

WKB approximation under a Killingbeck potential and its application to double-heavy-meson spectroscopy

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It is well known that nonrelativistic or semirelativistic constituent quark models can describe a large number of the meson and baryon properties with surprising accuracy. In this work, we study the Killingbeck potential using the WKB approximation in the constituent quark model. The implicit approximate solutions for the energy eigenvalues, together with analytical expressions for the root-mean-square radii and the probability densities at the origin, of the Schrödinger equation are obtained by using appropriate quantization conditions. Finally, we discuss applications to double-heavy-meson spectroscopy within the Killingbeck potential.

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I. INTRODUCTION

In the past decades, many potential energy functions that play important roles in describing physical phenomena have been proposed. Some wave equations with special potentials are exactly solvable, whereas many physically relevant potentials require numerical or approximate analytical methods [1–3]. As a semiclassical approximation method, the Wentzel-Kramers-Brillouin (WKB) approximation was originally developed by Wentzel, Kramers, and Brillouin in 1926 [4–6]. Later, it was successfully applied to particle energy spectra in high-energy physics by Kang and Schnitzer [1,7] and Quigg and Rosner [8]. This approximation can be used to analyze the Schrödinger equation and has been successfully applied to many important problems [3,9–12]. The WKB approximation based on the Schrödinger equation is expected to give reliable energy eigenvalues in the large-quantum-number limit [13]. For low-lying states the accuracy is model dependent, and the WKB results should be checked against exact or high-precision numerical solutions.

In 2005, Ma and Xu proposed an accurate quantization condition for one-dimensional quantum systems and three-dimensional spherically symmetric quantum systems [14,15]. Brau reported the relationship among energy, radial quantum number, and orbital angular momentum [2]. Lalus *et al.* used the WKB method to analyze the physical conditions that the system must satisfy in the process of tunneling [16]. Omugbe obtained the nonrelativistic rovibrational energy spectra, expectation values, and thermodynamic properties of the Deng-Fan potential within the framework of the WKB approximation method [17]. Moreover, the authors of Refs. [18–24] have used the WKB method to study mass spectra under various potential energies. The bound-state solution to the wave equations under quark-antiquark interaction potential functions, such as the Killingbeck or the Cornell potential, has attracted much research interest in high-energy physics [20]. In Refs. [2,25,26], the authors used the WKB method to study the trajectory relation with the assumption of the linear confinement potential. Compared with the Cornell potential, the Killingbeck potential contains an additional quadratic term and has been used to study double-heavy-meson spectra [27,28]. The advantage of the present WKB treatment lies in its analytical nature: it allows the energy spectrum and wave-function observables, such as the root-mean-square (rms) radius and the probability density at the origin, to be followed for arbitrary radial and orbital excitations. It also provides a controlled way to examine how a quadratic correction affects heavy-quark potentials. However, direct WKB studies of the mass spectra and wave-function observables generated by the Killingbeck potential remain limited. This work is intended to address that problem.

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Reference [29] presents several exactly solvable forms of the Killingbeck potential. These exact analytical results provide valuable benchmarks for assessing the accuracy of approximate methods, such as the WKB method. This motivates the use of complete elliptic integrals to obtain analytical expressions for physical observables and to track hadron properties without repeatedly solving the differential equation numerically. The main focus of this paper is to obtain WKB solutions for the Killingbeck potential in the constituent quark model. Furthermore, the rms radii and the probability densities at the origin, which are closely related to short-distance decay properties, are obtained and applied to double-heavy mesons.

This paper is organized as follows. In Sec. II, the WKB eigenvalues, rms radii, and probability densities at the origin of the Killingbeck potential are introduced. In Sec. III, we compare the WKB approximate solutions with quasisexactly solvable benchmarks for the Killingbeck potential. In Sec. IV, we analyze the mass spectra and spatial observables of double-heavy mesons and compare the WKB results with other theoretical calculations and available experimental data. A discussion and conclusion are given in Sec. V.

II. WKB APPROXIMATION

Brau calculated the energy spectrum under the Cornell potential in the quark model and reported the relation among energy, radial quantum number, and orbital angular momentum [2]. Following this strategy, we solve the spectra and spatial observables of double-heavy mesons under the Killingbeck potential.

The basic quantities in the Bohr-Sommerfeld quantization are the action variables [2]:

$$J_s = \oint p_s dq_s, \quad (2.1)$$

where s labels the degrees of freedom of the system, and q_s and p_s are the generalized coordinate and the corresponding conjugate momentum, respectively. The integral is performed over one cycle of the motion. The action variables are quantized according to the prescription

$$J_s = (n_s + c_s)h, \quad (2.2)$$

where h is Planck's constant, $n_s \geq 0$ is an integer quantum number, and c_s is a real constant that should be set to $\frac{1}{2}$ according to Langer [2].

Under the WKB approximation, the nonrelativistic Hamiltonian corresponding to the Killingbeck potential reads, in natural units ($\hbar = \frac{h}{2\pi} = c = 1$), as

$$H = \frac{1}{2\mu} \left(p_r^2 + \frac{p_\phi^2}{\rho^2} \right) - \frac{\kappa}{\rho} + \tau\rho + \nu\rho^2 + m_1 + m_2 + C + \frac{\sigma}{m_1 m_2} \mathbf{S}_1 \cdot \mathbf{S}_2, \quad (2.3)$$

with the spin-spin expectation value being

$$\langle \mathbf{S}_1 \cdot \mathbf{S}_2 \rangle = \frac{1}{2}S(S+1) - \frac{3}{4}. \quad (2.4)$$

Here, μ is the reduced mass; κ , τ , ν , C , and σ are the parameters to be fitted; and m_1 and m_2 are the masses of the two quarks. In addition, p_ϕ can be identified with L ($p_\phi = L$), where $L = l + c_\phi$ and l is the orbital angular momentum quantum number. The radial momentum p_ρ is derived from the conservation of the shifted energy λ' ($\lambda' = \lambda - m_1 - m_2 - C - \frac{\sigma}{m_1 m_2} [\frac{1}{2}S(S+1) - \frac{3}{4}]$, where λ is the total energy of the system):

$$p_\rho = \frac{1}{\rho} \sqrt{-2\mu\nu\rho^4 - 2\mu\tau\rho^3 + 2\mu\lambda'\rho^2 + 2\mu\kappa\rho - L^2}. \quad (2.5)$$

With the replacements

$$\rho \rightarrow \frac{r}{2\mu\kappa}, \quad \tau \rightarrow 4\mu^2\kappa^3a, \quad \nu \rightarrow 8\mu^3\kappa^4b, \quad \lambda' \rightarrow 2\mu\kappa^2E, \quad (2.6)$$

the Hamiltonian given in Eq. (2.3) can be transformed into the dimensionless form

$$H' = p_r^2 + \frac{p_\phi^2}{r^2} - \frac{1}{r} + ar + br^2. \quad (2.7)$$

A. WKB approximation eigenvalue of the Killingbeck potential

The radial momentum p_r derived from the effective Hamiltonian H' is given by

$$p_r(r) = \pm \sqrt{E - \left(\frac{L^2}{r^2} - \frac{1}{r} + ar + br^2 \right)}. \quad (2.8)$$

