

Temperature dependence of the masses of various meson states: A comparative study in the SU(3) and SU(4) extended linear- σ models

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In the extended Linear- σ Model (eLSM), the chiral phase structure of meson states, including pseudoscalars ($J^{pc} = 0^{-+}$), scalars ($J^{pc} = 0^{++}$), vectors ($J^{pc} = 1^{--}$), and axial vectors ($J^{pc} = 1^{++}$), is investigated with the mean-field approximation. A systematic comparison between SU(3) and SU(4) configurations is provided. It has been found that the estimations of meson masses derived from SU(4) eLSM are more congruent with experimental values than those derived from SU(3) eLSM. Consequently, we conclude that an increase in quark degrees of freedom significantly enhances the accuracy of meson mass simulations. We investigate the effect of temperature on the masses of various meson states calculated in the SU(3) and SU(4) eLSM. After establishing all the fitting parameters, the temperature dependence of meson masses shows that although various meson states exhibit unique patterns in their mass changes with temperature, they all seem to share a similar range of dissolution temperatures. This means that the critical temperature that marks the phase transition from hadrons to quarks appears to vary slightly depending on the meson states. In this regard, we find that the quarkonium states, formed by a quark and its antiquark, are largely unaffected by variations in the temperature.

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I. INTRODUCTION

To analyze different properties of QCD, several low-energy effective QCD-like models have been established, thereby providing insights into the phase structure at finite temperatures and densities and in the presence of magnetic and electric fields [1–4]. In the limit of vanishing quark masses, chiral symmetry emerges as a fundamental symmetry in QCD [5–8]. On the other hand, in the limit of finite quark masses, chiral symmetry becomes spontaneously broken. In this limit, the chiral condensate represents the associated order parameter. With small quark masses, the explicit breaking of chiral symmetry for the up and down quarks [9] causes the pseudo-Goldstone bosons to acquire finite masses, as in the case of light pseudoscalar mesons, which include pions [10,11]. For heavier quark flavors, including strange and charm quarks, the explicit breaking intensifies, resulting in the emergence of both hidden and open charmed mesons such as the D

[12] and χ_{c0} mesons [13], respectively. Under extreme conditions of high temperature and/or density, the in-medium chiral condensate seems to vanish, demonstrating the restoration of chiral symmetry and the emergence of meson state degeneracy [14,15].

The QCD phase structure can be explored using low-energy QCD methods, such as Dyson-Schwinger equations [16–18], chiral perturbation theory [19], etc. Different statistical thermal models are also employed in examining QCD phase structure. These include, for instance, the Hadron Resonance Gas (HRG) model [20,21], the Nambu–Jona-Lasinio (NJL) model [22,23], and the Linear- σ Model (LSM) [3,24,25]. In effective QCD-like models, such as LSM [3,24,25], pion mesons can be contracted with two quark flavors, specifically in an SU(2) configuration [26–28]. This framework also facilitates a systematic examination of light quark condensates [29]. The SU(3) configuration enables the construction of nonet meson masses [3,25], the study of the QCD phase transition [30,31], and the investigation of strange condensate [1,3]. The SU(4) configuration evidently incorporates the charm quark, allowing for the analysis of the corresponding condensate. A systematic investigation of charmed meson masses was analytically derived and presented in Ref. [32]. This paper is devoted to exploring the in-medium modifications of various charmed meson masses. A systematic analysis comparing the temperature dependence of various meson masses of SU(3) and SU(4) is also conducted.

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Recent advancements in experimental high-energy particle physics—particularly at the Large Hadron Collider (LHC) at CERN and the Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Laboratory (BNL), and especially the Beam Energy Scan program of the STAR experiment [33]—have brought considerable prominence among researchers to the in-medium modifications of various hadron states [34–36]. The analysis of in-medium modifications of various hadron properties is anticipated in future facilities including the Facility for Antiproton and Ion Research (FAIR), at GSI, Darmstadt, Germany [37–39] and the Nuclotron-based Ion Collider fAcility (NICA) at JINR, Dubna, Russia [40].

In-medium modifications of charmed mesons are particularly interesting because they serve as probes for the hot and dense medium created during heavy-ion collisions. However, modeling the charm sector using effective models like the eLSM comes with its own set of challenges, especially due to the necessity of an extra mass term $-2 \text{Tr}[\epsilon \Phi^\dagger \Phi]$ [41], along with the complex interactions between light and heavy flavor condensates. We would like to emphasize that our calculations incorporate the $U(1)_A$ anomaly. This is the $c(\det \Phi + \det \Phi^\dagger)$ term. Moreover, the influence of the anomaly on the pseudo-critical temperatures is also considered. More comprehensive information about the anomaly terms in the Lagrangian concerning phase transitions can be found in Refs. [42,43].

To integrate the finite-temperature behavior of the condensates into the model, we apply the mean-field approximation and account for finite-temperature effects via the grand potential, which includes contributions from quark-antiquark pairs and the Polyakov loop. We define a temperature-dependent mass parameter m_0^* as $m_0^2(1 - T^2/T_c^2)$. This change is motivated by the fact that, in the linear- σ model, the critical temperature scales as $N_c^{1/2}$ in the limit of $N_c \rightarrow \infty$, which is inconsistent with the N_c -independent behavior found in the NJL model [44]. The inclusion of this temperature-dependent factor appears to restore the expected large- N_c scaling behavior [45,46].

The quadratic dependence on temperature T is motivated by the low-temperature results of chiral perturbation theory, where the chiral condensate decreases in proportion to T^2 due to thermal fluctuations of Goldstone bosons [47]. In this regard, the factor introduced seems to model the chiral dynamics of the system. Nonetheless, this factor alone appears inadequate to describe the chiral phase transition or to identify the critical temperature. On the contrary, the factor rather illustrates the gradual decrease of the chiral condensates as temperature increases. The gluonic sector, which is responsible for confinement dynamics, is effectively incorporated through the Polyakov loop. By integrating both elements, we can simultaneously recover the correct large- N_c behavior and enforce the chiral effects and backreaction of quarks in the model.

The effectiveness of the SU(3) eLSM in examining light meson states and their chiral phase structure has been established [24,25,48]. Nonetheless, there exists a gap in the systematic and comparative analysis of meson masses in SU(4) configurations at finite temperature. The present study aims to present a comprehensive description of various meson states and their temperature dependence, all

within the eLSM framework for both SU(3) and SU(4) flavor symmetries.

This paper is structured as follows. In Sec. II, the extended Linear- σ Model (eLSM) is reviewed. The configurations for SU(3) and SU(4) are discussed in Secs. II A and II B, respectively. Section II C provides a detailed explanation of the finite temperature formalism for the masses of various meson states. The parametrization along with the numerical results are examined in Sec. III. The final conclusions are presented in Sec. IV.

II. EXTENDED LINEAR- σ MODEL

The LSM [49–54] provides a remarkably accurate description of various meson states [24,25,32]. As the degrees of freedom related to the quark flavors N_f increase, so does the number of meson states that can be created. The SU(3) meson states were derived and introduced in Refs. [24,25]. The analytical derivation of SU(4) meson states is introduced in Refs. [32,55].

The Lagrangian for the mesonic sector, which contains scalar, pseudoscalar, vector, and axial-vector mesons, together with their interactions and anomalies, is built as follows [32]:

$$\mathcal{L} = \mathcal{L}_{SP} + \mathcal{L}_{VA} + \mathcal{L}_{\text{Int}} + \mathcal{L}_{U(1)_A}, \quad (1)$$

$$\begin{aligned} \mathcal{L}_{SP} = & \text{Tr}[(D^\mu \Phi)^\dagger (D^\mu \Phi) - m^2 \Phi^\dagger \Phi] - \lambda_1 [\text{Tr}(\Phi^\dagger \Phi)]^2 \\ & - \lambda_2 \text{Tr}(\Phi^\dagger \Phi)^2 + \text{Tr}[H(\Phi + \Phi^\dagger)], \end{aligned} \quad (2)$$

$$\begin{aligned} \mathcal{L}_{AV} = & -\frac{1}{4} \text{Tr}(L_{\mu\nu}^2 + R_{\mu\nu}^2) + \text{Tr}\left[\left(\frac{m_1^2}{2} + \Delta\right)(L_\mu^2 + R_\mu^2)\right] \\ & + i\frac{g_2}{2} (\text{Tr}\{L_{\mu\nu}[L^\mu, L^\nu]\} + \text{Tr}\{R_{\mu\nu}[R^\mu, R^\nu]\}) \\ & + g_3 [\text{Tr}(L_\mu L_\nu L^\mu L^\nu) + \text{Tr}(R_\mu R_\nu R^\mu R^\nu)] \\ & + g_4 [\text{Tr}(L_\mu L^\mu L_\nu L^\nu) + \text{Tr}(R_\mu R^\mu R_\nu R^\nu)] \\ & + g_5 \text{Tr}(L_\mu L^\mu) \text{Tr}(R_\nu R^\nu) + g_6 [\text{Tr}(L_\mu L^\mu) \text{Tr}(L_\nu L^\nu) \\ & + \text{Tr}(R_\mu R^\mu) \text{Tr}(R_\nu R^\nu)], \end{aligned} \quad (3)$$

$$\begin{aligned} \mathcal{L}_{\text{Int}} = & \frac{h_1}{2} \text{Tr}(\Phi^\dagger \Phi) \text{Tr}(L_\mu^2 + R_\mu^2) + h_2 \text{Tr}[|L_\mu \Phi|^2] \\ & + |\Phi R_\mu|^2 + 2h_3 \text{Tr}(L_\mu \Phi R^\mu \Phi^\dagger), \end{aligned} \quad (4)$$

$$\mathcal{L}_{U(1)_A} = c[\det(\Phi) + \det(\Phi^\dagger)]. \quad (5)$$

The complex matrices for scalar σ_a (i.e., $J^{PC} = 0^{++}$), pseudoscalar π_a (i.e., $J^{PC} = 0^{-+}$), vector V_a^μ (i.e., $J^{PC} = 1^{--}$), and axial-vector A_a^μ (i.e., $J^{PC} = 1^{+-}$) meson states can be constructed as

$$\begin{aligned} \Phi = & \sum_{a=0}^{N_f^2-1} T_a(\sigma_a + i\pi_a), \quad L^\mu = \sum_{a=0}^{N_f^2-1} T_a(V_a^\mu + A_a^\mu), \\ R^\mu = & \sum_{a=0}^{N_f^2-1} T_a(V_a^\mu - A_a^\mu). \end{aligned} \quad (6)$$

The various generators are defined according to the number of quark flavors N_f , while T_a are the corresponding generators

of $U(N_f)$ expressed as $T_a = \hat{\lambda}_a/2$ with $a = 0, \dots, (N_f^2 - 1)$, and $\hat{\lambda}$ are the Gell-Mann matrices.

The covariant derivative

$$D^\mu \Phi \equiv \partial^\mu \Phi - i g_1 (L^\mu \Phi - \Phi R^\mu) - ie A^\mu [T_3, \Phi], \quad (7)$$

is associated with the degrees of freedom for (pseudo)scalar and (axial-)vector mesons and couples them through the coupling constant g_1 :

$$L^{\mu\nu} \equiv \partial^\mu L^\nu - ie A^\mu [T_3, L^\nu] - \{\partial^\nu L^\mu - ie A^\nu [T_3, L^\mu]\}, \quad (8)$$

$$R^{\mu\nu} \equiv \partial^\mu R^\nu - ie A^\mu [T_3, R^\nu] - \{\partial^\nu R^\mu - ie A^\nu [T_3, R^\mu]\}, \quad (9)$$

where $A^\mu = g A_\mu^\alpha \lambda^\alpha/2$ is the electromagnetic field. The constant g is the Yukawa coupling, which is fixed from the non-strange constituent quark mass as $g = 2m_q/\bar{\sigma}_x$. The field Φ is expressed as

$$\Phi = \sum_{a=0}^{N_f^2-1} T_a (\sigma_a + i\pi_a). \quad (10)$$

Expressions for $T_a \sigma_a$ and $T_a \pi_a$ can be found in Refs. [32,55]. According to Refs. [32,55], the global chiral invariance of the Lagrangian for $N_f = 4$ is identical to that of $N_f = 3$. However, for $N_f = 4$, the mass term $\mathcal{L}_{\text{mass}} = -2 \text{Tr}[\epsilon \Phi^\dagger \Phi]$ must be included [41]. The origin of this term can be understood from the equivalence between the symmetry-breaking Hamiltonians of the $SU(4) \times SU(4)$ and $SU(3) \times SU(3)$ groups [56,57].

