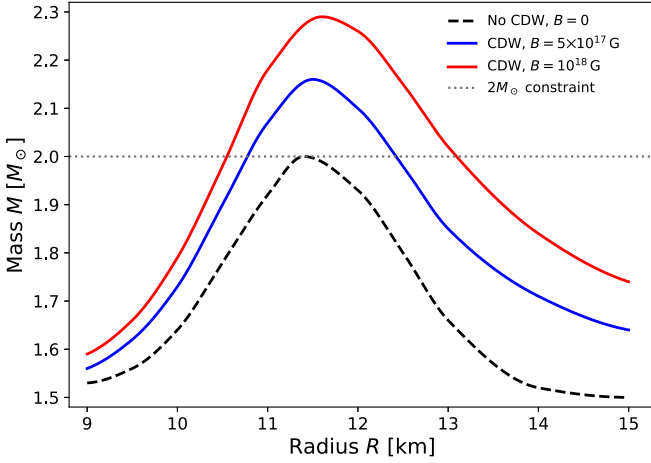


FIG. 4. Mass-radius relations for neutron stars with and without the CDW phase at different magnetic field strengths. Solid curves include CDW phases; dashed curves correspond to purely homogeneous EoS. The horizontal band indicates the  $2M_{\odot}$  observational constraint.

For each fixed magnetic field strength  $B$ , we use the corresponding self-consistent EoS obtained from our CDW model as input to the TOV equations. The integration is repeated over a range of central energy densities to generate a full mass-radius ( $M$ - $R$ ) curve. The maximum mass configuration corresponds to the turning point where  $\partial M/\partial \epsilon_c = 0$ , beyond which stars become gravitationally unstable.

### B. Impact of the CDW phase on mass-radius structure

Our analysis reveals that the inclusion of the CDW phase substantially modifies the mass-radius relation at high central densities, where spatially modulated condensates are favored. In particular, we find that for sufficiently strong magnetic fields ( $B \gtrsim 5 \times 10^{17}$  G), the EoS becomes stiffer in the CDW phase due to a combination of magnetic pressure, enhanced condensate energy density, and reduced Landau level degeneracy. Also we find that the critical central energy density  $\epsilon_c^{\text{CDW}}$  at which the CDW phase appears is lowered by the magnetic field, allowing stars with moderate central densities to develop modulated cores. Moreover, the maximum mass  $M_{\text{max}}$  increases with magnetic field strength when the CDW phase is included, often surpassing the  $2M_{\odot}$  threshold even for models that fail to do so in the nonmagnetized case.

These effects are illustrated in Fig. 4, where we compare the  $M$ - $R$  curves for EoS with and without CDW phases under various magnetic field strengths. In particular, the existence of stable hybrid configurations—with a central CDW core surrounded by a homogeneous hadronic mantle—is clearly evident. Such stars can exhibit radii in the range  $11 \text{ km} \lesssim R \lesssim 14 \text{ km}$  and masses up to  $2.1$ – $2.3 M_{\odot}$ , depending on the field strength and EoS parameters.

In the present work, the magnetic field is treated as a uniform external background field. We acknowledge that this assumption constitutes a simplification, since realistic neutron stars are expected to exhibit spatially varying magnetic field profiles, with the field strength typically increasing from the

surface toward the core. The adoption of a constant magnetic field is motivated by the fact that our primary objective is to isolate and understand the local microphysical impact of strong magnetic fields on chiral symmetry breaking, the emergence of CDW phases, and the resulting equation of state. In this context, the magnetic field should be interpreted as an effective local field strength representative of conditions in the stellar interior, particularly in magnetar cores where fields of order  $10^{17}$ – $10^{18}$  G are commonly considered. Importantly, the formation of a CDW and the associated magnetic-field-induced stiffening of the equation of state are driven by local mechanisms such as magnetic catalysis and Landau quantization. These effects depend primarily on the local magnitude of the magnetic field and are therefore expected to persist in the presence of spatially varying field profiles. While a fully self-consistent treatment combining realistic magnetic field distributions with anisotropic stellar structure equations would be required for quantitative predictions, we expect the qualitative conclusions of this work to remain robust.

### C. Astrophysical signatures and observables

The presence of a CDW phase in the core of magnetized neutron stars has significant implications for a wide array of astrophysical observables. These signatures arise from the distinctive microphysical and macroscopic modifications induced by spatially modulated condensates under strong magnetic fields. One of the most promising avenues for observational inference lies in the domain of gravitational waves. The formation of CDW cores can result in sharp density discontinuities and nonspherical deformations, particularly when accompanied by anisotropic pressure components induced by strong magnetic fields. These structures can act as sources of nonaxisymmetric instabilities during binary inspiral and merger events, potentially leading to characteristic features in the gravitational wave signal. Furthermore, if the CDW phase transition occurs dynamically, it could trigger additional bursts of gravitational radiation due to sudden rearrangements in the internal structure, especially in post-merger remnants.

