

# Impossibility of obtaining time-independent, three-dimensional, spherically symmetric densities of confined systems of relativistically moving constituents

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The quantum-mechanical definition of probability, the uncertainty principle, and Poincaré invariance provide strong basic restrictions on the ability to define spatial densities associated with form factors describing the properties of confined systems of relativistically moving constituents. Despite this, many papers ignore one or more of these restrictions. Here I show how to obtain *time-independent*, two-dimensional densities that are consistent with the stated restrictions. This is done using the light-front, infinite momentum frame formalism. Two-dimensional density interpretations of the axial-vector form factor and all three gravitational form factors is obtained. The resulting mass radius is smaller than the charge radius. Additionally, an expression of a two-dimensional mass density related to the trace of the energy momentum tensor is obtained. I also show that all known methods for finding three-dimensional densities—using the Breit frame, Abel transformations, Wigner distributions, and spherically symmetric wave packets with vanishing spatial extent—violate the basic restrictions in different ways. Furthermore, the use of the latter leads to densities that vanish almost everywhere in space as time increases from an initial value.

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## I. INTRODUCTION

Due to recent exciting experimental developments, see the review in Ref. [1] and significant advances in lattice QCD calculations [2], interest in gravitational form factors of hadrons and the related spatial densities has grown. Motivation for this interest continues to originate from the attempts to understand the proton mass puzzle [3–8] and the proton spin puzzle [9–11].

Matrix elements of the EMT between plane-wave states define gravitational form factors [12] that provide information about both the quark-gluon decomposition and the spatial distribution of energy, momentum, and angular momentum. Those form factors are in principle measurable using reactions such as deeply virtual Compton scattering [13–15] at facilities such as Jefferson Lab [16–18] and the upcoming Electron Ion Collider [19], as well as in  $\gamma\gamma^* \rightarrow \text{hadrons}$  at Belle [20]. Such studies complement the long-standing efforts, see, e.g., the reviews in Refs. [21–23], to obtain spatial densities of electromagnetic properties.

A large number of papers have been and are devoted to obtaining time-independent densities that obey spherical symmetry, meaning functions only of the spherical length coordinate  $r$ . The principal statement of the present paper is that such densities can only be obtained for nonrelativistic systems—those that obey invariance under Galileo trans-

formations. It is worthwhile to define nonrelativistic in the present context: The constituents move at speeds such that kinetic energies may be expressed as  $\frac{p^2}{2m}$  and that the hadronic wave function is normalized as unity (instead of, e.g., as twice the energy).

For some hadrons the constituents move at speeds large enough so that Galileo transformations are not relevant. Nucleons, composed predominantly of  $u$  and  $d$  quarks with very small masses, provide the most important example. In such cases, one must consider invariance under the Poincaré group instead of that of Galileo transformations. The Poincaré group contains Lorentz transformations and rotations. A related problem is that form factors depend on Lorentz-invariant quantities, but the spatial coordinate  $r$  is part of a four-vector. A relativistic treatment of densities demands a consistent treatment of the time variable as well as the spatial variable. The net result is that it not possible to legitimately obtain a density that depends on  $r$  and is independent of time. This paper is devoted to explaining, in an indisputable fashion, the meaning of “legitimate” that appears in the previous sentence.

I begin at the beginning with the postulate that probability density is determined by the absolute square of a wave function  $|\Psi|^2 = \Psi^*\Psi$ . The key point is that  $\Psi^*$  is the complex conjugate of  $\Psi$ . The *same* wave function appears twice.

If a system obeys Galilean invariance, then the Hamiltonian can be expressed as a sum of two separate terms, one involving internal coordinates and the other describing the motion of the center of mass [24]. In that case, the wave function factorizes into a product of a wave function depending on internal, relative coordinate times a wave function depending on the motion of the center of mass.

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Next consider a Galileo-invariant system probed with a momentum transfer,  $\Delta$ . The action of the probe can be separated into terms depending on internal coordinates and the center of mass, and the resulting transition amplitude is proportional to the three-dimensional Fourier transform of an appropriate density. This elegant result has been ingrained in the physics consciousness of many physicists since the Nobel prize winning work of Hofstadter [25,26]. Initial applications were to the charge densities of nuclei, for which the treatment is valid [27]. But problems with the interpretation of proton form factors as three-dimensional Fourier transforms of densities, arising from relativistic effects, were pointed out almost immediately by Yennie, Lévy, and Ravenhall [28]. A modern perspective regarding the nonrelativistic limit is given by Freese [29].

The quarks inside the proton and neutron move at speeds such that effects of special relativity are important [27] so that Galileo invariance must be replaced by Poincaré invariance. The Poincaré group comprises all geometric transformations that leave the quadratic form  $s^2(x_1, x_2) = c^2(t_1^2 - t_2^2) - (\mathbf{r}_1 - \mathbf{r}_2)^2$  invariant [30]. These transformations include rotations, Lorentz transformations, or translations. The Lorentz group accounts for three pure Lorentz transformations and three rotations. The group of space-time translations has four generators, so that the Poincaré group consists of 10 generators. In the usual instant form, translation generators are said to be dynamic because they contain interactions [31]. The front form presents an advantage because only three translation generators contain interactions [31,32]. There is an additional kinematic translation generator. This allows an important result to be obtained: The front form “is shown to have Galilean invariance with respect to motions transverse to the direction” of the front [33]. So there is two-dimensional Galileo invariance and two-dimensional densities may legitimately be obtained from two-dimensional Fourier transforms of form factors under two conditions.

The first condition involves the ability to define a position variable. For example, the proton to neutron axial-vector current matrix element is given by

$$\langle p', \lambda' | A_+^\mu(0) | p, \lambda \rangle_n = \bar{u}_p(p', \lambda) [G_A(\Delta^2) \gamma^\mu \gamma_5 + \frac{G_P(\Delta^2)}{2M} \gamma_5 \Delta^\mu] u_n(p, \lambda), \quad (1)$$

where  $\Delta^\mu = p'^\mu - p^\mu$  if G parity is conserved. The related two-dimensional densities are discussed in Sec. II. The difficulty is that the space-time position (0) can be anywhere, everywhere, and anywhen. Plane-wave states, with equal probability to exist anywhere in space, appear in Eq. (1) so that defining an origin is problematic. The spread in momentum is zero, so according to the uncertainty principle, the spread in coordinate space is infinite. One may instead use a wave packet that allows a definite position in coordinate space to be defined.

The second condition is that only kinematic boosts, those in the transverse direction to that of the light-front defined above, must be used. The light front coordinates are defined so that  $x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^3)$  and  $A \cdot B = A^+ B^- + A^- B^+ - \mathbf{A}_\perp \cdot \mathbf{B}_\perp$ . I choose  $\sqrt{2}\Delta^+ = \Delta^0 + \Delta^3$  to vanish, the Drell-Yan

frame [34]. This condition may be used for spacelike values of  $\Delta$ . If  $\Delta^+ = 0$ , then  $\Delta^2 = -\Delta_\perp^2$ , and one may define an appropriate wave packet [35,36]. This means that the wave packet must be a superposition of nucleon states with the same value of the plus component of momentum for  $\Psi$  and  $\Psi^*$ .