Starting with the $SU(3)$ configuration of eLSM, the following section goes over the essential characteristics.

A. $SU(3)$ configuration

The tree-level mesonic potential for the scalar-pseudoscalar states is presented in Appendix A, Eq. (A1). For $SU(3)$, this is reduced to

$$U(\sigma) = \frac{m^2}{2} \sigma_a^2 - \mathcal{G}_{abc} \sigma_c \sigma_a \sigma_b + \frac{1}{3} \mathcal{F}_{abcd} \sigma_a \sigma_b \sigma_c \sigma_d - h_a \sigma_a, \quad (11)$$

where the coefficients $\mathcal{G}_{ab}$, $\mathcal{G}_{abc}$, $\mathcal{G}_{abcd}$, $\mathcal{F}_{abcd}$, $\mathcal{H}_{abcd}$ are detailed in Appendix A.

It is more convenient to undertake the subsequent analysis in terms of the pure nonstrange and strange fields, which are obtained respectively from the following transformation:

$$\begin{pmatrix} \sigma_x \\ \sigma_y \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} & 1 \\ 1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} \sigma_0 \\ \sigma_8 \end{pmatrix}. \quad (12)$$

The explicit chiral symmetry breaking parameters in the pseudoscalar sector, i.e., the expression $\text{Tr}[\mathbf{H}(\Phi + \Phi^\dagger)]$ in Eq. (1), allowing the transformation of h_0 and h_8 using the same transformation basis:

$$H_{SU(3)} = T_0 h_0 + T_8 h_8 \equiv \frac{1}{2} \begin{pmatrix} h_x & 0 & 0 \\ 0 & h_x & 0 \\ 0 & 0 & \sqrt{2} h_y \end{pmatrix}. \quad (13)$$

Then the tree-level mesonic potential for the $SU(3)$ configuration reads

$$\begin{aligned} U(\sigma_x, \sigma_y) = & \frac{m^2}{2} (\sigma_x^2 + \sigma_y^2) - \frac{c}{2\sqrt{2}} \sigma_x^2 \sigma_y + \frac{\lambda_1}{2} \sigma_x^2 \sigma_y^2 \\ & + \frac{1}{8} (2\lambda_1 + \lambda_2) \sigma_x^4 + \frac{1}{4} (\lambda_1 + \lambda_2) \sigma_y^4 \\ & - h_x \sigma_x - h_y \sigma_y. \end{aligned} \quad (14)$$

The global minimization of the grand potential determines the h_x and h_y as

$$h_x = m^2 \sigma_x - \frac{c}{\sqrt{2}} \sigma_x \sigma_y - \lambda_1 \sigma_x \sigma_y^2 + \frac{1}{2} (2\lambda_1 + \lambda_2) \sigma_x^3, \quad (15)$$

$$h_y = m^2 \sigma_y - \frac{c}{2\sqrt{2}} \sigma_x^2 - \lambda_1 \sigma_x^2 \sigma_y + (\lambda_1 + \lambda_2) \sigma_y^3. \quad (16)$$

The tree-level masses of mesons are defined from the quadratic terms of the Lagrangian and are presented in detail in Appendix B.

The following section introduces the $SU(4)$ configuration.

B. $SU(4)$ configuration

The tree-level mesonic potential for the scalar and pseudoscalar states in the $SU(4)$ configuration can be derived from Eq. (A1):

$$U(\bar{\sigma}) = \frac{m^2}{2} \bar{\sigma}_a^2 + \frac{1}{3} [\mathcal{F}_{abcd} + \mathcal{G}_{abcd}] \bar{\sigma}_a \bar{\sigma}_b \bar{\sigma}_c \bar{\sigma}_d - h_a \bar{\sigma}_a + \epsilon_a \sigma_a^2. \quad (17)$$

The corresponding basis transformation to pure nonstrange, strange, and charm fields, respectively, reads

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_c \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{2} & -\sqrt{\frac{2}{3}} & \frac{1}{2\sqrt{3}} \\ \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \sigma_0 \\ \sigma_8 \\ \sigma_{15} \end{pmatrix}. \quad (18)$$

The explicit chiral symmetry breaking parameters allow for the transformation of h_0 , h_8 , and h_{15} using the same basis:

$$\begin{aligned} H_{SU(4)} = & T_0 h_0 + T_8 h_8 + T_{15} h_{15} \\ = & \frac{1}{2} \begin{pmatrix} h_x & 0 & 0 & 0 \\ 0 & h_x & 0 & 0 \\ 0 & 0 & \sqrt{2} h_s & 0 \\ 0 & 0 & 0 & \sqrt{2} h_c \end{pmatrix}. \end{aligned} \quad (19)$$

Finally, the tree-level $SU(4)$ mesonic potential in new basis reads

$$\begin{aligned} U(\sigma_x, \sigma_y, \sigma_c) = & \frac{1}{2} m^2 (\sigma_x^2 + \sigma_y^2 + \sigma_c^2) - \frac{c}{4} \sigma_x^2 \sigma_y \sigma_c \\ & + \frac{\lambda_1}{2} (\sigma_x^2 \sigma_y^2 + \sigma_x^2 \sigma_c^2 + \sigma_y^2 \sigma_c^2) \\ & + \frac{1}{8} (2\lambda_1 + \lambda_2) \sigma_x^4 \\ & + \frac{1}{4} (\lambda_1 + \lambda_2) \sigma_y^4 + \frac{1}{4} (\lambda_1 + \lambda_2) \sigma_c^4 \\ & - h_x \sigma_x - h_y \sigma_y - h_c \sigma_c + \epsilon_c \sigma_c^2. \end{aligned} \quad (20)$$

The global minima, defined by the absence of partial derivatives related to σ_x , σ_y , and σ_c , define the values of h_x , h_y , h_c ,

$$h_x = m^2\sigma_x - \frac{c}{2}\sigma_x\sigma_y\sigma_c + \lambda_1\sigma_x(\sigma_y^2 + \sigma_c^2) + \frac{1}{2}(2\lambda_1 + \lambda_2)\sigma_x^3, \quad (21)$$

$$h_y = m^2\sigma_y - \frac{c}{4}\sigma_x^2\sigma_c + \lambda_1\sigma_y(\sigma_x^2 + \sigma_c^2) + (\lambda_1 + \lambda_2)\sigma_y^3, \quad (22)$$

$$h_c = m^2\sigma_c + 2\epsilon_c\sigma_c - \frac{c}{4}\sigma_x^2\sigma_y + \lambda_1\sigma_c(\sigma_x^2 + \sigma_y^2) + (\lambda_1 + \lambda_2)\sigma_c^3. \quad (23)$$

The tree-level masses of mesons, defined from the quadratic terms of the Lagrangian are presented in detail in Appendix B. As previously stated, the derivation for the masses of the charmed mesonic states requires one to take into account the mass term $-2 \text{Tr}[\epsilon \Phi^\dagger \Phi]$ [41], where

$$\epsilon = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_c \end{pmatrix}, \quad (24)$$

with $\epsilon_c = m_c^2$ and m_c is the mass of the charm quark. This term provides an additional contribution specifically to the masses of charmed mesons.

The formalism for the temperature dependence shall be elaborated in the following section. The derivation of analytical formulas for meson masses, with a focus on their behavior at limited temperatures, will be presented.

C. Finite temperature formalism

To investigate how various in-medium modifications behave, we utilize the well-established Polyakov loop extension of the LSM. In this regard, we apply the chiral Lagrangian together with the Polyakov-loop potential, formulated as $\mathcal{L} = \mathcal{L}_{\text{chiral}} - \mathcal{U}(\phi, \phi^*, T)$. The grand potential can be derived then in the mean field approximation [58]. Given the assumption of thermal equilibrium, the grand partition function is expressed via a path integral that includes the quark, antiquark, and meson fields

$$\mathcal{Z} = \text{Tr} \exp[-\hat{\mathcal{H}}/T] = \int \prod_a \mathcal{D}\sigma_a \mathcal{D}\pi_a \int \mathcal{D}q \mathcal{D}\bar{q} \exp \left[\int_x \mathcal{L} \right], \quad (25)$$

where t is the time at which the system with volume V evolves and $\int_x \equiv i \int_0^{1/T} dt \int_V d^3x$. The partition function can be derived using the mean field approximation [59–61]. In this context, the meson fields are substituted with their expectation values, specifically $\bar{\sigma}_a$, within the action [62,63]. Employing standard techniques [63], the integration over the fermionic contributions can be performed. This leads to the derivation of the effective potential for the mesons,

$$\Omega(T) = \frac{-T \ln \mathcal{Z}}{V} = U(\bar{\sigma}) + \mathcal{U}(\phi, \phi^*, T) + \Omega_{\bar{q}q}, \quad (26)$$

where the fields ϕ or ϕ^* are a complex matrix of dimensions $N_f \times N_f$. This effective potential has three terms which can be elaborated as follows.

- (1) The first term is the purely mesonic potential presented in previous sections that describes the mass spectra in vacuum.
- (2) The second term is the Polyakov loop potential, which in this study is used in classic logarithmic form [25]. Additional information on the other potentials $\mathcal{U}(\phi, \phi^*, T)$ can be found in Refs. [58,64,65].
- (3) The third term is the mean-field quark-antiquark potential, which gives an in-medium modification on masses:

$$\Omega_{\bar{q}q} = -2v_c T \sum_f \int \frac{dp}{(2\pi)^3} E_f - 2v_c T \sum_f \int \frac{dp}{(2\pi)^3} (\ln g_f^+ + \ln g_f^-), \quad (27)$$

where $v_c = 2N_c$ and f runs over all flavors, $E_f = (p^2 + m_f^2)^{1/2}$ is the flavor dependent single particle energy of quark with mass m_f , and

$$g_f^+ = 1 + 3(\phi + \phi^* e^{-E_f^+/T}) e^{-E_f^+/T} + e^{-3E_f^+/T}, \quad (28)$$

$$g_f^- = 1 + 3(\phi^* + \phi e^{-E_f^-/T}) e^{-E_f^-/T} + e^{-3E_f^-/T}. \quad (29)$$

The quark and antiquark dispersion relations are $E_f^\pm(T, \mu_f) = E_f \mp \mu_f$, respectively. The quark masses are defined by values of the σ fields:

$$m_x = \frac{g}{2}\sigma_x, \quad m_s = \frac{g}{\sqrt{2}}\sigma_y, \quad m_c = \frac{g}{\sqrt{2}}\sigma_c.$$

