

Transition balance in QCD nucleation

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(Received 24 January 2025; revised 3 November 2025; accepted 13 March 2026; published 27 July 2026)

As an extended and more complete version of the primary QCD nucleation model presented in [Phys. Rev. C **95**, 024907 (2017)], the new model introduces explicitly the transition balance and formulates it in both macroscopic and microscopic descriptions. The microscopic description of the transition balance in QCD nucleation is implemented in a kinetic parton cascade model and tested for a first-order phase transition from gluons to pions in a one-dimensional expansion with Bjorken boost invariance.

DOI: [10.1103/gg8v-z7pn](https://doi.org/10.1103/gg8v-z7pn)

I. INTRODUCTION

One of the common scientific purposes of relativistic heavy-ion collision experiments at Relativistic Heavy Ion Collider (RHIC) [1], Facility for Antiproton and Ion Research (FAIR) [2,3], and Nuclotron-based Ion Collider fAcility (NICA) [4,5] is to find the QCD critical point, which is the end point of the first-order phase transition line in the temperature-baryon chemical-potential plane [6–15]. At this point, the largest fluctuation will occur, which will become a signal of the existence of the critical point in experiments. Abundant investigations have been made from both theoretical [16–28] and experimental sides [29–47], see Refs. [48–50] for an overview.

While the transition from QGP to hadron gas in an experiment with a certain collider energy may not meet the critical point, more probably it may cross the first-order phase transition line. The experimental evidence of the existence of the first-order phase transition is equivalent to that of the critical point. Therefore, it is important to study the dynamics of QCD nucleation [51,52] and the experimental signature of the first-order phase transition in relativistic heavy-ion collisions. For this purpose, we plan to build up a hydrodynamic and a kinetic transport model, which can describe the evolution of the separate phases in a first-order phase transition.

The first step has been made in Ref. [53], where we have established a framework of hydrodynamic description of separated phases in the first-order phase transition from gluons

to pions. Based on this, we have developed a microscopic scheme of the transition within the kinetic parton cascade model, Boltzmann Approach of MultiParton Scatterings (BAMPS) [54–56]. The dynamical scheme has been proven by comparisons of the numerical results with the analytical solutions in a one-dimensional expansion of a dissipative fluid with Bjorken boost invariance.

However, the established framework needs improvements. First, the first-order phase transition from gluons to pions at the fixed transition temperature is not realistic for the QGP created in relativistic heavy-ion collisions. The transition from all parton species to all hadron species at finite baryon chemical potential should be considered. Moreover, one has to take into account that, during the first-order transition, the temperature and baryon chemical potential change accordingly along the first-order phase-transition line [57]. Second, the framework applies only to transitions in an expansion, but not to those in a compression, in which transitions from hadrons to partons occur. Thus, the introduced hydrodynamic description is not complete.

In the present paper, we extend our previous work [53] to be able to describe QCD nucleation in both an expansion and a compression. We assume, as done in Ref. [53], constant temperature and baryon chemical potential during the first-order phase transition. In Sec. II we introduce our nucleation model. Compared with the primary model in Ref. [53], we formally take the particle or heat flow into account. The hydrodynamic description of the transition balance is presented in Sec. III. The main idea is that both the transition from partons to hadrons and the inverse transition from hadrons to partons occur simultaneously. This gives a new picture of QCD nucleation. Within this picture, not only the growth of hadron bubbles in an expansion, but also their shrinkage in a compression can be described. Based on the hydrodynamic description, in Sec. IV we establish an extended microscopic scheme of QCD nucleation by using a similar procedure as given in Ref. [53]. The microscopic scheme will be imple-

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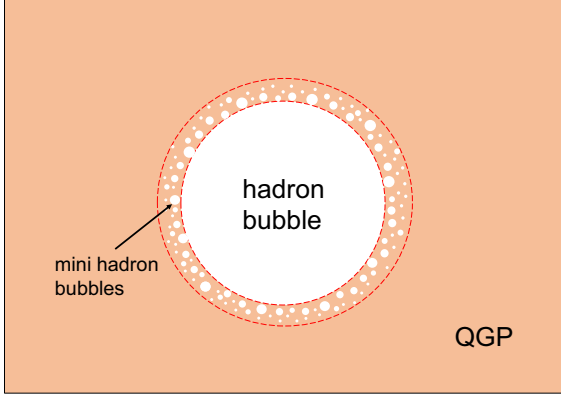


FIG. 1. The illustration of a hadron bubble with its interface.

mented in BAMPS and tested in Sec. V for the first-order phase transition from gluons to pions in a longitudinal expansion with Bjorken boost invariance. The numerical results are shown and compared with analytical ones derived in Ref. [53]. Finally, we give a summary and an outlook in Sec. VI.

II. HYDRODYNAMIC DESCRIPTION OF QCD NUCLEATION

Since the degrees of freedom of partons are more than those of hadrons, the first-order phase transition from the partonic phase to the hadronic phase can only happen in such a spatial region where matter expands. Vice versa, the first-order phase transition from the hadronic phase to the partonic phase can only happen in compressing regions. Only in this way does the entropy increase or maintain, so that the second law of thermodynamics holds.

QCD nucleation will occur when the temperature reaches the transition temperature T_c . Like the boiling water, bubbles containing hadrons will appear. If the temperature fluctuates spatially, we can imagine the occurrence of a number of hadron bubbles with different sizes. Each hadron bubble is separated from the QGP by an interface, which is a mixture of partons and hadrons. Transitions from partons to hadrons will enlarge the hadron bubble, while transitions from hadrons to partons will shrink the hadron bubble. In both cases, the physical process of the phase transition occurs in the interface, where the Gibbs condition holds.

How the phase transition in the interface proceeds is not clear yet, the same as the physical understanding for the creation of hadron bubbles [51]. Our idea is the following: We assume that there exist mini hadron bubbles in the interface, as illustrated in Fig. 1. The formation of mini hadron bubbles is due to statistical fluctuations [51]. Although the mini hadron bubbles are much smaller than hadron bubbles, they can still be regarded as macroscopic objects. The mini hadron bubbles may evolve with increasing sizes and merge with the hadron bubble, which makes it bigger. Also, they may evolve with decreasing sizes and disappear, which makes the hadron bubble smaller. Different from hadron bubbles, which only appear during the phase transition, mini hadron bubbles may have a certain probability to appear in QGP even

at higher temperatures than T_c . Once a mini hadron bubble appears at $T > T_c$, it will be squeezed until it disappears, since the pressure inside the mini hadron bubble is smaller than that outside. The small size of the mini hadron bubble makes the period from the appearance to its disappearance so short that the mini hadron bubble may not be visible in a macroscopic view. When the temperature decreases to T_c , mini hadron bubbles can eventually get larger to form visible hadron bubbles. Similarly, mini parton bubbles may also appear in hadron gas at lower temperatures than T_c .