Here is an outline of the remainder of this paper. Section II shows how to obtain time-independent two-dimensional densities and presents illustrative examples. Section III explains why the widely used Breit-frame interpretation is severely flawed because it does not satisfy the property that the density is the absolute square of a wave function [37] and because the uncertainty principle prevents a meaningful definition of position. Section IV shows how the use of the Wigner distribution does not alleviate the problems associated with using the Breit frame because the fluctuations in both position and momentum are infinite. Section V explains that using the Abel transformation does not generally lead to a correct three-dimensional density [38] because of length contraction effects for moving systems. Section VI shows that the use of wave packets that are spherically symmetric and of vanishing spatial extent leads to densities that vanish for times  $t > r/c$  and is inconsistent with standard calculations of form factors of relativistic systems. Section VII summarizes this paper.

## II. LIGHT FRONT EXAMPLES OF TIME-INDEPENDENT DENSITIES

I begin this section with some background information. Much of this is known, see, e.g., Ref. [36], but the treatment of the time ( $x^+$ ) dependence is new, and I start by explaining the key point.

The sole advantage of using the Breit frame (Sec. III) is that the transferred energy vanishes, so that the four-momentum transfer becomes effectively a three-momentum transfer. The same feature is obtained by using the infinite momentum frame along with the Drell-Yan frame. To see this, note that in light-front coordinates the Einstein energy-momentum relation for a free particle of four-momentum  $p^\mu$  is given by

$$p^- = \frac{\mathbf{p}_\perp^2 + M^2}{2p^+}, \quad (2)$$

with  $p^-$  playing the role of the energy. In the Drell-Yan frame only the perpendicular component of the target momentum is changed. Thus, by taking the limit  $p^+ \rightarrow \infty$  the transferred energy vanishes. This is one example of Feynman’s statement that “moving at large momentum  $\cdots$  (interaction) times are dilated by the relativistic transformation so as  $P$  rises things change more and more slowly” in Lecture 27 of Ref. [39].

The limit must be taken with care, with  $\mathbf{p}_\perp$  remaining finite. The necessary procedure will be shown below.

Many examples of two-dimensional densities that arise from using the infinite momentum frame along with the Drell-Yan frame have appeared, including Refs. [40–66]. The focus here is on obtaining time-independent results and presenting two examples that have not yet been discussed. These are the axial-vector density and a mass density defined from the energy momentum tensor.

Spatial densities in quantum field theories are given by expectation values of local operators. Given a local op-

erator  $\hat{O}(x)$ —such as the electromagnetic current or the energy-momentum tensor—and a physically realizable single-hadron state  $|\Psi\rangle$ , the relevant density is given by  $\langle\Psi|\hat{O}(x)|\Psi\rangle$ .

Form factors,  $F_O$ , are associated with the local operator  $\hat{O}(x)$  through the matrix element between one-particle plane-wave states  $\langle p', \lambda' | \hat{O}(0) | p, \lambda \rangle$  for spin-1/2 nucleons. The connection between these two matrix elements is obtained by using the relation

$$\hat{O}(x) = e^{i\hat{P}\cdot x} \hat{O}(0) e^{-i\hat{P}\cdot x}, \quad (3)$$

and evaluating matrix elements using a basis of plane-wave states that are eigenfunctions of  $\hat{P}$ , the four-momentum operator.

The time ( $x^+$ ) dependence is kept in addition to the usual spatial dependence. Using light front coordinates gives

$$\hat{P}\cdot x = \hat{P}^- x^+ + \hat{P}^+ x^- - \hat{\mathbf{P}}_\perp \cdot \mathbf{x}_\perp. \quad (4)$$

I use a change of variables  $P = \frac{1}{2}(p + p')$  and  $\Delta = p' - p$ , with  $\Delta^+ = 0$ , twice use the completeness relation for single-nucleon states,

$$\sum_\lambda \int \frac{d^4 p}{2p^+ (2\pi)^3} |p^+, \mathbf{p}, \lambda\rangle \langle p^+, \mathbf{p}_\perp, \lambda| = 1, \quad (5)$$

and then evaluate  $\langle p', \lambda' | \hat{O}(x) | p, \lambda \rangle$ . This procedure leads to

$$\begin{aligned} & \int dx^- \langle\Psi|\hat{O}(x^+, x^-, \mathbf{x}_\perp)|\Psi\rangle \\ &= \sum_{\lambda\lambda'} \int \frac{d^4 P}{2P^+ (2\pi)^3} \int \frac{d^2 \Delta_\perp}{(2\pi)^2} e^{i\frac{\mathbf{P}_\perp \cdot \Delta_\perp x^+}{P^+}} \langle\Psi|P^+, \mathbf{p}'_\perp, \lambda'\rangle \\ & \times \frac{\langle P^+, \mathbf{p}'_\perp, \lambda' | \hat{O}(0) | P^+, \mathbf{p}_\perp, \lambda \rangle}{2P^+} \langle P^+, \mathbf{p}_\perp, \lambda | \Psi \rangle e^{-i\Delta_\perp \cdot \mathbf{x}_\perp}, \end{aligned} \quad (6)$$

after integrating over all values of  $x^-$ , necessary because boosts in the longitudinal direction are dynamic. This equation is the general space-time expression for an arbitrary nucleon state  $|\Psi\rangle$ . For such a general state, the density associated with  $\hat{O}(x)$  depends not only on the hadron's internal structure but also on its wave-function spread.

A wave function that is completely localized at the origin is needed to obtain a density depending only on internal structure. Localization may be achieved by considering wave packets with arbitrarily narrow width in the position representation, though this width must be kept nonzero until after all other calculations have been performed [37]. A specific example—which is also a  $P^+$  plane-wave state, and a state with definite light front helicity  $\lambda$ —is given by:

$$\begin{aligned} & \langle p^+, \mathbf{p}_\perp, \lambda | \Psi \rangle \\ &= \sqrt{2\pi} (2\sigma) e^{-\sigma^2 \mathbf{p}_\perp^2} \sqrt{2p^+ (2\pi)} \delta_\epsilon(p^+ - P^+) \delta_{\lambda\Lambda}, \end{aligned} \quad (7)$$

where  $\delta_\epsilon(x)$  is an arbitrary representation of the Dirac delta function (such that  $\delta_\epsilon(x) \rightarrow \delta(x)$  as  $\epsilon \rightarrow 0$ ). The  $\sigma \rightarrow 0$  limit should not be taken until all momentum integrals have been performed.