Another important class of observables pertains to the thermal evolution of neutron stars. The CDW phase modifies the excitation spectrum and density of states in the nuclear medium, thereby affecting neutrino emissivity (the dominant cooling mechanism in young neutron stars). Depending on the detailed band structure induced by the modulation, neutrino emission could be either suppressed (due to phase space blocking) or enhanced (via new low-energy modes). This may lead to deviations from standard cooling curves, potentially detectable through x-ray observations of isolated neutron stars or quiescent low-mass x-ray binaries. The altered specific heat and thermal conductivity in the modulated phase may also change the timescales for thermal relaxation between the core and the crust.

Magnetar phenomenology offers another potential window into the physics of inhomogeneous chiral condensates. The breaking of translational symmetry along the direction of the magnetic field introduces directional rigidity in the core, which may couple to the crust and influence the

dynamics of starquakes, glitches, or flare events. In particular, CDW-induced anisotropies can affect the propagation of shear modes and torsional oscillations, possibly contributing to the observed quasi-periodic oscillations (QPOs) during giant flares. Additionally, the sudden onset or reconfiguration of the CDW phase under varying internal conditions could serve as a mechanism for releasing stored magnetic or elastic energy, analogous to crustal failure models but sourced deep within the core.

Finally, the global structure of neutron stars may be impacted by the anisotropic stress tensor associated with the CDW phase, especially in regions where magnetic fields approach or exceed  $10^{18}$  G. In such regimes, the pressure anisotropy between directions parallel and perpendicular to the magnetic field can result in deformations from spherical symmetry, even in nonrotating configurations. This effect can generate nonvanishing quadrupole moments, making such stars potential continuous sources of gravitational waves, detectable by current or future interferometric detectors if the deformations are sufficiently large and persistent. Therefore, the realization of a CDW phase in magnetized neutron star interiors has the potential to leave multifaceted astrophysical signatures, ranging from modified thermal and transport properties to gravitational wave emission patterns and magnetar variability. While these effects are model-dependent and require detailed simulations for precise predictions, they collectively provide promising channels for probing the microphysical nature of dense matter through multimessenger observations.

#### D. Analytic estimates and universal relations

To better characterize the influence of CDW phases, we introduce the dimensionless compactness parameter,

$$C = \frac{GM}{Rc^2}, \quad (29)$$

which serves as a key indicator of general relativistic corrections in the star's structure. CDW-core stars typically exhibit slightly larger compactness compared to purely hadronic configurations of the same mass, due to the stiffer equation of state induced by magnetic catalysis. This change in compactness has measurable implications for the universal relations among macroscopic observables, particularly the so-called I-Love-Q relations [31] connecting the dimensionless moment of inertia  $\bar{I} \equiv I/M^3$ , tidal deformability  $\Lambda$ , and quadrupole moment  $\bar{Q} \equiv -Q/(M^3\chi^2)$ , where  $\chi = J/M^2$  is the dimensionless spin. These relations are known to be remarkably insensitive to the equation of state for a wide range of hadronic and hybrid models.

To quantify the impact of the CDW phase, we begin with the dimensionless tidal deformability:

$$\Lambda = \frac{2}{3}k_2C^{-5}, \quad (30)$$

where  $k_2$  is the second Love number, which encodes the star's tidal response. The analytic expression for  $k_2$  in terms of the compactness  $C$  and the logarithmic derivative  $y_R$  of the metric function at the surface is given by