In the rest of this section, we give the hydrodynamic description of QCD nucleation by assuming that the evolution of mini hadron bubbles is governed by hydrodynamic equations. As an improvement to our primary work in Ref. [53], we take the particle flow (or heat flow) into account.

We now recall the basic hydrodynamic equations. The most general decomposition of the particle four-flow and the energy-momentum tensor can be written as [58]

$$N^\mu = nU^\mu + Q^\mu, \quad (1)$$

$$T^{\mu\nu} = eU^\mu U^\nu - P\Delta^{\mu\nu} + W^\mu U^\nu + W^\nu U^\mu + \pi^{\mu\nu}, \quad (2)$$

where $\Delta^{\mu\nu} = g^{\mu\nu} - U^\mu U^\nu$ with $g^{\mu\nu} = (1, -1, -1, -1)$. U^μ is the fluid four-velocity, $n = U_\mu N^\mu$ is the particle number density, and $e = U_\mu T^{\mu\nu} U_\nu$ is the energy density. $P = -\frac{1}{3}\Delta_{\mu\nu} T^{\mu\nu}$ denotes the pressure. $\pi^{\mu\nu}$ is the shear-stress tensor. The bulk pressure is not taken into account. Q^μ and W^μ denote the flow of particle number and energy-momentum, respectively. W^μ vanishes in the Landau's definition of the fluid four-velocity:

$$U^\mu = \frac{T^{\mu\nu} U_\nu}{\sqrt{U_\alpha T^{\alpha\beta} T_{\beta\gamma} U^\gamma}}, \quad (3)$$

which is used throughout this paper. With the energy-momentum conservation $\partial_\mu T^{\mu\nu} = 0$, we obtain

$$De = -(e + P)\partial_\mu U^\mu + \pi^{\mu\nu} \nabla_{<\mu} U_{>\nu}, \quad (4)$$

where

$$D = U^\mu \partial_\mu, \quad (5)$$

$$\nabla^\mu = \Delta^{\mu\nu} \partial_\nu, \quad (6)$$

$$A^{(\mu\nu)} = \left[\frac{1}{2} (\Delta_\alpha^\mu \Delta_\beta^\nu + \Delta_\alpha^\nu \Delta_\beta^\mu) - \frac{1}{3} \Delta^{\mu\nu} \Delta_{\alpha\beta} \right] A^{\alpha\beta}. \quad (7)$$

With the definition of the shear pressure

$$\tilde{\pi} = -\frac{\pi^{\mu\nu} \nabla_{<\mu} U_{>\nu}}{\partial_\mu U^\mu}, \quad (8)$$

Eq. (4) is reduced to

$$De = -(e + P + \tilde{\pi})\partial_\mu U^\mu. \quad (9)$$

For the number density, we have

$$\partial_\mu N^\mu = J, \quad (10)$$

where J is the source of the particle production. Putting Eq. (1) in the above equation, we obtain

$$Dn = -n\partial_\mu U^\mu - \partial_\mu Q^\mu + J. \quad (11)$$

Considering the occurrence of the first-order phase transition in a piece of a fluid (in the interface of a hadron bubble)

with a volume of V in its rest frame at the proper time τ . The total volume of mini hadron bubbles is V_h , while the volume of partons is V_p . We have $V = V_p + V_h$, and $f_p = V_p/V$ is defined as the partonic fraction. Thus, the total particle number and energy density read

$$n = f_p n_p + (1 - f_p) n_h, \quad (12)$$

$$e = f_p e_p + (1 - f_p) e_h, \quad (13)$$

where n_p and n_h (e_p and e_h) denote the number (energy) density of partons and hadrons, respectively. Furthermore, Q^μ and $\tilde{\pi}$ can be separated into the partonic and hadronic part,

$$Q^\mu = f_p Q_p^\mu + (1 - f_p) Q_h^\mu, \quad (14)$$

$$\tilde{\pi} = f_p \tilde{\pi}_p + (1 - f_p) \tilde{\pi}_h. \quad (15)$$

After a time $d\tau$, the volumes are changed to $V + dV$, $V_p + dV_p$, and $V_h + dV_h$ with $dV = dV_p + dV_h$. We assume that during the phase transition, both the parton and hadron phase have the same fluid four-velocity U^μ . The time evolution of total particle number and energy density is governed by Eqs. (11) and (9). Taking the derivative Dn in Eq. (11) in the local rest frame, we have

$$Dn = \frac{dn}{d\tau} = \frac{d}{d\tau} \frac{N}{V} = -n \frac{1}{V} \frac{dV}{d\tau} + \frac{1}{V} \frac{dN}{d\tau}. \quad (16)$$

Thus,

$$\frac{1}{V} \frac{dV}{d\tau} = \partial_\mu U^\mu, \quad (17)$$

$$\frac{1}{V} \frac{dN}{d\tau} = -\partial_\mu Q^\mu + J. \quad (18)$$

J corresponds to the number changing in the parton and hadron phase as well as in the conversion between partons and hadrons, while $-\partial_\mu Q^\mu$ indicates the particle flow. We denote dN^F as the number of the flowing partons and hadrons during $d\tau$, which gives

$$\frac{1}{V} \frac{dN^F}{d\tau} = -\partial_\mu Q^\mu. \quad (19)$$

The volumes occupied by the flowing partons and hadrons are, respectively,

$$dV_p^F = f_p dV^F = f_p \frac{dN^F}{n}, \quad (20)$$

$$dV_h^F = (1 - f_p) dV^F = (1 - f_p) \frac{dN^F}{n}. \quad (21)$$

Correspondingly, the volume changes due to the phase transition are denoted by dV_p^T and dV_h^T , respectively. We have then $dV_p = dV_p^F + dV_p^T$ and $dV_h = dV_h^F + dV_h^T$.