Using the wave function of Eq. (7) in Eq. (6) gives:

$$\begin{aligned} & \int dx^- \langle\Psi|\hat{O}(x^+, x^-, \mathbf{x}_\perp)|\Psi\rangle \\ &= (2\pi)(2\sigma)^2 \int \frac{d^2 \mathbf{P}_\perp}{(2\pi)^2} e^{-2\sigma^2 \mathbf{P}_\perp^2} \int \frac{d^2 \Delta_\perp}{(2\pi)^2} e^{i\frac{\mathbf{P}_\perp \cdot \Delta_\perp x^+}{P^+}} \\ & \times \frac{\langle P^+, \mathbf{p}'_\perp, \Lambda | \hat{O}(0) | P^+, \mathbf{p}_\perp, \Lambda \rangle}{2P^+} e^{-i\Delta_\perp \cdot \mathbf{x}_\perp} e^{-\frac{\sigma^2}{2} \Delta_\perp^2}, \end{aligned} \quad (8)$$

the most general result for the  $x^-$ -integrated density at  $(x^+, \mathbf{x}_\perp)$ .

Note that the  $x^+$  dependence is contained in the factor  $e^{i\frac{\mathbf{P}_\perp \cdot \Delta_\perp x^+}{P^+}}$ , as expected. This and the Gaussian limit on the magnitude of  $\mathbf{P}_\perp$  suggests that the necessary frame to observe a time-independent density is the infinite momentum frame with  $p^+, p'^+, P^+ \rightarrow \infty$ .

Therefore I take the limit of the infinite momentum frame ( $P^+ \rightarrow \infty$ ) to find:

$$\begin{aligned} & \int dx^- \langle\Psi|\hat{O}(x^+, x^-, \mathbf{x}_\perp)|\Psi\rangle_{P^+ \rightarrow \infty} \equiv \rho_O(\mathbf{x}_\perp) \\ &= (2\pi)(2\sigma)^2 \int \frac{d^2 \mathbf{P}_\perp}{(2\pi)^2} e^{-2\sigma^2 \mathbf{P}_\perp^2} \\ & \times \int \frac{d^2 \Delta_\perp}{(2\pi)^2} \frac{\langle P^+, \mathbf{p}'_\perp, \Lambda | \hat{O}(0) | P^+, \mathbf{p}_\perp, \Lambda \rangle}{2P^+} e^{-\frac{\sigma^2}{2} \Delta_\perp^2}. \end{aligned} \quad (9)$$

The time- $(x^+)$  dependence has disappeared. The coordinate  $x^+$  does not appear in the remainder of this paper.

The ability to proceed from Eq. (9) depends on the form of the specific matrix element  $\langle P^+, \mathbf{p}'_\perp, \lambda' | \hat{O}(0) | P^+, \mathbf{p}_\perp, \lambda \rangle$ . If there are no such factors of  $\mathbf{P}_\perp$ , then doing the  $\mathbf{P}_\perp$  integration gives the result:

$$\begin{aligned} & \rho_O(\mathbf{x}_\perp) \\ &= \int \frac{d^2 \Delta_\perp}{(2\pi)^2} \frac{\langle P^+, \mathbf{p}'_\perp, \Lambda | \hat{O}(0) | P^+, \mathbf{p}_\perp, \Lambda \rangle}{2P^+} e^{-i\Delta_\perp \cdot \mathbf{x}_\perp} e^{-\frac{\sigma^2}{2} \Delta_\perp^2}. \end{aligned} \quad (10)$$

This general density is a two-dimensional Fourier transform of a form factor,

$$F_O(\Delta) = \frac{\langle P^+, \mathbf{p}'_\perp, \Lambda | \hat{O}(0) | P^+, \mathbf{p}_\perp, \Lambda \rangle}{2P^+}. \quad (11)$$

If  $F_O$  depends only on the magnitude of  $\Delta_\perp$ , then one may define a mean-squared transverse radius,

$$\langle \mathbf{x}_\perp^2 \rangle_O = \int d^2 \mathbf{x}_\perp x_\perp^2 \rho_O(\mathbf{x}_\perp) \quad (12)$$

$$= -\frac{1}{4} \frac{d}{d\Delta_\perp^2} F_O(\Delta_\perp^2). \quad (13)$$

The mean-squared transverse size  $\langle \mathbf{x}_\perp^2 \rangle_O$  is a moment of a well-defined density, unlike the  $\langle r^2 \rangle$  of the Breit frame. The transverse squared radii related to a given form factor are two-thirds of the oft-used spherical radii. An example of Eq. (12) was derived in 1968 [40].

On the other hand, if the matrix element of  $O$  contains factors of  $\mathbf{P}_\perp$ , then the resulting integral could contain a factor

$\sigma^{-1}$  or worse, causing the result to diverge in the  $\sigma \rightarrow 0$  limit. This essentially occurs because of the uncertainty principle: An observable that depends on  $\mathbf{P}_\perp$  cannot be well defined in the limit of the hadron's transverse position being arbitrarily well known.

Two sets of examples of densities that depend only on the internal structure of hadrons (do not depend on the wave packet) are provided in the remainder of this section.

### A. Transverse axial current densities

Given the preceding background, proceed to the case where  $\hat{O}(0)$  is the axial vector current of Eq. (1). Lepage-Brodsky [67] spinors are used in the evaluations. Take  $\mu = +$ , so that the second term on the right-hand side vanishes and use the matrix element  $\langle P^+, \mathbf{p}'_\perp, \Lambda | \gamma^+ \gamma_5 | P^+, \mathbf{p}_\perp, \Lambda \rangle = 2P^+$  to obtain:

$$\begin{aligned} \int dx^- \langle \Psi | A^+_+(0)(x^+, x^-, \mathbf{x}_\perp) | \Psi \rangle_{P^+ \rightarrow \infty} &\equiv \rho_A(x_\perp) \\ &= \int \frac{d^2 \Delta_\perp}{(2\pi)^2} G_A(-\Delta_\perp^2) e^{-i\Delta_\perp \cdot \mathbf{x}_\perp}. \end{aligned} \quad (14)$$

Equation (14) shows that the transverse axial charge density is simply the two-dimensional Fourier transform of the axial form factor,  $G_A$ .

Obtaining the density associated with  $G_P$  is more involved. First, concentrate on the second term of Eq. (1) and define the resulting density to be  $\rho_P(\mathbf{x}_\perp)$ . Take the light front helicity state to be a superposition of positive and negative helicity states:  $|\Lambda\rangle = 1/\sqrt{2}(|+\rangle + |-\rangle)$ , meaning polarization in the  $x$  direction. The matrix element of  $\gamma_5$  between  $x$ -polarized spinors is  $-\Delta_x$ . Then take  $\mu = x$ . The result is that

$$\begin{aligned} \rho_P(\mathbf{x}_\perp) &= \nabla_x^2 \int \frac{d^2 \Delta_\perp}{(2\pi)^2} G_P(-\Delta_\perp^2) e^{-i\Delta_\perp \cdot \mathbf{x}_\perp} \\ &= \tilde{G}'_P(x_\perp) \cos^2 \phi + \frac{\tilde{G}_P(x_\perp)}{x_\perp} \sin^2 \phi, \end{aligned} \quad (15)$$

where  $\tilde{G}_P(x_\perp)$  is the two-dimensional Fourier transform of  $G_P$ , and  $\phi$  is the angle between  $\mathbf{x}_\perp$  and the direction of polarization.