The masses of the various states can be determined from the second derivative of the grand potential with respect to the corresponding fields, evaluated at the potential minimum [24,25,32]. In the present work, these minima are defined by vanishing expectation values for all scalar, pseudoscalar, vector, and axial-vector fields [59],

$$(m_i^2)_{ab} = \left. \frac{\partial^2 \Omega}{\partial \varphi_{i,a} \partial \varphi_{i,b}} \right|_{\min}, \quad (30)$$

where $\varphi_{i,a}$ and $\varphi_{i,b}$ are the corresponding mass fields of the i th hadron state. Assuming that the contribution of the quark-antiquark potential to the Lagrangian vanishes in the vacuum, the mass matrix in hot and dense matter should be extended by the additional term [59]

$$(m_i^2)_{ab} = \left. \frac{\partial^2 U(\bar{\sigma})(T, \mu)}{\partial \varphi_{i,a} \partial \varphi_{i,b}} \right|_{\min} + \left. \frac{\partial^2 \Omega_{\bar{q}q}(T, \mu)}{\partial \varphi_{i,a} \partial \varphi_{i,b}} \right|_{\min}, \quad (31)$$

where i stands for (pseudo)scalar and (axial)vector mesons and a and b are integers ranging from 0 to $N_f^2 - 1$. The first term in Eq. (31) is related to the vacuum, where the meson masses are developed from the σ fields. The second term gives the in-medium modification on the masses of various meson

states:

$$\frac{\partial^2 \Omega_{q\bar{q}}(T, \mu)}{\partial \varphi_{i,a} \partial \varphi_{i,b}} = v_c \sum_f \int_0^\infty \frac{dp}{(2\pi)^3} \frac{1}{2E_f} \times \left[(n_f^+ + n_f^-) \left(m_{f,ab}^2 - \frac{m_{f,a}^2 m_{f,b}^2}{2E_f^2} \right) + (b_f^+ + b_f^-) \left(\frac{m_{f,a}^2 m_{f,b}^2}{2E_f T} \right) \right], \quad (32)$$

the expression (32) contains some notations for the quark mass first derivative with respect to the meson fields $m_{f,a}^2 \equiv \partial m_f^2 / \partial \varphi_{i,a}$ and the second derivative of the quark mass with respect to meson fields $m_{f,ab}^2 \equiv \partial^2 m_f^2 / \partial \varphi_{i,a} \partial \varphi_{i,b}$. The values of these derivatives for SU(3) configuration can be found, for example, in Refs. [59,66], and for the SU(4) case they are presented in Appendix C, Table IX. The notations $n_f^\pm$ and $b_f^\pm$ have the following definitions [24,25,32]:

$$n_f^+ = \frac{\phi e^{-E_f^+/T} + 2\phi^* e^{-2E_f^+/T} + e^{-3E_f^+/T}}{1 + 3(\phi + \phi^* e^{-E_f^+/T}) e^{-E_f^+/T} + e^{-3E_f^+/T}}, \quad (33)$$

$$n_f^- = \frac{\phi^* e^{-E_f^-/T} + 2\phi e^{-2E_f^-/T} + e^{-3E_f^-/T}}{1 + 3(\phi^* + \phi e^{-E_f^-/T}) e^{-E_f^-/T} + e^{-3E_f^-/T}}. \quad (34)$$

The normalization factors for quarks and antiquarks are $b_f^\pm = 3(n_f^\pm)^2 - c_f^\pm$, where [24,25,32]

$$c_f^+ = \frac{\phi e^{-E_f^+/T} + 4\phi^* e^{-2E_f^+/T} + 3e^{-3E_f^+/T}}{1 + 3(\phi + \phi^* e^{-E_f^+/T}) e^{-E_f^+/T} + e^{-3E_f^+/T}}, \quad (35)$$

$$c_f^- = \frac{\phi^* e^{-E_f^-/T} + 4\phi e^{-2E_f^-/T} + 3e^{-3E_f^-/T}}{1 + 3(\phi^* + \phi e^{-E_f^-/T}) e^{-E_f^-/T} + e^{-3E_f^-/T}}. \quad (36)$$

The numerical results and the procedure for parameter fitting are introduced in the following section.

III. RESULTS

The Lagrangian of eLSM, given in Eqs. (1)–(5), contains a large number of parameters:

$$m_0^2, m_1^2, c, \delta_x, \delta_y, \delta_c, g_1, g_2, g_3, g_4, g_5, g_6, h_x, h_y, h_c, h_1, h_2, h_3, \lambda_1, \lambda_2.$$

In this study, the coupling of the glueball to other mesons is neglected. Furthermore, the parameters g_2, g_3, g_4, g_5 , and g_6 in Eq. (3) are not discussed in this work and therefore are not considered in the fit. Fitting these parameters requires fixing some meson masses and decay constants. The remaining masses and decay constants are then obtained from the fit itself.

The condensate values in vacuum are computed from the meson decay constants using the partially conserved axial-vector current (PCAC) relation [54]

$$f_a = d_{aab} \sigma_b, \quad (37)$$

with summation over b , and d_{abc} is the standard symmetric structure constants of SU(N). σ_x and σ_y are defined by the pion

TABLE I. The data used for parameters fitting. In brackets are shown results for fit with $c \neq 0$ (if different).

Observable	Experiment (MeV)	SU(3) fit (MeV)	SU(4) fit (MeV)
f_π	92.2 ± 4.6	95.02	92.0
f_K	110.4 ± 5.5	106.9	109.1
f_D	203 ± 5.5		225.0
m_π	137.3 ± 6.9	139.85	140.79
m_K	495.6 ± 24.8	420.49	490.3
m_η	547.9 ± 27.4	140.0 (531.9.0)	140.79 (531.13)
m'_η	957.8 ± 47.9	644.93 (965.44)	640.75 (965.56)
m_ϕ	1019.5 ± 51	1019.19	1014.9
m_ρ	775.5 ± 38.8	770.11	749.62
m_{f_1}	1426.4 ± 71.3	1424.77	1460.39
m_{a_1}	1230 ± 62	1069.76	1078.68
m_{K_1}	1253 ± 7	1253.46	1276.45
m_σ	500–1200	600 (700)	745(700)
m_{D_1}	2420 ± 9		2615.77

and kaon decay constants. The value of σ_c can be expressed in terms of the D -meson decay constant or, alternatively, the η_c meson decay constant as $f_{\eta_c} = 2\sigma_c$ [32,67,68]:

$$f_\pi = \frac{\sigma_x}{Z_\pi}, \quad f_K = \frac{\sqrt{2}\sigma_y + \sigma_x}{2Z_K}, \quad f_D = \frac{\sqrt{2}\sigma_c + \sigma_x}{2Z_D}.$$

As justified in Appendix B, the factors Z_i are included in the expressions above. Table I presents the experimental data used in the fit alongside the corresponding best-fit results. We conclude that the estimation grounded in SU(4) eLSM is more consistent with experimental values than that grounded in SU(3) eLSM. This finding draws a key conclusion that an increase in quark degrees of freedom enhances the simulation accuracy of meson masses.

The rest of the parameters are defined as follows:

- (1) $\delta_x, \delta_y, \delta_c$ define explicit symmetry breaking in the vector and axial-vector channels, which arises from nonvanishing quark masses. In this regard, we find the following:

$$\Delta_{\text{SU}(3)} = T_0 \delta_0 + T_8 \delta_8 \equiv \frac{1}{2} \begin{pmatrix} \delta_x & 0 & 0 \\ 0 & \delta_x & 0 \\ 0 & 0 & \sqrt{2}\delta_s \end{pmatrix}, \quad (38)$$

$$\Delta_{\text{SU}(4)} = T_0 \delta_0 + T_8 \delta_8 + T_{15} \delta_{15} \equiv \frac{1}{2} \begin{pmatrix} \delta_x & 0 & 0 & 0 \\ 0 & \delta_x & 0 & 0 \\ 0 & 0 & \sqrt{2}\delta_s & 0 \\ 0 & 0 & 0 & \sqrt{2}\delta_c \end{pmatrix}. \quad (39)$$

The transition from the original octet-singlet basis to the non-strange, strange, and charm quark flavor basis is performed using the transformation matrices in Eqs. (12) and (18). These symmetry-breaking parameters are proportional to the squares of the corresponding quark masses: $\delta_x \propto m_x^2$, $\delta_y \propto m_y^2$, and $\delta_c \propto m_c^2$. In the isospin-symmetric limit, we set $\delta_x = 0$ and

TABLE II. The best fit of the parameters $g_1, c_1, h_2, h_3, c, m_0^2, \lambda_1, \lambda_2$, and g .

	g_1	c_1 (GeV ²)	h_2	h_3	c	m_0^2 (GeV ²)	λ_1	λ_2	g
SU(3)	5.58	0.44	40.975	-15.216	0	-(0.282) ²	-10.515	55.576	5.5
					2.828	-(0.258) ²	3.15	36.253	
SU(4)	5.632	0.395	39.215	-13.907	0	-(0.135) ²	-2.115	45.54	5.217
					11.24	-(0.707) ²	4.055	29.19	

determine the remaining parameters δ_y and δ_c from the fit. The quark masses are defined by the Yukawa coupling g and the corresponding condensates. The Yukawa coupling g is fixed from the nonstrange constituent quark mass as $g = 2m_q/\sigma_x$, where $m_q = 0.3$ GeV (see Table II).

- (2) The parameters h_x, h_y, h_c define the Explicit Chiral Symmetry Breaking (ESB) in the (pseudo)scalar sector via the term $\text{Tr}[H(\Phi + \Phi^\dagger)]$, and they are defined by the global minimum of the mesonic potential of the model (see Sec. II).
- (3) The parameters $m_1^2, g_1, h_1, h_2, h_3$ define the vector and axial-vector sectors. As can be seen from the mass expressions in Tables VI and VII, the meson masses depend more significantly on the value of c_1 , which can be written as

$$c_1 = m_1^2 + \frac{h_1}{2}(\sigma_x^2 + \sigma_y^2), \quad \text{SU(3)}, \quad (40)$$

$$c_1 = m_1^2 + \frac{h_1}{2}(\sigma_x^2 + \sigma_y^2 + \sigma_c^2), \quad \text{SU(4)}, \quad (41)$$

so that we can reduce the fit to g_1, h_2, h_3, c_1 . The parameters obtained from the best fit are listed in Table II.

- (4) The parameters $m_0^2, c = 0, \lambda_1, \lambda_2$ define the scalar-pseudoscalar sector. The parameter c define the presence of the $U(1)_A$ anomaly. For the case when the $U(1)_A$ anomaly is absent, $c = 0$. Accordingly, the mass of η' meson is identical to the pion mass. The parameter λ_2 is given by the kaon and pion masses. The well-known classical relation for λ_2 reads

$$\lambda_2 = \frac{M_K^2 - M_\pi^2}{(2f_K - f_\pi)(f_K - f_\pi)}. \quad (42)$$

It is identical for both the SU(3) and SU(4) configurations. The parameters λ_1 and m_0 then can be obtained using equations for masses of the scalar σ meson m_σ and the pion mass m_π .

- (5) In the presence of the $U(1)_A$ anomaly, i.e., $c \neq 0$, determining the model parameters requires a system of equations constrained by the following experimental inputs:
 - (i) the pion and kaon masses,
 - (ii) the averaged squared mass of η and η' mesons ($m_\eta^2 + m_{\eta'}^2 = m_{\eta_N}^2 + m_{\eta_S}^2$), and
 - (iii) the mass of the scalar σ meson.

Using this solution, we can then predict the parameters, the masses of the scalar mesons m_{a_0}, m_{f_0} , as well as the scalar and pseudoscalar mixing angles θ_S, θ_P .

The parameters $g_1, c_1, h_2, h_3, c, m_0^2, \lambda_1, \lambda_2$, and g for both SU(3) and SU(4) configurations are specified in Table II. It is important to point out that the vacuum masses for the other mesons, which are not included in the fit, can be found in Table V in Appendix B.

The Polyakov potential, as introduced in Sec. II C, depends on the parameter T_0 , which represents the deconfinement critical temperature in the pure gauge sector with $T_0 = 0.27$ GeV. Following the argument from the Introduction and our previous research [32], we incorporate an additional temperature dependence, making the substitution of the parameter m_0^2 by $\tilde{m}_0^2$ with

$$\tilde{m}_0^2 = m_0^2 \left(1 - \frac{T^2}{T_c^2}\right), \quad (43)$$

where T_c is proportional to one of the fundamental quantities of QCD, the scale for the hadron-quark confinement, i.e., Λ_{QCD} ; we keep T_c for the first step as 0.27 GeV. Figure 1 shows the temperature dependence of quark condensates in the eLSM configuration with and without the factor $F(T) = (1 - T^2/T_c^2)$ in Eq. (43), and $T_0 = T_c = 0.27$ GeV. The figure clearly shows the restoration of chiral symmetry in the light quark sector (σ_x) and strange (σ_y) sector at higher temperature, but and charm quark condensates (σ_c) stays modestly temperature dependent over a large temperature range.