Taking the derivative De in Eq. (9) in the local rest frame and using Eq. (17), we obtain $dE = -(P + \tilde{\pi})dV$, which indicates that the total energy in V changes during $d\tau$ due to the work done by the pressure. dE is negative in an expansion ($dV > 0$), while it is positive in a compression ($dV < 0$). We have required here that $P + \tilde{\pi} > 0$. According to the Gibbs condition, the parton and hadron energy densities are constant

during the phase transition. Thus, we have $dE = e_p dV_p + e_h dV_h$ and obtain the energy balance

$$-e_p dV_p = e_h dV_h + (P + \tilde{\pi})dV. \quad (22)$$

It shows that the energy lost in the parton phase is transferred in the hadron phase and compensates the energy loss due to the work done by the pressure in an expansion ($dV > 0$) for instance. With $dV_h = dV - dV_p$, we obtain

$$dV_p = dV_p^T + dV_p^F = -\frac{e_h + P + \tilde{\pi}}{e_p - e_h} dV. \quad (23)$$

This equation determines the change of the spatial occupation of the partonic phase (and of the mini hadron bubbles dV_h) during a time $d\tau$. Equivalently, from the definition of f_p , we have

$$\frac{df_p}{d\tau} = \frac{d}{d\tau} \frac{V_p}{V} = \left(-\frac{e_h + P + \tilde{\pi}}{e_p - e_h} - f_p \right) \partial_\mu U^\mu. \quad (24)$$

It shows that the evolution of the partonic fraction depends on the rate of expansion (or compression) $\partial_\mu U^\mu$ and the shear pressure $\tilde{\pi}$ only when the temperature, pressure, and chemical potential are fixed during the phase transition.

The actual transition between partons and hadrons is determined by dV_p^T and dV_h^T . From Eqs. (19)–(21), and (23), we obtain

$$dV_p^T = \left(-\frac{e_h + P + \tilde{\pi}}{e_p - e_h} + \frac{f_p}{n} \frac{\partial_\nu Q^\nu}{\partial_\mu U^\mu} \right) dV, \quad (25)$$

$$dV_h^T = \left(\frac{e_p + P + \tilde{\pi}}{e_p - e_h} + \frac{1 - f_p}{n} \frac{\partial_\nu Q^\nu}{\partial_\mu U^\mu} \right) dV. \quad (26)$$

In an expansion ($dV > 0$), where partons are converted into hadrons, dV_p^T should be negative and dV_h^T should be positive. Vice versa, in a compression ($dV < 0$), where hadrons are converted into partons, dV_p^T should be positive and dV_h^T should be negative. This provides the conditions for the applicability of our QCD nucleation model,

Condition:

$$-\frac{e_h + P + \tilde{\pi}}{e_p - e_h} + \frac{f_p}{n} \frac{\partial_\nu Q^\nu}{\partial_\mu U^\mu} < 0 \quad (27)$$

$$\text{and } \frac{e_p + P + \tilde{\pi}}{e_p - e_h} + \frac{1 - f_p}{n} \frac{\partial_\nu Q^\nu}{\partial_\mu U^\mu} > 0. \quad (28)$$

These conditions are obviously fulfilled in ideal fluids, since $\tilde{\pi}$ and $\partial_\nu Q^\nu$ vanish. Therefore, these conditions shall be fulfilled if the particle system is not too far from thermal equilibrium.

III. TRANSITION BALANCE IN QCD NUCLEATION

Suppose a QCD nucleation via the first-order phase transition in an expanding medium. In the previous section, we derived the volume changes of the parton and hadron phase for the conversion, i.e., dV_p^T and dV_h^T from Eqs. (25) and (26). In the case considered, dV_p^T is negative, while dV_h^T is positive. Partons are converted into hadrons. In the expansion, both the parton and hadron phases should gain energy to balance the energy loss due to the work outwards in order to maintain the temperature and pressure in both phases. It is true for

the hadron phase, because accompanying the conversion from partons to hadrons, energy will enter the hadron phase and can eventually balance the energy loss of the hadron phase. On the other hand, it is not obvious for the parton phase how to gain enough energy to balance its energy loss. In our primary model [53], we implemented microscopic back reactions (from hadrons to partons) by hand to balance the energy loss in the parton phase. Thus, the primary model is established only for the transition from partons to hadrons in an expansion, but not for the one from hadrons to partons in a compression.

In this section, we extend our primary model to a complete hydrodynamic form, which can describe both the transitions from partons to hadrons as well as from hadrons to partons. To this end, we propose simultaneous transitions from partons to hadrons and from hadrons to partons. We imagine that, during a time interval, some mini hadron bubbles expand and some other mini hadron bubbles shrink. If the whole system is expanding, the increase of the mini hadron bubbles is dominant, while if the whole system is shrinking, the decrease of the mini hadron bubbles is dominant. The net change of all mini hadron bubbles is governed by the description in the previous section, see Eqs. (25) and (26). The shrinking mini hadron bubbles guide the hydrodynamic description of the transition from hadrons to partons, which is lacking in the primary model [53].

We denote dV_p^{PtH} and dV_h^{PtH} as the volume changes of the parton and hadron phase in the transition from partons to hadrons (PtH stands for the conversion from partons to hadrons), while we denote dV_p^{HtP} and dV_h^{HtP} as those of the transition from hadrons to partons. Thus, per definition, dV_p^{PtH} and dV_h^{PtH} are always negative, while dV_p^{HtP} and dV_h^{HtP} are always positive. They are related by

$$dV_p^T = dV_p^{PtH} + dV_p^{HtP}, \quad dV_h^T = dV_h^{PtH} + dV_h^{HtP}. \quad (29)$$

We define $\alpha \equiv dV_h^T/dV_p^T$, $\alpha_{PtH} \equiv dV_h^{PtH}/dV_p^{PtH}$, and $\alpha_{HtP} \equiv dV_h^{HtP}/dV_p^{HtP}$. In the case that there is no net transition, we have $dV_p^T = dV_h^T = 0$, which leads to $\alpha_{PtH} = \alpha_{HtP}$. We assume that $\alpha_{PtH} = \alpha_{HtP}$ should still hold when there is a net transition between partons and hadrons. Then it follows that $\alpha = \alpha_{PtH} = \alpha_{HtP}$. For ideal fluids, where $\tilde{\pi}$ and $\partial_\nu Q^\nu$ vanish, we obtain

$$\alpha_{PtH} = \alpha_{HtP} = \alpha = -\frac{e_p + P}{e_h + P} > -\frac{e_p}{e_h} \quad (30)$$

according to Eqs. (25) and (26). Since dV in Eqs. (25) and (26) cancel out, equation and inequality (30) hold, no matter in an expansion or a compression. We assume that the inequality $\alpha_{PtH} = \alpha_{HtP} = \alpha > -e_p/e_h$ still holds in viscous fluids.

