

# Riemannian geometry of the moment manifold in relativistic kinetic theory and the approach to hydrodynamics

Craig S. Wright 

*University of Exeter, Exeter, England, United Kingdom*



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The kinetic-to-hydrodynamic crossover in the quark-gluon plasma is conventionally diagnosed by scale ratios such as the Knudsen and inverse Reynolds numbers, which are not metrical on the space of distributions. I equip the Denicol–Niemi–Molnár–Rischke (DNMR) 14-moment manifold with the Fisher-Rao metric—the local Hessian of the Kullback-Leibler divergence—and study the resulting Riemannian geometry for conformal Bjorken flow in the relaxation-time approximation. The equilibrium Fisher metric fixes the shear viscosity as  $\eta = 4P\tau_R/5$ , identifying near-equilibrium transport as an equilibrium statistical quantity. The full geodesic distance exceeds its linearized approximation by 15–35% at moderate shear, quantifying where Navier-Stokes-type expansions under-resolve the state-space distance. The equilibrium Fisher-Rao Ricci scalar of the five-dimensional shear sector is computed exactly within the truncation to be  $R = -5/21$ ; the negative sign identifies the equilibrium statistical manifold as locally hyperbolic in the Riemannian-geometry sense, which on a statistical model corresponds to diverging geodesic deviation between nearby moment states—a geometric statement about the truncated exponential family, not a prediction for physical relaxation rates. All results are obtained within the DNMR 14-moment truncation for conformal Bjorken flow; the equilibrium relations are closure independent within the truncation, and finite-shear quantities are properties of the Jaynes maximum-entropy completion.

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## I. INTRODUCTION

### A. The problem this paper addresses

The transition from a kinetic description to a hydrodynamic description in the quark-gluon plasma (QGP) produced at RHIC and the LHC [1,2] is conventionally characterized by scalar diagnostics. The most common is the Knudsen number  $\text{Kn} = \tau_R/\tau$ ; the inverse Reynolds number  $\text{Re}^{-1} = |\pi|/P$  is used alongside it; and the relative entropy  $D_{\text{KL}}(f \| f_{\text{eq}})$  appears through the  $H$  theorem in the standard derivations [3–5]. Each is useful and each fails to do one particular job.  $\text{Kn}$  is a ratio of external scales and does not respond to the distribution function at all.  $\text{Re}^{-1}$  responds to shear only and depends on the frame in which  $\pi$  is defined. Relative entropy is sensitive to the distribution but is a scalar divergence, not a distance: it is not symmetric, does not satisfy a triangle inequality, and does not provide geodesics or curvature on the space of states.

The context for this problem is well established. Hydrodynamic attractors [3,6] explain why hydrodynamics works far from equilibrium, with the gradient expansion itself known to diverge factorially [7] and the attractor picture sharpened

through resurgence [8], adiabatic hydrodynamization [9,10], and bootstrap methods [11]. At the level of formal theory, the Bemfica-Disconzi-Noronha-Kovtun (BDNK) framework [12–14] provides causal, stable first-order viscous hydrodynamics by exploiting the gauge freedom in defining hydrodynamic variables. Transport coefficients themselves are extracted through distinct channels—Kubo formulas [15], holographic models [16,17], lattice QCD [18,19], and Bayesian inference from heavy-ion data [20–22]—but no single formalism treats them as components of one geometric object. The present paper does so within the specific setting defined below.

*Why QGP makes the geometric question physically meaningful.* Three features of the QGP produced at RHIC and the LHC make a metrical, coordinate-invariant diagnostic of the kinetic-to-hydrodynamic crossover physically useful rather than purely formal. First, the system hydrodynamizes extremely fast ( $\tau_{\text{hydro}} \lesssim 1$  fm/cycle), so scale-ratio diagnostics like  $\text{Kn}$  and  $\text{Re}^{-1}$  are evaluated in the regime where they are marginal; a distance diagnostic that responds to the full moment content of  $f$ , rather than to external scale ratios, is a concrete observational need. Second, the hydrodynamic attractor is a curve in moment space whose geometric status (geodesic or nongeodesic) is a physically meaningful question with no answer in the scale-ratio language. Third, Bayesian inference of transport coefficients from heavy-ion data [20–22] varies the 14-moment manifold globally; a metric on that manifold turns such variation into geometric statements about sensitivity and distinguishability. The Fisher-Rao metric is the unique coordinate-invariant local

\*Contact author: cw881@exeter.ac.uk

answer to this class of question on a statistical model (Čencov’s theorem).

This paper addresses one concrete problem: in the conformal Bjorken-flow QGP described by the relaxation-time approximation (RTA) Boltzmann equation in the Denicol–Niemi–Molnár–Rischke (DNMR) (DNMR) 14-moment truncation, construct a coordinate-invariant, metrical scalar measure of the distance *within the truncated exponential family spanned by the DNMR moments* between a nonequilibrium moment state and the local-equilibrium projection onto that truncation, determine what that distance says about near-equilibrium transport and about the breakdown of linearized (tangent-space) expansions, and compute the intrinsic curvature of the resulting moment-projected statistical manifold. The three levels at which the argument operates are kept separate throughout: (1) Fisher-Rao as the canonical local metric on a statistical model; (2) the DNMR 14-moment family as a chosen *Ansatz* space; and (3) the geometric quantities computed on that *Ansatz* space, which are diagnostics of the projection, not of the full kinetic theory. The paper addresses levels 2 and 3 only within the stated setting.

The object that answers this problem is the Fisher-Rao metric [23,24] on the DNMR moment manifold. Its relationship to the relative entropy already used in kinetic theory is local and precise: the Fisher metric is the Hessian of the Kullback-Leibler divergence at equilibrium,

$$D_{\text{KL}}[f(\theta) \| f(\theta + \delta\theta)] = \frac{1}{2} g_{AB}(\theta) \delta\theta^A \delta\theta^B + O(\delta\theta^3), \quad (1)$$

so the Fisher-Rao metric is not an alternative to relative entropy but its canonical *infinitesimal* form. The present construction relies only on this local identity; no equivalence between Fisher-Rao and  $D_{\text{KL}}$  is assumed beyond it. By a theorem of Čencov [23], the Fisher-Rao metric is the unique Riemannian metric on a statistical manifold invariant under sufficient statistics. This fixes Fisher-Rao as the canonical *local* metrical object induced by relative entropy; it does not exclude other geometric constructions on the same space (for instance the Wasserstein geometry discussed in Sec. V D), and no claim of physical uniqueness is made here.

## B. What is computed in this paper

Three consequences follow from equipping the DNMR moment manifold with the Fisher-Rao metric in conformal Bjorken flow under RTA.

(1) *Shear viscosity from the equilibrium metric* (Sec. IV, Theorem 2). The dissipative Fisher metric component is the exact dimensionless constant  $g_{\alpha\alpha}|_{\text{eq}} = 2/15$ , and the shear viscosity is

$$\eta = 6P \tau_R g_{\alpha\alpha}|_{\text{eq}} = \frac{4}{3} P \tau_R, \quad (2)$$

with  $P$  the equilibrium pressure. Near-equilibrium transport is an equilibrium statistical geometric quantity in this setting.

(2) *A finite-shear deviation reconstructible from the exact kinetic solution within the model* (Sec. V). The full geodesic Fisher-Rao distance  $\mathcal{D}_{\text{full}}(\alpha_\pi) = \int_0^{\alpha_\pi} \sqrt{g_{\alpha\alpha}(\alpha)} d\alpha$  differs from its linearized form  $\mathcal{D}_{\text{lin}}(\alpha_\pi) = \sqrt{g_{\alpha\alpha}|_{\text{eq}}} \alpha_\pi$  by 15–35% over the moderate-shear regime, as measured directly against the exact RTA Boltzmann solution

(Sec. IX, Table II). This is the model-internal observable used to test the geometry on the moment-projected manifold against the exact RTA Boltzmann solution projected onto the same 14-moment basis: the 15–35% gap is reconstructible from that projection and quantifies where the tangent-space (linearized) expansion under-resolves the geometric distance within the projected manifold. The 15–35% figure is a property of the projection; it is not a statement about distances in the full Boltzmann phase space, which the 14-moment truncation does not resolve.

(3) *Intrinsic curvature of the shear sector* (Sec. VI, Theorem 4). The five-dimensional (5D) conformal shear sector of the moment manifold has Ricci scalar

$$R|_{\text{eq}} = -\frac{5}{21}, \quad (3)$$

a rational number fixed by the  $\text{SO}(3)$  representation structure of the moment basis (rank-2 traceless symmetric) and the angular structure of the shear moment function on  $S^2$ . Any one-dimensional Bjorken restriction is flat: the curvature is an intrinsically multidimensional feature of the shear sector.

Two auxiliary results are used to connect the geometry to the dynamics but are not claims of dynamical universality: Sec. VIII computes the geodesic curvature of Israel-Stewart (IS) trajectories on the manifold and finds  $\kappa_g = \mathcal{O}(1)$  throughout the evolution, establishing that the hydrodynamic attractor is *not* a geodesic of the Fisher-Rao metric; Sec. IX verifies the finite-shear distance gap against exact RTA numerically and calibrates the IS-RTA convergence scale.