What do  $\rho_A$  and  $\rho_P$  look like? The dipole form of Ref. [68] is used to provide an illustration. Thus

$$G_A(\Delta^2) = \frac{G_A(0)}{(1 - \Delta^2/M_A^2)^2}, \quad (16)$$

with  $G_A(0) = 1.2723$  and  $M_A = 1.014$  GeV. Then

$$\rho_A(x_\perp) = G_A(0) M_A^2 \frac{M_A x_\perp K_1(M_A x_\perp)}{4\pi}, \quad (17)$$

where  $K_1$  is a modified Bessel function. A simple form of  $G_P(Q^2)$  is obtained if this induced pseudoscalar term has a pion pole. Then the Goldberger-Treiman relation is generalized to nonzero values of momentum transfer [69,70]:

$$G_P(\Delta^2) = \frac{4M^2 G_A(\Delta^2)}{-\Delta^2 + m_\pi^2}. \quad (18)$$

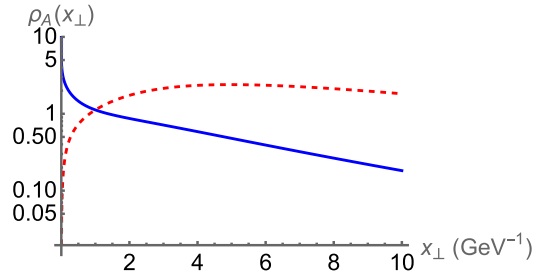


FIG. 1. Solid-  $\tilde{\rho}_A(b)$ , Dashed-  $b\tilde{\rho}_A(b)$ .

Then taking the two-dimensional Fourier transform leads to the result:

$$\begin{aligned} \tilde{G}_P(x_\perp) &= G_A(0) \frac{M^2 (2K_0(m_\pi x_\perp) + M_A x_\perp (m_\pi^2/M_A^2 - 2) K_1(M_A x_\perp))}{4\pi (1 - m_\pi^2/M_A^2)^2}. \end{aligned} \quad (19)$$

The transverse axial charge density  $\tilde{G}_A$  and  $x_\perp \tilde{G}_A$  are shown in Fig. 1. The induced pseudoscalar density,  $\rho_P(\mathbf{x}_\perp)$ , is shown in Fig. 2 to have a very strong spatial anisotropy.

### B. Densities of gravitational form factors

The proton gravitational form factors  $A(t)$ ,  $J(t)$ , and  $D(t)$  are defined [12] as

$$\begin{aligned} \langle p', \lambda | T^{\mu\nu}(0) | p, \lambda \rangle &= \bar{u}(p', \lambda) \left\{ \frac{P^\mu P^\nu}{M} A(t) + \frac{\Delta^\mu \Delta^\nu - \Delta^2 g^{\mu\nu}}{4M} D(t) \right. \\ &\quad \left. + \frac{i P^{[\mu} \sigma^{\nu]\rho} \Delta_\rho}{2M} J(t) \right\} u(p, \lambda), \end{aligned} \quad (20)$$

where  $P = (p + p')/2$ ,  $\Delta = p' - p$ ,  $t = \Delta^2$ ,  $\sigma^{\mu\nu} = i/2[\gamma^\mu, \gamma^\nu]$ , and the curly brackets  $\{\}$  signify symmetrization over the indices between them (without a factor 1/2).  $T^{\mu\nu}$  is the renormalization-scale-independent symmetric energy momentum tensor of QCD.

I wish to obtain spatial densities related to each of the form factors appearing in Eq. (20). One may isolate  $A(t)$  by

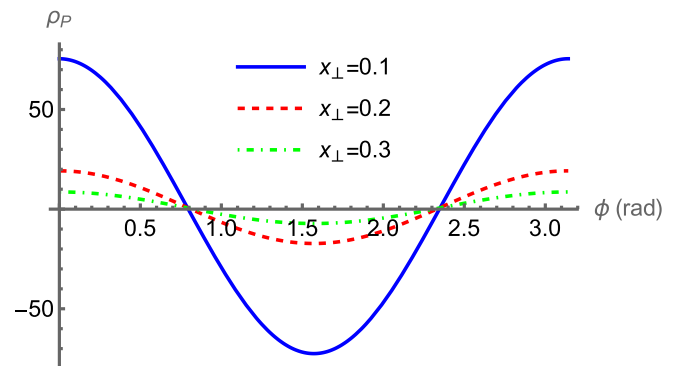


FIG. 2.  $\rho_P(\phi)$  for  $b = 0.1, 0.2, 0.3$  GeV $^{-1}$ .

considering the matrix element of  $T^{++}(0)$ . The result is

$$\langle p', \lambda | T^{++}(0) | p, \lambda \rangle = 2P^+ P^+ A(t). \quad (21)$$

Using this matrix element for  $\hat{\mathcal{O}}(0)$  in Eq. (9) and taking  $\sigma \rightarrow 0$  leads to the density:

$$\begin{aligned} \rho_A(\mathbf{x}_\perp) &\equiv \frac{1}{P^+} \int dx^- \langle \Psi | \hat{T}^{++}(x^-, \mathbf{x}_\perp) | \Psi \rangle \\ &= \int \frac{d^2 \Delta_\perp}{(2\pi)^2} A(\Delta^2) e^{-i\Delta_\perp \cdot \mathbf{x}_\perp}. \end{aligned} \quad (22)$$

The term  $\rho_A(\mathbf{x}_\perp)$  has been denoted the momentum density [46]. Its integral is unity.