Following the energy balance in Eq. (22), we build up the energy balance in the transition from partons to hadrons and in the inverse transition, respectively,

$$-e_p dV_p^{PtH} = e_h dV_h^{PtH} + e_h dV_h^F + (P + \tilde{\pi}_h)(1 - f_p)dV + dq, \quad (31)$$

$$-e_h dV_h^{HtP} = e_p dV_p^{HtP} + e_p dV_p^F + (P + \tilde{\pi}_p)f_p dV - dq. \quad (32)$$

The sum of Eqs. (31) and (32) gives Eq. (22), so that the evolution of the total energy is guaranteed. dV is either positive in an expansion or negative in a compression. Also, dV_p^F and dV_h^F can be positive or negative for the particle flow from or into the neighborhood of V .

Equation (31) shows the energy transfer from the partonic phase to the expanding mini hadron bubbles. Because of Eq. (30), we have

$$-e_p dV_p^{PtH} > e_h dV_h^{PtH}. \quad (33)$$

The energy of lost partons should be larger than the energy of gained hadrons, no matter in an expansion or a compression. It can easily be understood in an expansion, since the energy excess $-e_p dV_p^{PtH} - e_h dV_h^{PtH}$ must at least compensate for the energy loss of the hadron phase due to the work outwards, $(P + \tilde{\pi}_h)(1 - f_p)dV > 0$. In a compression, this term is negative. To hold the inequality (33) mathematically, a positive energy dq must be present and sufficiently large.

To complete the energy balance, we add the term $e_h dV_h^F$, which can be positive or negative. A positive dV_h^F indicates that hadrons with number $n_h dV_h^F$ flow from the neighborhood into the volume V . To compensate for the accompanying energy win, an energy of $e_h dV_h^F$ will flow out of V , so that there is no net energy flow in the Landau definition of the flow velocity. Analogously, for a negative dV_h^F , hadrons with number $-n_h dV_h^F$ flow out of V . To compensate for the energy loss, an energy of $-e_h dV_h^F$ from the neighborhood of V will flow into V .

Analogous to Eqs. (31) and (32) describes the energy transfer from the shrinking mini hadron bubbles to the partonic phase. We have

$$-e_h dV_h^{HtP} < e_p dV_p^{HtP}. \quad (34)$$

Moving $-dq$ to the left-hand side of Eq. (32), we see that an energy of dq must be added into the hadron phase during the conversion, in order to provide enough energy for the converted partons to compensate for the energy loss due to the particle flow and the work done by the pressure in the parton phase in an expansion.

From Eqs. (31) and (32) with $\alpha_{PtH} = \alpha_{HtP} = \alpha$, we obtain

$$-dV_p^{PtH} = \frac{e_h dV_h^F + (P + \tilde{\pi}_h)(1 - f_p)dV + dq}{e_p + \alpha e_h}, \quad (35)$$

$$-dV_h^{HtP} = \alpha \frac{e_p dV_p^F + (P + \tilde{\pi}_p)f_p dV - dq}{e_p + \alpha e_h}. \quad (36)$$

Thus, dV_p^{PtH} , $dV_h^{PtH} = \alpha dV_p^{PtH}$, dV_h^{HtP} , and $dV_p^{HtP} = dV_h^{HtP}/\alpha$ are determined when dq is fixed. To hold both the inequalities (33) and (34), dq in Eqs. (31) and (32) must be larger than $-e_h dV_h^F - (P + \tilde{\pi}_h)(1 - f_p)dV$ and $e_p dV_p^F + (P + \tilde{\pi}_p)f_p dV$.

Some comments on the meaning of the quantity dq are given below. First, dq is a positive parameter, which is mathematically needed to hold the inequalities (33) and (34). With dq , in any transitions, no matter whether they are from partons to hadrons or from hadrons to partons, and they occur in an expansion or a compression, the energy of partons is always larger than that of hadrons. Second, from Eqs. (31) and (32)

we see that dq is the energy which is released from those partons during their transition to hadrons, while it is absorbed by those hadrons, before they are converted to partons. Thus, dq can be understood as a kind of latent heat, which is exchanged between the parton and hadron phase during the simultaneous transitions. Different from the latent heat from textbooks, dq is a quantity in a microscopic sense. Third, the amount of dq affects how many partons and hadrons are involved in the phase transition. From Eqs. (35) and (36) we see that the larger dq , the more partons are converted into hadrons (larger $-dV_p^{PiH}$) and the more hadrons are converted into partons (larger $-dV_h^{HiP}$). dq can also be considered as a phenomenological parameter, which affects the probability that a microscopic transition process occurs. Fourth, following the third point, dq affects the fluctuation in particle number during the phase transition, which is a macroscopic effect. On the other hand, on average, dq has no macroscopic effect on the phase transition, since the sum of Eqs. (31) and (32) should be equal to Eq. (22) and the dqs cancel.

IV. MICROSCOPIC SCHEME OF QCD NUCLEATION

Based on Eqs. (31), (32), (35), and (36) given in the previous section, we now establish a microscopic scheme for QCD nucleation. The primary scheme has been presented in Ref. [53], where the microscopic processes from hadrons to partons were implemented by hand.

In microscopic calculations, we do not simulate the increase or decrease of mini hadron bubbles. In the interface, partons and hadrons are mixed randomly. The microscopic transition processes will give the correct number and kinetic energy of the converted particles.

The space-time evolution of partons and hadrons is described by the kinetic Boltzmann equations,

$$p^\mu \partial_\mu f_i(x, p) = C_i[f], \quad (37)$$

where f_i is the phase-space distribution function of a sort of particles. i stands for a parton species or a hadron species. $C_i[f]$ denotes the collision term, which determines the interactions and transitions involving particles of i at the space-time point $x \equiv x^\mu$ with the energy-momentum $p \equiv p^\mu$.