## C. Scope, observables, and what this paper does not claim

This paper makes no claims about general reformulations of relativistic kinetic theory. All results are obtained within one specific setting: the RTA collision kernel, the DNMR 14-moment truncation, and conformal Bjorken flow. The Limitations section (Sec. XII) states failure modes explicitly.

Two categories of result must be distinguished, and the language throughout the paper enforces the distinction.

*Definitional results* are formulated independently of the specific collision kernel and truncation used here: the Fisher-Rao metric itself (1), its positive definiteness (Theorem 1), the relation between the equilibrium Fisher metric in a dissipative direction and the corresponding transport coefficient (Theorem 2), the geodesic and curvature framework, and the finite-shear metric cumulant expansion. Whether each of these constructions carries over to a given alternative kernel or truncation is a question that must be answered case by case; the paper does not make that claim generically.

*Equilibrium geometric values* are the specific numbers:  $g_{\alpha\alpha}|_{\text{eq}} = 2/15$ ,  $R = -5/21$ , the ten sectional curvatures, the Jacobi-deviation rates, and the 15–35% finite-shear correction. These values are fixed by angular integrals against the equilibrium distribution and by Schur’s lemma applied to the  $\text{SO}(3)$  irreducible blocks of the DNMR 14-moment basis. Within the restricted class of truncation extensions considered here—those that add only  $\text{SO}(3)$ -irreducible blocks orthogonal to the shear block at equilibrium—the values are unchanged at the level of the argument given. Within extensions that couple the shear block (for example nonconformal

TABLE I. Summary of moment manifolds and associated results. Each row identifies the manifold, its coordinates, dimensionality, and the theorems or results that are proved on it.

| Manifold                   | dim | Coordinates                                         | Results                                                                                                              |
|----------------------------|-----|-----------------------------------------------------|----------------------------------------------------------------------------------------------------------------------|
| Bjorken conformal          | 2   | $(T, \alpha_\pi)$                                   | Theorems 1–3; geodesic curvature $\kappa_g$ ; numerical Sec. IX                                                      |
| Conformal shear            | 5   | $\alpha^{(a)}, a = 1, \dots, 5$                     | Theorem 4 ( $R = -5/21$ ); sectional curvatures; $\text{SO}(3) \rightarrow \text{SO}(2)$ ; Jacobi hierarchy (Sec. X) |
| Conformal (therm. + shear) | 6   | $(T, \alpha^{(a)})$                                 | Block structure; Theorem 2 ( $g_{\alpha\alpha} _{\text{eq}} = 2/15$ )                                                |
| Nonconformal Bjorken       | 2   | $(T, \alpha_\pi)$                                   | $R_{2D}(T)$ numerical; $g_{\alpha\alpha}(T, m)$                                                                      |
| Full DNMR                  | 14  | $(T, u^i, \alpha_\pi, \alpha_\pi^{ij}, \alpha_n^i)$ | Formal definitions only; future work                                                                                 |

bulk-shear mixing), the argument does not hold and the values change. No claim is made that these values survive extensions beyond the 14-moment basis at higher angular momentum  $\ell > 2$ .

The observable of the paper is result (ii): the 15–35% gap between  $\mathcal{D}_{\text{full}}$  and  $\mathcal{D}_{\text{lin}}$ , which is reconstructible from the exact kinetic solution within the model and provides a geometric quantification of near-equilibrium expansion breakdown.

- (1) *Three-layer separation.* The paper operates at three clearly distinct levels, and the remaining sections keep them separate.
  - (a) The Fisher-Rao metric is the canonical local quadratic form induced by relative entropy on any statistical model; this is established independently of which model is chosen.
  - (b) The DNMR 14-moment expansion defines a specific truncated exponential family—a modeling choice—on which the Fisher-Rao metric of level (a) is induced.
  - (c) All geometric quantities computed in this paper (distance, sectional curvatures, Ricci scalar, Jacobi-deviation rates) are intrinsic properties of the truncated family of level (b) under the metric of level (a). They are not derived properties of the full kinetic theory, and they are not claimed to be stable under enlargement of the *Ansatz* beyond the class specified in Sec. III E.

In particular, the geometric distance  $\mathcal{D}$  introduced below is a distance *within the projected Ansatz space of level (b)*, not within the full space of admissible distribution functions; two distributions with identical DNMR moments but different high-momentum tails are not distinguished by  $\mathcal{D}$ . Conversely, a given DNMR state  $(\varepsilon, u^\mu, \pi^{\mu\nu})$  does not fix a unique distribution: it is compatible with a family of distributions

that share the same 14 moments but differ at higher orders. The exponential family (19) (Sec. III D) is one such completion—the Jaynes maximum-entropy completion—and its choice is a hypothesis, not a theorem. Quantities that depend only on equilibrium integrals against  $f_{\text{eq}}$  are closure independent within the 14-moment truncation; quantities that depend on  $O(\alpha_\pi^2)$  and higher content of the completion are closure dependent, and are reported in this paper as properties of the maximum-entropy completion.

Finally, and as a separate point of scope: Fisher-Rao is a geometry of statistical distinguishability on a space of distributions, not a geometry of physical state space. The paper does not claim these coincide, and does not use the Fisher-Rao structure to derive properties of the physical state space beyond what is explicitly labeled as a property of the statistical family.

- (2) *Positioning.* The contribution of this paper is not a general reformulation of relativistic kinetic theory. It is a controlled geometric description of the conformal Bjorken QGP in the RTA 14-moment truncation, in which near-equilibrium transport, finite-shear distance, and intrinsic curvature appear as consequences of a single object—the Fisher-Rao metric on the truncated moment manifold—with one observable accessible from the exact kinetic solution *projected onto the same 14-moment basis* within the model, against which the construction is tested. All geometric quantities are properties of the moment-projected exponential-family manifold used here as a diagnostic embedding; they are not claimed to correspond to invariant features of the full kinetic theory beyond that embedding.

## D. Organization

Section II fixes the RTA Boltzmann setup and the DNMR moment expansion used throughout. Section III constructs the Fisher-Rao metric on the moment manifold, proves its positive definiteness (Theorem 1), and establishes the internal consistency and truncation robustness of the construction (Sec. III D). Section IV derives the shear-viscosity relation (Theorem 2). Section V defines the Fisher-Rao distance, relates it to relative entropy, and proves its leading-order monotonicity (Theorem 3). Section VI computes the intrinsic geometry of the shear sector (Theorem 4). Section VII records the nonconformal behavior of the dissipative metric in

TABLE II. Full vs linearized Fisher-Rao distance at  $T = 0.5$  GeV.

| $\alpha$ | $\mathcal{D}_{\text{full}}$ | $\mathcal{D}_{\text{lin}}$ | Ratio |
|----------|-----------------------------|----------------------------|-------|
| 0.1      | 0.036                       | 0.037                      | 0.994 |
| 0.5      | 0.177                       | 0.183                      | 0.968 |
| 1.0      | 0.340                       | 0.365                      | 0.932 |
| 2.0      | 0.625                       | 0.730                      | 0.856 |
| 3.0      | 0.856                       | 1.095                      | 0.781 |
| 5.0      | 1.194                       | 1.826                      | 0.654 |

a classical massive gas (not QCD). Section VIII computes the geodesic curvature of Israel-Stewart trajectories. Section IX gives the numerical comparison of  $\mathcal{D}_{\text{full}}$  against exact RTA. Section X derives the Jacobi-deviation pattern on the shear manifold. Section XI discusses relations to existing geometric frameworks and the extension structure. Section XII states the failure modes of the construction explicitly. The proof of Theorem 2 is given in the Appendix.

## II. SETUP: RTA BOLTZMANN EQUATION AND DNMR MOMENTS

### A. Bjorken flow

I consider a system undergoing boost-invariant longitudinal expansion in the central rapidity region [25]. I work in Milne coordinates  $(\tau, x, y, \eta_s)$  with metric  $ds^2 = d\tau^2 - dx^2 - dy^2 - \tau^2 d\eta_s^2$ , where  $\tau = \sqrt{t^2 - z^2}$  is the proper time and  $\eta_s = \frac{1}{2} \ln[(t+z)/(t-z)]$  the spacetime rapidity. The flow velocity is  $u^\mu = (1, 0, 0, 0)$  in Milne coordinates.

The energy-momentum tensor decomposes as

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + (P + \Pi) \Delta^{\mu\nu} + \pi^{\mu\nu}, \quad (4)$$

where  $\epsilon$  is the energy density,  $P$  the equilibrium pressure,  $\Pi$  the bulk viscous pressure,  $\pi^{\mu\nu}$  the shear stress tensor, and  $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$  the spatial projector. In Bjorken flow, the expansion rate is  $\theta \equiv \nabla_\mu u^\mu = 1/\tau$ , the vorticity vanishes, and the shear tensor has a single independent component.

For a conformal system ( $\Pi = 0$ ,  $P = \epsilon/3$ ), I define the normalized shear

$$\bar{\pi} \equiv \frac{\pi}{\epsilon + P} = \frac{3\pi}{4\epsilon}, \quad (5)$$

where  $\pi \equiv P_T - P_L$  is the pressure anisotropy with  $P_T$  and  $P_L$  the transverse and longitudinal pressures satisfying  $2P_T + P_L = 3P$ .