The spatial densities associated with  $D(t)$  are those of the pure stress tensor. Keeping the second term on the right-hand side of Eq. (20), defining its contribution to be that of the matrix element of  $T_D^{\mu\nu}(0)$  and taking the  $i, j = 1, 2$  components allows one to use Eq. (9) to define spatial densities [36]. [Using any term that involves a minus component would lead to divergent terms in the integral over  $\mathbf{P}_\perp$  that appears in Eq. (9).] The result is

$$\begin{aligned} \rho_D^{ij}(\mathbf{x}_\perp) &\equiv P^+ \int dx^- \langle \Psi | \hat{T}_D^{ij}(x^-, \mathbf{x}_\perp) | \Psi \rangle \\ &= \frac{1}{4} \int \frac{d^2 \Delta_\perp}{(2\pi)^2} (\Delta^i \Delta^j + \Delta_\perp^2 \delta^{ij}) D(\Delta^2) e^{-i\Delta_\perp \cdot \mathbf{x}_\perp}. \end{aligned} \quad (23)$$

Next we turn to the transverse density associated with the proton's angular momentum,  $J(t)$ . Keeping the third term on the right-hand side of Eq. (20), defining its contribution to be that of the matrix element of  $T_J^{\mu\nu}(0)$  and taking the component  $T_J^{+j}$  with  $j = 1, 2$  allows one to use Eq. (9) to define spatial densities [36]. This is because

$$\bar{u}(p', \lambda) \frac{iP^+ \sigma^{j\rho} \Delta_\rho}{2M} u(p, \lambda) = i(\hat{\lambda} \times \Delta_\perp)^j P^+, \quad (24)$$

where  $\hat{\lambda}$  indicates the direction of the light-front helicity in the light-front direction. This result is obtained using the Gordon identity and Lepage-Brodsky spinors. Then using  $T_D^{+i}(0)$  in Eq. (9) leads to the result:

$$\begin{aligned} \rho_J^j(\mathbf{x}_\perp, \lambda) &\equiv \int dx^- \langle \Psi | T_J^{+j}(x^+ = 0, x^-, \mathbf{x}_\perp) | \Psi \rangle \\ &= \frac{1}{2} (\hat{\lambda} \times \nabla_\perp)^j \tilde{J}(|\mathbf{x}_\perp|), \end{aligned} \quad (25)$$

where  $\tilde{J}(|\mathbf{x}_\perp|)$  is the two-dimensional Fourier transform of  $J(\Delta^2)$ .

The three densities have been determined by using components of  $T^{\mu\nu}$  that allow the integral over  $\mathbf{P}_\perp$  to be performed with a divergence-free result.

There is one more combination to consider here, the trace of the energy momentum tensor,  $T_\mu^\mu$ . Take  $\mathcal{O}(0)$  to be  $T_\mu^\mu(0)$  and use it in Eq. (9) along with [71]

$$\begin{aligned} \bar{u}(p', \lambda) T_\mu^\mu(0) u(p, \lambda) &= 2MG_s(\Delta^2) \\ &= 2(M^2 - \Delta^2/4)A(\Delta^2) + J(\Delta^2)\Delta^2 \\ &\quad - \frac{3\Delta^2}{4}D(\Delta^2), \end{aligned} \quad (26)$$

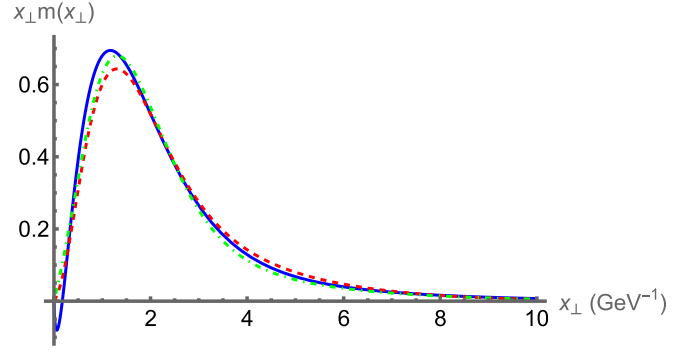


FIG. 3. Mass (squared) distribution  $m(x_\perp)$  Solid- dipole, dashed- tripole, and dot-dashed four-pole.

to find

$$\begin{aligned} m(x_\perp) &\equiv \frac{P^+}{M} \int dx^- \langle \Psi | T_\mu^\mu(x^-, \mathbf{x}_\perp) | \Psi \rangle \\ &= \int \frac{d^2 \Delta_\perp}{(2\pi)^2} 2MG_s(\Delta^2) e^{-i\Delta_\perp \cdot \mathbf{x}_\perp}. \end{aligned} \quad (27)$$

The key feature in obtaining this result is that the matrix element of  $T_\mu^\mu$  is independent of  $\mathbf{P}_\perp$ . Note that  $\int d^2 x_\perp m(x_\perp) = 2M^2$ , so that  $m(x_\perp)$  may be thought of as a mass (squared) density.

The resulting values of  $x_\perp m(x_\perp)$ , Fig. 3, are obtained by using the lattice QCD  $n$ -pole form factors, with  $n = 2, 3, 4$ , in Ref. [2] and the supplementary material [25]. The short distance values are very sensitive to the high-momentum form factors, and the use of the dipole leads to negative values of  $m(x_\perp)$  for small values of  $x_\perp$ . The dominant change between using  $n = 2$  and  $n = 3$  comes from the change in  $D(\Delta^2)$ . Calculations of perturbative QCD favor using a dipole for  $A$  and  $J$  and a tripole for  $D$  [72,73].

The root-mean-square transverse radii [square root of the mean-square transverse radii of Eq. (13)] are 0.60, 0.54 and 0.52 fm for  $n = 2, 3, 4$  respectively. These seem to be small, but recall that these are smaller by a factor of  $\sqrt{2/3}$  than conventional expressions of radii, due to using two dimensions instead of three. Even so these radii are smaller than the electromagnetic charge radius.

The remainder of this paper is devoted to showing that obtaining three-dimensional densities related to form factors of confined systems of relativistically moving constituents is not justified. Recent unfortunate examples include Refs. [2,74].

### III. WHAT'S WRONG WITH THE BREIT FRAME INTERPRETATION OF DENSITIES

The standard formulation [75,76] of the Breit frame interpretation is discussed first. The Breit frame is defined for spacelike probes by having the initial and final proton momentum being equal in magnitude and opposite in direction, so that no energy is transferred,  $q^0 = 0$ . This enables  $q^2$  to be expressed as  $q^2 = -\mathbf{q}^2$ , and taking a three-dimensional Fourier transform is possible. The matrix element of the 0th component of the electromagnetic current is proportional to

the Sachs electric form factor  $G_E$  if the helicities of the initial and final states are opposite.

Then one makes a nonrelativistic treatment (see Ref. [75], chap. 8.1) to take a three-dimensional Fourier transform of  $G_E(\mathbf{q})$  to obtain a function of the three-dimensional distance coordinate. Using Galileo invariance allows the separation of relative and internal variables so that this procedure is valid [54]. However, this separation is not valid for a system in which the component particles move relativistically. An attempted derivation of this procedure for relativistic systems was presented in Ref. [76], but this was fatally flawed because an infinite term was neglected [37].