We solve the Boltzmann equation by employing the parton transport model BAMPS [54]. In BAMPS, particles are sampled in phase space at an initial time. The time evolution of f_i in phase space is the result of the free streaming of particles and collisions among particles. Collisions of particles in a spatial local cell have certain collision probabilities, which can be derived from the collision term $C_i[f]$. With these collision probabilities, we simulate collisions in a stochastic manner. In this section, microscopic processes of the conversion from partons to hadrons and their inverse processes will be discussed, and the probabilities of these processes will be given.

For simplicity, we consider a phase transition from gluons to pions in a one-dimensional expansion with Bjorken boost invariance [59], the same as considered in Ref. [53]. The simulation for transitions involving quarks and other hadron species is more complicated and postponed until future work. Furthermore, we assume that there is no particle flow in the considered case, i.e., $\partial_\mu Q_{g/\pi}^\mu = 0$ and $dV_g^F = dV_\pi^F = 0$. From

Eqs. (25) and (26) we obtain

$$\alpha = \frac{dV_\pi^T}{dV_g^T} = -\frac{e_g + P + \tilde{\pi}}{e_\pi + P + \tilde{\pi}}. \quad (38)$$

Assuming massless pions and Boltzmann statistics for gluons and pions, the energy densities and pressures are

$$e_g = 3n_g T_c + B, \quad e_\pi = 3n_\pi T_c, \quad (39)$$

$$P_g = n_g T_c - B, \quad P_\pi = n_\pi T_c, \quad (40)$$

where $B^{1/4} = 0.23$ GeV is the bag constant [60]. The Gibbs condition gives $P_g = P_\pi$. The ratio of the number of gained pions to that of lost gluons in the conversion from gluons to pions is then

$$r = \frac{n_\pi dV_\pi^{PiH}}{-n_g dV_g^{PiH}} = -\frac{n_\pi}{n_g} \alpha = 1 + \frac{(n_\pi - n_g)\tilde{\pi}}{n_g(e_\pi + P + \tilde{\pi})}. \quad (41)$$

Due to the assumption $\alpha_{PiH} = \alpha_{HiP} = \alpha$, r is the same as the ratio of the number of lost pions to that of gained gluons in the conversion from pions to gluons. According to the definition of the shear pressure [Eq. (8)], $\tilde{\pi}$ is negative for an expansion. $P + \tilde{\pi}$ should be positive for our hydrodynamic description of QCD nucleation. Therefore, we have $r > 1$, which indicates that gained (lost) pions are more than lost (gained) gluons in the (inverse) conversion. We need processes such as $2g \rightarrow 3\pi$, $2g \rightarrow 4\pi$, etc., and their inverse processes. Moreover, due to $P + \tilde{\pi} > 0$, we have

$$\alpha > -\frac{e_g}{e_\pi} = -\frac{4n_g - n_\pi}{3n_\pi}. \quad (42)$$

Thus,

$$1 < r = -\frac{n_\pi}{n_g} \alpha < \frac{4n_g - n_\pi}{3n_g} = \frac{61}{48}, \quad (43)$$

where the degeneracy factors of gluons ($d_g = 16$) and pions ($d_\pi = 3$) are used. Due to $r < 3/2$, the processes $g + g \leftrightarrow \pi + \pi$ and $g + g \leftrightarrow \pi + \pi + \pi$ are sufficient to complete the conversion. We note that the consideration of these microscopic processes is not from the first principles but is necessary to realize the macroscopic volume change according to Eq. (23).

We denote the probability of $g + g \rightarrow \pi + \pi$ and $g + g \rightarrow \pi + \pi + \pi$ by P_{22} and P_{23} , and the probability of $\pi + \pi \rightarrow g + g$ and $\pi + \pi + \pi \rightarrow g + g$ by P_{22b} and P_{32} , respectively. The angular distribution of all the transition processes is assumed to be isotropic. In the following, we derive the explicit form of these transition probabilities.

We assume that the transition probabilities are independent of the momenta of particles involved in the processes. Suppose N_g is the number of gluons and N_π is the number of pions in the volume V . Thus, the number of permutations for a two-gluon pair is $N_g(N_g - 1)/2$. The numbers for a two-pion pair and a three-pion triplet are $N_\pi(N_\pi - 1)/2$ and $N_\pi(N_\pi - 1)(N_\pi - 2)/6$. We then have the numbers of lost gluons and gained pions in the conversion from gluons to pions, i.e., in the processes $g + g \rightarrow \pi + \pi$

and $g + g \rightarrow \pi + \pi + \pi$,

$$\frac{1}{2}N_g(N_g - 1)(2P_{22} + 2P_{23}) = -n_g dV_g^{PiH}, \quad (44)$$

$$\frac{1}{2}N_g(N_g - 1)(2P_{22} + 3P_{23}) = n_\pi dV_\pi^{PiH}. \quad (45)$$

Solving the above equations, we obtain

$$\begin{aligned} P_{22} &= \frac{-3n_g dV_g^{PiH} - 2n_\pi dV_\pi^{PiH}}{N_g(N_g - 1)} \\ &= -\frac{(3 - 2r)n_g dV_g^{PiH}}{N_g(N_g - 1)} \\ &\approx -\frac{3 - 2r}{f_g^2 n_g} \frac{dV_g^{PiH}}{V^2}, \end{aligned} \quad (46)$$

$$\begin{aligned} P_{23} &= 2 \frac{n_g dV_g^{PiH} + n_\pi dV_\pi^{PiH}}{N_g(N_g - 1)} \\ &= \frac{2(1 - r)n_g dV_g^{PiH}}{N_g(N_g - 1)} = \frac{2(r - 1)}{3 - 2r} P_{22}. \end{aligned} \quad (47)$$

Analogously, the equations for the conversion from pions to gluons ($\pi + \pi \rightarrow g + g$ and $\pi + \pi + \pi \rightarrow g + g$ processes) read

$$\begin{aligned} \frac{1}{2}N_\pi(N_\pi - 1)2P_{22b} + \frac{1}{6}N_\pi(N_\pi - 1)(N_\pi - 2)3P_{32} \\ = -n_\pi dV_\pi^{HiP}, \end{aligned} \quad (48)$$

$$\begin{aligned} \frac{1}{2}N_\pi(N_\pi - 1)2P_{22b} + \frac{1}{6}N_\pi(N_\pi - 1)(N_\pi - 2)2P_{32} \\ = n_g dV_g^{HiP}. \end{aligned} \quad (49)$$