### B. Relaxation-time approximation

The Boltzmann equation in the RTA reads

$$p^\mu \partial_\mu f + \Gamma_{\alpha\beta}^\mu p^\alpha p^\beta \frac{\partial f}{\partial p^\mu} = -\frac{up}{\tau_R} (f - f_{\text{eq}}), \quad (6)$$

where  $f(x, p)$  is the one-particle distribution function on the mass shell  $\Sigma_m = \{p^\mu : g_{\mu\nu} p^\mu p^\nu = m^2, p^0 > 0\}$ , and  $f_{\text{eq}}$  is the equilibrium (Jüttner) distribution. For a massless classical (Boltzmann-statistics) gas,

$$f_{\text{eq}}(p; T, u) = \frac{g}{(2\pi)^3} \exp\left(-\frac{p^\mu u_\mu}{T}\right), \quad (7)$$

where  $g$  is the degeneracy factor and  $T$  the temperature.

The relaxation time is

$$\tau_R = \frac{5\eta/s}{T}, \quad (8)$$

where  $\eta/s$  is the specific shear viscosity, fixed as a constant. The factor 5 is specific to the conformal RTA with Boltzmann statistics.

### C. DNMR 14-moment expansion

The DNMR method [26] expands the nonequilibrium distribution around  $f_{\text{eq}}$  using irreducible tensor moments:

$$f = f_{\text{eq}}(1 + \phi), \quad \phi = \sum_r a_r H_r(p), \quad (9)$$

where  $H_r(p)$  are orthogonal polynomial functions on the mass shell, constructed from irreducible projections of  $p^{\mu_1} \dots p^{\mu_n}$ , and  $a_r$  are moment coefficients.

For Bjorken flow of a conformal gas, the only active dissipative channel is shear. The relevant moment function is

$$H_\pi(p) = \frac{p^{(\mu} p^{\nu)} \hat{\sigma}_{\mu\nu}}{(up)^2}, \quad (10)$$

where  $\hat{\sigma}_{\mu\nu} = \sigma_{\mu\nu}/\sqrt{\sigma^{\alpha\beta}\sigma_{\alpha\beta}}$  is the unit shear tensor and  $p^{(\mu} p^{\nu)} = \Delta_{\alpha\beta}^{\mu\nu} p^\alpha p^\beta$  is the traceless-symmetric spatial projection.

The shear stress is related to the moment coefficient by

$$\pi^{\mu\nu} = \beta_\pi \alpha_\pi \hat{\sigma}^{\mu\nu}, \quad (11)$$

where  $\beta_\pi = 4P/5 = 4\epsilon/15$  is the DNMR thermodynamic integral for the shear channel (specific to the conformal RTA). The scalar  $\alpha_\pi$  serves as the dissipative coordinate on the moment manifold.

### D. Israel-Stewart equations

The DNMR truncation [26] extends the Israel-Stewart theory of transient relativistic thermodynamics [27,28] by deriving the second-order transport coefficients directly from kinetic theory via the method of moments [29]. The resulting equations for conformal Bjorken flow [2,26] are

$$\frac{d\epsilon}{d\tau} = -\frac{4\epsilon/3 - \frac{2}{3}\pi}{\tau}, \quad (12)$$

$$\frac{d\pi}{d\tau} = -\frac{\pi}{\tau_R} + \frac{4\beta_\pi}{3\tau} - \frac{38}{21} \frac{\pi}{\tau}, \quad (13)$$

where the coefficient  $38/21 = \delta_{\pi\pi} + \tau_{\pi\pi}/3 = 4/3 + 10/21$  collects the two second-order transport coefficients  $\delta_{\pi\pi} = 4\tau_R/3$  and  $\tau_{\pi\pi} = 10\tau_R/7$  from Table II in Ref. [26] (conformal massless classical gas, 14-moment approximation). The Bjorken reduction of the  $\tau_{\pi\pi}$  tensor coupling  $\pi^{(\mu} \lambda^{\nu)} \sigma^{\lambda)}$  uses the identity  $\hat{\sigma}^{(\mu} \lambda^{\nu)} \hat{\sigma}^{\lambda)} = \hat{\sigma}^{\mu\nu}/\sqrt{6}$  together with  $|\sigma| = \sqrt{2/3}/\tau$ . No nonlinear  $\pi^2$  term appears: the  $O(\text{Re}^{-2})$  collision-integral coefficients  $\phi_1 - \phi_8$  are not computed in the DNMR framework [26].

## III. FISHER-RAO METRIC ON THE MOMENT MANIFOLD

### A. The moment manifold

At each spacetime point, the distribution function  $f(p; \theta)$  is parametrized by a finite set of coordinates  $\theta^A$  on a manifold  $\mathcal{M}$ . For the 14-moment truncation in Bjorken flow (conformal, massless), the relevant coordinates are

$$\theta^A = \{T, \alpha_\pi\}, \quad (14)$$

where  $T$  is the temperature and  $\alpha_\pi$  is the shear moment coefficient. This is a two-dimensional (2D) manifold. For

the general (nonconformal, non-boost-invariant) case, the coordinates extend to  $\theta^A = \{T, u^i, \alpha_\Pi, \alpha_\pi^{ij}, \alpha_n^i\}$ , spanning a 14-dimensional manifold.

Table I summarizes the different manifold truncations used in this paper and which results are established on each.

### B. Definition of the Fisher-Rao metric

To define a probability measure on the mass shell, I use the Lorentz-invariant one-particle measure

$$d\mu_p \equiv (up) d\Gamma_p, \quad d\Gamma_p = \frac{d^3p}{(2\pi)^3 p^0}. \quad (15)$$

The normalized probability density is

$$\hat{f}(p; \theta) = \frac{f(p; \theta)}{n(\theta)}, \quad n(\theta) \equiv \int_{\Sigma_m} f(p; \theta) d\mu_p. \quad (16)$$

The Fisher-Rao metric on  $\mathcal{M}$  is then

$$g_{AB}(\theta) = \int_{\Sigma_m} \hat{f}(p; \theta) \partial_A \ln \hat{f} \partial_B \ln \hat{f} d\mu_p. \quad (17)$$

With this convention, all expectations  $\langle \dots \rangle$  below are taken with respect to the normalized measure  $\hat{f} d\mu_p$ .

This definition is the direct relativistic extension of the Fisher information matrix from mathematical statistics. It has the following defining properties.

- (1) *Riemannian*.  $g_{AB}$  is symmetric and positive semidefinite by construction (it is an expectation of a product of score functions).
- (2) *Invariant*.  $g_{AB}$  is the unique (up to a constant) Riemannian metric on statistical manifolds that is invariant under sufficient statistics [23].
- (3) *Contractive (in the full distribution space)*. The Fisher-Rao distance between two distributions in the full distribution space decreases under Markov maps—including the action of the collision operator—by the data-processing inequality [24]. This property applies to pairs of full distributions; projection onto the DNMR truncation and evolution under the projected dynamics do not in general commute, so the Fisher-Rao distance between projected states need not decrease monotonically under the projected dynamics. Where monotonicity is invoked later in this paper (Theorem 3), it is proved directly within the linearized DNMR/IS setting, not imported from the full-space Markov-contraction property.

### C. Exponential-family structure

The equilibrium distribution (7) belongs to the exponential family:  $f_{\text{eq}} \propto \exp(-p \times u/T)$ , with natural parameter  $\beta_\mu = u_\mu/T$  and sufficient statistic  $p^\mu$ . For the exponential family, the Fisher-Rao metric reduces to the Hessian of the log-partition function [24]:

$$g_{AB}|_{\text{eq}} = \frac{\partial^2 \ln Z}{\partial \theta^A \partial \theta^B}, \quad (18)$$

where  $Z(\theta) = \int \exp[-\psi(p; \theta)] d\Gamma_p$  with  $\psi$  the negative log likelihood.

For the DNMR parametrization, the nonequilibrium distribution takes the exponential form

$$f(p; \theta) = f_{\text{eq}}(p; T) \exp[\alpha_\pi H_\pi(p) - \Psi(\theta)], \quad (19)$$

where  $\Psi(\theta) = \ln[\int f_{\text{eq}} e^{\alpha_\pi H_\pi} d\Gamma_p / \int f_{\text{eq}} d\Gamma_p]$  ensures normalization. This exponential parametrization [rather than the linear approximation  $f = f_{\text{eq}}(1 + \phi)$ ] is essential: it ensures  $f > 0$  everywhere and places the family within the exponential-family framework where the Fisher metric has its simplest form.

*Exponential family as a closure choice.* The exponential family defined by (19) is the Jaynes maximum-entropy completion of the DNMR 14-moment truncation: among all distributions with the given moments  $\{1, p^\mu, p^\mu p^\nu, \dots\}$ , it is the unique distribution that maximizes the Gibbs entropy. This is a *modeling choice*, not a derivation from the Boltzmann equation. A given DNMR state  $(\varepsilon, u^\mu, \pi^{\mu\nu})$  fixes only the 14 moments and is compatible with a family of distributions that differ at higher moments (tails, skewness, kurtosis); the maximum-entropy completion singles out one representative from that family by hypothesis. In consequence, the following hold.