The temperature of the deconfinement transition T_d can be defined as the inflection point $\max |\partial\Phi/\partial T|_{\mu=\text{const}}$ and the pseudocritical temperature of the chiral phase transition T_χ can be defined as $\max |\partial\sigma_f/\partial T|_{\mu=\text{const}}$. Table III presents the obtained values of T_χ in the light sector for all considered modifications of the model in the case $T_0 = T_c = 0.27$ GeV. The temperature behavior of the condensates is mainly governed by the background gauge field Φ , and the value of the parameter T_0 plays crucial role in determining the pseudocritical temperature of the chiral phase transition, T_χ . It is clearly seen that the resulting pseudocritical temperature is higher than that suggested by the latest lattice QCD results [69]. One way to improve this situation is to rescale the value of T_0 , thereby simulating the backreaction of quarks [70]. Nevertheless, it is seen clearly that the factor $F(T)$, Eq. (43), effectively plays the role of such a backreaction, leading to a reduction of T_χ for both SU(3) and SU(4) configurations.

The illustrations of the thermal evolution of the meson mass spectrum are presented in Fig. 2 for the scalar mesons, in Fig. 3 for pseudoscalar mesons, in Fig. 4 vector and 5 axial-vector mesons, and in Fig. 5 for open/hidden charmed mesons. All are given at vanishing chemical potential μ_f without the factor $F(T)$. For the pseudoscalar and scalar charmed mesons we presented only D, D_s, D_0^*, D_{s0}^* states,

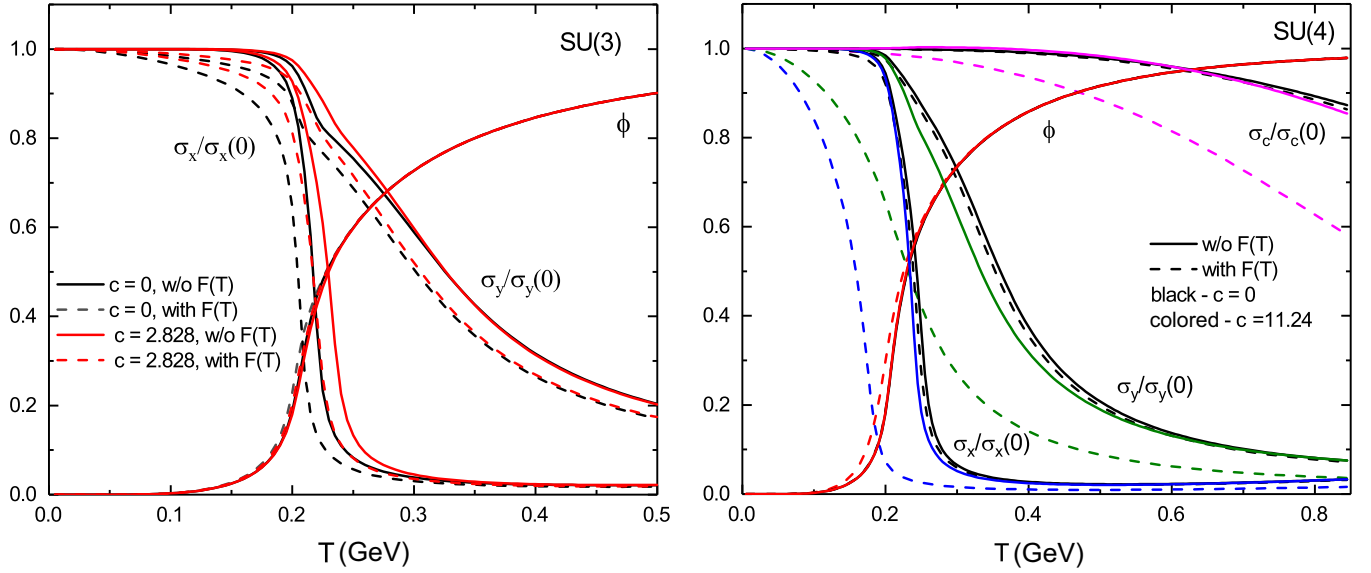


FIG. 1. The quark condensates for SU(3) (shown in the left panel) and SU(4) (presented in the right panel) are displayed as a function of temperature in GeV units. The solid curves indicate calculations performed at $c = 0$ with a vanishing $F(T)$, while the dashed curves represent calculations at $c = 0$ with a finite $F(T)$. The dash-dotted curves illustrate the calculations at finite c and vanishing $F(T)$. Calculations conducted at finite c and finite $F(T)$ are depicted with dotted curves. All calculations are executed at $T_0 = T_c = 0.27$ GeV.

because masses of the η_c meson with $m_{\eta_c} = 4.844$ GeV and the χ_{c0} meson with $m_{\chi_{c0}} = 3.551$ GeV behave as a constant as functions of temperature in the considered temperature range. These figures allow us to conclude that, while various mesonic states display specific patterns in their mass evolution with temperature, their dissociation tends to happen within a similar temperature range close to the critical temperature, with minor variations depending on the type of meson. On the other hand, the quarkonium states seem to be largely unaffected by temperature changes.

IV. CONCLUSION

Using the Polyakov-loop extended Linear- σ Model, we studied the temperature behavior of mesonic states and of the characteristic fields $\sigma_x, \sigma_y, \sigma_c$, which are the order parameters of the chiral phase transition. Our results are consistent with the standard picture and demonstrate that temperature significantly influences the light quark condensates σ_x , while the strange quark sector σ_y undergoes the chiral symmetry restoration at higher temperature. In contrast, the charm quark condensate σ_c is almost independent of temperature. This observation suggests that the pseudocritical temperature of the chiral phase transition increases when moving from light to charm quark degrees of freedom.

As can be seen from Table III, the pseudocritical temperature of the chiral phase transition in the light sector is higher than that predicted by lattice QCD, $T_\chi \in [0.150, 0.170]$ MeV [69]. This feature is common in quark-meson models that incorporate the Polyakov loop (see, e.g., Ref. [71] for the PNJL model, Ref. [70] for the quark-meson model, and Ref. [48] for the eLSM). One possible way to improve this situation is to rescale the parameter T_0 , thereby simulating the backreaction of quarks [70]. In the present work we also propose to account for this backreaction by introducing a temperature dependence of the parameter m_0^2 , which affects the condensates. As shown in Table III, this modification reduces the value of the pseudocritical temperature, with the effect of the factor $F(T)$ being more pronounced in the SU(4) configuration. Nevertheless, it is possible to choose the parameters T_0 and T_c in such a way as to reproduce the lattice QCD results. The corresponding parameter sets are presented in Table IV below.

We found that the reduced values for T_0 are in good agreement with the N_f dependence of T_0 suggested in the work [70]. It is also clearly seen that the effect of the $U(1)_A$ anomaly differs between the SU(3) and SU(4) configurations.

The various meson states including pseudoscalars ($J^{pc} = 0^{-+}$), scalars ($J^{pc} = 0^{++}$), vectors ($J^{pc} = 1^{--}$), and axial vectors ($J^{pc} = 1^{++}$) are investigated in SU(3) and SU(4) eLSM

TABLE III. The pseudo-critical temperature of the chiral phase transition (T_χ) for SU(3) and SU(4) configurations with and without the factor $F(T)$ with $T_0 = T_c = 0.27$ GeV.

	$c = 0$	$c = 0$ with $F(T)$	$c \neq 0$	$c \neq 0$ with $F(T)$
SU(3)	0.2175	2.075	0.2325	0.2175
SU(4)	0.2475	2.425	0.2375	0.1725

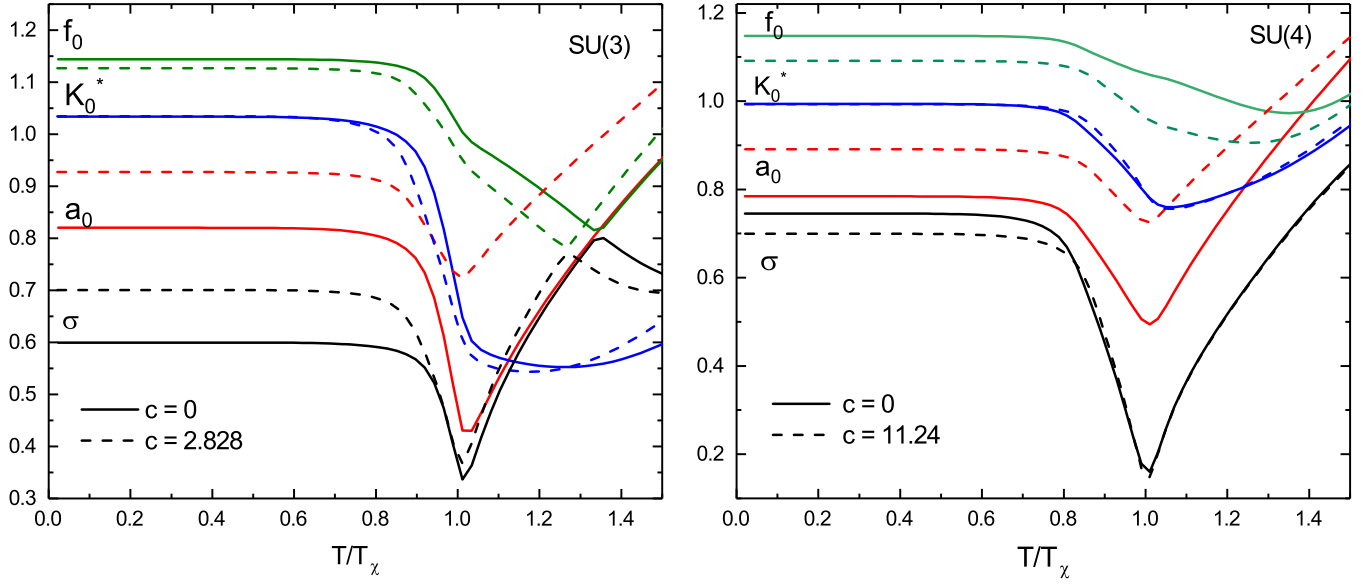


FIG. 2. Scalar mesons for SU(3) (left panel) and SU(4) (right panel). Meson masses are plotted both for $c = 0$ (solid lines) and $c \neq 0$ (dashed lines) cases without involving the factor $F(T) = (1 - T^2/T_0^2)$.

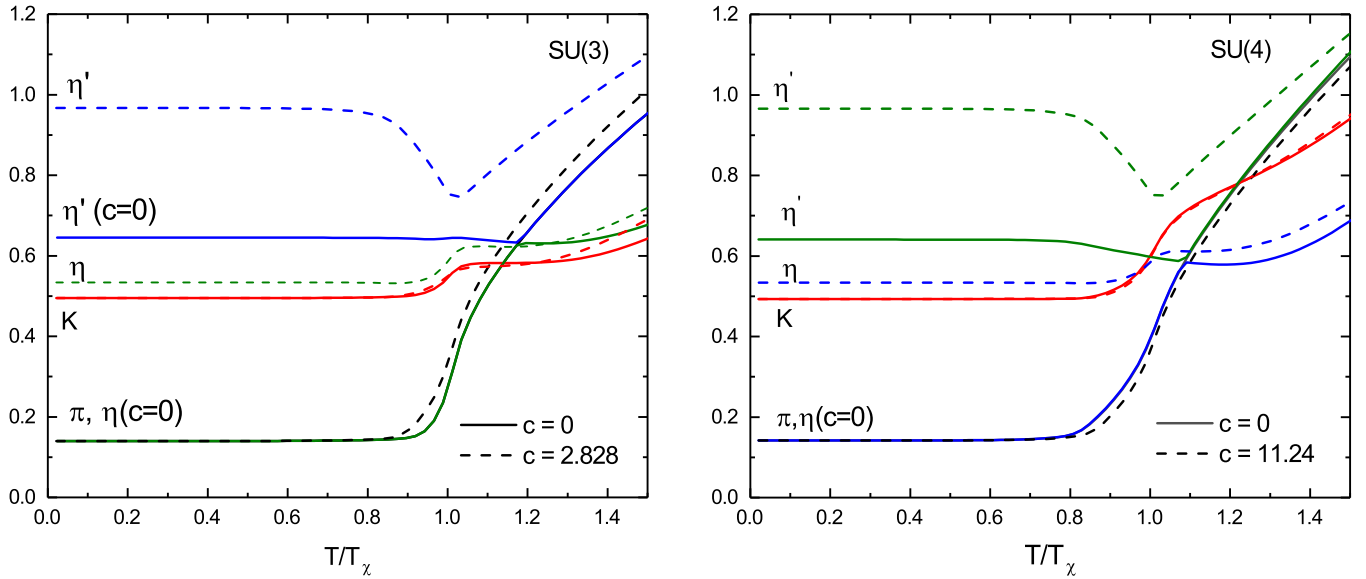


FIG. 3. Masses of pseudoscalar mesons for SU(3) (left panel) and SU(4) (right panel) cases. Meson masses are plotted for $c = 0$ (solid lines) and $c \neq 0$ (dashed line) cases.