The solutions are

$$\begin{aligned} P_{22b} &= \frac{3n_g dV_g^{HiP} + 2n_\pi dV_\pi^{HiP}}{N_\pi(N_\pi - 1)} \\ &= -\frac{(3/r - 2)n_\pi}{N_\pi(N_\pi - 1)} dV_\pi^{HiP} \\ &\approx -\frac{(3/r - 2)}{(1 - f_g)^2 n_\pi} \frac{dV_\pi^{HiP}}{V^2}, \end{aligned} \quad (50)$$

$$\begin{aligned} P_{32} &= 6 \frac{-n_g dV_g^{HiP} - n_\pi dV_\pi^{HiP}}{N_\pi(N_\pi - 1)(N_\pi - 2)} \\ &= -6 \frac{(1 - 1/r)n_\pi}{N_\pi(N_\pi - 1)(N_\pi - 2)} dV_\pi^{HiP} \\ &= \frac{6(1 - 1/r)}{(3/r - 2)(N_\pi - 2)} P_{22b} \\ &\approx \frac{6(r - 1)}{3 - 2r} \frac{1}{(1 - f_g)n_\pi V} P_{22b}. \end{aligned} \quad (51)$$

All the transition probabilities are fixed when dq in Eqs. (35) and (36) is given. Its setup will be shown in the next section.

We now turn to discuss the momentum-energy transfer in processes $g + g \rightarrow \pi + \pi$, $g + g \rightarrow \pi + \pi + \pi$ and their inverse processes. In the BAMPS, a gluon takes only its kinetic energy. But in the transition processes, the potential energy is also involved. According to the energy balance Eq. (31) and the energy density Eq. (39), the energy of gluons before the conversion is the sum of the kinetic energy $-3n_g T_c dV_g^{PiH}$ and

the potential energy $-B dV_g^{PiH}$. The total energy is equal to the sum of the to be released energy dq and the kinetic energy of pions after the conversion. We take $-(3n_g T_c + B) dV_g^{PiH} - dq$ as the gluon energy in the processes $g + g \rightarrow \pi + \pi$ and $g + g \rightarrow \pi + \pi + \pi$ with energy-momentum conservation. The average energy per gluon to be converted is then

$$\begin{aligned} \epsilon_g &= \frac{-(3n_g T_c + B) dV_g^{PiH} - dq}{-n_g dV_g^{PiH}} \\ &= 3T_c + \frac{B}{n_g} - \frac{dq}{-n_g dV_g^{PiH}} \\ &= 3T_c + \frac{B}{n_g} - \frac{dq}{(P + \tilde{\pi}_h)(1 - f_g)dV + dq} \frac{e_g + \alpha e_\pi}{n_g} \\ &= 3T_c + \frac{B}{n_g} - \frac{dq}{(P + \tilde{\pi}_h)(1 - f_g)dV + dq} \\ &\quad \times \left[\frac{B}{n_g} - (r - 1)3T_c \right]. \end{aligned} \quad (52)$$

It is obvious that $\epsilon_g > 3T_c$. Since transition processes with momentum and kinetic energy conservation are numerically implemented in a standard routine, we amplify the momentum (also the kinetic energy) of each gluon by $\epsilon_g/(3T_c)$ before performing $g + g \rightarrow \pi + \pi$ and $g + g \rightarrow \pi + \pi + \pi$ processes.

For the conversion from pions to gluons, we rewrite Eq. (32) as

$$\begin{aligned} -e_\pi dV_\pi^{HiP} &= (3n_g T_c + B) dV_g^{HiP} \\ &\quad + (n_g T_c - B + \tilde{\pi}_g) f_g dV - dq \\ &= 3n_g T_c dV_g^{HiP} + (n_g T_c + \tilde{\pi}_g) f_g dV \\ &\quad + B(dV_g^{HiP} - f_g dV) - dq. \end{aligned} \quad (53)$$

The potential energy $B(-dV_g^{HiP} + f_g dV)$ and the to be absorbed energy dq should be added to the kinetic energy of pions in the conversion. Thus, we take $-e_\pi dV_\pi^{HiP} + B(-dV_g^{HiP} + f_g dV) + dq$ as the energy of pions in the processes $\pi + \pi \rightarrow g + g$ and $\pi + \pi + \pi \rightarrow g + g$. The average energy per pion to be converted is

$$\begin{aligned} \epsilon_\pi &= \frac{-e_\pi dV_\pi^{HiP} + B(-dV_g^{HiP} + f_g dV) + dq}{-n_\pi dV_\pi^{HiP}} \\ &= 3T_c + \frac{B}{\alpha n_\pi} - \frac{dq + B f_g dV}{dq - (P + \tilde{\pi}_g) f_g dV} \frac{e_g + \alpha e_\pi}{\alpha n_\pi} \\ &= 3T_c - \frac{B}{r n_g} + \frac{dq + B f_g dV}{dq - (P + \tilde{\pi}_g) f_g dV} \\ &\quad \times \left[\frac{B}{r n_g} - \frac{r - 1}{r} 3T_c \right]. \end{aligned} \quad (54)$$

Because $B = (n_g - n_\pi)T_c$ and $1 < r < (4n_g - n_\pi)/(3n_g)$ [see Eq. (43)], ϵ_π is larger than $3T_c$, when r approaches to 1, while ϵ_π becomes smaller than $3T_c$, when r approaches to $(4n_g - n_\pi)/(3n_g)$. In the latter case, $\pi + \pi + \pi \rightarrow g + g$ processes have more contributions than in the former case. We multiply the momentum (also the kinetic energy) of each pion by $\epsilon_\pi/(3T_c)$ before performing $\pi + \pi \rightarrow g + g$ and $\pi + \pi + \pi \rightarrow g + g$ processes.

On average, a pion (gluon) after the conversion takes a larger energy than $3T_c$. Through collisions with other pions (gluons), these converted pions (gluons) will lose the energy excess, which just compensates the energy loss in the pion (gluon) phase due to the work in the expansion.