- (1) The equilibrium Fisher metric  $g_{\alpha_\pi \alpha_\pi}|_{\text{eq}} = 2/15$  and the resulting Navier-Stokes (NS) relation  $\eta = 4P\tau_R/5$  are evaluated at  $\alpha_\pi = 0$ , where all admissible completions coincide to leading order; these leading-order results are therefore closure independent within the 14-moment truncation.
- (2) The finite-shear metric correction, the full geodesic distance  $\mathcal{D}_{\text{full}}$ , the finite-shear deviation, the Amari-Chentsov tensor, the sectional curvatures, and the Ricci scalar are evaluated at  $\alpha_\pi > 0$  or use  $O(\alpha_\pi^2)$  and higher structure of the exponential family; these results are *closure dependent*: they are properties of the maximum-entropy completion of the DNMR truncation and are not uniquely determined by the 14-moment state alone.

The paper reports closure-dependent quantities as diagnostics of the chosen (maximum-entropy) completion of the DNMR moments; it does not claim they are uniquely determined by the physical state. Other admissible completions of the same moments would in general produce different values for these quantities. This closure dependence is restated as a limitation in Sec. XII(7).

### D. Scope and internal consistency

Two questions bear on the interpretation of the construction. The first is a consistency question: the geometry is defined on the exponential-family manifold of the DNMR truncation and therefore depends on the higher-order structure of the distribution function in  $\alpha_\pi$ , while the dynamics used throughout this paper is the standard linearized DNMR/Israel-Stewart closure. The second is a robustness question: to what extent are the specific numerical results (such as  $R = -5/21$ ) intrinsic to the physics rather than features of the low-dimensional truncation? Both are addressed explicitly.

*Consistency: Geometry is kinematic, and dynamics is linearized.* The exponential parametrization (19) and the standard DNMR linear Ansatz  $f = f_{\text{eq}}(1 + \phi)$  with  $\phi = \alpha_\pi H_\pi$  are related by a reparametrization exact to  $O(\alpha_\pi)$  and differing at  $O(\alpha_\pi^2)$ . Expanding,

$$f = f_{\text{eq}} e^{\alpha_\pi H_\pi - \Psi} = f_{\text{eq}} [1 + \alpha_\pi H_\pi + O(\alpha_\pi^2)], \quad (20)$$

reproduces the DNMR linear approximation at leading order. The difference at  $O(\alpha_\pi^2)$  modifies the non-Gaussian tails of the distribution but does not change the linearized IS equations, which depend only on  $f$  at  $O(\alpha_\pi)$  through the moment  $\pi = 2\beta_\pi \alpha_\pi$ . The higher-order structure of the exponential family is never inserted into the dynamics. Four statements make the internal consistency precise.

- (1) The IS Eqs. (12) and (13) are derived from the linearized Boltzmann equation and are independent of the distribution-function parametrization at  $O(\alpha_\pi^2)$  and beyond.
- (2) The equilibrium metric  $g_{\alpha\alpha}|_{\text{eq}} = 2/15$  and the Amari-Chentsov tensor  $T_{abc}$  are evaluated at  $\alpha_\pi = 0$ , where the exponential and linear Ansätze coincide exactly. All theorems in this paper use equilibrium quantities and are therefore insensitive to the parametrization beyond leading order.
- (3) The finite-shear metric correction [Eq. (34)] is a geometric property of the exponential-family manifold as a Riemannian space. It is compared against the exact RTA Boltzmann solution in Sec. IX, not against IS trajectories. The 15–35% correction is a statement about the manifold itself, not a prediction for any dynamics.
- (4) When IS trajectories are studied on the manifold (Sec. VIII), the trajectory  $(T(\tau), \alpha_\pi(\tau))$  is a curve in the moment manifold, and its geodesic curvature is a geometric property of that curve. The geometry does not feed back into the IS equations, and no variational principle, gradient flow, or geodesic evolution law linking the Fisher-Rao geometry to the physical dynamics is invoked or assumed anywhere in this paper. Geodesics of the Fisher-Rao metric are curves defined by the metric on the Ansatz space; they are not trajectories of any physical flow.

The interpretation adopted throughout is therefore the following: the geometry is kinematic, defined independently of truncation order in the dynamics; the linearized DNMR/IS dynamics is used solely to generate explicit trajectories on the manifold. A fully nonlinear closure using the exponential-family structure to compute nonlinear transport coefficients is a separate project, beyond the scope of this paper.

*Consequence.* Because the IS dynamics used here does not see the  $O(\alpha_\pi^2)$  content of the exponential family, any geometric quantity that depends on that content—the finite-shear metric correction, the full geodesic distance  $\mathcal{D}_{\text{full}}$ , the 15–35% deviation, the Amari-Chentsov tensor, the sectional curvatures and Ricci scalar—is an intrinsic property of the exponential-family manifold used as a diagnostic embedding, not a quantity probed by the dynamics. Such quantities are interpreted in this paper as properties of the projected Ansatz

space; they are not claimed to correspond to invariant features of the full kinetic theory, and they are not claimed to be reconstructible from the linearized IS evolution alone. Wherever these quantities are compared against the exact RTA Boltzmann solution (Sec. IX, Table II), the comparison is performed between the exponential-family geometry and the exact RTA solution *projected onto the same 14-moment basis*; the projection step is not commutative with the full-theory evolution, and the resulting gap is a property of the projection.

*Robustness: Structural vs truncation-specific results.* The Fisher-Rao metric is defined on any statistical manifold, for any collision kernel, for any moment truncation. Structural results—the metric definition, its positive definiteness (Theorem 1), the relation (36) to relative entropy, the geodesic and curvature framework—carry over to any extension of the setup. Specific numerical results (the exact value  $g_{\alpha\alpha}|_{\text{eq}} = 2/15$ , the Ricci scalar  $R = -5/21$ , the Jacobi-deviation rates, and the 15–35% finite-shear correction) are features of the 14-moment truncation and the equilibrium angular structure of  $H_\pi$  on  $S^2$ .

Three robustness properties make the numerical results less fragile than the truncation would suggest.

- (1) *Reparametrization invariance.* The Fisher-Rao metric transforms as a tensor under smooth reparametrizations of the moment manifold; the Ricci scalar is a coordinate-invariant scalar. The value  $R = -5/21$  is independent of the coordinate choice within the truncation.
- (2) *Equilibrium invariance within DNMR.* The equilibrium metric  $g_{ab}|_{\text{eq}} = C_2 \delta_{ab}$  and the Amari-Chentsov tensor  $T_{abc}$  are defined by angular integrals of basis functions against  $f_{\text{eq}}$ . For the five-dimensional shear sector, the basis elements  $H^{(a)}$  carry the same SO(3) irreducible representation (rank-2 traceless symmetric), so the equilibrium integrals are fixed by Schur’s lemma. This provides a purely *kinematic* statement about the equilibrium geometry: it involves only  $f_{\text{eq}}$ , not the collision operator. The *dynamics* on the manifold does of course depend on the collision kernel; what does not is the equilibrium metric and the curvature derived from it. This is the proper scope of the present claim: equilibrium curvature values in the shear sector are determined at the level of the equilibrium distribution alone.
- (3) *Submanifold invariance under orthogonal truncation extension.* The DNMR 14-moment basis decomposes at equilibrium into SO(3)-irreducible blocks: scalar (bulk), vector (heat flux), and rank-2 traceless symmetric (shear). Under the Fisher metric, inner products between blocks of different irreducible type at equilibrium vanish by Schur’s lemma. Within the restricted class of truncation extensions considered here—those that add only orthogonal SO(3)-irreducible blocks, such as the scalar and vector channels of the full DNMR 14-moment basis—the block structure of the extended metric is unchanged at the level of the argument given: the shear sector remains a Riemannian submanifold of the extension, with its induced met-

ric, Amari-Chentsov tensor, and sectional curvatures unchanged. In particular, the value  $R = -5/21$  for the shear sector is unchanged within this restricted class. *The word “unchanged” here and elsewhere in this subsection refers specifically to equilibrium angular integrals against  $f_{\text{eq}}$ : these integrals vanish between inequivalent SO(3)-irreducible blocks at equilibrium (Schur’s lemma), so the induced metric, Amari-Chentsov tensor, and sectional curvatures of the shear block coincide before and after the enlargement. This is a statement about the equilibrium statistical manifold within the specified class of extensions, not a general claim of robustness under change of truncation, closure, or collision kernel.* The argument does not apply to extensions that break SO(3) (e.g., anisotropic backgrounds) or that couple the shear block to other blocks at equilibrium (e.g., nonconformal couplings once bulk and shear mix at finite mass), and no claim is made about such extensions. The  $(T, \alpha_\pi)$  Bjorken submanifold curvature changes once nonconformal bulk and heat-flux couplings enter; this is addressed numerically in Sec. VII.

- (4) *Limits of the truncation.* The 14-moment truncation breaks down when the distribution develops structure at angular momentum  $\ell > 2$ . The Fisher-Rao analysis inherits this limitation: the moment manifold only captures what the truncation can resolve. I do not claim  $R = -5/21$  is a property of the full Boltzmann phase space; I claim it is an exact property of the DNMR 14-moment manifold, which is itself a well-defined and widely used approximation to the full kinetic theory. The results of this paper are therefore a proof of principle within the truncation: the definitional constructions (metric, Hessian identity, geodesic framework) are formulated at a level that does not depend on the collision kernel, while the specific numerical values are properties of the stated truncation.