TABLE IV. The parameters T_0 , T_c and the corresponding pseudocritical temperature of the chiral phase transition (T_χ) in GeV for SU(3) and SU(4) configurations with and without the factor $F(T)$.

	(GeV)	$c = 0$	$c \neq 0$	$c = 0$ with $F(T)$	$c \neq 0$ with $F(T)$
SU(3)			$T_0 = 0.178, T_c = 0.156$		
	T_χ	0.198	0.208	0.153	0.165
	T_d	0.143	0.148	0.143	0.143
SU(3)			$T_0 = 0.148, T_c = 0.148$	$T_0 = 0.22, T_c = 0.22$	
	T_χ	0.198	0.208	0.218	0.143
	T_d	0.2	0.213	0.223	0.163

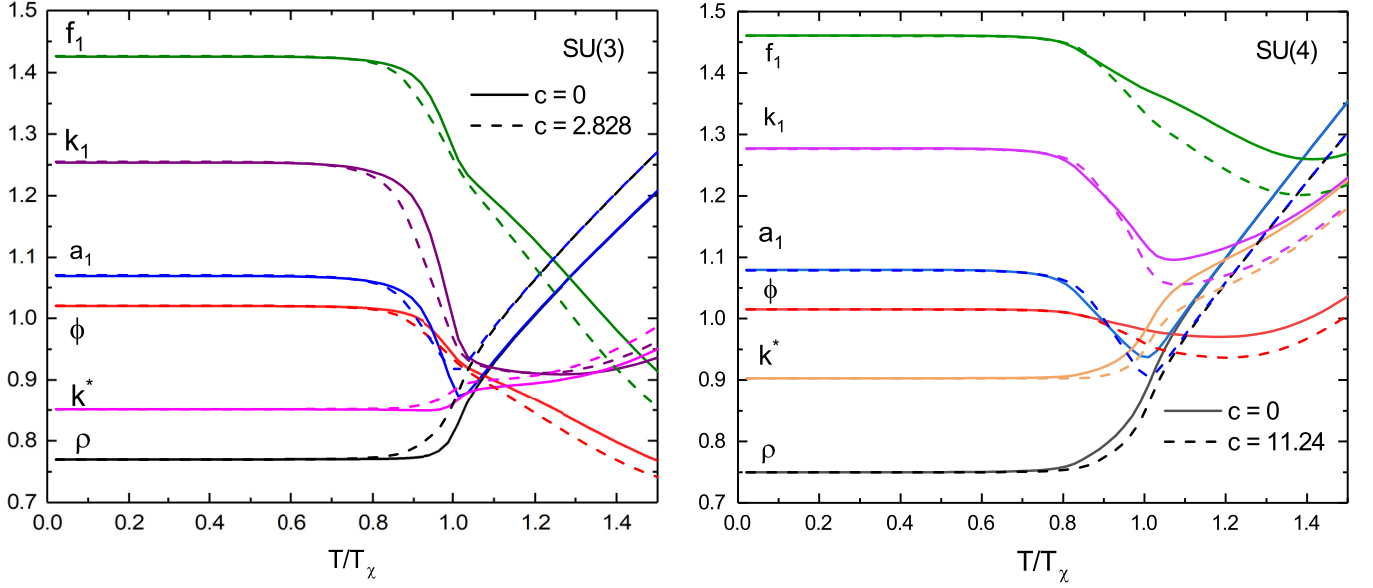


FIG. 4. Vector and axial-vector mesons for SU(3) (left panel) and SU(4) (right panel). In the left panel the different meson masses are illustrated for the cases of both $c = 0$ (solid lines) and $c \neq 0$ (dashed lines), whereas in the right panel the results are obtained at a finite value of c .

both in vacuum and at finite temperatures. The vacuum masses of the mesons are listed in Table V for both configurations and compared with experimental values [72]. It is obvious that the masses involving open charm in the scalar and pseudoscalar sectors agree better with experimental values than those in the vector sector. However, the model parameters obtained here do not allow for equally good agreement for quarkonium states.

The examination of the chiral phase-structure properties for all of these mesonic states allows one to determine the in-medium modifications of hadronic matter as well as the

critical temperature at which each mesonic state splits into its free quarks. This facilitates a comparison between the masses of strange and charm mesons relative to the masses of lighter mesons. The comparative analyses in SU(3) and SU(4) eLSM helps in determining the influences that brought about additional degrees of freedom and thereby the accurate estimation of the various meson states.

We conclude that the inclusion of heavy quark in SU(4) improves the precision of meson mass simulations compared with SU(3). The temperature behavior of meson masses appears to show a striking change in the critical temperature

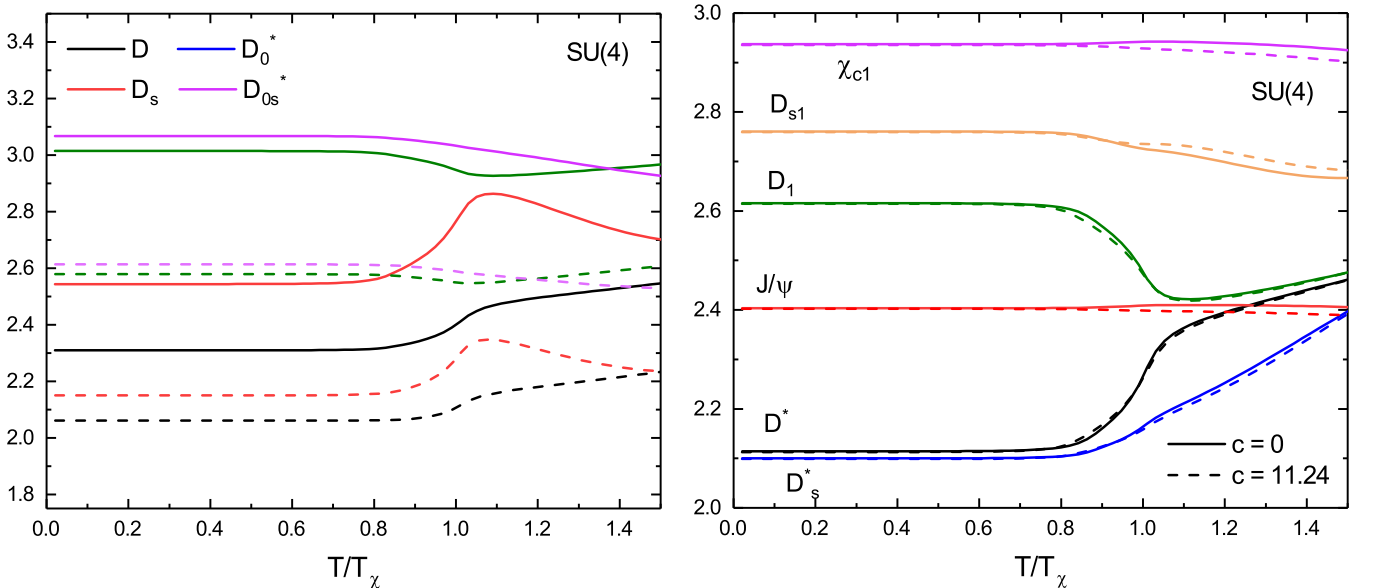


FIG. 5. Left panel: scalar and pseudoscalar mesons with open charm and quarkonia. Right panel: vector and axial-vector mesons with open charm and quarkonia. Meson masses are plotted for both $c = 0$ (solid lines) and $c \neq 0$ (dashed lines) cases.

TABLE V. A comparison between meson masses within SU(3) and SU(4) eLSM.

Pseudoscalar sector ($J^P = 0^-$)			Scalar sector ($J^P = 0^+$)		
Expt. (GeV)	SU(3) (GeV)	SU(4) (GeV)	Expt. (GeV)	SU(3) (GeV)	SU(4) (GeV)
π (0.139)	0.139	0.142	a_0 (0.98)	0.926	0.89
K (0.497)	0.42	0.493	K_0^* (0.63–1.425)	1.032	0.993
η (0.547)	0.531	0.533	σ (0.5–1.2)	0.7	0.7
η' (0.958)	0.965	0.966	f_0 (0.98–1.5)	1.124	1.091
Vector mesons ($J^P = 1^-$)			Axial-vector mesons ($J^P = 1^+$)		
ρ (0.761)	0.77	0.749	a_1 (1.26)	1.067	1.015
K^* (0.89)	0.851	0.903	K_1 (1.27)	1.253	1.276
ϕ (1.02)	1.019	1.015	f_1 (1.428)	1.425	1.46
Charmed mesons					
Scalar-pseudoscalar sector			Vector, axial-vector sector		
Expt. (GeV)	J^P	SU(4) (GeV)	Expt. (GeV)	J^P	SU(4) (GeV)
η_c (2S) (3.637)	0^-	4.844	J/ψ (3.096)	1^-	2.403
D (1.864)	0^-	2.03	D^* (2.007)	1^-	2.114
D_s (1.968)	0^-	2.116	D_s^* (2.112)	1^-	2.1
χ_{c0} (3.86)	0^+	3.551	χ_{c1} (3.414)	1^+	2.937
D_0^* (2.343)	0^+	2.578	D_1 (2.422)	1^+	2.615
D_{s0}^* (2.317)	0^+	2.293	D_{s1} (2.45)	1^+	2.760

region. Our results indicate that different mesonic states display a qualitatively similar behavior near the dissociation region, although the corresponding dissociation temperatures depend on the particular meson state. In addition, we observe that quarkonium states, composed of a heavy quark-antiquark pair, are affected by temperature only gradually, indicating a higher stability against thermal effects.

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The authors declare that there are no conflicts of interest regarding the publication of this published article.

DATA AVAILABILITY

The authors confirm that the data supporting the findings of this study are available within the article.

APPENDIX A: TREE-LEVEL MASSES

Let us now consider the scalar-pseudoscalar term of the Lagrangian equation (2) that includes the determinant term $c[\det \Phi + \det \bar{\Phi}]$ as shown in Eq. (5) with $c_0 = c_1 = 0$. If we

consider that σ_0 and σ_8 fields have nonzero vacuum expectation values ($\bar{\sigma}_0$ and $\bar{\sigma}_8$), and we adjust these fields according to their expectation values [54], then the tree-level potential of the LSM in a general form can be expressed as

$$\begin{aligned}
 U(\bar{\sigma}) = & \frac{m^2}{2} \bar{\sigma}_a^2 - [3\delta(N_f, 2)\mathcal{G}_{ab} + \delta(N_f, 3)\mathcal{G}_{abc}\bar{\sigma}_c]\bar{\sigma}_a\bar{\sigma}_b \\
 & + \frac{1}{3}[\mathcal{F}_{abcd} + \delta(N_f, 4)\mathcal{G}_{abcd}]\bar{\sigma}_a\bar{\sigma}_b\bar{\sigma}_c\bar{\sigma}_d \\
 & - h_a\bar{\sigma}_a.
 \end{aligned} \tag{A1}$$