#### Frame-dependent internal wave function

A specific simple example of the frame dependence of an internal wave function is obtained by considering a hadronic bound-state system of two light spin-0 particles. Then one may compute a form factor using a Bethe-Salpeter wave function  $\Phi(k; P)$ , where  $k$  is the relative four-momentum and  $K$  is the total four-momentum. Such can be written in a compact form if the interaction kernel is obtained by summing a set of Feynman graphs [77,78]. Then the Nakanishi integral representation: Reference [79] provides an explicit expression of the wave function:

$$\Phi(k; K) = -\frac{i}{\sqrt{4\pi}} \int_{-1}^1 dz \times \int_0^\infty d\gamma \frac{g(\gamma, z)}{[\gamma + m^2 - \frac{1}{4}M^2 - k^2 - K \cdot k z - i\epsilon]^3}, \quad (28)$$

where  $m$  is the mass of each constituent scalar particle and  $M$  is the hadronic mass. The weight function  $g(\gamma, z)$  itself is not singular, but the singularities of the BS amplitude are reproduced by using Eq. (28) [80].

Consider the simplest case that  $g(\gamma, z) = 1$  to illustrate the key point. Then integration reveals that

$$\Phi(k; K) = -\frac{i}{\sqrt{4\pi}} \frac{1}{(m^2 - M^2/4 - k^2 - i\epsilon)^2 - (K \cdot k)^2}, \quad (29)$$

which can be shown to be a product of propagators [81]. Note that this expression cannot be factored into a product of a function of  $K$  with another function of the relative momentum,  $k$ , so that one obtains different  $k$  dependence for every value of  $K$ .

This explicit dependence on  $K$  dramatically influences current understanding of form factors because the initial and final hadrons have different momentum,  $K$  and  $K + q$ , and therefore different relative wave functions. The presence of different wave functions of the initial and final nucleons invalidates a probability or density interpretation. In the Breit frame the nucleon initial and final momentum are opposite, so that the associated wave functions of the two states are different.

That the initial momenta is different from the final value is a severe problem, but there is something even worse—the

violation of the uncertainty principle. A plane-wave state of definite momentum has a constant probability density to be anywhere and everywhere in space. Defining a position requires that one use a wave packet that is a superposition of momentum eigenstates. Since  $\mathbf{P} = (\mathbf{K} + \mathbf{K}')/2 = 0$  appears with vanishing probability in such superpositions, the Breit frame is not relevant.

#### IV. WHAT'S WRONG WITH USING THE WIGNER DISTRIBUTION

In Ref. [82] the authors present the three-dimensional spatial distributions defined in the Breit frame where the system is on *average* at rest. The aim of this Sec. IV is to show that this criteria is not sufficient to define spatial positions of a system that is at rest because the fluctuations in the values of position and momentum are infinite.

I begin by reviewing the procedure of Ref. [82] for distributions in the instant form, using its notation. The phase-space density for an average position  $R$  and average momentum  $P$  is given by

$$\rho_{R,P} = \int \frac{d^3\Delta}{(2\pi)^3 2P^0} e^{-i\Delta \cdot R} \left| P - \frac{\Delta}{2} \right\rangle \left\langle P + \frac{\Delta}{2} \right|, \quad (30)$$

where the initial and final target energies are given by on-mass-shell values. The aim of using the Wigner distribution is to “localize the system in both position and momentum space.” This is impossible if one accepts the uncertainty principle.

Matrix elements of position-dependent operators  $\hat{O}$  using the Wigner distribution are given by

$$\langle \hat{O}(X) \rangle_{R,P} = \text{Tr}[\hat{O}(X) \rho_{R,P}], \quad (31)$$

with translation invariance implying that

$$\langle O(X) \rangle_{R,P} = \langle O(x) \rangle_{0,P}, \quad (32)$$

where  $x = X - R$  is the relative average position.

Specifying to the energy momentum tensor, setting the origin at the average position of the system  $\mathbf{R} = \mathbf{0}$  and denoting  $x^\mu = (0, \mathbf{r})$ , the static EMT encoding the distribution of energy and momentum inside the system with canonical polarization  $\mathbf{s}$  and average momentum  $\mathbf{P}$ , is *defined* to be the following Fourier transform:

$$\mathcal{T}^{\mu\nu}(\mathbf{r}; \mathbf{P}) = \int \frac{d^3\Delta}{(2\pi)^3} e^{-i\Delta \cdot \mathbf{r}} \frac{\langle p', \mathbf{s} | T^{\mu\nu}(0) | p, \mathbf{s} \rangle}{2P^0}. \quad (33)$$

Thus, the resulting densities are simply a definition. Note that  $x^0 = 0$  because the positions  $\mathbf{X}$  and  $\mathbf{R}$  are chosen to be at the same time  $X^0 = R^0$ .

I next show that although the average of the momentum is indeed 0, the fluctuations are infinite. This is done by using Eq. (30) and the trace, Eq. (31), to compute the average momentum and the average-squared momentum. Then the average momentum is given by

$$\begin{aligned} \langle \mathbf{K} \rangle &= \int d^3K \mathbf{K} \int \frac{d^3\Delta}{(2\pi)^3 2P^0} e^{-i\Delta \cdot R} \\ &\quad \times \langle K | P - \Delta/2 \rangle \langle P + \Delta/2 | K \rangle \\ &= \int d^3K \frac{\mathbf{K}}{E(K)} = 0, \end{aligned} \quad (34)$$

where  $E(K) = \sqrt{K^2 + M^2}$ . One indeed obtains the result that the average value of the momentum vanishes but only as a consequence of parity. So it is no surprise that using the same formalism yields

$$\langle \mathbf{K}^2 \rangle = \int d^3 K \frac{\mathbf{K}^2}{E(K)} = \infty. \quad (35)$$

The fluctuations in the momentum are infinite. The statement that the average momentum vanishes is true, but that is not sufficient to provide a meaningful state at rest.

Next I turn to coordinate space with the average position obtained as:

$$\langle \mathbf{r} \rangle = \int d^3 \mathbf{r} \mathbf{r} \frac{d^3 \Delta}{(2\pi)^3 2P^0} \langle \mathbf{r} | P - \Delta/2 \rangle \langle P + \Delta/2 | \mathbf{r} \rangle e^{i\Delta \cdot \mathbf{r}}. \quad (36)$$

The necessary overlap matrix element is given by

$$\langle \mathbf{r} | P - \Delta/2 \rangle = e^{i(\mathbf{P} - \Delta/2) \cdot \mathbf{r}}, \quad (37)$$

obtained by using the normalization of  $|x\rangle$  of Eq (2) of [82],

$$|x\rangle = \int \frac{d^3 p}{2E(p)(2\pi)^3} e^{-i\mathbf{p} \cdot \mathbf{r}} |p\rangle \quad (38)$$

with  $|x\rangle$  having dimension of inverse length squared. Then

$$\langle \mathbf{r} \rangle = \int d^3 \mathbf{r} \mathbf{r} \int \frac{d^3 \Delta}{(2\pi)^3} \frac{1}{E(\mathbf{P} + \Delta/2) + E(\mathbf{P} - \Delta/2)} = 0, \quad (39)$$

where again spatial inversion is used to obtain the vanishing. The promise Ref. [82] of a vanishing quantity is kept.