### E. Block structure

The coordinates split into a thermodynamic sector  $\theta^a = \{T\}$  and a dissipative sector  $\theta^\alpha = \{\alpha_\pi\}$ . At equilibrium ( $\alpha_\pi = 0$ ), the cross terms vanish by isotropy:

$$g_{T\alpha_\pi}|_{\text{eq}} = \left\langle \frac{\partial \ln f_{\text{eq}}}{\partial T} H_\pi \right\rangle_{\text{eq}} = 0, \quad (21)$$

because  $\partial_T \ln f_{\text{eq}} = p \times u/T^2$  is a scalar under rotations while  $H_\pi$  is a rank-2 traceless tensor. The metric at equilibrium is therefore block diagonal:

$$g_{AB}|_{\text{eq}} = \begin{pmatrix} g_{TT}|_{\text{eq}} & 0 \\ 0 & g_{\alpha_\pi\alpha_\pi}|_{\text{eq}} \end{pmatrix}. \quad (22)$$

### F. Theorem 1: Positive definiteness

*Theorem 1.* For  $T > 0$  and for all values of the dissipative coordinates for which the cumulant-generating function  $\Psi(\theta)$  is finite, the Fisher-Rao metric  $g_{AB}$  is positive definite on the moment manifold  $\mathcal{M}$ .

*Proof.* For any tangent vector  $v^A$ ,

$$g_{AB} v^A v^B = \int_{\Sigma_m} \hat{f} (v^A \partial_A \ln \hat{f})^2 d\mu_p \geq 0. \quad (23)$$

The equality holds only if  $v^A \partial_A \ln \hat{f} = 0$  almost everywhere on  $\Sigma_m$ . In the Bjorken conformal truncation, the score functions  $\partial_T \ln \hat{f}$  and  $H_\pi - \partial_{\alpha_\pi} \Psi$  are linearly independent on  $\Sigma_m$ , so  $v^A = 0$ . Hence  $g_{AB}$  is positive definite. Smoothness follows from the smoothness of  $f_{\text{eq}}$  and of the cumulant-generating function on its domain. ■

## IV. TRANSPORT COEFFICIENTS FROM GEOMETRY

### A. Thermodynamic sector

The thermodynamic block of the Fisher-Rao metric at equilibrium is the standard Ruppeiner metric [30]:

$$g_{TT}|_{\text{eq}} = \left\langle \left( \frac{\partial \ln f_{\text{eq}}}{\partial T} \right)^2 \right\rangle_{\text{eq}} = \frac{C_V}{T^2}, \quad (24)$$

where  $C_V = d\epsilon/dT$  is the specific heat at constant volume. For the conformal gas,  $\epsilon \propto T^4$ , so  $C_V = 4\epsilon/T$  and

$$g_{TT}|_{\text{eq}} = \frac{4\epsilon}{T^3}. \quad (25)$$

This establishes the connection to equilibrium thermodynamic geometry: the Fisher-Rao metric restricted to the thermodynamic sector is Ruppeiner’s metric, a well-known result in information geometry [24].

### B. Dissipative sector: Theorem 2

*Theorem 2.* For the conformal massless Boltzmann gas in the Anderson-Witting RTA, the dissipative-sector Fisher-Rao metric component at equilibrium is the exact dimensionless constant

$$g_{\alpha_\pi\alpha_\pi}|_{\text{eq}} = \frac{2}{15}, \quad (26)$$

and the shear viscosity is determined by the metric through

$$\eta = 6P \tau_R g_{\alpha_\pi\alpha_\pi}|_{\text{eq}} = \frac{4P \tau_R}{5}, \quad (27)$$

where the equilibrium pressure  $P$  sets the scale. In the physical shear coordinate  $\pi$ , the metric is  $g_{\pi\pi}|_{\text{eq}} = g_{\alpha_\pi\alpha_\pi}/(2T\beta_\pi)^2 = 1/(30T^2\beta_\pi^2)$ .

*Closure independence at this order.* Both Eqs. (26) and (27) are evaluated at  $\alpha_\pi = 0$  against the equilibrium distribution  $f_{\text{eq}}$ . All admissible completions of the DNMR 14-moment state coincide at  $\alpha_\pi = 0$  to leading order, so the equilibrium values  $g_{\alpha_\pi\alpha_\pi}|_{\text{eq}} = 2/15$  and  $\eta = 4P\tau_R/5$  are closure independent within the 14-moment truncation. The closure-dependent regime, in which different admissible completions of the same 14 moments yield different Fisher metrics and hence different geometric quantities, is entered only when  $\alpha_\pi > 0$  or when  $O(\alpha_\pi^2)$  and higher structure of the chosen exponential family is used; that regime is flagged explicitly in Sec. III D and restated as L7 in the Limitations.

The factor  $6P$  connecting the Fisher metric to  $\beta_\pi = 4P/5$  arises because the Fisher inner product weights by the number

measure  $E_p f_{\text{eq}}/n_{\text{eq}}$ , while the transport coefficient involves the energy-weighted measure  $E_p^3 f_{\text{eq}}$  from the collision operator (see the Appendix). Concretely, define the Fisher moment

$$\mathcal{I}_k = \int d\Gamma_p (up)^k f_{\text{eq}} [H_\pi(p)]^2. \quad (28)$$

Then  $g_{\alpha\alpha}|_{\text{eq}} = \mathcal{I}_1/n_{\text{eq}} = 2/15$  and  $\beta_\pi = \mathcal{I}_3/(15T)$ , with the ratio  $\mathcal{I}_3/\mathcal{I}_1 = \langle (up)^2 \rangle_{\text{eq}} = 12T^2$  (the mean-square energy per particle). Hence

$$\beta_\pi = \frac{12T^2}{15T} \frac{n_{\text{eq}} g_{\alpha\alpha}|_{\text{eq}}}{1} = \frac{4T}{5} n_{\text{eq}} g_{\alpha\alpha}|_{\text{eq}}. \quad (29)$$

*Proof.* See the Appendix for the explicit momentum-space integration. The key steps are the following: (i) express  $H_\pi$  in spherical coordinates as a Legendre polynomial  $P_2(\cos \theta)$ ; (ii) evaluate the angular integral using the orthogonality relation  $\int_{-1}^1 [P_2(\mu)]^2 d\mu = 2/5$ ; (iii) evaluate the radial integral  $\int_0^\infty p^{n+2} e^{-p/T} dp = (n+2)! T^{n+3}$ . The result  $g_{\alpha\alpha} = 2/15$  is exact; Fig. 13 illustrates the associated geometric recovery of  $\eta/s$  along the numerical evolution. ■

### C. Geometric unification of transport coefficients

The main structural consequence of Theorem 2 is that, in the conformal shear channel, the transport coefficient is obtained from the equilibrium Fisher metric up to a thermodynamic prefactor fixed by the equation of state. Analogous relations for the bulk and diffusion channels on the full 14-moment manifold,

$$\eta = c_\eta(T) \tau_R g_{\alpha_\pi \alpha_\pi}|_{\text{eq}}, \quad (30)$$

$$\zeta = c_\zeta(T) \tau_R g_{\alpha_\Pi \alpha_\Pi}|_{\text{eq}}, \quad (31)$$

$$\kappa = c_\kappa(T) \tau_R g_{\alpha_n \alpha_n}|_{\text{eq}}, \quad (32)$$

where  $c_\eta = \langle E_p^2 \rangle_{\text{eq}}/T$ ,  $c_\zeta$ , and  $c_\kappa$  are known thermodynamic functions, are proposed as a conjectural generalization rather than a theorem in the present paper. Explicit verification for the bulk and diffusion channels requires the analogous momentum-space integrations on the 14-dimensional manifold, which I defer to future work.

The block-diagonal structure (22) at equilibrium reflects the orthogonality of the scalar ( $\Pi$ ), tensor ( $\pi^{\mu\nu}$ ), and vector ( $n^\mu$ ) channels under the  $\text{SO}(3)$  rotation symmetry of the local rest frame (LRF). Off-diagonal components in the general case encode cross-channel couplings constrained by Onsager reciprocity.