By factoring out $1/r$, the momentum can be expressed via a quartic polynomial $P_4(r)$:

$$p_r(r) = \frac{\sqrt{P_4(r)}}{r}, \quad P_4(r) = -br^4 - ar^3 + Er^2 + r - L^2. \quad (2.9)$$

The four analytical roots R_1, R_2, R_3 , and R_4 of the momentum polynomial $P_4(r) = 0$ can be determined using algebraic invariants. We define the fundamental resolvent parameters D_2, D_1, Δ_0 , and Δ_1 as follows:

$$D_2 = \frac{-8bE - 3a^2}{8b^2}, \quad (2.10)$$

$$D_1 = \frac{a^3 + 4abE - 8b^2}{8b^3}, \quad (2.11)$$

$$\Delta_0 = E^2 + 3a + 12bL^2, \quad (2.12)$$

$$\Delta_1 = 2E^3 + 9aE - 27a^2L^2 - 27b - 72bEL^2. \quad (2.13)$$

Q and Z can be given as

$$Q = \sqrt[3]{\frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}}, \quad (2.14)$$

$$Z = \frac{1}{2} \sqrt{-\frac{2}{3}D_2 - \frac{1}{3b} \left(Q + \frac{\Delta_0}{Q} \right)}. \quad (2.15)$$

Then, the exact analytical roots are explicitly formulated as

$$R_{1,2,3,4} = -\frac{a}{4b} + \varepsilon_1 Z + \frac{\varepsilon_2}{2} \sqrt{-4Z^2 - 2D_2 - \varepsilon_1 \frac{D_1}{Z}}, \quad (2.16)$$

with $\varepsilon_1 = \pm 1$ and $\varepsilon_2 = \pm 1$.

If $P_4(r)$ possesses four real roots ordered as $R_1 > R_2 > R_3 > R_4$, the two non-negative roots define the classical turning points, $r_+ = R_1$ and $r_- = R_2$.

The Bohr-Sommerfeld quantization condition, $\int_{r_-}^{r_+} p_r dr = (n + c_r)\pi$, can be expressed as

$$-bZ_3 - aZ_2 + EZ_1 + Z_0 - L^2 Z_{-1} = (n_r + c_r)\pi, \quad (2.17)$$

where $Z_k = \int_{r_-}^{r_+} r^k P_4^{-1/2}(r) dr$ and n_r is the radial quantum number.

In order to express Z_k as special functions, we define the purely algebraic geometric parameters as follows:

$$\delta = (R_2 - R_3), \quad (2.18)$$

$$g = \frac{2}{\sqrt{b(R_1 - R_3)(R_2 - R_4)}}, \quad (2.19)$$

$$\eta = \frac{(R_1 - R_2)(R_3 - R_4)}{(R_1 - R_3)(R_2 - R_4)}, \quad (2.20)$$

$$\gamma = \frac{R_1 - R_2}{R_1 - R_3}, \quad (2.21)$$

and let $W_0 = K(\eta)$, $E_0 = E(\eta)$, and $W_1(\gamma, \eta) = \Pi(\gamma, \eta)$ denote the complete elliptic integrals of the first, second, and third kind, respectively. The higher-order integral sequence $W_m(\gamma, \eta)$ is fundamentally generated by taking partial derivatives of the third-kind elliptic integral with respect to the characteristic parameter γ :

$$W_2 = W_1 + \gamma \frac{\partial W_1}{\partial \gamma}, \quad (2.22)$$

$$W_3 = W_2 + \frac{\gamma}{2} \frac{\partial W_2}{\partial \gamma}. \quad (2.23)$$

By carrying out the differentiation of the complete elliptic integrals using their analytical partial derivative identities, the recurrence relations in Eqs. (2.22) and (2.23) are exactly reduced to the following explicit algebraic forms:

$$W_2 = W_1 + \frac{\gamma E_0 + (\eta - \gamma)W_0 + (\gamma^2 - \eta)W_1}{2(\gamma - 1)(\eta - \gamma)}, \quad (2.24)$$

$$W_3 = W_2 + \frac{\gamma E_0 - \gamma W_0 + 2\gamma^2 W_1 + 3(\gamma^2 - \eta)(W_2 - W_1)}{4(\gamma - 1)(\eta - \gamma)}. \quad (2.25)$$

Then, the exact special-function representations of Z_k are as follows:

$$Z_0 = gW_0, \quad (2.26)$$

$$Z_1 = g[R_3 W_0 + \delta W_1], \quad (2.27)$$

$$Z_2 = g[R_3^2 W_0 + 2R_3 \delta W_1 + \delta^2 W_2], \quad (2.28)$$

$$Z_3 = g[R_3^3 W_0 + 3R_3^2 \delta W_1 + 3R_3 \delta^2 W_2 + \delta^3 W_3], \quad (2.29)$$

$$Z_{-1} = g \left[\frac{1}{R_3} W_0 - \frac{\delta}{R_3(R_3 + \delta)} W_1 \left(\frac{R_3}{R_3 + \delta} \gamma, \eta \right) \right]. \quad (2.30)$$

Solving the implicit algebraic equation, Eq. (2.17), provides the WKB energy eigenvalues E . Accurate semiclassical results are obtained with the Langer prescription $c_r = c_\phi = \frac{1}{2}$.

In the high-energy or large-quantum-number limit, the outer turning point grows as the excitation energy increases. The terms $-br^4$, Er^2 , and $-L^2$ then control the leading radial action, whereas the odd-power terms $-ar^3$ and r enter as subleading corrections. The polynomial in Eq. (2.9) can thus be asymptotically truncated into an even-powered function of r :

$$P_4(r) \approx -br^4 + Er^2 - L^2. \quad (2.31)$$

For the truncated polynomial $-br^4 + Er^2 - L^2 = 0$, we can substitute $x = r^2$ to yield a standard quadratic equation:

$$bx^2 - Ex + L^2 = 0. \quad (2.32)$$

The two roots of this quadratic equation are analytically given by

$$x_{\pm} = \frac{E \pm \sqrt{E^2 - 4bL^2}}{2b}. \quad (2.33)$$

Because $x = r^2$, the original four roots of the quartic equation lose their initial asymmetry (previously denoted as R_1, R_2, R_3 , and R_4). Instead, they degenerate into two perfectly symmetric pairs:

$$R_1 = \sqrt{x_+} \equiv r_+, \quad (2.34)$$

$$R_2 = \sqrt{x_-} \equiv r_-, \quad (2.35)$$

$$R_3 = -\sqrt{x_-} = -r_-, \quad (2.36)$$

$$R_4 = -\sqrt{x_+} = -r_+, \quad (2.37)$$

where r_+ and r_- naturally emerge as the physical turning points corresponding to a pure harmonic oscillator potential.

As the effective potential transforms into a symmetric harmonic well, the odd-powered terms vanish, allowing us to entirely bypass the computationally expensive complete elliptic integrals $K(\eta)$, $E(\eta)$, and $\Pi(\gamma, \eta)$. The integral can now be evaluated using elementary functions.