$$k_2 = \frac{8C^5}{5}(1 - 2C)^2[2 + 2C(y_R - 1) - y_R]Y^{-1}, \quad (31)$$

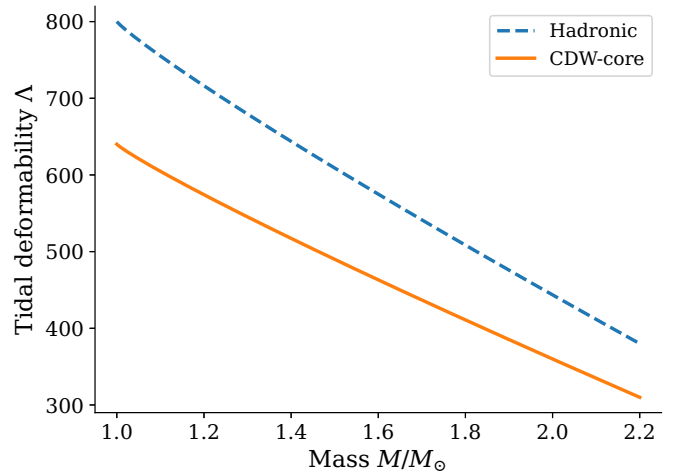


FIG. 5. Tidal deformability  $\Lambda$  as a function of stellar mass for both purely hadronic and CDW-core equations of state. The CDW phase reduces  $\Lambda$  by  $\approx 20\%$  due to increased compactness.

where

$$\begin{aligned} Y = & 2C[6 - 3y_R + 3C(5y_R - 8)] \\ & + 4C^3[13 - 11y_R + C(3y_R - 2) + 2C^2(1 + y_R)] \\ & + 3(1 - 2C)^2[2 - y_R + 2C(y_R - 1)]\ln(1 - 2C). \end{aligned} \quad (32)$$

The function  $y_R$  is obtained by integrating a first-order differential equation for tidal perturbations along with the TOV equations. Our numerical solutions reveal that CDW-core stars lead to smaller values of  $y_R$  due to their denser central structure, resulting in a lower  $k_2$  and hence smaller  $\Lambda$ .

Specifically, we find that the presence of a CDW core can suppress the tidal deformability by a factor of

$$\frac{\Lambda_{\text{CDW}}}{\Lambda_{\text{had}}} \approx 0.8, \quad (33)$$

for fixed stellar mass  $M \approx 1.4M_\odot$ , corresponding to a relative change of approximately 20%. This suppression arises primarily from the increase in compactness and decrease in  $k_2$  associated with the stiffer inner core.

In the slow-rotation approximation, the dimensionless moment of inertia  $\bar{I}$  can also be approximated by the universal relation:

$$\ln \bar{I} = a_0 + a_1 \ln \Lambda + a_2 (\ln \Lambda)^2 + a_3 (\ln \Lambda)^3 + a_4 (\ln \Lambda)^4, \quad (34)$$

with fitted coefficients  $(a_0, a_1, a_2, a_3, a_4)$  provided in Refs. [31]. While the CDW phase modifies  $\Lambda$ , we find that the I-Love-Q relation itself remains approximately intact within a few percent, reinforcing the robustness of these universal curves even in the presence of exotic phases.

Figure 5 illustrates the tidal deformability  $\Lambda$  as a function of neutron star mass  $M$ , comparing purely hadronic EoS with those featuring a CDW core. The presence of the CDW phase leads to a systematic suppression of  $\Lambda$  across the mass range  $1.0M_\odot \leq M \leq$

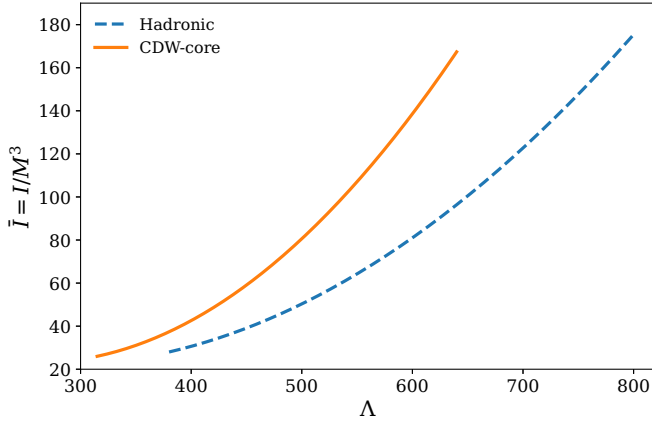


FIG. 6. Universal I-Love relation showing the dimensionless moment of inertia  $\bar{I} = I/M^3$  versus tidal deformability  $\Lambda$ . CDW-core stars follow the same universal curve with a slight offset due to changes in internal structure.

$2.2M_{\odot}$ . This suppression arises due to two primary effects: (i) the increased compactness of the star induced by the stiffer CDW EoS at high density, and (ii) the reduction in the second Love number  $k_2$  resulting from a denser and more centrally concentrated mass distribution. Notably, the reduction in  $\Lambda$  is most pronounced near  $M \approx 1.4M_{\odot}$ , where gravitational wave constraints from binary neutron star mergers (e.g., GW170817) are most stringent. The  $\approx 20\%$  decrease in  $\Lambda$  brought by CDW cores allows models with softer outer layers to remain compatible with current observational bounds on tidal deformability.