For the sake of simplicity, the bar notation will be omitted hereafter. The constants $\mathcal{G}_{ab}$, $\mathcal{G}_{abc}$, $\mathcal{G}_{abcd}$, $\mathcal{F}_{abcd}$, $\mathcal{H}_{abcd}$ are defined in the following way:

$$\mathcal{G}_{ab} = \frac{c}{6}[\delta_{a0}\delta_{b0} - \delta_{a1}\delta_{b1} - \delta_{a2}\delta_{b2} - \delta_{a3}\delta_{b3}], \tag{A2}$$

$$\begin{aligned}
 \mathcal{G}_{abc} = & \frac{c}{6}\left[d_{abc} - \frac{3}{2}(\delta_{a0}d_{0bc} + \delta_{b0}d_{a0c} + \delta_{c0}d_{ab0})\right. \\
 & \left. + \frac{9}{2}d_{000}\delta_{a0}\delta_{b0}\delta_{c0}\right],
 \end{aligned} \tag{A3}$$

$$\begin{aligned}
 \mathcal{G}_{abcd} = & \frac{c}{16}[\delta_{ab}\delta_{cd} + \delta_{ad}\delta_{bc} + \delta_{ac}\delta_{bd} - (d_{abn}d_{ncd} + d_{adn}d_{nbc} \\
 & + d_{acn}d_{nbd}) + 16\delta_{a0}\delta_{b0}\delta_{c0}\delta_{d0}] \\
 & - 4(\delta_{a0}\delta_{b0}\delta_{cd} + \delta_{a0}\delta_{c0}\delta_{bd} + \delta_{a0}\delta_{d0}\delta_{bc} \\
 & + \delta_{b0}\delta_{c0}\delta_{ad} + \delta_{b0}\delta_{d0}\delta_{ac} + \delta_{c0}\delta_{d0}\delta_{ab}) \\
 & + \sqrt{8}(\delta_{a0}d_{bcd} + \delta_{b0}d_{cda} + \delta_{c0}d_{dab} + \delta_{d0}d_{abc}),
 \end{aligned} \tag{A4}$$

$$\begin{aligned}
 \mathcal{F}_{abcd} = & \frac{\lambda_1}{4}(\delta_{ab}\delta_{cd} + \delta_{ad}\delta_{bc} + \delta_{ac}\delta_{bd}) \\
 & + \frac{\lambda_2}{8}(d_{abn}d_{ncd} + d_{adn}d_{nbc} + d_{acn}d_{nbd}),
 \end{aligned} \tag{A5}$$

$$\mathcal{H}_{abcd} = \frac{\lambda_1}{4} \delta_{ab} \delta_{cd} + \frac{\lambda_2}{8} (d_{abn} d_{ncd} + f_{acn} f_{nbd} + f_{bcn} f_{nad}). \quad (\text{A6})$$

The standard antisymmetric and symmetric structure constants of $SU(N)$ are represented by f_{abc} and d_{abc} respectively, where $a, b, c = 1, \dots, N_c^2 - 1$, with N_c being the degrees of freedom for colors.

The tree-level masses of pseudoscalars $(m_P^2)_{ab}$ and scalars $(m_S^2)_{ab}$ are determined from the quadratic components of the Lagrangian and, in the general $SU(2)$ – $SU(4)$ configurations, can be expressed as

$$(m_S^2)_{ab} = m^2 \delta_{ab} - 6[\delta(N_f, 2) \mathcal{G}_{ab} + \delta(N_f, 3) \mathcal{G}_{abc} \bar{\sigma}_c] + 4[\mathcal{F}_{abcd} + \delta(N_f, 4) \mathcal{G}_{abcd}] \bar{\sigma}_c \bar{\sigma}_d, \quad (\text{A7})$$

$$(m_P^2)_{ab} = m^2 \delta_{ab} + 6[\delta(N_f, 2) \mathcal{G}_{ab} + \delta(N_f, 3) \mathcal{G}_{abc} \bar{\sigma}_c] + 4[\mathcal{F}_{abcd} - \delta(N_f, 4) \mathcal{G}_{abcd}] \bar{\sigma}_c \bar{\sigma}_d. \quad (\text{A8})$$

Here, the summation runs over the index n only. Equations for vector and axial-vector mesons are defined as

$$(m_V^2)_{ab} = m_1^2 + J_{abmn} \sigma_m \sigma_n, \quad (\text{A9})$$

$$(m_A^2)_{ab} = m_1^2 + J'_{abmn} \sigma_m \sigma_n, \quad (\text{A10})$$

where a, b refer to the meson state, and the summation runs over $m, n \in \{1, 2, \dots, (N_c^2 - 1)\}$ [31]. The coefficients J_{abcd} and J'_{abcd} are given as follows:

$$J_{abcd} = g_1^2 f_{acn} f_{bdn} + \frac{h_1}{2} \delta_{ab} \delta_{cd} + \frac{h_2}{2} d_{abn} d_{cdn} + \frac{h_3}{2} (d_{acn} d_{bdn} + d_{adn} d_{bcn} - f_{acn} f_{bdn} - f_{adn} f_{bcn}), \quad (\text{A11})$$

$$J'_{abcd} = g_1^2 d_{acn} d_{bdn} + \frac{h_1}{2} \delta_{ab} \delta_{cd} + \frac{h_2}{2} d_{abn} d_{cdn} - \frac{h_3}{2} (d_{acn} d_{bdn} + d_{adn} d_{bcn} - f_{acn} f_{bdn} - f_{adn} f_{bcn}). \quad (\text{A12})$$

Again, the sum operates solely over the index n , while d_{abc} and f_{abc} represent the symmetric and antisymmetric structure constants of $U(N_f)$, respectively. It should be noticed that all the resulting expressions are converted to the nonstrange-strange and the nonstrange-strange-charm basis as discussed in Secs. II A and II B.

Furthermore, it should be emphasized that due to the mixing between the (pseudo)scalar and (axial)vector sector

through the covariant derivative, Eq. (7), the tree-level expressions of the pseudoscalars and some scalars are not mass eigenstates. It was reported that, when the corresponding wave functions are renormalized to the constants Z_i , such a mixing can be resolved [73]. The factors Z_i are now emerging in mass expressions [32]:

$$\begin{aligned} Z_\pi &= Z_{\eta_N} = \frac{m_{a_1}}{\sqrt{m_{a_1}^2 - g_1^2 \sigma_x^2}}, \\ Z_K &= \frac{2m_{K_1}}{\sqrt{4m_{K_1}^2 - g_1^2 (\sigma_x + \sqrt{2} \sigma_y)^2}}, \\ Z_{\eta_S} &= \frac{m_{f_{1S}}}{\sqrt{m_{f_{1S}}^2 - 2g_1^2 \sigma_y^2}}, \\ Z_{K_0^*} &= \frac{2m_{K^*}}{\sqrt{4m_{K^*}^2 - g_1^2 (\sigma_x - \sqrt{2} \sigma_y)^2}}. \end{aligned} \quad (\text{A13})$$

And for charmed mesons the factors Z_i read [32]

$$\begin{aligned} Z_{\eta_c} &= \frac{m_{\chi_{c1}}}{\sqrt{m_{\chi_{c1}}^2 - 2g_1^2 \sigma_c^2}}, \\ Z_D &= \frac{2m_{D_1}}{\sqrt{4m_{D_1}^2 - g_1^2 (\sigma_x + \sqrt{2} \sigma_c)^2}}, \\ Z_{D_s} &= \frac{\sqrt{2}m_{D_{s1}}}{\sqrt{2m_{D_{s1}}^2 - g_1^2 (\sigma_y + \sigma_c)^2}}, \\ Z_{D_0^*} &= \frac{2m_{D^*}}{\sqrt{4m_{D^*}^2 - g_1^2 (\sigma_x - \sqrt{2} \sigma_c)^2}}, \\ Z_{D_0^{*0}} &= \frac{2m_{D^{*0}}}{\sqrt{4m_{D^{*0}}^2 - g_1^2 (\sigma_x - \sqrt{2} \sigma_c)^2}}, \\ Z_{D_s^*} &= \frac{\sqrt{2}m_{D_s^*}}{\sqrt{2m_{D_s^*}^2 - g_1^2 (\sigma_y - \sigma_c)^2}}. \end{aligned}$$

APPENDIX B: MESON STATES

For $N_f = 3$ the identification of the physical scalar and pseudoscalar fields is given as

$$T_a \sigma_a = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} a_0^0 + \frac{1}{\sqrt{6}} \sigma_8 + \frac{1}{\sqrt{3}} \sigma_0 & a_0^+ & K_0^{*+} \\ a_0^- & -\frac{1}{\sqrt{2}} a_0^0 + \frac{1}{\sqrt{3}} \sigma_0 + \frac{1}{\sqrt{6}} \sigma_8 & K_0^{*0} \\ K_0^{*-} & \bar{K}_0^{*0} & -\frac{2}{\sqrt{6}} \sigma_8 + \frac{1}{\sqrt{3}} \sigma_0 \end{pmatrix}, \quad (\text{B1})$$

$$T_a \pi_a = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{6}} \pi_8 + \frac{1}{\sqrt{3}} \pi_0 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{6}} \pi_8 + \frac{1}{\sqrt{3}} \pi_0 & K^- \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}} \pi_8 + \frac{1}{\sqrt{3}} \pi_0 \end{pmatrix}. \quad (\text{B2})$$

TABLE VI. The meson masses for SU(3) configuration.

Scalar mesons	Pseudoscalar mesons
$m_{a_0}^2 = m_0^2 + \frac{c}{\sqrt{6}}\sigma_y + (\lambda_1 + \frac{3}{2}\lambda_2)\sigma_x^2 + \lambda_1\sigma_y,$ $m_{K_0^*}^2 = Z_{K_0}^2(m_0^2 + \frac{c}{\sqrt{2}}\sigma_y + (\lambda_1 + \frac{1}{2}\lambda_2)\sigma_x^2 + \frac{\lambda_2}{\sqrt{2}}\sigma_x\sigma_y + (\lambda_1 + \lambda_2)\sigma_y^2),$ $m_{\sigma_N}^2 = m_0^2 - \frac{c\sigma_y}{\sqrt{2}} + \frac{3}{2}(2\lambda_1 + \lambda_2)\sigma_x^2 + \lambda_1\sigma_y^2,$ $m_{\sigma_S}^2 = m_0^2 + 3(\lambda_1 + \lambda_2)\sigma_y^2 + \lambda_1\sigma_x^2,$ $m_{\sigma_{SN}}^2 = -\frac{c\sigma_y}{\sqrt{2}} + 2\lambda_1\sigma_x\sigma_y.$	$m_\pi^2 = Z_\pi^2(m_0^2 - \frac{c}{\sqrt{2}}\sigma_y + (\lambda_1 + \frac{\lambda_2}{2})\sigma_x^2 + \lambda_1\sigma_y^2),$ $m_K^2 = Z_K^2(m_0^2 - \frac{c}{2}\sigma_x + (\lambda_1 + \frac{\lambda_2}{2})\sigma_x^2 + (\lambda_1 + \lambda_2)\sigma_y^2 - \frac{\lambda_2}{\sqrt{2}}\sigma_x\sigma_y),$ $m_{\eta_N}^2 = Z_\pi^2(m_0^2 + \frac{c}{\sqrt{2}}\sigma_y + (\lambda_1 + \frac{\lambda_2}{2})\sigma_x^2 + \lambda_1\sigma_y^2),$ $m_{\eta_S}^2 = Z_{\eta_S}^2(m_0^2 + \lambda_1\sigma_x^2 + (\lambda_1 + \lambda_2)\sigma_y^2),$ $m_{\eta_{NS}}^2 = Z_\pi^2 Z_{\eta_S}^2 \frac{c}{2}\sigma_x.$
Vector mesons	Axial-vector mesons
$m_\rho^2 = m_1^2 + \frac{h_1}{2}(\sigma_x^2 + \sigma_y^2) + \frac{1}{2}(h_2 + h_3)\sigma_x^2 + 2\delta_x,$ $m_{K^*}^2 = m_1^2 + \frac{h_1}{2}(\sigma_x^2 + \sigma_y^2) + \frac{1}{4}(g_1^2 + h_2)(\sigma_x^2 + 2\sigma_y^2)$ $\quad + \frac{\sigma_x\sigma_y}{\sqrt{2}}(h_3 - g_1^2) + \delta_x + \delta_y,$ $m_{\omega_S}^2 = m_1^2 + \frac{h_1}{2}(\sigma_x^2 + \sigma_y^2) + (h_2 + h_3)\sigma_y^2 + 2\delta_y,$ $m_{\omega_N}^2 = m_\rho^2.$	$m_{a_1}^2 = m_1^2 + g_1^2\sigma_x^2 + \frac{h_1}{2}(\sigma_x^2 + \sigma_y^2) + \frac{1}{2}(h_2 - h_3)\sigma_x^2 + 2\delta_x,$ $m_{K_1}^2 = m_1^2 + \frac{h_1}{2}(\sigma_x^2 + \sigma_y^2) + \frac{1}{4}(g_1^2 + h_2)(\sigma_x^2 + 2\sigma_y^2)$ $\quad - \frac{\sigma_x\sigma_y}{\sqrt{2}}(h_3 - g_1^2) + \delta_x + \delta_y,$ $m_{f_{1S}}^2 = m_1^2 + 2g_1^2\sigma_y^2 + \frac{h_1}{2}(\sigma_x^2 + \sigma_y^2) + (h_2 - h_3)\sigma_y^2 + 2\delta_y,$ $m_{f_{1N}}^2 = m_{a_1}^2.$

Here $\pi^\pm = (\pi_1 \mp i\pi_2)/\sqrt{2}$ and $\pi^0 = \pi_3$ are the pions, $K^\pm = (\pi_4 \mp i\pi_5)/\sqrt{2}$, $K^0 = (\pi_6 - i\pi_7)/\sqrt{2}$, $\bar{K}^0 = (\pi_6 + i\pi_7)/\sqrt{2}$ are the kaons. The π_0 and the π_8 are admixtures of the η and η' mesons and the fields σ_0 and σ_8 correspond to admixtures of the σ and f_0 mesons.