The Bohr-Sommerfeld quantization condition is expressed as

$$\int_{r_-}^{r_+} \frac{\sqrt{-br^4 + Er^2 - L^2}}{r} dr = \left(n_r + \frac{1}{2} \right) \pi. \quad (2.38)$$

We obtain the exact closed-form solution for the integral:

$$E = 2\sqrt{b}(2n_r + L + 1). \quad (2.39)$$

Ultimately, this yields the explicit physical energy expression:

$$E_{n,l} = \left(2n_r + l + \frac{3}{2} \right) \omega, \quad (2.40)$$

where $\omega = 2\sqrt{b}$.

In summary, the Killingbeck-type potential leads to an implicit WKB eigenvalue equation that can be written entirely in terms of the roots of the quartic momentum polynomial and complete elliptic integrals. This form is useful because it avoids repeated direct integration of the radial equation and gives a transparent connection between the potential parameters and the quantum numbers. In the large-excitation limit, the quadratic confinement term controls the leading radial action, and the spectrum approaches the harmonic-oscillator-like behavior $E_{n,l} \approx (2n + l + 3/2)\omega$, with $\omega = 2\sqrt{b}$.

Thus, the WKB treatment converts the Killingbeck-type potential problem into an implicit algebraic eigenvalue equation expressed through complete elliptic integrals. The same formulation also shows that, for highly excited states, the quadratic term drives the spectrum toward the harmonic-oscillator-like asymptotic form $E_{n,l} \approx (2n + l + 3/2)\omega$.

B. Calculation of observables: rms radius and probability density at the origin

One significant advantage of the WKB method is that it yields analytical expressions for physical observables. We explicitly evaluate the mean square radius $\langle r^2 \rangle$ and, for S -wave states, the probability density at the origin $|\Psi(0)|^2$ of the Killingbeck potential.

Using the definition of the expectation value in the WKB framework, $\langle f(r) \rangle = \frac{1}{T} \int_{r_-}^{r_+} \frac{f(r)}{p_r} dr$ [2], the period corresponds to an integral:

$$T = \int_{r_-}^{r_+} \frac{1}{p_r} dr = \int_{r_-}^{r_+} \frac{r}{\sqrt{P_4(r)}} dr = Z_1. \quad (2.41)$$

Thus, the expectation values can be expressed in terms of the previously derived Z_k .

The mean square radius is given by $\langle r^2 \rangle = Z_3/Z_1$, which yields

$$\langle r^2 \rangle = \frac{R_3^3 W_0 + 3R_3^2 \delta W_1 + 3R_3 \delta^2 W_2 + \delta^3 W_3}{R_3 W_0 + \delta W_1}. \quad (2.42)$$

For S -wave hadron-decay processes, the probability density at the origin is evaluated via the generalized Feynman-Hellmann theorem [30,31], $|\Psi(0)|^2 = \frac{1}{4\pi} \langle \frac{dV(r)}{dr} \rangle$. For our effective potential, $V'(r) = 1/r^2 + a + 2br$. Taking the expectation value gives

$$|\Psi(0)|^2 = \frac{1}{4\pi} \frac{Z_{-1} + aZ_1 + 2bZ_2}{Z_1}. \quad (2.43)$$

For states with $l > 0$, the centrifugal barrier gives $|\Psi(0)|^2 = 0$, and this constraint is imposed in the numerical tables below.

III. QUASIEXACT SOLVABILITY AND EFFECTIVE HAMILTONIAN

We use cases of the Killingbeck potential with quasisolvability (QES) to test the WKB results. The simplified Hamiltonian is given by (2.7)

$$H' = p^2 - \frac{1}{r} + ar + br^2, \quad (3.1)$$

which satisfies the equation

$$H'\psi = E\psi. \quad (3.2)$$

With the transformation

$$\psi(\mathbf{r}) = \frac{\Phi(\theta, \phi)u(r)}{r}, \quad (3.3)$$

one obtains

$$-u''(r) + \left(\frac{l(l+1)}{r^2} - \frac{1}{r} + ar + br^2 - E \right) u(r) = 0. \quad (3.4)$$

Although the WKB method generally gives approximate eigenvalues, the Killingbeck potential admits QES solutions under specific parameter constraints. These solutions provide useful benchmarks for the WKB formalism. With a wavefunction ansatz composed of the correct near-origin behavior, a Gaussian asymptotic factor, and a finite polynomial, the Schrödinger equation is mapped into a three-term recurrence relation. The ansatz is

$$u(r) = N_0 r^{l+1} e^{-\alpha r^2 - \beta r} f(r), \quad (3.5)$$

where

$$f(r) = \sum_{k=0}^n c_k r^k. \quad (3.6)$$

To match the asymptotic behavior of the potential $V(r) \sim br^2 + ar$, we define the Gaussian decay parameters as

$$\alpha = \frac{\sqrt{b}}{2} \quad (3.7)$$

and

$$\beta = \frac{a}{2\sqrt{b}}. \quad (3.8)$$

Substituting the ansatz into the radial equation yields a three-term recurrence relation for the polynomial expansion coefficients c_k :

$$c_k = \frac{-[1 - 2\beta(k+l)]c_{k-1} - [E + \beta^2 - \sqrt{b}(2k+2l-1)]c_{k-2}}{k(k+2l+1)}, \quad (3.9)$$

with the initial conditions $c_0 = 1$ and $c_{-1} = 0$.

To ensure normalizability of the wave function, the infinite series must truncate exactly at the n_r th term (i.e., $c_{n_r+1} = c_{n_r+2} = 0$). This truncation condition is mathematically equivalent to the vanishing determinant of a tridiagonal Bender-Dunne matrix M of dimension $(n_r + 1) \times (n_r + 1)$:

$$\det(M) = 0. \quad (3.10)$$

The nonzero matrix elements of M are explicitly given by

$$M_{i,i} = 1 - 2\beta(i + l), \quad (3.11)$$

$$M_{i,i+1} = i(i + 2l + 1), \quad (3.12)$$

$$M_{i+1,i} = 2\sqrt{b}(n_r - i + 1), \quad (3.13)$$

where the indices i correspond to the elements in the series. For a specific state, such as $n_r = 3$ and $l = 2$, solving the algebraic equation $\det(M) = 0$ in the real domain yields the physical roots for β , which then determine the wave-function coefficients and the exact energy.

For the QES state, the energy is fixed by the truncation constraint:

$$E_{n_r,l}^{\text{exact}} = 2\alpha[2(l + n_r) + 3] - \beta^2. \quad (3.14)$$

The algebraic normalization constraint requires

$$N_0 = \left[\sum_{i=0}^{n_r} \sum_{j=0}^{n_r} c_i c_j I_{2l+2+i+j}(\alpha, 2\beta) \right]^{-\frac{1}{2}}. \quad (3.15)$$

The exact squared wave function at the origin is

$$|\Psi_{n_r,l}(0)|^2 = \begin{cases} \frac{N_0^2 c_0^2}{4\pi}, & l = 0, \\ 0, & l > 0, \end{cases} \quad (3.16)$$

where the basic integral is defined as

$$I_k(\alpha, \beta) = \int_0^\infty r^k e^{-2\alpha r^2 - \beta r} dr \quad (3.17)$$

and can be expressed as

$$I_k(\alpha, \beta) = \Gamma(k + 1)(4\alpha)^{-(k+1)/2} \exp\left(\frac{\beta^2}{16\alpha}\right) D_{-k-1}\left(\frac{\beta}{2\sqrt{\alpha}}\right), \quad (3.18)$$

where $D_\nu(z)$ is the parabolic-cylinder function.