Next consider the fluctuations in position. Using the previous manipulations one finds that

$$\langle \mathbf{r}^2 \rangle = \int d^3 \mathbf{r} \mathbf{r}^2 \int \frac{d^3 \Delta}{(2\pi)^3} \frac{1}{E(\mathbf{P} + \Delta/2) + E(\mathbf{P} - \Delta/2)} = \infty. \quad (40)$$

The fluctuations in position are also infinite in any reference frame.

Another noteworthy feature of the preceding equation is that the dimensions on the right-hand side are of length cubed. This peculiarity arises from the definition of the position eigenstate, Eq. (38). Replacing the factor  $\frac{1}{2E(p)}$  by  $\frac{1}{\sqrt{2E(p)}}$  would lead to expressions with the proper dimensions and remove the energy denominator from Eq. (40). The fluctuations in position are still infinite.

Although the average position and momentum vanishes, the fluctuations in both quantities are infinite. Thus, there is no state that is both at fixed position and is at rest.

## V. THE ABEL TRANSFORMATION DOES NOT YIELD A PROPER THREE-DIMENSIONAL DENSITY

Several authors including those of Refs. [83–88] have used the Abel transformation in an attempt to obtain three-dimensional spherically symmetric densities. Reference [38] convincingly showed that “the application of the inverse Abel transform to the light front densities does not produce a

physically meaningful result.” I include a few remarks in the interest of presenting a complete description of why three-dimensional, spherically symmetric densities of confined systems of relativistically moving constituents do not exist.

The Abel transformation is an integral transform that is used in analyzing spherically symmetric functions. Suppose there is a system that has a density,  $\rho$ , that is spherically symmetric. Then one may integrate along a chosen direction,  $z$ , to obtain a function of the coordinates transverse to  $z$ :

$$\rho(b) = \int_{-\infty}^{\infty} dz \rho(r) = 2 \int_b^{\infty} dr \frac{\rho(r)}{\sqrt{b^2 - r^2}}, \quad (41)$$

in which  $r = \sqrt{b^2 + z^2}$ . The first term is familiar from the Glauber (eikonal) theory of scattering. It gives the thickness of matter traversed by a beam moving through matter on a straight line trajectory with an impact parameter  $b$ , and indeed  $\rho(b)$  is denoted as the thickness function in many applications.

The immediate problem with applying an Abel transformation to understand confined systems of relativistically moving constituents, is that such a moving system cannot have a spherically symmetric density because of the effects of Lorentz contraction.

Note also that there are significant EIC-related efforts to determine the three-dimensional structure of the nucleon. It is universally known that the three directions are the longitudinal momentum,  $x = k^+/p^+$ , and the position of quarks and gluons in the transverse spatial plane, see, e.g., p. 99 of Ref. [89] and also, e.g., Ref. [90]. The longitudinal and transverse directions are described using different variables, so that there can be no spherical symmetry. More generally, one may say that the nonspherical nature of the nucleon structure is due to the requirements of Poincaré invariance.

It is possible to obtain wave functions and densities in three spatial directions by Fourier transforming the longitudinal momentum to the Miller-Brodsky [91] longitudinal spatial coordinate,  $\tilde{z} = p^+ x^-$ , conjugate to  $x$ . Some examples show that a two-particle wave function can be a product of a function of  $\mathbf{x}_\perp$  with another different function of  $\tilde{z}$ . The lack of spherical symmetry renders the Abel transform inapplicable.

## VI. USING SPHERICAL WAVE PACKETS WITH VANISHING SPATIAL EXTENT DOES NOT HELP

The work of Ref. [92] states “that the matrix element of a local operator between hadronic states can be used to unambiguously define the associated spatial density.” The relation between the charge density of a spinless particle and the electric form factor is used as an example in that paper.

The analysis of Sec. II, based on work done since 1968, shows already how to obtain spatial densities for systems having relativistic dynamics. Nevertheless, it is worthwhile to examine the interesting claim of the previous paragraph by examining the details of Ref. [92].

The authors of Ref. [92], use a frame in which the average momentum of the initial and final hadrons is zero (ZAMF), meaning  $\mathbf{P} = 0$ . Such a frame is, of course, a Breit frame. This is because the initial and final nucleons must have the same energy. These authors improve on the standard use of

the Breit frame by using wave packets. However, they state, “we define the charge density distribution in the ZAMF by employing spherically symmetric wave packets.” Thus, their resulting densities are simply a definition. Furthermore, such densities explicitly depend on the chosen wave packet. There is no reason why spherically symmetric wave packets must be used. See, for example, Eq. (7).

### A. The vanishing spherically symmetry density

But there is a more fundamental problem with the approach of Ref. [92], starting with the second equation of that paper, reproduced here:

$$\langle p' | \hat{\rho}(\mathbf{r}, 0) | p \rangle = e^{-i(\mathbf{p}' - \mathbf{p}) \cdot \mathbf{r}} (E + E') F(q^2). \quad (42)$$

In the ZAMF  $E' = E = \sqrt{\mathbf{q}^2/4 + M^2}$  and  $\mathbf{q}^2 = -(\mathbf{p}' - \mathbf{p})^2$ . One may then Fourier transform Eq. (42) to find

$$F(q^2) = \frac{1}{2E(2\pi)^3} \int d^3r e^{i(\mathbf{p}' - \mathbf{p}) \cdot \mathbf{r}} \langle p' | \hat{\rho}(\mathbf{r}) | p \rangle. \quad (43)$$

This expression looks suspect because the left-hand side is a scalar, independent of reference frame, but the right-hand side depends on  $\mathbf{q} = \mathbf{p}' - \mathbf{p}$ , a three-vector and  $E$ , the time-component of a four-vector.

Perhaps this seeming contradiction could be resolved by considering time dependence. Consider the matrix element of the time component of the current as a function of the four-position  $x = (t, \mathbf{r})$ ,  $j^0(x)$ , in the wave packet used by Ref. [92]. Then the space-time density is given by

$$\begin{aligned} \rho(x) &= \int \frac{d^3p}{2E(p)(2\pi)^3} \\ &\times \int \frac{d^3p'}{2E(p')(2\pi)^3} \frac{\langle p' | j^0(0) | p \rangle}{E(p) + E(p')} \phi(p) \phi(p') e^{i(p' - p) \cdot x}. \end{aligned} \quad (44)$$

This is the same as Eq. (6) of Ref. [92] except for keeping the time dependence. Next, I follow Ref. [92] and take their limit of the vanishing spatial extent of the wave packet. This leads to the result:

$$\rho(t, r) = \int \frac{d^3q}{(2\pi)^3} e^{-i\mathbf{q} \cdot \mathbf{r}} \int_{-1}^1 d\alpha \frac{1}{2} F((\alpha^2 - 1)\mathbf{q}^2) e^{i\alpha|q|t}, \quad (45)$$

which is the same as Eq. (11) of Ref. [92] in the limit that  $t = 0$ . The only difference with that reference arises from appearance of the exponential  $e^{i\alpha|q|t}$  that appears as a result of the same logic used to obtain the  $\alpha^2$  term of Eq. (45).