For $N_f = 4$ we use the following matrices:

$$T_a \sigma_a = \frac{1}{\sqrt{2}} \begin{pmatrix} A_S & a_0^+ & K_0^{*+} & \bar{D}_0^0 \\ a_0^- & B_S & K_0^{*0} & \bar{D}_0^- \\ K_0^{*-} & \bar{K}_0^{*0} & C_S & D_{s0}^- \\ D_0^0 & D_0^+ & D_{s0}^+ & D_S \end{pmatrix},$$

$$T_a \pi_a = \frac{1}{\sqrt{2}} \begin{pmatrix} A_P & \pi^+ & K^+ & D^0 \\ \pi^- & B_P & K^0 & D^- \\ K^- & \bar{K}^0 & C_P & D_s^- \\ D^0 & D^+ & D_s^+ & D_P \end{pmatrix}, \quad (B3)$$

where the diagonal elements are given as

$$A_S = \frac{1}{2}\sigma_0 + \frac{1}{\sqrt{2}}a_0^0 + \frac{1}{\sqrt{6}}\sigma_8 + \frac{1}{\sqrt{12}}\sigma_{15},$$

$$A_P = \frac{1}{2}\pi_0 + \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\pi_8 + \frac{1}{\sqrt{12}}\pi_{15},$$

$$B_S = \frac{1}{2}\sigma_0 - \frac{1}{\sqrt{2}}a_0^0 + \frac{1}{\sqrt{6}}\sigma_8 + \frac{1}{\sqrt{12}}\sigma_{15},$$

$$B_P = \frac{1}{2}\pi_0 - \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\pi_8 + \frac{1}{\sqrt{12}}\pi_{15},$$

$$C_S = \frac{1}{2}\sigma_0 - \frac{2}{\sqrt{6}}\sigma_8 + \frac{1}{\sqrt{12}}\sigma_{15},$$

$$C_P = \frac{1}{2}\pi_0 - \frac{2}{\sqrt{6}}\pi_8 + \frac{1}{\sqrt{12}}\pi_{15},$$

$$D_S = \frac{1}{2}\sigma_0 - \frac{3}{\sqrt{12}}\sigma_{15},$$

$$D_P = \frac{1}{2}\pi_0 - \frac{3}{\sqrt{12}}\pi_{15}. \quad (B4)$$

The nondiagonal pseudoscalar elements are defined as

$$D^0 = \frac{1}{\sqrt{2}}(\pi_9 + i\pi_{10}),$$

$$\bar{D}^0 = \frac{1}{\sqrt{2}}(\pi_9 - i\pi_{10}),$$

$$D^\pm = \frac{1}{\sqrt{2}}(\pi_{11} \pm i\pi_{12}),$$

$$D_s^\pm = \frac{1}{\sqrt{2}}(\pi_{13} \pm i\pi_{14}). \quad (B5)$$

The nondiagonal scalar mesons are represented as

$$D_0^0 = \frac{1}{\sqrt{2}}(\sigma_9 + i\sigma_{10}),$$

$$\bar{D}_0^0 = \frac{1}{\sqrt{2}}(\sigma_9 - i\sigma_{10}),$$

$$D_0^\pm = \frac{1}{\sqrt{2}}(\sigma_{11} \pm i\sigma_{12}),$$

$$D_{s0}^\pm = \frac{1}{\sqrt{2}}(\sigma_{13} \pm i\sigma_{14}). \quad (B6)$$

The mixing of isoscalar states in pseudoscalar and scalar SU(3) multiplets is described in detail in Refs. [59,73]. The π_0 , π_8 , and π_{15} fields are admixtures of the η , η' , and η_c mesons and the scalar fields σ_0 , σ_8 , and σ_{15} are admixtures of the σ , f_0 , and χ_{c0} mesons. In the SU(4) eLSM, the interpretation of mixing pattern is assumed to be similar to that of its SU(3) configuration. Accordingly, the isoscalar states in the 15-plet singlet (η_0 , η_8 , η_{15}) are defined as

$$|\eta_0\rangle = \frac{1}{2}(|u\bar{u}\rangle + |d\bar{d}\rangle + |s\bar{s}\rangle + |c\bar{c}\rangle),$$

$$|\eta_8\rangle = \frac{1}{\sqrt{6}}(|u\bar{u}\rangle + |d\bar{d}\rangle - 2|s\bar{s}\rangle),$$

$$|\eta_{15}\rangle = \frac{1}{\sqrt{12}}(|u\bar{u}\rangle + |d\bar{d}\rangle + |s\bar{s}\rangle - 3|c\bar{c}\rangle). \quad (B7)$$

Defining the eigenstates for the nonstrange-strange-charm basis as $\eta_N = (|u\bar{u}\rangle + |d\bar{d}\rangle)/\sqrt{2}$, $\eta_S = |s\bar{s}\rangle$, $\eta_c = |c\bar{c}\rangle$, the

TABLE VII. The non-charmed meson masses for SU(4) configuration.

Scalar noncharmed mesons	Pseudoscalar noncharmed mesons
$m_{a_0}^2 = m_0^2 + \frac{c}{2}\sigma_c\sigma_y + \lambda_1(\sigma_x^2 + \sigma_y^2 + \sigma_c^2) + \frac{3\lambda_2}{2}\sigma_x^2$, $m_{K_0^*}^2 = Z_{K_0^*}^2[m_0^2 - \frac{c}{2\sqrt{2}}\sigma_x\sigma_c + (\lambda_1 + \frac{\lambda_2}{2})\sigma_x^2 + \lambda_1(\sigma_y^2 + \sigma_c^2)$ $+ \frac{\lambda_2}{\sqrt{2}}\sigma_x\sigma_y + \lambda_2\sigma_y^2]$, $m_{\sigma_N}^2 = m_0^2 - \frac{c}{2}\sigma_c\sigma_y + 3(\lambda_1 + \frac{\lambda_2}{2})\sigma_x^2 + \lambda_1(\sigma_y^2 + \sigma_c^2)$, $m_{\sigma_S}^2 = m_0^2 + \lambda_1(\sigma_x^2 + \sigma_c^2) + 3(\lambda_1 + \lambda_2)\sigma_y^2$, $m_{\sigma_{NS}}^2 = 2\lambda_1\sigma_x\sigma_y - \frac{c}{2}\sigma_c\sigma_x$.	$m_\pi^2 = Z_\pi^2[m_0^2 - \frac{c}{2}\sigma_y\sigma_c + (\lambda_1 + \frac{\lambda_2}{2})\sigma_x^2 + \lambda_1\sigma_y^2 + \lambda_1\sigma_c^2]$, $m_K^2 = Z_K^2[m_0^2 - \frac{c}{2\sqrt{2}}\sigma_x\sigma_c + (\lambda_1 + \frac{\lambda_2}{2})\sigma_x^2 + \lambda_1(\sigma_y^2 + \sigma_c^2)$ $- \frac{\lambda_2}{\sqrt{2}}\sigma_x\sigma_y + \lambda_2\sigma_y^2]$, $m_{\eta_N}^2 = Z_\pi^2[m_0^2 + \frac{c}{2}\sigma_c\sigma_y + (\lambda_1 + \frac{\lambda_2}{2})\sigma_x^2 + \lambda_1(\sigma_y^2 + \sigma_c^2)]$, $m_{\eta_S}^2 = Z_{\eta_S}^2[m_0^2 + \lambda_1(\sigma_x^2 + \sigma_c^2) + (\lambda_1 + \lambda_2)\sigma_y^2]$, $m_{\eta_{NS}}^2 = -Z_\pi Z_{\eta_S} \frac{c}{2}\sigma_x\sigma_c$.
Vector non-charmed mesons	Axial-vector non-charmed mesons
$m_\rho^2 = m_1^2 + \frac{h_1}{2}(\sigma_x^2 + \sigma_y^2 + \sigma_c^2) + \frac{1}{2}(h_2 + h_3)\sigma_x^2 + 2\delta_x$, $m_{K^*}^2 = m_1^2 + \frac{h_1}{2}(\sigma_x^2 + \sigma_y^2 + \sigma_c^2) + \frac{1}{4}(g_1^2 + h_2)(\sigma_x^2 + 2\sigma_y^2)$ $+ \frac{\sigma_x\sigma_y}{\sqrt{2}}(h_3 - g_1^2) + \delta_x + \delta_y$, $m_{\omega_S}^2 = m_1^2 + \frac{h_1}{2}(\sigma_x^2 + \sigma_y^2 + \sigma_c^2) + (h_2 + h_3)\sigma_y^2 + 2\delta_y$, $m_{\omega_N}^2 = m_\rho^2$.	$m_{a_1}^2 = m_1^2 + g_1^2\sigma_x^2 + \frac{h_1}{2}(\sigma_x^2 + \sigma_y^2 + \sigma_c^2) + \frac{1}{2}(h_2 - h_3)\sigma_x^2 + 2\delta_x$, $m_{K_1}^2 = m_1^2 + \frac{h_1}{2}(\sigma_x^2 + \sigma_y^2 + \sigma_c^2) + \frac{1}{4}(g_1^2 + h_2)(\sigma_x^2 + 2\sigma_y^2)$ $- \frac{\sigma_x\sigma_y}{\sqrt{2}}(h_3 - g_1^2) + \delta_x + \delta_y$, $m_{f_{1S}}^2 = m_1^2 + 2g_1^2\sigma_y^2 + \frac{h_1}{2}(\sigma_x^2 + \sigma_y^2 + \sigma_c^2) + (h_2 - h_3)\sigma_y^2 + 2\delta_y$, $m_{f_{1N}}^2 = m_{a_1}^2$.

transition matrix for SU(4) can be written as

$$\begin{pmatrix} \eta_N \\ \eta_S \\ \eta_C \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{2} & -\sqrt{\frac{2}{3}} & \frac{1}{2\sqrt{3}} \\ \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \eta_0 \\ \eta_8 \\ \eta_{15} \end{pmatrix}. \quad (\text{B8})$$

This full 3×3 mixing matrix can be considered as block diagonal, specifically, a 2×2 block for the light η/η' system and a 1×1 block for the heavy η_c [74]. The physical particles can be realized in the same way as in SU(3): $|\eta\rangle \approx \cos\theta_P|\eta_8\rangle - \sin\theta_P|\eta_0\rangle$, $|\eta'\rangle \approx \sin\theta_P|\eta_8\rangle + \cos\theta_P|\eta_0\rangle$, where the mixing angle θ_P is experimentally found to be about -41° [75,76]. Using the basis in Eq. (B8), the mass elements for η mesons can be written as [59]

$$m_{\eta'/\eta}^2 = \frac{1}{2}(m_{\eta_N}^2 + m_{\eta_S}^2 \pm \sqrt{(m_{\eta_N}^2 - m_{\eta_S}^2)^2 - 4m_{\eta_{NS}}^4}), \quad (\text{B9})$$

with m_{η_N} , m_{η_S} , $m_{\eta_{NS}}$ defined in the tables below. We keep the same structure for physical particles as in SU(3) because of the heavy mass of the charm quark. It is assumed to be decoupled almost completely from the light quarks. This means that the off-diagonal elements $\eta_8\eta_{15}$ and $\eta_0\eta_{15}$ become relatively small relative to $\eta_{15}\eta_{15}$ and can be neglected also due to the

mixing angles θ_{015} and θ_{815} being small [74,77]. It should be noted that the state η_{15} can be identified the charmonium state $\eta_c(1S)$ [74,78,79].