The rms radius

$$\sqrt{\langle r^2 \rangle} = N_0 \sqrt{\sum_{i=0}^{n_r} \sum_{j=0}^{n_r} c_i c_j I_{2l+4+i+j}(\alpha, 2\beta)}. \quad (3.19)$$

To test the accuracy and stability of the analytical WKB framework, we perform a comprehensive comparison against the exact QES benchmarks and high-precision finite-element method (FEM) calculations, as summarized in Table I.

As shown in Table I, the analytical WKB energy eigenvalues agree well with the exact QES results. For the deeply bound ground state ($n_r = 0$, $l = 0$), the WKB approximation yields a relative error of 1.73%. The error generally decreases as the radial node n or the orbital angular momentum l increases, reaching the $10^{-4}\%$ level for highly excited states such as $n_r = 9$ and $l = 9$. This asymptotic convergence embodies the semiclassical nature of the Bohr-Sommerfeld

quantization, which mathematically approaches the exact solution in the large quantum number limit.

The physical observables derived from the algebraic moment integrals Z_k are also stable. The WKB predictions for the rms radius deviate from the exact values by less than 1% across all tested states. For the tested S -wave states, the relative error of $|\Psi(0)|^2$ remains within about 3% and decreases for higher radial excitations. Although the FEM gives more accurate energies and rms radii in most cases, the WKB estimate of $|\Psi(0)|^2$ is closer to the exact QES value than the FEM result for the radially excited S -wave states. This indicates that the algebraic WKB expression for short-distance observables can be competitive with, and in some highly excited S -wave cases slightly more accurate than, the numerical FEM extraction.

The FEM numerical results reproduce the QES values with negligible error (relative errors $\approx 10^{-4}\%$). Overall, Table I shows that the algebraic WKB method provides a compact semianalytical way to estimate both energies and spatial observables, complementing high-precision numerical solutions. Regarding the comparison between WKB and numerical results, we emphasize that the Bohr-Sommerfeld quantization is not a variational method and therefore does not provide an upper bound to the exact energy. Its truncation error can have either sign, so a WKB eigenvalue may slightly undershoot the exact numerical value, as seen for several entries in Table I.

IV. THE SPECTROSCOPY OF DOUBLE-HEAVY MESONS UNDER THE KILLINGBECK POTENTIAL

We compute the mass spectra and physical observables of three double-heavy systems: charmonium ($c\bar{c}$), bottomonium ($b\bar{b}$), and the B_c meson ($b\bar{c}$). The mass spectra obtained with the WKB approximation and the FEM are compared with available experimental data and with results from other theoretical methods. Since the establishment of the quark model, many B_c , ψ , and Υ states have been reported experimentally. To test the WKB approximate solution under the Killingbeck potential, we fit 19 experimental values of B_c , $c\bar{c}$, and $b\bar{b}$ mesons and obtained the parameters shown in Table II. The fitted parameters are then used consistently in the subsequent WKB and FEM calculations of the mass spectra, rms radii, and probability densities at the origin. We also give the predictions for higher bottomonium, charmonium, and B_c states in Tables IV–VI.

A. Mass spectrum of B_c mesons under the Killingbeck potential

There are two experimental values of B_c mesons shown in Table III. The experimental masses of 1^1S_0 and 2^1S_0 states are 6.274 and 6.871 GeV [32], respectively. Comparing the WKB results for the B_c mesons with the two experimental values, one sees that the masses of the 1^1S_0 and 2^1S_0 states predicted in this work are 6.31 and 6.89 GeV, respectively, which are slightly higher than the experimental values. The low-lying WKB masses are broadly comparable with those from the Godfrey-Isgur (GI) model, the modified GI (MGI) model, and the results in the relativistic quark model in Ref. [33], although the present model is not intended to

TABLE I. Comprehensive comparison among the exact QES results with $b = 2$ and $a = 2\sqrt{b}\beta$, numerical FEM results, and WKB approximations for varying radial nodes n_r and orbital angular momenta l .

State (n_r, l)	β	Method	Physical quantities			Relative error (%)		
			Energy E	$ \Psi(0) ^2$	RMS Radius	δE	$\delta \Psi(0) ^2$	δR_{rms}
$n_r = 0, l = 0$	$\frac{1}{2}$	Exact	3.992641	0.723618	0.881227			
		FEM	3.992647	0.709215	0.881228	1.52×10^{-4}	1.99	1.34×10^{-4}
		WKB	3.923429	0.743204	0.875652	1.73	2.71	0.633
$n_r = 0, l = 3$	$\frac{1}{8}$	Exact	12.712297	0	1.741369			
		FEM	12.712300	0	1.741369	1.92×10^{-5}		1.96×10^{-5}
		WKB	12.710982	0	1.741328	0.0103		0.00237
$n_r = 0, l = 6$	$\frac{1}{14}$	Exact	21.208101	0	2.278191			
		FEM	21.208104	0	2.278191	1.05×10^{-5}		1.50×10^{-5}
		WKB	21.207795	0	2.278184	0.00144		3.02×10^{-4}
$n_r = 0, l = 9$	$\frac{1}{20}$	Exact	29.695985	0	2.707405			
		FEM	29.695987	0	2.707405	7.21×10^{-6}		1.00×10^{-5}
		WKB	29.695862	0	2.707402	4.14×10^{-4}		8.34×10^{-5}
$n_r = 3, l = 0$	-1.686704	Exact	9.882950	0.551012	2.592124			
		FEM	9.882983	0.540045	2.592130	3.38×10^{-4}	1.99	2.37×10^{-4}
		WKB	9.847205	0.559709	2.592377	0.362	1.58	0.00975
$n_r = 3, l = 3$	-1.312870	Exact	19.489576	0	2.953067			
		FEM	19.489613	0	2.953072	1.93×10^{-4}		1.72×10^{-4}
		WKB	19.488735	0	2.953021	0.00431		0.00154
$n_r = 3, l = 6$	-1.100886	Exact	28.486534	0	3.280754			
		FEM	28.486575	0	3.280758	1.43×10^{-4}		1.27×10^{-4}
		WKB	28.486453	0	3.280737	2.83×10^{-4}		4.97×10^{-4}
$n_r = 3, l = 9$	-0.966824	Exact	37.249018	0	3.583439			
		FEM	37.249060	0	3.583443	1.15×10^{-4}		1.16×10^{-4}
		WKB	37.249039	0	3.583431	5.80×10^{-5}		2.12×10^{-4}
$n_r = 6, l = 0$	-2.930225	Exact	12.626983	0.536628	3.641682			
		FEM	12.627065	0.525949	3.641693	6.49×10^{-4}	1.99	2.92×10^{-4}
		WKB	12.593548	0.544352	3.642645	0.265	1.44	0.0264
$n_r = 6, l = 3$	-2.378684	Exact	24.040348	0	3.859066			
		FEM	24.040451	0	3.859076	4.31×10^{-4}		2.55×10^{-4}
		WKB	24.039153	0	3.859018	0.00497		0.00126
$n_r = 6, l = 6$	-2.049826	Exact	33.981978	0	4.090185			
		FEM	33.982096	0	4.090194	3.46×10^{-4}		2.29×10^{-4}
		WKB	33.981800	0	4.090162	5.24×10^{-4}		5.44×10^{-4}
$n_r = 6, l = 9$	-1.829721	Exact	43.321170	0	4.321990			
		FEM	43.321296	0	4.322000	2.93×10^{-4}		2.31×10^{-4}
		WKB	43.321171	0	4.321979	3.34×10^{-6}		2.68×10^{-4}
$n_r = 9, l = 0$	-3.890146	Exact	14.565246	0.534598	4.464345			
		FEM	14.565398	0.523961	4.464360	0.00104	1.99	3.28×10^{-4}
		WKB	14.532847	0.541861	4.465652	0.222	1.359	0.0293
$n_r = 9, l = 3$	-3.258419	Exact	27.566473	0	4.612814			
		FEM	27.566667	0	4.612831	7.06×10^{-4}		3.69×10^{-4}
		WKB	27.564952	0	4.612768	0.00552		9.94×10^{-4}
$n_r = 9, l = 6$	-2.864117	Exact	38.465883	0	4.787142			
		FEM	38.466107	0	4.787160	5.82×10^{-4}		3.79×10^{-4}
		WKB	38.465564	0	4.787117	8.30×10^{-4}		5.18×10^{-4}
$n_r = 9, l = 9$	-2.589559	Exact	48.448515	0	4.972654			
		FEM	48.448759	0	4.972671	5.05×10^{-4}		3.29×10^{-4}
		WKB	48.448452	0	4.972640	1.29×10^{-4}		2.80×10^{-4}