V. TEST OF THE NEW NUCLEATION MODEL

To test the new nucleation scheme that we have introduced in the previous sections, we demonstrate in this section the first-order phase transition from gluons to pions by employing BAMPS for a one-dimensional expansion with Bjorken boost invariance. The formal transition probabilities and the details of the numerical implementations of the processes $g + g \rightarrow \pi + \pi$, $g + g \rightarrow \pi + \pi + \pi$, and their inverse processes are given in Sec. IV. Except for these, all other setups are almost the same as those made in Ref. [53]. Below, we repeat some important setups and give the heat dq explicitly.

Since the test focuses on the nucleation model during the phase transition, the interactions of gluons (pions) in the gluon (pion) phase are considered to be elastic with an isotropic distributed scattering angle for simplicity. The cross section of gluonic scatterings is set to be $\sigma_g = 40$ mb, and the test particle number per a real particle is set to be $N_{\text{test}} = 20\,000$. Both are larger than those given in Ref. [53]. At the initial time $\tau_0 = 0.5$ fm/c, gluons are distributed according to the Boltzmann distribution with the temperature $T_g = 0.3$ GeV. According to Eqs. (52) and (58)–(60) in Ref. [53], the gluon system reaches the temperature at the phase transition $T_c = 0.2227$ GeV at $\tau_c = 1.415$ fm/c. The nucleation begins. At τ_c the shear viscosity to the entropy density ratio is $\eta_g/s_g = 0.0406$. The ratio η_g/s_g should be maintained during the phase transition. We set the cross section of pionic scatterings $\sigma_\pi = \sigma_g s_g/s_\pi = \sigma_g d_g/d_\pi$, so that the shear viscosity to the entropy density ratio of pions takes the same value as that of gluons [61–65]. With this setup, we can obtain analytical $f_g(\tau)$ [see Eq. (45) in Ref. [53]], with which the numerical result will be compared.

We note that the shear pressure $\tilde{\pi}$ is not extracted from the particle distribution but is obtained by employing the first-order theory of hydrodynamics. Then the shear tensor reads

$$\pi_{g/\pi}^{\mu\nu} = 2\eta_{g/\pi} \nabla^{<\mu} U^{\nu>}. \quad (55)$$

For the Bjorken expansion, we have

$$\frac{1}{V} \frac{dV}{d\tau} = \nabla_\mu U^\mu = \frac{1}{\tau}, \quad (56)$$

$$\tilde{\pi}_{g/\pi} = -2\eta_{g/\pi} \frac{\nabla^{<\mu} U^{\nu>} \nabla_{<\mu} U_{\nu>}}{\nabla_\mu U^\mu} = -\frac{4\eta_{g/\pi}}{3\tau}. \quad (57)$$

According to Eqs. (49), (56), and (57) in Ref. [53], we obtain $\eta_g = 0.339$ fm^{−3} and $P = 0.084$ GeV fm^{−3} at τ_c . The shear pressure is then $\tilde{\pi}_g = -0.0629$ GeV fm^{−3}. Thus, $P + \tilde{\pi}_g > 0$ and condition (27) is fulfilled. This is the reason that we had to use an unrealistically large cross section σ_g to get a small η_g , which leads to an unrealistically small η_g/s_g at τ_c . In reality, the phase transition may begin at a much larger time than that given here. Smaller cross sections (larger η_g), with which $P + \tilde{\pi}_g > 0$ [see Eq. (57)], can apply to our nucleation model.

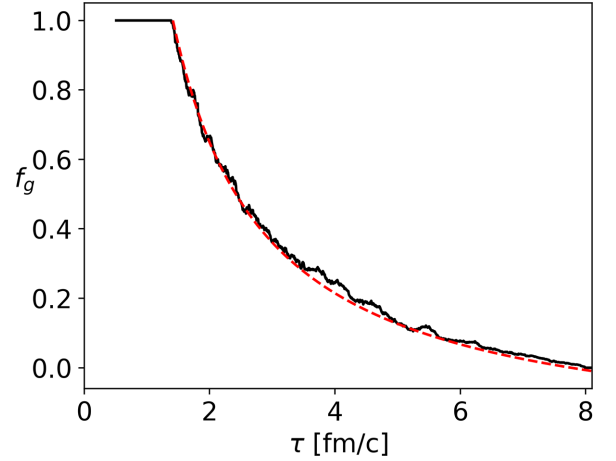


FIG. 2. The time evolution of the gluon fraction. The solid curve and the dashed curve depict the numerical result and the expected function, respectively.

With $\tilde{\pi}_g$ and P we calculate r in Eq. (41) at τ_c and get $r = 1.1872 < 61/48$, which meets the condition (43). Since the absolute value of $\tilde{\pi}$ is inversely proportional to τ , r becomes smaller for $\tau > \tau_c$ and condition (27) and the inequality (43) always hold. This ensures the applicability of our nucleation model.

Recall that dq must be larger than $-e_\pi dV_\pi^F - (P + \tilde{\pi}_\pi)(1 - f_g)dV$ and $e_g dV_g^F + (P + \tilde{\pi}_g)f_g dV$. For our case, $dV_g^F = dV_\pi^F = 0$ and $dV > 0$, so that dq must be larger than $(P + \tilde{\pi}_g)f_g dV$. We set $dq = (1 + \beta)(P + \tilde{\pi}_g)f_g dV$ with $\beta = 0.2$. Then dV_g^{PiH} and dV_π^{HiP} in Eqs. (35) and (36) are known for a given time step $d\tau$. With these, all the transition probabilities P_{22} , P_{23} , P_{22b} , and P_{32} are fixed. We note that, with larger β (also larger dq), more particles will be involved in the transitions [see Eqs. (35) and (36)]. This will reduce the fluctuation in particle number.

In the following, we focus on the local region at zero space-time rapidity with a small interval of 0.03 and show the nucleation in this region. We have checked all the numerical results with the new nucleation model and found that they agree with the analytical results, such as those demonstrated in Ref. [53]. Below, we show some of these results, the time evolution of the gluon fraction, the number and energy density, and the total entropy. Other results, such as the time evolution of the pressure, temperature, chemical potential, and entropy density, can be obtained from the number and energy densities and will not be shown in this presentation.