For the scalar sector, considering the light sector only (ignoring charm completely), the mixing among the three relevant states can be parametrized in the same way as discussed above with corresponding replacements (σ_0, σ_8) , (σ_N, σ_S) and the scalar mixing angle θ_S [59]:

$$m_{\sigma/f_0}^2 = \frac{1}{2}(m_{\sigma_N}^2 + m_{\sigma_S}^2 \pm \sqrt{(m_{\sigma_N}^2 - m_{\sigma_S}^2)^2 - 4m_{\sigma_{NS}}^4}). \quad (\text{B10})$$

The vector V^μ and the axial-vector A^μ sectors are described by the following matrices:

$$V^\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}}(\omega_N + \rho^0) & \rho^+ & K^{*+} & D^{*0} \\ \rho^- & \frac{1}{\sqrt{2}}(\omega_N - \rho^0) & K^{*0} & D^{*-} \\ K^{*-} & \bar{K}^{*0} & \omega_S & D_s^{*-} \\ \bar{D}^{*0} & D^{*+} & D_s^{*+} & J/\psi \end{pmatrix}, \quad (\text{B11})$$

TABLE VIII. The masses of mesons with open (hidden) charm.

Scalar charmed mesons	Pseudoscalar charmed mesons
$m_{\chi_{c0}}^2 = m_0^2 + \lambda_1(\sigma_x^2 + \sigma_y^2) + 3(\lambda_1 + \lambda_2)\sigma_c^2 + 2\epsilon_c$, $m_{D_0^*}^2 = Z_{D_0^*}^2[m_0^2 - \frac{c}{2\sqrt{2}}\sigma_x\sigma_y + (\lambda_1 + \frac{\lambda_2}{2})\sigma_x^2 + \lambda_1\sigma_y^2$ $+ (\lambda_1 + \lambda_2)\sigma_c^2 + \frac{\lambda_2}{\sqrt{2}}\sigma_x\sigma_c + \epsilon_c]$, $m_{D_{s0}^*}^2 = Z_{D_{s0}^*}^2[m_0^2 - \frac{c}{4}\sigma_x^2 + \lambda_1\sigma_x^2 + (\lambda_1 + \lambda_2)\sigma_y^2$ $+ \lambda_2\sigma_y\sigma_c + (\lambda_1 + \lambda_2)\sigma_c^2 + \epsilon_c]$.	$m_D^2 = Z_D^2[m_0^2 + \frac{c}{2\sqrt{2}}\sigma_x\sigma_y + (\lambda_1 + \frac{\lambda_2}{2})\sigma_x^2 + \lambda_1\sigma_y^2$ $+ (\lambda_1 + \lambda_2)\sigma_c^2 - \frac{\lambda_2}{\sqrt{2}}\sigma_x\sigma_c + \epsilon_c]$, $m_{\eta_c}^2 = Z_{\eta_c}^2[m_0^2 + \lambda_1(\sigma_x^2 + \sigma_y^2) + (\lambda_1 + \lambda_2)\sigma_c^2 + 2\epsilon_c]$, $m_{D_s}^2 = Z_{D_s}^2[m_0^2 + \frac{c}{4}\sigma_x^2 + \lambda_1\sigma_x^2 + (\lambda_1 + \lambda_2)\sigma_y^2$ $+ (\lambda_1 + \lambda_2)\sigma_c^2 - \lambda_2\sigma_y\sigma_c + \epsilon_c]$.
Vector charmed mesons	Axial-vector charmed mesons
$m_{D^*}^2 = m_1^2 + \frac{h_1}{2}(\sigma_x^2 + \sigma_y^2 + \sigma_c^2) + \frac{1}{4}(g_1^2 + h_2)(\sigma_x^2 + 2\sigma_c^2)$ $+ \frac{\sigma_x\sigma_c}{\sqrt{2}}(h_3 - g_1^2) + \delta_x + \delta_c$, $m_{D_{s1}^*}^2 = m_1^2 + \frac{h_1}{2}(\sigma_x^2 + \sigma_y^2 + \sigma_c^2) + (h_2 + h_3)\sigma_c^2 + 2\delta_c$, $m_{D_s^*}^2 = m_1^2 + \frac{h_1}{2}(\sigma_x^2 + \sigma_y^2 + \sigma_c^2) + \frac{1}{2}(g_1^2 + h_2)(\sigma_y^2 + \sigma_c^2)$ $+ (h_3 - g_1^2)\sigma_y\sigma_c + \delta_y + \delta_c$.	$m_{D_1}^2 = m_1^2 + \frac{h_1}{2}(\sigma_x^2 + \sigma_y^2 + \sigma_c^2) + \frac{1}{4}(g_1^2 + h_2)(\sigma_x^2 + 2\sigma_c^2)$ $+ \frac{\sigma_x\sigma_c}{\sqrt{2}}(g_1^2 - h_3) + \delta_x + \delta_c$, $m_{\chi_{c1}}^2 = m_1^2 + 2g_1^2\sigma_c^2 + \frac{h_1}{2}(\sigma_x^2 + \sigma_y^2 + \sigma_c^2) + (h_2 - h_3)\sigma_c^2 + 2\delta_c$, $m_{D_{s1}}^2 = m_1^2 + \frac{h_1}{2}(\sigma_x^2 + \sigma_y^2 + \sigma_c^2) + \frac{1}{2}(g_1^2 + h_2)(\sigma_y^2 + \sigma_c^2)$ $+ \sigma_y\sigma_c(g_1^2 - h_3) + \delta_y + \delta_c$.

TABLE IX. First and second derivatives of the squared quark mass with respect to the meson fields.

	$m_{f,a}^2 m_{f,b}^2 / g^4$	$m_{f,ab}^2 / g^2$	$m_{y,a}^2 m_{y,b}^2 / g^4$	$m_{y,ab}^2 / g^2$	$m_{c,a}^2 m_{c,b}^2 / g^4$	$m_{c,ab}^2 / g^2$
$\sigma_0 \sigma_0$	$\frac{1}{4} \sigma_x^2$	$\frac{1}{2}$	$\frac{1}{4} \sigma_y^2$	$\frac{1}{4}$	$\frac{1}{4} \sigma_c^2$	$\frac{1}{4}$
$\sigma_1 \sigma_1$	0	1	0	0	0	0
$\sigma_4 \sigma_4$	0	$\frac{Z_{k_0}^2 \sigma_x}{\sigma_x - \sqrt{2} \sigma_y}$	0	$\frac{Z_{k_0}^2 \sqrt{2} \sigma_y}{\sqrt{2} \sigma_y - \sigma_x}$	0	0
$\sigma_9 \sigma_9$	0	$\frac{Z_{D_0}^2 \sigma_x}{\sigma_x - \sqrt{2} \sigma_c}$	0	0	0	$\frac{Z_{D_0}^2 \sqrt{2} \sigma_c}{\sqrt{2} \sigma_c - \sigma_x}$
$\sigma_{13} \sigma_{13}$	0	0	0	$\frac{Z_{D_{s0}}^2 \sigma_y}{\sigma_y - \sigma_c}$	0	$\frac{Z_{D_{s0}}^2 \sigma_c}{\sigma_c - \sigma_y}$
$\sigma_{15} \sigma_{15}$	$\frac{1}{12} \sigma_x^2$	$\frac{1}{6}$	$\frac{1}{12} \sigma_y^2$	$\frac{1}{12}$	$\frac{3}{4} \sigma_c^2$	$\frac{3}{4}$
$\sigma_8 \sigma_8$	$\frac{1}{6} \sigma_x^2$	$\frac{1}{3}$	$\frac{2}{3} \sigma_y^2$	$\frac{2}{3}$	0	0
$\sigma_0 \sigma_8$	$\frac{1}{2\sqrt{6}} \sigma_x^2$	$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{6}} \sigma_y^2$	$-\frac{1}{\sqrt{6}}$	0	0
$\sigma_0 \sigma_{15}$	$\frac{1}{4\sqrt{3}} \sigma_x^2$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{4\sqrt{3}} \sigma_y^2$	$\frac{1}{4\sqrt{3}}$	$-\frac{\sqrt{3}}{4} \sigma_c^2$	$-\frac{\sqrt{3}}{4}$
$\sigma_8 \sigma_{15}$	$\frac{1}{6\sqrt{2}} \sigma_x^2$	$-\frac{1}{3\sqrt{2}}$	$\frac{1}{3\sqrt{2}} \sigma_y^2$	$-\frac{1}{3\sqrt{2}}$	0	0
$\pi_0 \pi_0$	0	$\frac{1}{2}$	0	$\frac{1}{4}$	0	$\frac{1}{4}$
$\pi_1 \pi_1$	0	Z_π^2	0	0	0	0
$\pi_4 \pi_4$	0	$\frac{Z_K^2 \sigma_x}{\sigma_x + \sqrt{2} \sigma_y}$	0	$\frac{Z_K^2 \sqrt{2} \sigma_y}{\sqrt{2} \sigma_y + \sigma_x}$	0	0
$\pi_9 \pi_9$	0	$\frac{Z_D^2 \sigma_x}{\sigma_x + \sqrt{2} \sigma_c}$	0	0	0	$\frac{Z_D^2 \sqrt{2} \sigma_c}{\sigma_x + \sqrt{2} \sigma_c}$
$\pi_{13} \pi_{13}$	0	0	0	$\frac{Z_{D_s}^2 \sigma_y}{\sigma_y + \sigma_c}$	0	$\frac{Z_{D_s}^2 \sigma_c}{\sigma_y + \sigma_c}$
$\pi_{15} \pi_{15}$	0	$\frac{1}{6}$	0	$\frac{1}{12}$	0	$\frac{3}{4}$
$\pi_8 \pi_8$	0	$\frac{1}{3}$	0	$\frac{2}{3}$	0	0
$\pi_0 \pi_8$	0	$\frac{1}{\sqrt{6}}$	0	$-\frac{1}{\sqrt{6}}$	0	0
$\pi_0 \pi_{15}$	0	$\frac{1}{2\sqrt{3}}$	0	$\frac{1}{4\sqrt{3}}$	0	$-\frac{\sqrt{3}}{4}$
$\pi_8 \pi_{15}$	0	$\frac{1}{3\sqrt{2}}$	0	$-\frac{1}{3\sqrt{2}}$	0	0

$$A^\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}}(f_{1,N} + a_1^0) & a_1^+ & K_1^+ & \bar{D}_1^0 \\ a_1^- & \frac{1}{\sqrt{2}}(f_{1,N} - a_1^0) & K_1^0 & D_1^- \\ K_1^- & \bar{K}_1^0 & f_{1S} & D_{s1}^- \\ \bar{D}_1^0 & D_1^+ & D_{s1}^+ & \chi_{c1} \end{pmatrix}. \quad (\text{B12})$$

The variation between strange and nonstrange isoscalars is disregarded in Ref. [68,76].

The mass expressions obtained for the listed states are presented in Tables VI–VIII below. As noted above, we express all meson masses in term of the strange-nonstrange(-charm) bases.

APPENDIX C: DERIVATIONS OF SQUARED QUARK MASS

In Sec. II C, it was noted that the in-medium meson masses are influenced by the quark contribution, Eq. (32). The results for the first and second derivatives of the quark mass with respect to the meson fields for the SU(3) configuration can be found, for example, in Refs. [59,66], and those in presence of (axial-)vector fields in [48]. Our results for SU(4) configuration are presented in Table IX. The degenerate light flavors are presented in second and third columns, $m_{f,a}^2 m_{f,b}^2 / g^4$, $m_{f,ab}^2 / g^2$, the strange quark in fourth and fifth columns, $m_{y,a}^2 m_{y,b}^2 / g^4$, $m_{y,ab}^2 / g^2$, and the charm quark in last two columns, $m_{c,a}^2 m_{c,b}^2 / g^4$, $m_{c,ab}^2 / g^2$.

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