TABLE II. The parameters fitted in this work.

Parameter	Value
m_c (GeV)	2.00235
m_b (GeV)	5.38954
τ (GeV ²)	0.140838
ν (GeV ³)	0.00463394
σ (GeV ³)	0.0881383
C_{cc} (GeV)	-1.01759
κ	0.6012

improve the mass descriptions of the GI model and the MGI model. Besides, the masses obtained by the WKB approximation are in good agreement with those computed by the FEM.

Furthermore, the spatial observables, such as the rms radii $\sqrt{\langle r^2 \rangle}$ and the probability density at the origin $|\Psi(0)|^2$, calculated via the WKB approximation are consistent with the numerical FEM results as shown in Table III. For instance, the WKB and FEM values for $|\Psi(0)|^2$ of the 1^1S_0 state are 0.37365 and 0.36509 GeV³, respectively.

In addition, we also predict the masses, the rms radii $\sqrt{\langle r^2 \rangle}$, and the probability density at the origin $|\Psi(0)|^2$ of highly excited states such as P , D , F , G , and H waves in Table IV under the Killingbeck potential. Comparing our mass predictions with other well-established theoretical frameworks, the results for the low-lying S -, P -, and D -wave states are generally consistent with those from the GI model [34], the MGI model [35], and the relativistic quark model in Ref. [33]. For instance, our predicted mass for the 1^3S_1 state (around 6.315 GeV) is slightly lower but very close to the GI and MGI predictions (≈ 6.338 GeV). For the 1^1P_1 and 1^1D_2 states, our results are slightly higher than those of the GI model by approximately 30–40 MeV, which remains within a reasonable theoretical uncertainty range for phenomenological potential models.

It is also noteworthy to discuss the asymptotic behavior of the highly excited energy spectra induced by the Killingbeck potential. Unlike pure linear confinement models (e.g., the Cornell potential), which yield Regge trajectory behavior ($M^2 \propto n + l$) [36,37], the Killingbeck potential explicitly incorporates a harmonic oscillator term (br^2). As the states become highly excited, their geometric sizes expand significantly, as reflected in the increasing rms radii, causing the r^2 term to eventually dominate over the linear r term at large distances. Consequently, the mass spectra of these highly excited states transition towards harmonic-oscillator-like behavior, growing more rapidly with n and l . This intrinsic potential feature explains why our predicted masses for the highest excitations tend to be larger than those obtained from models lacking an r^2 dependence.

Besides the mass spectrum, Table IV lists the rms radii $\sqrt{\langle r^2 \rangle}$ and the squared wave functions at the origin $|\Psi(0)|^2$, providing deeper insights into the internal structure of the B_c mesons. As expected, the rms radius monotonically increases with both the radial quantum number n and the orbital quantum number l . This indicates that highly excited states possess significantly larger geometric sizes, expanding from

≈ 1.32 GeV⁻¹ for the $1S$ state to over 9.36 GeV⁻¹ for the $6H$ state.

Meanwhile, the squared wave function at the origin, $|\Psi(0)|^2$, strictly vanishes for all $l > 0$ states (P , D , F , G , and H waves) due to the centrifugal barrier $l(l+1)/r^2$. For the S -wave states, $|\Psi(0)|^2$ continuously decreases from 0.374 GeV³ ($1S$) down to 0.129 GeV³ ($6S$) as n increases.

B. Mass spectrum of charmonium under the Killingbeck potential

In this part, we obtain the spectroscopy information of the heavy meson $c\bar{c}$ and compare our results with those in Refs. [38,39] and those from the MGI model [40]. Tables III and V present the mass spectrum, $\sqrt{\langle r^2 \rangle}$, and $|\Psi(0)|^2$ for the $c\bar{c}$ system.

As is shown in Table III, the WKB approach yields results that generally match the FEM numerical solutions. In Table V, we systematically present the calculated mass spectra, rms radii, and squared wave functions at the origin for the highly excited charmonium ($c\bar{c}$) states under the Killingbeck potential, focusing on radial and orbital excitations up to $n_r = 6$ and $l = 5$ (H -wave state). Comparing with established theoretical benchmarks, our results show reasonable consistency for several orbital excitations. For example, our predicted mass for the 1^1D_2 state is 3.796 GeV, which is very close to the prediction of 3.799 GeV by Barnes *et al.* [38]. Similarly, the 1^1F_3 state mass (4.032 GeV) lies in agreement with the existing literature values (4.026–4.074 GeV). For certain S -wave and P -wave states, slight deviations are observed, which naturally arise from the specific parametrization of the Killingbeck potential.

A defining feature of this work is the prediction of high-spin and higher-radial excitations that are not fully tabulated in previous literature. As seen in Table V, earlier theoretical models leave extensive blanks for G -wave, H -wave, and $n \geq 4$ states. By contrast, our model provides mass predictions for these multiplets, extending all the way to 6^1H_5 at roughly 5.997 GeV. These predictions offer useful theoretical benchmarks for upcoming experiments.

Regarding the spatial observables, the rms radii and $|\Psi(0)|^2$ in the $c\bar{c}$ system mirror the physical trends observed in the B_c meson system. Due to the lighter mass of the charm quark compared to the bottom quark, the $c\bar{c}$ bound states are generally more extended in space. The rms radius monotonically increases with higher quantum numbers, reaching 10.490 GeV⁻¹ for the 6^1H_5 state.