## V. FISHER-RAO DISTANCE AND MONOTONICITY

### A. Definition

The *Fisher-Rao distance within the DNMR 14-moment exponential family, from the projected nonequilibrium state to its local-equilibrium projection*, at proper time  $\tau$ , is defined as

$$\mathcal{D}(\tau) \equiv \sqrt{g_{\alpha\alpha}(\tau) \alpha_\pi(\tau)^2}, \quad (33)$$

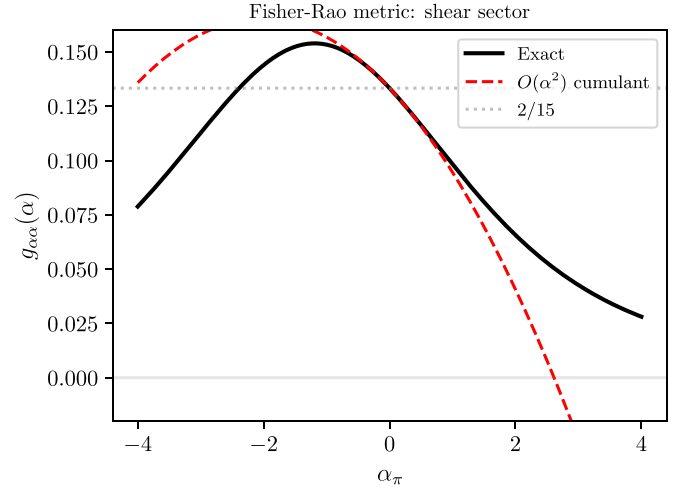


FIG. 1. Fisher-Rao metric component  $g_{\alpha\alpha}(\alpha)$  as a function of the shear coordinate: exact (black solid) vs  $O(\alpha^2)$  cumulant approximation (red dashed). Gray dotted: equilibrium value  $2/15$ . The metric decreases with increasing  $|\alpha|$  for the physically relevant Bjorken branch ( $\alpha > 0$ ); the asymmetry between positive and negative  $\alpha$  reflects the nonzero third cumulant  $\kappa_3 \neq 0$ .

where  $\alpha_\pi = \pi/\beta_\pi$  is the natural parameter. At equilibrium,  $\mathcal{D} = 0$ . Using the correction formula from the Appendix,

$$g_{\alpha\alpha}(\alpha) = \frac{2}{15} \left[ 1 - \frac{3}{7} \left( \frac{2}{3} \right)^{3/2} \alpha - \frac{2}{35} \alpha^2 + O(\alpha^3) \right], \quad (34)$$

where  $c_1 = -3(2/3)^{3/2}/7 \approx -0.233$  and  $c_2 = -2/35 \approx -0.057$  are exact, derived from the third and fourth cumulants of  $H_\pi$  under the equilibrium distribution. The corrections are significant in the crossover regime  $|\alpha| \sim O(1)$  and must be retained for quantitative accuracy. Figure 1 compares the exact  $g_{\alpha\alpha}(\alpha)$  (computed numerically from the full cumulant-generating function) with the  $O(\alpha^2)$  approximation.

### B. Full geodesic distance vs linearized approximation

The distance (33) uses the local metric  $g_{\alpha\alpha}(\alpha)$ , but in practice one often linearizes:  $\mathcal{D}_{\text{lin}} = \sqrt{2/15} |\alpha|$ . The full geodesic distance is the proper integral

$$\mathcal{D}_{\text{full}}(\alpha) = \int_0^{|\alpha|} \sqrt{g_{\alpha\alpha}(\alpha')} d\alpha'. \quad (35)$$

Since  $g_{\alpha\alpha}(\alpha) < g_{\alpha\alpha}|_{\text{eq}}$  for  $\alpha \neq 0$ , the linearized distance overestimates:  $\mathcal{D}_{\text{lin}} > \mathcal{D}_{\text{full}}$  always. At  $\alpha = 1$  the overestimation is 7%; at  $\alpha = 3$ , 22%; at  $\alpha = 5$ , 35% (Table II and Fig. 2).

### C. Comparison with Knudsen and inverse Reynolds numbers

The standard measures of hydrodynamic applicability are the Knudsen number  $\text{Kn} = \tau_R/\tau$  (ratio of microscopic to macroscopic timescales) and the inverse Reynolds number  $\text{Re}^{-1} = |\pi|/P$  (ratio of dissipative to equilibrium stresses). Both are coordinate dependent and nongeometric:  $\text{Kn}$  depends on the arbitrary choice of macroscopic scale  $\tau$ , and  $\text{Re}^{-1}$  depends on the decomposition of  $T^{\mu\nu}$  into equilibrium and nonequilibrium parts.

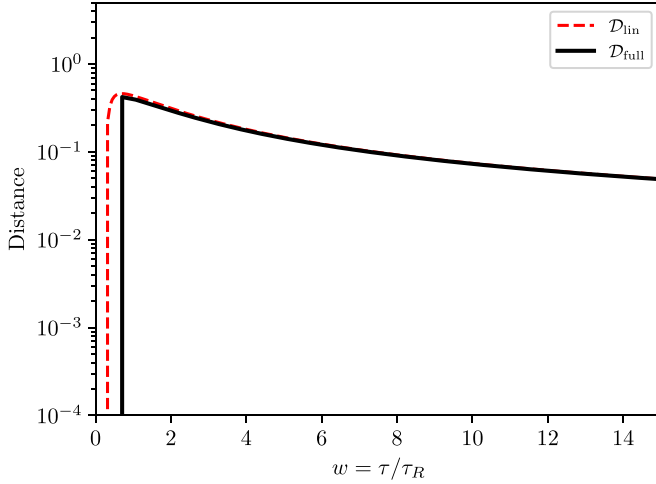


FIG. 2. Full (geodesic) Fisher-Rao distance  $\mathcal{D}_{\text{full}}$  (black) vs linearized  $\mathcal{D}_{\text{lin}}$  (red dashed) along the exact RTA Boltzmann solution in conformal Bjorken flow. The linearized version overestimates by 15–35% at moderate to large shear.

The Fisher-Rao distance  $\mathcal{D}$  is distinguished by three properties that  $\text{Kn}$  and  $\text{Re}^{-1}$  lack.

- (1) *Coordinate invariance.*  $\mathcal{D}$  is a Riemannian distance, invariant under reparametrization of the moment manifold.
- (2) *Information-theoretic meaning.*  $\mathcal{D}$  measures the statistical distinguishability of the current distribution from equilibrium, in the sense of the Cramér-Rao bound.
- (3) *Monotonicity.*  $\mathcal{D}$  contracts under the collision operator by the data-processing inequality, providing a natural Lyapunov-like structure.

#### D. Relation to relative entropy and uniqueness of the metric

Because relative entropy already plays a natural role in kinetic theory, it is important to state precisely what the Fisher-Rao metric adds. The two objects are not competitors. For a smooth family of distributions  $f(x, p; \theta)$  parametrized by coordinates  $\theta^A$ , the Kullback-Leibler divergence between nearby states has the expansion

$$D_{\text{KL}}[f(\theta) \| f(\theta + \delta\theta)] = \frac{1}{2} g_{AB}(\theta) \delta\theta^A \delta\theta^B + O(\delta\theta^3), \quad (36)$$

with  $g_{AB}$  the Fisher information matrix. The Fisher-Rao metric is therefore the unique local quadratic form induced by relative entropy. By the theorem of Čencov (see Refs. [23,24]), it is also the unique Riemannian metric on a statistical manifold that is invariant under sufficient statistics.

This distinction matters conceptually. Relative entropy is a scalar divergence between finite states. By itself it does not define an intrinsic Riemannian geometry on the manifold of moments: it does not supply geodesics, curvature tensors, or sectional-curvature information. The Fisher-Rao tensor provides precisely that local geometric structure. In the present context this is the reason for introducing the metric language: not to replace entropy-based notions, but to pass from a scalar distinguishability measure to a coordinate-invariant differential geometry on the state space.

Accordingly, the distance constructions above should be read as the local geometric refinement of the entropy-based framework already familiar from kinetic theory. In particular, the full geodesic distance  $\mathcal{D}_{\text{full}}$  integrates the Fisher metric along the geodesic connecting two states, whereas  $D_{\text{KL}}$  is a nonsymmetric scalar divergence that does not satisfy a triangle inequality. The 15–35% discrepancy between  $\mathcal{D}_{\text{full}}$  and  $\mathcal{D}_{\text{lin}}$  quantified in Table II is a geometric statement about the manifold that  $D_{\text{KL}}$  alone cannot access. The Fisher-Rao metric is also distinct from the Wasserstein geometry on distributions used in the Jordan-Kinderlehrer-Otto framework [31] for gradient-flow formulations of dissipative dynamics; the two define different geometries on the same space and are not compared here.

#### E. Theorem 3: Evolution and monotonicity

*Theorem 3 (leading order).* Under the Israel-Stewart equations (12) and (13) in Bjorken flow, the *linearized* Fisher-Rao distance  $\mathcal{D}_{\text{lin}}(\tau) = \sqrt{2/15} |\alpha_\pi|$  satisfies, to leading order in  $\bar{\pi} = \pi/(\epsilon + P)$ ,

$$\frac{d\mathcal{D}_{\text{lin}}}{d\tau} = -\frac{\mathcal{D}_{\text{lin}}}{\tau_R} + \frac{4\sqrt{2}\beta_\pi/15}{3\tau} \text{sgn}(\pi) + O(\bar{\pi}^2), \quad (37)$$

where the first term is the collisional relaxation (always negative, driving  $\mathcal{D}_{\text{lin}}$  toward zero) and the second is the expansion-driven source.

*Proof.* Evaluate  $d\mathcal{D}_{\text{lin}}/d\tau$  using  $\mathcal{D}_{\text{lin}} = \sqrt{2/15} |\pi|/\beta_\pi$  at leading order, the IS equation (13) for  $d\pi/d\tau$ , and the evolution of  $\beta_\pi = 4\epsilon/15$  from (12). The collisional term  $-\pi/\tau_R$  in (13) maps directly to  $-\mathcal{D}_{\text{lin}}/\tau_R$ . The NS source term  $4\beta_\pi/(3\tau)$  produces the expansion-driven contribution. ■

*Remark.* For the full geodesic distance  $\mathcal{D}_{\text{full}}$ , the evolution equation acquires additional terms from the  $\alpha$  dependence of  $g_{\alpha\alpha}(\alpha)$ . The monotonicity statement below applies only to  $\mathcal{D}_{\text{lin}}$ .