Let us consider the limit of very large times. In that case, the integrand oscillates rapidly and the integral vanishes. Thus one expects that

$$\lim_{t \rightarrow \infty} \rho(t, r) = 0. \quad (46)$$

The density seems to vanish as time increases. This surely represents a problem with using Eq. (43) to compute form factors. The result is a vanishing density for  $r \neq t$ .

Further understanding may be obtained by examining Eq. (45) using an asymptotic expansion in large, positive

values of  $t$ . This is done by using  $e^{i\alpha|q|t} = \frac{1}{i|q|t} \frac{\partial e^{i\alpha|q|t}}{\partial \alpha}$  and integrating over  $\alpha$  by parts. Keeping only the leading term in powers of  $1/t$  leads to

$$\rho(t, r) \sim \frac{1}{4\pi^2 r} \int \frac{d^3q}{(2\pi)^3} \frac{1}{|q|t} e^{-i\mathbf{q} \cdot \mathbf{r}} \sin |q|t \quad (47)$$

because  $F(0) = 1$ . The integration over angles of  $\mathbf{q}$  yields

$$\rho(t, r) \sim \frac{1}{\pi r t} \int dq \sin |q| r \sin |q| t \quad (48)$$

$$= \frac{1}{8\pi r t} \delta(r - t). \quad (49)$$

At large times,  $t$ , the density vanishes except at  $r = t$ .

It is useful to study the full time dependence by using the specific example of a Gaussian form factor,

$$F_G((\alpha^2 - 1)\mathbf{q}^2) = e^{-(1 - \alpha^2)\mathbf{q}^2 R^2/6}. \quad (50)$$

A closed form of the expression for the resulting density at  $t = 0$  is obtained by doing the integral of Eq. (45) to find

$$\rho_G(0, r) = \frac{3e^{-\frac{3r^2}{2R^2}}}{2\pi r R^2}. \quad (51)$$

The  $1/r$  behavior at the origin seems to be a remarkably unphysical way to account for a simple Gaussian form factor.

Next let us examine the time dependence of  $\rho_G(t, r)$ . The integral over  $q$  in Eq. (45) may be done in closed form with the result

$$\begin{aligned} \rho_G(t, r) &= \frac{3}{4rR^3} \sqrt{\frac{3}{2\pi^3}} \int_{-1}^1 \frac{d\alpha}{(1 - \alpha^2)^{3/2}} (F(r - \alpha t) \\ &+ F(r + \alpha t)), \end{aligned} \quad (52)$$

$$F(x) \equiv x e^{\frac{-3x^2}{\sqrt{1 - \alpha^2}}}. \quad (53)$$

Proceed by numerical integration to find  $\rho_G(r, t)$  with  $r$  and  $t$  in units of  $R$ . The results in Fig. 4 show that indeed the density vanishes for  $t > r$ . Note the sharp falloff when  $t = r$ . The decreasing width as the value of  $r$  increases indicates that the limit of a delta function is approached.

The weird behavior at  $r = 0$  and the rapid fall off with time indicate that Eq. (43) is simply not an expression that may be correctly used to compute a form factor of a system with relativistic dynamics.

### B. Relativistic calculations of form factors

I therefore briefly pursue the computation of form factors for systems with relativistic dynamics by considering the example of a spinless system made of two spin-0 constituents, where the form factor is given by [80]:

$$\begin{aligned} (K + K')^\mu F(Q^2) &= i \int \frac{d^4k}{(2\pi)^4} (K + K' + k)^\mu (k^2 - m^2) \\ &\times \Phi\left(\frac{1}{2}K - k; K\right) \Phi\left(\frac{1}{2}K' - k; K'\right), \end{aligned} \quad (54)$$

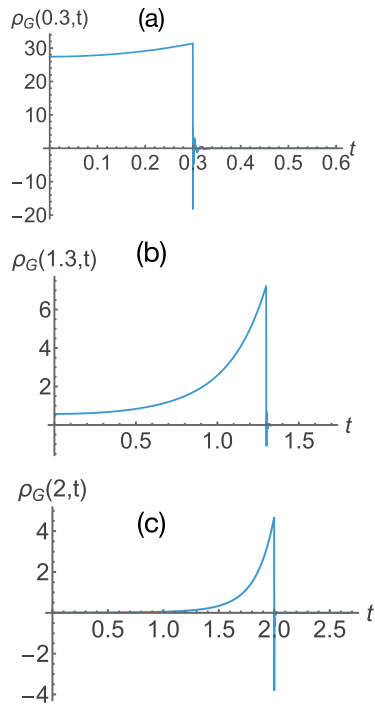


FIG. 4. Time-dependent densities (a)  $\rho_G(0.3, t)$  (b)  $\rho_G(1.3, t)$  (c)  $\rho_G(2, t)$ . Units of  $r$  and  $t$  are relative to  $R$ .

where  $K' = K + q$ . An example of an expression for  $\Phi$  is given in Eq. (29). The only way in which Eq. (54) can be massaged into the form of Eq. (43) is in the nonrelativistic limit defined in the Introduction [27,81]. An example of the disparate forms, note that one obtains instant-form expressions by integrating over  $k^0$ . There are poles for both positive and negative values of  $k^0$ . Poles at negative energy cannot appear in Eq. (43).

Once again, I find that there is no time-independent spherically symmetric density for confined systems of relativistically moving constituents.

## VII. SUMMARY

A method for obtaining time-independent two-dimensional densities is described in Sec. II. The axial vector current and mass density related to the trace of the energy momentum tensor (defining a mass density) are used as examples. The mass radius is found to be smaller than the electromagnetic charge radius.

That using the Breit frame violates the quantum-mechanical definition of probability, the uncertainty principle, and Poincaré invariance is shown in Sec. III.

The use of the Abel transformation to relate two- and three-dimensional distributions depends on a density having three-dimensional spherical symmetry. But densities of light hadrons cannot be spherically symmetric; see Sec. IV.

Wigner distributions provide wave packets with both vanishing average size and average momentum. But the fluctuations in both position and momentum are infinite, as shown in Sec. V.

The use of spherical wave packets of vanishing spatial extent leads to densities,  $\rho(r, t)$ , that vanish almost everywhere in space as time increases from an initial value; see Sec. VI.

The net result is that the use of the infinite momentum frame, light front formalism seems to be the only way to obtain densities consistent with the quantum-mechanical definition of probability, the uncertainty principle, and the principle of relativity.

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## DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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