In the nonrelativistic approximation [8,31],

$$\frac{\Gamma_{ee}(1^3S_1)}{\Gamma_{ee}(n_r^3S_1)} = \frac{E_{n_r^3S_1}^2 |\Psi_{1S}(0)|^2}{E_{1^3S_1}^2 |\Psi_{n_rS}(0)|^2}. \quad (4.1)$$

The measured results $\Gamma_{ee}(J/\psi) = 5.53 \pm 0.1$ keV and $\Gamma_{ee}[\psi(3686)(2S)] = 2.29 \pm 0.06$ keV [32] yield $\frac{\Gamma_{ee}(J/\psi)}{\Gamma_{ee}[\psi(2S)]} = 2.41 \pm 0.08$, while the corresponding WKB estimate is $\frac{\Gamma_{ee}(J/\psi)}{\Gamma_{ee}[\psi(2S)]} = \frac{E_{2S}^2 |R_{1S}(0)|^2}{E_{1S}^2 |R_{2S}(0)|^2} \approx 2.4$. For higher-vector

TABLE III. Mass spectra and physical observables of mesons under the Killingbeck potential. All mass-related values are in MeV. Parameters are adopted from Table II.

					Mass M		$\sqrt{\langle r^2 \rangle}$ [10^{-3} MeV $^{-1}$]		$ \Psi(0) ^2$ [10^8 MeV ^{3}]		
	State	Expt. ^a mass	GI ^b mass	MGI ^c mass	Ref. ^d mass	WKB	FEM	WKB	FEM	WKB	FEM
$c\bar{c}$	1^1S_0	2984.1 ± 0.4	2975	2981	3000	3064	3075	1.6935	1.7901	1.6433	1.5904
	2^1S_0	3637.7 ± 0.9	3623	3642	3670	3659.25	3663.87	3.6549	3.6809	0.9765	0.9534
	1^3S_1	3096.9 ± 0.006	3090	3089	3100	3086.14	3096.76	1.6935	1.7901	1.6433	1.5904
	2^3S_1	3686.1 ± 0.011	3676	3684	3730	3681.23	3685.86	3.6549	3.6809	0.9765	0.9534
	3^3S_1	4039.6 ± 4.3	4100	4035	4180	4110.62	4114.22	5.1096	5.1248	0.8481	0.8293
	4^3S_1	4415 ± 5	4450	4274		4486.90	4489.73	6.3201	6.3302	0.7957	0.7790
	5^3S_1	4641 ± 10	4756	4443	4560	4833.88	4836.93	7.3783	7.3877	0.7687	0.7535
	1^1P_1	3525.4 ± 0.14	3516	3538	3510	3514.60	3515.88	2.9498	2.9666	0	0
$b\bar{b}$	1^1S_0	9398.7 ± 2.0	9402	9398	9410	9389.49	9397.96	0.8224	0.9241	16.914	16.798
	2^1S_0	9999.0 ± 3.5	9976	9989	10000	10034.9	10039.3	2.2958	2.3252	5.7775	5.7077
	1^3S_1	9460.4 ± 0.09	9465	9463	9460	9392.53	9401.03	0.8224	0.9241	16.914	16.798
	2^3S_1	10023 ± 0.5	10003	10017	10020	10038.0	10042.4	2.2958	2.3252	5.7775	5.7077
	3^3S_1	10355 ± 0.5	10354	10356	10390	10387.7	10389.1	3.4641	3.4749	4.3039	4.2466
	4^3S_1	10579 ± 1.2	10635	10612	10680	10666.9	10670.0	4.4157	4.4288	3.7569	3.6928
	5^3S_1	10885 ± 2.6	10878	10822	10930	10917.8	10920.3	5.2505	5.2602	3.4626	3.3975
	6^3S_1	11000 ± 4	11102	11001	11150	11149.4	11152.5	5.9972	6.0075	3.2788	3.2126
	1^1P_1	9899.3 ± 0.8	9894	9881	9880	9960.78	9962.16	1.8677	1.8880	0	0
B_c	1^1S_0	6274.5 ± 0.32	6271	6271	6260	6306.99	6317.51	1.3237	1.4286	3.7369	3.6509
	2^1S_0	6871.2 ± 1.0	6855	6855	6850	6893.84	6898.26	3.1073	3.1345	1.8549	1.8162
$\chi^2_{\text{err}} = \sum_i \left[\frac{(M_{\text{th}} - M_{\text{expt}})}{\text{err}_i} \right]^2$			2.18×10^6	1.78×10^6	1.62×10^7	4.06×10^6	5.33×10^5				

^aData from Particle Data Group [32].^bResults from the GI model [34,38,41,42].^cResults from the MGI model [35,40,43]. In Ref. [40], the $\psi(4230)$ resonance, denoted $Y(4220)$ there, is associated with $\psi(4S)$ within a $4S$ - $3D$ mixing scheme.^dValues reported in Ref. [33].

charmonia, the electronic widths are further reduced to the sub-keV level, such as $\psi(4040)$, $\psi(4160)$, and $\psi(4415)$, $\Gamma_{ee}[\psi(3S)] = 0.83 \pm 0.2$ keV and $\Gamma_{ee}[\psi(4S)] = 0.35 \pm 0.12$ keV [32], which give $\frac{\Gamma_{ee}(J/\psi)}{\Gamma_{ee}[\psi(3S)]} = 6.7 \pm 1.6$ and $\frac{\Gamma_{ee}(J/\psi)}{\Gamma_{ee}[\psi(4S)]} = 15.8 \pm 5.4$, whereas our WKB estimates give $\frac{\Gamma_{ee}(J/\psi)}{\Gamma_{ee}[\psi(3S)]} \approx 3.4$ and $\frac{\Gamma_{ee}(J/\psi)}{\Gamma_{ee}[\psi(4S)]} \approx 4.1$. The comparison is consistent with identifying $\psi(3686)$ as the first radial excitation of J/ψ , while the larger deviations for $\psi(4040)$ and $\psi(4415)$ may indicate non-negligible S - D mixing. This behavior also implies suppressed short-distance annihilation amplitudes for highly excited charmonium states. The values of $|\Psi(0)|^2$ for higher S -wave states are predicted in Table V.

C. Mass spectrum of bottomonium under the Killingbeck potential

In recent years, improved collision energies and detector resolutions have led to more precise information on bottomonium spectroscopy. We therefore show the spectroscopy results of $b\bar{b}$ and provide predictions for states that have not yet been established experimentally in Tables III and VI.

This system has more experimental information and theoretical reference values [41–43] than B_c and $c\bar{c}$ mesons. The authors of Refs. [40,43] take the same experimental values as those we use, which facilitates the work of the comparison and verification.

In Table VI, we tabulate the theoretical predictions for the masses, rms radii, and squared wave functions at the origin for the highly excited bottomonium ($b\bar{b}$) states under the Killingbeck potential. Comparing our calculated masses with existing literature, such as the classic GI model [41], the updated Godfrey results [42], and the MGI model [43], we observe reasonable and systematic consistency. For relatively lower excited states within this table, such as the 3^1S_0 and 3^1P_1 states, our WKB predictions (10.383 and 10.593 GeV, respectively) are slightly higher than those of the GI and MGI models by approximately 40–50 MeV. This minor mass shift is a consequence of the harmonic oscillator term affecting the confinement potential at large distances.