*Corollary (approximate).* For  $\tau \gg \tau_R$  (i.e.,  $\text{Kn} \ll 1$ ), the collisional relaxation term  $-\mathcal{D}_{\text{lin}}/\tau_R$  dominates over the expansion source at leading order in  $\bar{\pi}$ , and  $\mathcal{D}_{\text{lin}}(\tau)$  is approximately monotonically decreasing. This provides a *leading-order* geometric criterion for hydrodynamization: the system has hydrodynamized when the collisional term dominates the distance evolution. The numerical results in Sec. IX confirm this behavior for the full distance  $\mathcal{D}_{\text{full}}$  as well, but a rigorous monotonicity proof for  $\mathcal{D}_{\text{full}}$  including finite-shear corrections has not been obtained.

### VI. INTRINSIC RIEMANNIAN GEOMETRY OF THE MOMENT MANIFOLD

For the equilibrium conformal shear analysis in this section, the relevant manifold is six-dimensional: one thermodynamic coordinate  $T$  and five shear-sector coordinates  $\alpha_{ij}$  parametrizing the symmetric traceless shear tensor  $\pi^{ij}$ . The metric at equilibrium is block diagonal (Theorem 1):  $g = g_{TT} dT^2 + g_{(ij)(kl)} d\alpha^{ij} d\alpha^{kl}$ . I now compute the intrinsic geometry of the five-dimensional shear sector.

### A. Orthonormal basis for symmetric traceless tensors

A symmetric traceless  $3 \times 3$  matrix has five independent components. I introduce an orthonormal basis  $\{E^{(a)}\}$ ,  $a = 1, \dots, 5$ , satisfying  $\text{tr}(E^{(a)}E^{(b)}) = \delta^{ab}$ :

$$\begin{aligned} E^{(1)} &= \frac{1}{\sqrt{2}} \text{diag}(1, -1, 0) \quad (v_2\text{-type}), \\ E^{(2)} &= \frac{1}{\sqrt{6}} \text{diag}(-1, -1, 2) \quad (\text{Bjorken}), \\ E_{12}^{(3)} &= E_{21}^{(3)} = \frac{1}{\sqrt{2}} \quad (\text{directed flow}), \\ E_{13}^{(4)} &= E_{31}^{(4)} = \frac{1}{\sqrt{2}} \quad (xz \text{ shear}), \\ E_{23}^{(5)} &= E_{32}^{(5)} = \frac{1}{\sqrt{2}} \quad (yz \text{ shear}). \end{aligned} \quad (38)$$

The shear moment function in this basis is  $H_a(\hat{\mathbf{p}}) = \text{tr}[H(\hat{\mathbf{p}})E^{(a)}]$ , where  $H_{ij}(\hat{\mathbf{p}}) = \hat{p}_i \hat{p}_j - \delta_{ij}/3$  for massless particles.

### B. Fisher-Rao metric on the shear sector

The metric on the five-dimensional shear manifold is the covariance matrix of the  $H_a$  under the equilibrium distribution:

$$g_{ab}|_{\text{eq}} = \langle H_a H_b \rangle_{\text{eq}} = C_2 \delta_{ab}, \quad (39)$$

where  $C_2 = \langle H_a^2 \rangle_{\text{eq}} = 2/15$  (Theorem 2) and the off-diagonal components vanish by  $\text{SO}(3)$  isotropy of  $f_{\text{eq}}$ . The metric is proportional to the identity on  $\mathbb{R}^5$ , with proportionality constant the exact value  $2/15$ .

### C. Theorem 4: Nonzero Riemann curvature of the shear manifold

*Theorem 4.* The five-dimensional shear sector of the moment manifold, equipped with the Fisher-Rao metric, has nonzero Riemannian curvature at equilibrium. The Ricci scalar is

$$R|_{\text{eq}} = -\frac{5}{21}, \quad (40)$$

computed from the Amari-Chentsov curvature formula and verified analytically. The manifold has mixed-sign sectional curvatures.

*Proof.* The Amari-Chentsov formula [24] gives the Riemann tensor of a statistical manifold in terms of the second and third cumulants of the sufficient statistics:

$$R_{abcd} = \frac{1}{4} \sum_{e,f} (T_{ace} g^{ef} T_{bdf} - T_{ade} g^{ef} T_{bcf}), \quad (41)$$

where  $T_{abc} = \langle H_a H_b H_c \rangle_{\text{eq}}$  is the third cumulant tensor and  $g^{ef} = \delta^{ef}/C_2$  is the inverse metric.

The third cumulant is *not* zero. Unlike a Gaussian distribution, the angular distribution of  $H_{ij}$  on the unit sphere is non-Gaussian: the cubic cumulant encodes the skewness of the Legendre decomposition. The nonzero independent components, computed by angular integration over the unit sphere

TABLE III. The ten independent sectional curvatures.

| Plane        | Physical content        | $K$       |
|--------------|-------------------------|-----------|
| $(E^1, E^2)$ | $v_2$ -Bjorken coupling | $-10/49$  |
| $(E^1, E^3)$ | $v_2$ -directed flow    | $+5/49$   |
| $(E^1, E^4)$ | $v_2$ -xz shear         | $-25/196$ |
| $(E^1, E^5)$ | $v_2$ -yz shear         | $-25/196$ |
| $(E^2, E^3)$ | Bjorken-directed flow   | $-10/49$  |
| $(E^2, E^4)$ | Bjorken-xz shear        | $+5/196$  |
| $(E^2, E^5)$ | Bjorken-yz shear        | $+5/196$  |
| $(E^3, E^4)$ | Directed-xz coupling    | $-25/196$ |
| $(E^3, E^5)$ | Directed-yz coupling    | $-25/196$ |
| $(E^4, E^5)$ | xz-yz coupling          | $-25/196$ |

and verified analytically, are

$$\begin{aligned} T_{112} &= -\frac{8}{105\sqrt{6}}, \quad T_{222} = \frac{16}{35 \times 6^{3/2}}, \\ T_{134} &= \frac{4}{35\sqrt{6}}, \quad T_{244} = T_{255} = \frac{8}{105\sqrt{6}}, \end{aligned} \quad (42)$$

with all other components related by permutation symmetry or vanishing. Note  $T_{112} = -T_{222}$ : the skewness of the Bjorken component has equal magnitude but opposite sign to that of the  $v_2$  component, reflecting the geometric duality between longitudinal and transverse deformations.

Substituting into Eq. (41) and contracting,  $\text{Ric}_{ab} = \sum_c R_{acbc}$  is proportional to  $\delta_{ab}$  [by  $\text{SO}(3)$  isotropy at equilibrium] with  $\text{Ric}_{aa} = -2/315$  for each  $a$  (verified to machine precision,  $<10^{-18}$ ), giving  $R = g^{ab} \text{Ric}_{ab} = (5/C_2) \times (-2/315) = (5 \times 15/2) \times (-2/315) = -5/21$ .

The Riemann tensor satisfies all algebraic symmetries to machine precision ( $<10^{-18}$ ): antisymmetry in the first and second pair of indices, pair symmetry  $R_{abcd} = R_{cdab}$ , and the first Bianchi identity. ■

*Remark.* Any one-dimensional restriction (e.g. Bjorken flow, which projects onto the single direction  $E^{(2)}$ ) trivially yields  $R = 0$  because all one-dimensional Riemannian manifolds are flat. The nonzero curvature is an intrinsically five-dimensional effect arising from the angular non-Gaussianity of  $H_{ij}$  on  $S^2$ .

### D. Sectional curvatures and physical interpretation

*Statistical, not dynamical.* The sectional curvatures computed in this subsection are intrinsic properties of the truncated exponential family with the Fisher-Rao metric. They are not predictions for physical relaxation rates and do not presuppose any geodesic flow on the manifold corresponding to kinetic evolution; no variational principle linking curvature to dynamics is invoked. The physical-interpretation remarks in the remainder of this subsection describe features of the equilibrium statistical manifold within the 14-moment truncation; whether they correspond to dynamical behavior of the underlying kinetic theory is not established here and is flagged as an open question in Sec. X.