In Fig. 6, we present the so-called I-Love relation, showing the dimensionless moment of inertia  $\bar{I} \equiv I/M^3$  as a function of  $\Lambda$ . This universal curve is known to be remarkably insensitive to the microphysical details of the EoS for a wide range of hadronic and hybrid models. Our results confirm that the inclusion of a CDW core introduces only a mild deviation from the hadronic curve, preserving the approximate universality of the I-Love relation to within a few percent. This robustness suggests that even in the presence of exotic inhomogeneous condensates, standard I-Love-Q universal relations may still be employed for gravitational waveform modeling and parameter estimation, with minor corrections. However, the systematic offset introduced by CDW phases could serve as an indirect diagnostic of exotic matter in future high-precision measurements of neutron star moments of inertia and tidal deformabilities.

These plots reinforce the astrophysical viability of CDW phases: they remain consistent with gravitational wave constraints while introducing novel, potentially observable corrections to global neutron star observables. These features offer promising targets for future multimessenger probes of QCD matter at extreme densities.

However, the CDW-induced suppression of  $\Lambda$  has important implications for gravitational wave astrophysics. In particular, the inferred constraints on the EoS from GW170817 and similar events—such as the upper bound  $\Lambda(1.4M_{\odot}) \lesssim 800$ , are more easily satisfied by EoS models

that incorporate a CDW phase, even when the outer layers are relatively soft. This may allow for the reconciliation of observational constraints with microphysically motivated inhomogeneous phases.

Finally, since  $\Lambda \propto R^5$ , even a modest reduction in stellar radius due to the CDW core can lead to a substantial drop in tidal deformability. This sensitivity emphasizes the importance of accurately modeling the star's interior composition in any effort to extract microphysical information from gravitational wave signals.

Altogether, our analysis demonstrates that magnetically catalyzed CDW phases introduce quantifiable corrections to compactness, tidal deformability, and potentially the moment of inertia, while still preserving the approximate validity of universal relations. These results strengthen the case for including spatially modulated condensates in future neutron star models and waveform templates for gravitational wave astronomy.

## V. DISCUSSION AND OUTLOOK

In this work, we have explored the possibility that strong magnetic fields can fundamentally alter the phase structure of dense nuclear matter, rendering the CDW phase energetically favorable in neutron star interiors. This possibility stands in contrast to previous conclusions based on nucleon-meson models without magnetic effects, which had found CDW configurations to be unfavorable due to the associated softening of the equation of state. By incorporating magnetic catalysis and Landau quantization, our extended model shows that spatially modulated chiral condensates can arise at lower baryon chemical potentials and support a stiffer equation of state, particularly in regimes of ultrastrong magnetic fields comparable to those inferred in magnetars.

Our results suggest that CDW phases are not merely theoretical curiosities but may play a genuine role in the microphysics of neutron star cores, especially in the presence of magnetic fields exceeding  $B \approx 10^{18}$  G. The emergence of the CDW phase modifies the thermodynamics of dense matter, influencing both the equilibrium composition and the macroscopic structure of neutron stars. Most notably, we find that stable configurations with CDW cores can exceed the observed two-solar-mass threshold, indicating that such phases are consistent with existing mass-radius constraints. Moreover, CDW cores impact universal relations such as the I-Love-Q curves, slightly reducing the tidal deformability and moment of inertia without strongly violating their approximate universality, thus offering potential observational signatures in gravitational wave astronomy.

Looking ahead, several promising directions remain for future work. The inclusion of finite temperature effects and neutrino trapping is essential for modeling the early stages of proto-neutron stars or post-merger remnants, where thermal and lepton number effects could alter the phase boundaries and transport properties of the CDW phase. Likewise, incorporating the magnetic field back-reaction on the star's equilibrium structure (currently treated as an external input) would allow a more realistic modeling of anisotropic stress and deformation in strong-field regimes. Additionally, a more

detailed analysis of collective excitations, transport coefficients, and neutrino emissivities in the CDW phase could yield valuable insights into cooling behavior, glitch dynamics, and electromagnetic variability in magnetars.