A useful contribution of the present work is the prediction of high-spin and high-radial excitation states, many of which are absent or incomplete in previous theoretical frameworks. Notably, we predict the masses of the F -, G -, and H -wave multiplets, extending up to 6^1H_5 at 11.773 GeV. These predictions provide theoretical benchmarks for ongoing and future experimental searches at high-luminosity facilities such as Belle II and LHCb.

The calculated physical observables in the bottomonium sector also follow the established monotonic trends. Because b quarks are much heavier, these states are more tightly bound, yielding smaller rms radii overall compared to charmonium. Nevertheless, the geometric size still expands significantly up

TABLE IV. Comprehensive mass spectrum and physical observables of the B_c mesons under the Killingbeck potential. Masses are given in GeV, rms radii in GeV^{-1} , and the probability density at the origin in GeV^3 .

State	Mass M (GeV)				rms $\sqrt{\langle r^2 \rangle}$ WKB	$ \Psi(0) ^2$ WKB
	GI [34]	Ref. [33]	MGI [35]	WKB		
1^3S_1	6.338	6.340	6.338	6.3152	1.3239	0.37365
3^1S_0	7.253	7.240	7.250	7.2832	4.4393	0.15333
3^3S_1	7.279	7.240	7.272	7.2914	4.4393	0.15333
4^1S_0	7.572		7.572	7.6166	5.5437	0.14016
4^3S_1	7.595		7.588	7.6247	5.5437	0.14016
5^1S_0			7.854	7.9212	6.5092	0.13307
5^3S_1			7.860	7.9294	6.5092	0.13307
6^1S_0				8.2079	7.3788	0.12872
6^3S_1				8.2160	7.3788	0.12872
1^1P_1	6.741	6.730	6.745	6.7789	2.5117	0
2^1P_1	7.145	7.140	7.125	7.1744	3.9593	0
3^1P_1	7.472		7.414	7.5113	5.1242	0
4^1P_1			7.650	7.8183	6.1296	0
5^1P_1				8.1068	7.0285	0
6^1P_1				8.3821	7.8497	0
1^1D_2	7.036	7.020	7.032	7.0431	3.3332	0
2^1D_2			7.335	7.3869	4.6097	0
3^1D_2			7.581	7.6986	5.6791	0
4^1D_2				7.9904	6.6215	0
5^1D_2				8.2684	7.4749	0
6^1D_2				8.5361	8.2611	0
1^1F_3	7.266	7.240	7.248	7.2567	4.0167	0
2^1F_3			7.505	7.5739	5.1784	0
3^1F_3				7.8696	6.1780	0
4^1F_3				8.1505	7.0715	0
5^1F_3				8.4206	7.8881	0
6^1F_3				8.6823	8.6454	0
1^1G_4				7.4468	4.6208	0
2^1G_4				7.7467	5.6968	0
3^1G_4				8.0309	6.6406	0
4^1G_4				8.3035	7.4937	0
5^1G_4				8.5672	8.2791	0
6^1G_4				8.8238	9.0113	0
1^1H_5				7.6227	5.1706	0
2^1H_5				7.9102	6.1788	0
3^1H_5				8.1854	7.0761	0
4^1H_5				8.4513	7.8946	0
5^1H_5				8.7098	8.6527	0
6^1H_5				8.9620	9.3626	0

to 7.752 GeV^{-1} for the highly excited 6^1H_5 state. The behavior of $|\Psi(0)|^2$ remains strictly governed by the centrifugal barrier for $l > 0$ states, while the S -wave probability densities systematically dilute with increasing radial nodes. The experimental electronic widths [32] are

$$\begin{aligned}
 \Gamma_{ee}[\Upsilon(1S)] &= 1.34 \pm 0.018 \text{ keV}, \\
 \Gamma_{ee}[\Upsilon(2S)] &= 0.612 \pm 0.011 \text{ keV}, \\
 \Gamma_{ee}[\Upsilon(3S)] &= 0.443 \pm 0.008 \text{ keV}, \\
 \Gamma_{ee}[\Upsilon(4S)] &= 0.272 \pm 0.029 \text{ keV}, \\
 \Gamma_{ee}[\Upsilon(5S)] &= 0.31 \pm 0.07 \text{ keV}, \\
 \Gamma_{ee}[\Upsilon(6S)] &= 0.130 \pm 0.030 \text{ keV}.
 \end{aligned} \tag{4.2}$$

These values give

$$\begin{aligned}
 \frac{\Gamma_{ee}[\Upsilon(1S)]}{\Gamma_{ee}[\Upsilon(2S)]} &= 2.19 \pm 0.05, \\
 \frac{\Gamma_{ee}[\Upsilon(1S)]}{\Gamma_{ee}[\Upsilon(3S)]} &= 3.02 \pm 0.07, \\
 \frac{\Gamma_{ee}[\Upsilon(1S)]}{\Gamma_{ee}[\Upsilon(4S)]} &= 4.93 \pm 0.53, \\
 \frac{\Gamma_{ee}[\Upsilon(1S)]}{\Gamma_{ee}[\Upsilon(5S)]} &= 4.32 \pm 0.98, \\
 \frac{\Gamma_{ee}[\Upsilon(1S)]}{\Gamma_{ee}[\Upsilon(6S)]} &= 10.31 \pm 2.38.
 \end{aligned} \tag{4.3}$$

TABLE V. Comprehensive mass spectrum and physical observables of the charmonium ($c\bar{c}$) mesons under the Killingbeck potential. Masses are given in GeV, rms radii in GeV^{-1} , and probability density at the origin in GeV^3 . All values are rounded to five significant figures.

State	Mass M (GeV)				rms $\sqrt{\langle r^2 \rangle}$ WKB	$ \Psi(0) ^2$ WKB
	Ref. [38]	Ref. [39]	MGI [40]	WKB		
3^1S_0	4.043	4.072	4.013	4.0887	5.1098	0.084812
4^1S_0	4.384	4.420	4.260	4.4647	6.3195	0.079569
5^1S_0			4.433	4.8122	7.3792	0.076869
6^1S_0				5.1413	8.3350	0.075338
6^3S_1				5.1633	8.3350	0.075338
2^1P_1	3.934	3.945	3.933	3.9527	4.5465	0
3^1P_1	4.279	4.334	4.200	4.3336	5.8303	0
4^1P_1		4.639	4.389	4.6844	6.9388	0
5^1P_1				5.0160	7.9302	0
6^1P_1				5.3340	8.8358	0
1^1D_2	3.799	3.807	3.848	3.7960	3.8379	0
2^1D_2	4.158	4.174	4.137	4.1850	5.2487	0
3^1D_2		4.560	4.343	4.5413	6.4304	0
4^1D_2			4.490	4.8769	7.4715	0
5^1D_2				5.1979	8.4140	0
6^1D_2				5.5081	9.2822	0
1^1F_3	4.026	4.041	4.074	4.0315	4.5843	0
2^1F_3	4.350	4.372	4.296	4.3939	5.8696	0
3^1F_3		4.746		4.7338	6.9751	0
4^1F_3				5.0583	7.9630	0
5^1F_3				5.3711	8.8655	0
6^1F_3				5.6750	9.7022	0
1^1G_4	4.225	4.247	4.250	4.2445	5.2475	0
2^1G_4				4.5891	6.4383	0
3^1G_4				4.9171	7.4825	0
4^1G_4				5.2328	8.4260	0
5^1G_4				5.5390	9.2943	0
6^1G_4				5.8375	10.103	0
1^1H_5				4.4434	5.8528	0
2^1H_5				4.7751	6.9686	0
3^1H_5				5.0938	7.9615	0
4^1H_5				5.4023	8.8667	0
5^1H_5				5.7029	9.7049	0
6^1H_5				5.9968	10.490	0

The corresponding theoretical ratios obtained from the predicted $|\Psi(0)|^2$ values in Table VI are 3.29, 4.71, 5.63, 6.44, and 6.99 for the $2S$, $3S$, $4S$, $5S$, and $6S$ states, respectively. Although the agreement is not exact for all highly excited states, the theoretical results reproduce the overall increasing trend of the experimental ratios. This supports the physical picture that the short-distance wave-function density, and hence the electronic annihilation amplitude, is progressively suppressed for higher bottomonium S -wave excitations.