Figure 2 depicts the numerical extracted gluon fraction f_g (solid curve) and the analytical result (dashed curve) according to Eq. (45) in Ref. [53]. We see a perfect agreement.

In Fig. 3, we show the time evolution of the number and the kinetic-energy density of gluons (black curves) and pions (red curves), respectively, evaluated by Eqs. (70), (71), and (74) in Ref. [53]. For comparison, we multiply the densities of pions by the ratio of the degeneracy factors d_g/d_π . We see that the densities remain almost constant during the phase transition. However, a slight decrease in the densities is also observed when comparing with the analytical

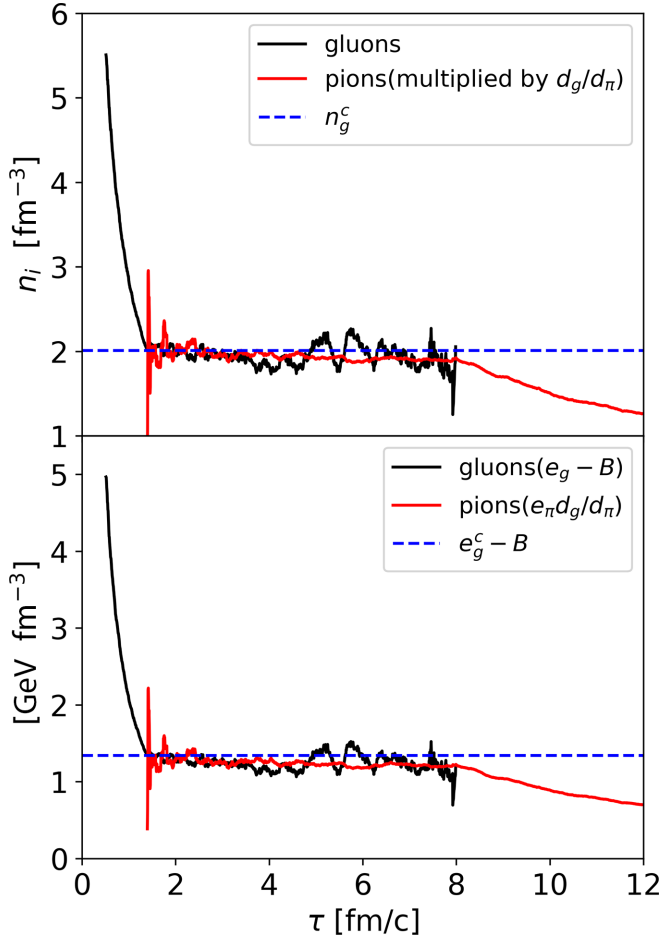


FIG. 3. The time evolution of the number and the kinetic-energy density. The black (red) curves are for gluons (pions). The dashed lines depict the analytical values during the phase transition.

values $n_g^c = n_g(\tau_c) = 2.013 \text{ fm}^{-3}$ and $e_g^c - B = e_g(\tau_c) - B = 1.3448 \text{ GeV fm}^{-3}$, which are denoted by the dashed lines. This is due to the presence of the particle flow of produced pions to higher rapidities, which is ignored in our present microscopic scheme. The particle flow lowers the densities and temperature of pions. Subsequently, the conversion from pions to gluons makes the densities and temperature of gluons smaller than the analytical values. The effect of particle flow was not visible in the former scheme [53] because the back reactions to gluons are made by hand.

Finally, the total entropy per space-time rapidity and per transverse area is calculated according to Eqs. (24) and (44) in Ref. [53] and is shown in Fig. 4. Due to the nonzero but small shear viscosity, the increase of the total entropy during the phase transition is not significant, but still visible despite large numerical uncertainties.

VI. SUMMARY AND OUTLOOK

In this article, we have improved our primary model of QCD nucleation in Ref. [53] by realizing the transition

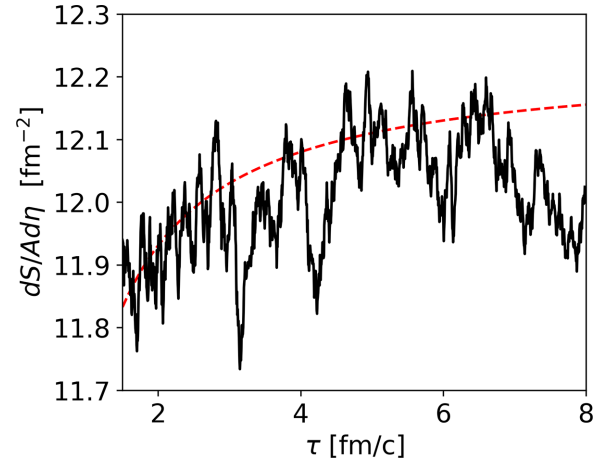


FIG. 4. The total entropy per space-time rapidity per transverse area.

balance in a formal hydrodynamic description. The main idea is that the QCD phase transition from one phase to another is a result of two reversed and simultaneous transitions: one is from partons to hadrons, and another is from hadrons to partons. We formulated the particle and energy transfer in both conversions. Based on these, we established a dynamic scheme for microscopic processes within the kinetic theory and implemented it in the parton cascade model BAMPS. As already demonstrated in Ref. [53], we also tested the improved scheme by comparing our numerical results with the analytical ones derived in Ref. [53] for a first-order phase transition from gluons to pions in a one-dimensional expansion with Bjorken boost invariance. The numerical and analytical results agree well with each other.

The advantage of the proposed model of QCD nucleation is that it can describe both the phase transition from partons to hadrons in an expansion and the one from hadrons to partons in a compression. The next step of improvement will be including quarks and more hadron species into the model and modeling the first-order phase transition with finite baryon chemical potential [57].

ACKNOWLEDGMENTS

Z.X. thanks P. Zhuang for helpful discussions. This work was financially supported by the National Natural Science Foundation of China under Grants No. 12035006 and No. 11890712. C.G. acknowledges support by the Deutsche Forschungsgemeinschaft (DFG) through the Grant No. CRC-TR211 “Strong-interaction matter under extreme conditions.” The BAMPS simulations were performed at Tsinghua National Laboratory for Information Science and Technology and on TianHe-1(A) at the National Supercomputer Center in Tianjin.

DATA AVAILABILITY

There are no publicly available research data or software supporting this paper. Requests for further information or data should be sent to the authors.

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