The interplay between inhomogeneous condensates, strong magnetic fields, and compact star observables is a fertile ground for future research. As multi-messenger astronomy continues to provide increasingly precise constraints on neutron star structure, particularly through gravitational wave detections and x-ray timing, the distinctive signatures of CDW phases—ranging from modified tidal deformabilities to suppressed cooling rates—may offer an indirect window into the symmetry-breaking patterns of dense QCD matter. Ultimately, the incorporation of spatially modulated phases into neutron star modeling may prove not only necessary but inevitable in light of forthcoming observational advances. An important extension of the present study would be the inclusion of anisotropic stellar structure equations and magnetic-field-induced deformations, which we leave for future work. Future extensions of this work should include spatially varying magnetic field profiles and anisotropic stellar structure equations to further refine the quantitative impact of strong magnetic fields on inhomogeneous chiral phases in neutron stars.

#### DATA AVAILABILITY

No data were created or analyzed in this study.

#### APPENDIX: LANDAU LEVEL SUMMATION DETAILS

In this Appendix, we provide technical details on how Landau quantization is implemented in the evaluation of the fermionic contribution to the thermodynamic potential under a uniform external magnetic field  $\vec{B} = B\hat{z}$ . Charged fermions (specifically protons, electrons, and muons) experience transverse motion quantized into Landau levels, significantly altering their density of states.

For a charged fermion species  $f$  with electric charge  $q_f$ , mass  $m_f$ , and spin  $s = \pm 1$ , the energy levels in the presence of a magnetic field oriented along the  $z$  axis are given by

$$E_{n,s}(p_z) = \sqrt{p_z^2 + 2n|q_f B| + m_f^2}, \quad (\text{A1})$$

where  $n \in \mathbb{Z}_{\geq 0}$  is the Landau level index. Each level (except for the lowest one,  $n = 0$ ) is doubly degenerate with respect to spin, provided  $m_f > 0$ . The spin degeneracy factor  $g_n$  is therefore

$$g_n = \begin{cases} 1 & n = 0, \\ 2 & n \geq 1. \end{cases} \quad (\text{A2})$$

The number of quantum states per unit area in the transverse plane for each Landau level is

$$\frac{|q_f B|}{2\pi}. \quad (\text{A3})$$

The contribution of a charged fermion species  $f$  to the matter part of the grand potential is given by

$$\Omega_f = -\frac{|q_f B|}{2\pi} \sum_{n=0}^{n_{\max}} g_n \int_{-\infty}^{\infty} \frac{dp_z}{2\pi} X, \quad (\text{A4})$$

where

$$X = E_{n,s}(p_z) + T \ln(1 + e^{-\beta(E_{n,s}(p_z) - \mu_f)}) + T \ln(1 + e^{-\beta(E_{n,s}(p_z) + \mu_f)}), \quad (\text{A5})$$

where  $\mu_f$  is the chemical potential,  $\beta = 1/T$  is the inverse temperature, and  $T$  is the temperature (set to zero in our neutron star application). In the zero-temperature limit, this expression reduces to

$$\Omega_f^{(T=0)} = -\frac{|q_f B|}{2\pi} \sum_{n=0}^{n_{\max}} g_n \int_{-p_{z,F}^{(n)}}^{p_{z,F}^{(n)}} \frac{dp_z}{2\pi} (\mu_f - E_{n,s}(p_z)), \quad (\text{A6})$$

where the Fermi momentum for each Landau level is defined by the condition

$$p_{z,F}^{(n)} = \sqrt{\mu_f^2 - m_f^2 - 2n|q_f B|}, \quad \text{if } \mu_f^2 \geq m_f^2 + 2n|q_f B|. \quad (\text{A7})$$

The upper limit  $n_{\max}$  on the Landau sum is determined dynamically by

$$n_{\max} = \left\lfloor \frac{\mu_f^2 - m_f^2}{2|q_f B|} \right\rfloor. \quad (\text{A8})$$

Neutral fermions such as neutrons do not couple directly to the magnetic field and hence retain their standard dispersion relation:

$$E(p) = \sqrt{p^2 + m_n^2}, \quad (\text{A9})$$

and their contribution to the thermodynamic potential is computed using conventional Fermi integrals.

In practice, to ensure numerical convergence and stability, the Landau level summation is truncated once  $n_{\max}$  exceeds the kinematic bound or the integrand becomes numerically negligible. At strong magnetic fields and low densities, only the lowest few Landau levels are populated, and in the extreme quantum limit ( $n_{\max} = 0$ ), the entire charged matter resides in the LLL, leading to an effective dimensional reduction to  $1 + 1$  dimensions. This enhances magnetic catalysis and can trigger the onset of spatially modulated condensates.

This formalism captures the essential modifications of fermion thermodynamics in the presence of a magnetic field and is implemented consistently throughout the model to compute the matter contribution to the total free energy, as well as the particle densities and magnetization. The same summation structure applies to electrons and muons, with appropriate choices for their masses and charges.

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