Finally, a comprehensive summary and comparison of the low-lying mass spectra and spatial observables for heavy quarkonia ($c\bar{c}$, $b\bar{b}$) and the B_c meson are presented in Tables III–VI. Regarding phenomenological applications, the current calculations based on the basic Killingbeck potential exhibit some deviations from the experimental data concerning the absolute energy levels of the lowest 1^1S_0 states and the hyperfine mass splittings between the 1^1S_0 and 3^1S_1 states. These deviations are more pronounced when compared to

the predictions of the more sophisticated GI or MGI models [41,43], which can be attributed to the relatively simplified spin-spin contact interaction term adopted in the present Hamiltonian.

Nevertheless, by expanding the framework to evaluate spatial observables alongside the energy spectra, we provide a more comprehensive test of the wave functions. The explicit derivation and calculation of $|\Psi(0)|^2$ throughout our tables offer direct analytical inputs that could be utilized for predicting leptonic decay widths and electromagnetic transition rates. This highlights the potential of the WKB method as a practical and transparent analytical tool for investigating heavy-quarkonium physics.

V. DISCUSSION AND CONCLUSION

In this work, we have investigated the WKB approximation for the Killingbeck potential and applied it to the mass

TABLE VI. Comprehensive mass spectrum and physical observables of the bottomonium ($b\bar{b}$) mesons under the Killingbeck potential. Masses are given in GeV, rms radii in GeV^{-1} , and the probability density at the origin in GeV^3 .

State	Mass M (GeV)				$\text{rms}\sqrt{\langle r^2 \rangle}$ WKB	$ \Psi(0) ^2$ WKB
	Ref. [42]	GI [41]	MGI [43]	WKB		
3^1S_0	10.336	10.333	10.336	10.383	3.4578	0.43087
4^1S_0	10.623	10.616	10.597	10.664	4.4159	0.37568
5^1S_0	10.869	10.806	10.810	10.915	5.2497	0.34628
6^1S_0	11.097	11.079	10.991	11.147	5.9985	0.32785
3^1P_1	10.541	10.540	10.530	10.593	4.1017	0
4^1P_1	10.790	10.790	10.751	10.845	4.9623	0
5^1P_1	11.016	11.013	10.938	11.078	5.7310	0
6^1P_1		11.218	11.101	11.298	6.4326	0
1^1D_2	10.148	10.148	10.163	10.214	2.5999	0
2^1D_2	10.450	10.450	10.450	10.502	3.6871	0
3^1D_2	10.706	10.706	10.681	10.757	4.5980	0
4^1D_2	10.935	10.934	10.876	10.992	5.4006	0
5^1D_2		11.143	11.046	11.215	6.1275	0
6^1D_2				11.427	6.7972	0
1^1F_3	10.355	10.354	10.366	10.403	3.1982	0
2^1F_3	10.619	10.619	10.609	10.662	4.1852	0
3^1F_3	10.853	10.853	10.812	10.901	5.0345	0
4^1F_3		11.066	10.988	11.126	5.7940	0
5^1F_3				11.341	6.4884	0
6^1F_3				11.547	7.1325	0
1^1G_4	10.530	10.530	10.534	10.565	3.7209	0
2^1G_4	10.770	10.770	10.747	10.807	4.6341	0
3^1G_4		10.988	10.929	11.035	5.4351	0
4^1G_4				11.251	6.1595	0
5^1G_4				11.460	6.8268	0
6^1G_4				11.662	7.4491	0
1^1H_5				10.712	4.1937	0
2^1H_5				10.942	5.0487	0
3^1H_5				11.161	5.8100	0
4^1H_5				11.371	6.5047	0
5^1H_5				11.575	7.1485	0
6^1H_5				11.773	7.7517	0

spectra and spatial observables of double-heavy mesons. By extracting analytical quantization conditions and algebraic expectation values, we reduced the reliance on direct numerical integration and provided a more stable semianalytical way to track highly excited states.

The WKB method gives reliable semiquantitative results in the heavy-quarkonium sector; notably, the truncation error generally decreases as the radial node n_r or the orbital angular momentum l increases. This asymptotic convergence supports the applicability of the Bohr-Sommerfeld quantization to highly excited hadron states.

Phenomenologically, our mass predictions for the low-lying S -, P -, and D -wave states are in reasonable agreement with experimental data and established theoretical models, such as the GI model and the MGI model. More importantly, we systematically mapped out the high-spin and highly excited multiplets (e.g., the F -, G -, and H -waves up to $n_r = 6$), which have not been systematically tabulated in many previous studies.

A distinctive physical feature of our results arises from the intrinsic asymptotic behavior of the Killingbeck potential. Unlike purely linear confinement models that produce standard Regge trajectories ($M^2 \propto n_r + l$), the inclusion of the harmonic oscillator term (br^2) modulates the long-range interaction. For spatially extended highly excited states, the r^2 term progressively dominates over the linear confinement, causing the mass spectrum to grow more rapidly and transition toward a harmonic oscillatorlike behavior.

Furthermore, the evaluation of spatial observables provides insights into the internal dynamics of these heavy mesons. The calculated rms radii ($\sqrt{\langle r^2 \rangle}$) demonstrate a monotonic expansion with increasing quantum numbers. For S -wave states, the probability density at the origin systematically decreases with increasing radial nodes due to spatial dispersion. The determination of $|\Psi(0)|^2$ throughout our tables offers analytical inputs for predicting leptonic decay constants, annihilation widths, and electromagnetic transition rates.

We acknowledge that the current calculations exhibit some deviations concerning the absolute energy levels of the ground states (e.g., 1^1S_0) and the corresponding hyperfine mass splittings when compared to the more sophisticated GI and MGI models. These discrepancies can be primarily attributed to the simplified contact spin-spin interaction adopted in the present effective Hamiltonian, which may require further relativistic or perturbative quantum chromodynamics corrections for deep-binding regimes.

Nevertheless, by unifying the energy-spectrum calculation with the derivation of spatial observables, this work demonstrates the potential of the WKB method as a transparent and practical analytical tool. The predictions for higher bottomonium, charmonium, and B_c states provided here can serve as useful theoretical references for ongoing and future experimental searches at major high-energy facilities. We expect that

our work will contribute to the understanding of the role of the quark interaction potential in highly excited states and hope that it is helpful to study and establish the mass spectrum of the double-heavy-meson family.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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