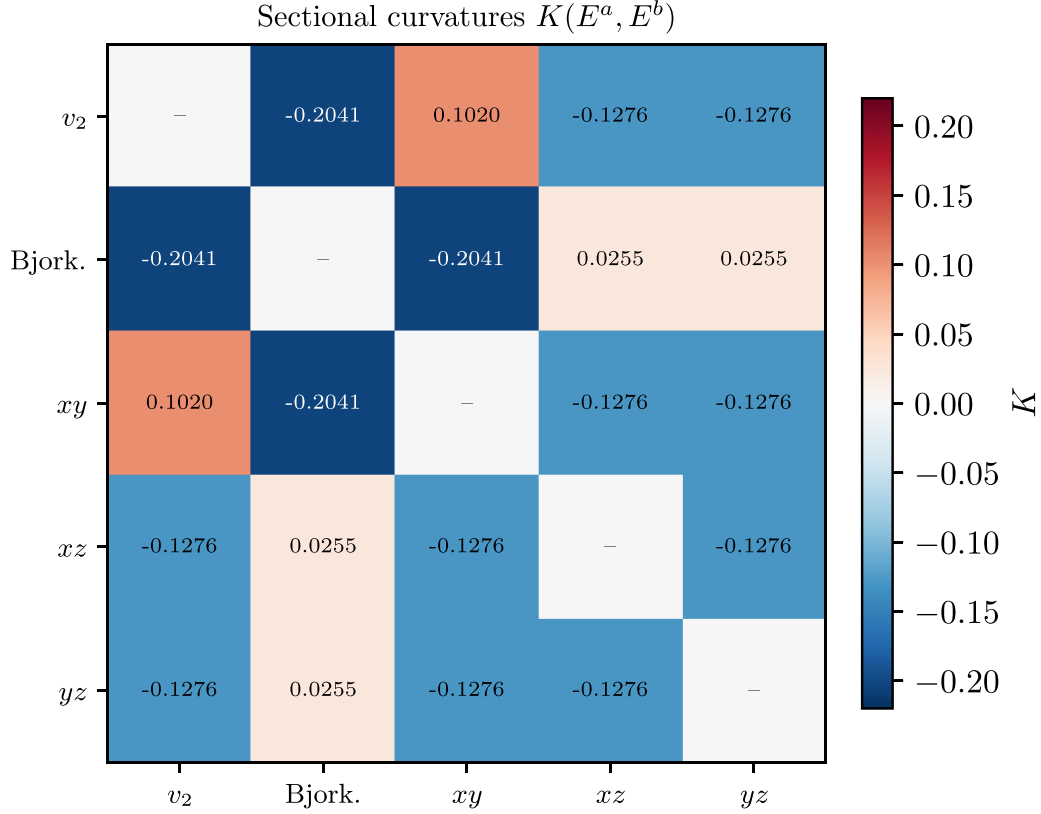


FIG. 3. Sectional curvatures  $K(E^a, E^b)$  of the five-dimensional conformal shear manifold at equilibrium. Blue: negative (diverging geodesics). Red: positive (focusing). The strongest negative curvature ( $K = -10/49$ ) is in the Bjorken- $v_2$  and Bjorken-directed-flow planes.

The sectional curvature  $K(E^{(a)}, E^{(b)})$  of the two-plane spanned by basis directions  $a$  and  $b$  is

$$K(a, b) = \frac{R_{abab}}{g_{aa}g_{bb} - g_{ab}^2} = \frac{R_{abab}}{C_2^2}. \quad (43)$$

The ten independent sectional curvatures are listed in Table III. The residual SO(2) azimuthal symmetry about the beam axis at equilibrium accounts for the degeneracies  $K(1, 4) = K(1, 5)$ ,  $K(2, 4) = K(2, 5)$ , and  $K(3, 4) = K(3, 5)$ . Figure 3 displays the sectional curvatures as a matrix.

The largest negative sectional curvature ( $K = -10/49$ ) is in the planes coupling the Bjorken direction  $E^{(2)}$  to the elliptic-flow direction  $E^{(1)}$  and to the directed-flow direction  $E^{(3)}$ . Negative curvature means nearby geodesics *diverge*: small perturbations in the  $v_2$  direction are geometrically amplified when combined with Bjorken shear. Conversely, the positive curvature  $K(E^{(1)}, E^{(3)}) = 5/49$  in the  $v_2$ -directed-flow plane means perturbations in that plane are geometrically focused.

*Physical interpretation.* The mixed-sign sectional curvatures encode the intrinsic geometric coupling between different shear channels in the QGP. I discuss each sign in turn.

*Negative curvature (diverging geodesics).* The strongest negative curvature  $K = -10/49 \approx -0.204$  occurs in two planes:  $(E^{(1)}, E^{(2)})$  (elliptic flow-Bjorken) and  $(E^{(2)}, E^{(3)})$  (Bjorken-directed flow). In these planes, infinitesimally sep-

arated geodesics diverge: the Fisher-Rao distance between nearby distributions grows faster than in flat space. Physically, this means that the statistical distinguishability of distributions that differ in both longitudinal (Bjorken) and transverse ( $v_2$  or  $v_1$ ) shear components grows faster than linearly in the geometric-distance sense used here. This is geometrically consistent, within the present truncation, with the fact that longitudinal and transverse dynamics decouple at different rates in heavy-ion collisions [1,2].

The off-diagonal shear planes also have negative curvature:  $K(E^{(1)}, E^{(4)}) = K(E^{(1)}, E^{(5)}) = -25/196 \approx -0.128$  for the  $v_2$ -off-diagonal coupling, and  $K(E^{(3)}, E^{(4)}) = K(E^{(3)}, E^{(5)}) = K(E^{(4)}, E^{(5)}) = -25/196$  for the remaining off-diagonal pairs. These degeneracies reflect the SO(2) azimuthal symmetry: the  $xz$  and  $yz$  shear components are related by a rotation about the beam axis.

*Positive curvature (focusing geodesics).* The positive curvature  $K(E^{(1)}, E^{(3)}) = 5/49 \approx 0.102$  in the  $v_2$ -directed-flow plane means that nearby geodesics in this plane *converge*: distributions that differ in both  $v_2$  and  $v_1$  approach each other geometrically. Geometrically, within the present truncation, a perturbation in the  $v_2$  direction on the equilibrium manifold is focused toward the  $v_1$  direction. The positive curvature  $K(E^{(2)}, E^{(4)}) = K(E^{(2)}, E^{(5)}) = 5/196 \approx 0.026$  in the Bjorken-off-diagonal planes has the same sign but weaker magnitude.

*Ordering of sectional curvatures.* The magnitudes  $10/49 > 25/196 > 5/49 > 5/196$  define an ordering among the shear

TABLE IV. The splitting of the metric eigenvalues.

| $\alpha_B$ | Anisotropy $\lambda_{\max}/\lambda_{\min}$ | Comment              |
|------------|--------------------------------------------|----------------------|
| 0          | 1.01                                       | Isotropic            |
| 0.5        | 1.26                                       | Mild breaking        |
| 1.0        | 1.53                                       | Moderate shear       |
| 2.0        | 2.17                                       | Strong shear         |
| 3.0        | 3.24                                       | Far from equilibrium |

two-planes within the equilibrium Fisher-Rao manifold of the present truncation. The Bjorken- $v_2$  coupling is the strongest ( $|K| = 0.204$ ), followed by the off-diagonal couplings ( $|K| = 0.128$ ), the  $v_2 - v_1$  focusing ( $|K| = 0.102$ ), and the Bjorken-off-diagonal focusing ( $|K| = 0.026$ ). This ordering is a geometric feature of the equilibrium Fisher-Rao manifold within the present truncation; at finite shear the  $SO(3) \rightarrow SO(2)$  breaking modifies it, as shown in the next subsection.

### E. Metric anisotropy at finite shear: $SO(3) \rightarrow SO(2)$ breaking

At equilibrium the metric is proportional to  $\delta_{ab}$  [ $SO(3)$  symmetry]. At finite shear  $\alpha = \alpha_B E^{(2)}$  (pure Bjorken deformation), the tilted distribution  $f \propto f_{\text{eq}} e^{\alpha_B H_2}$  breaks  $SO(3)$  to the residual  $SO(2)$  of azimuthal rotations about the beam axis. The metric eigenvalues split, as shown in Table IV. The two smallest eigenvalues correspond to the  $v_2$ -type and Bjorken directions (which contract as shear increases), while the three largest correspond to the off-diagonal shear components (which remain close to their equilibrium values). This symmetry-breaking pattern is the geometric counterpart of the kinematic fact that Bjorken flow preferentially populates the longitudinal shear mode while leaving transverse modes at their equilibrium level. Figure 4 shows the full eigenvalue splitting as a continuous function of  $\alpha_B$ .

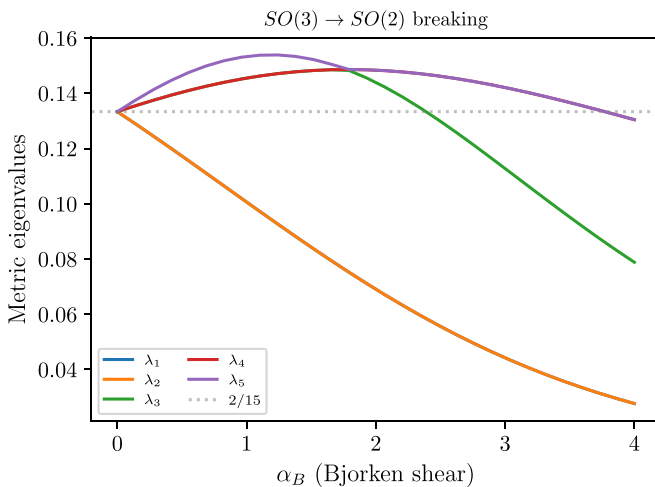


FIG. 4. Metric eigenvalues of the five-dimensional shear sector as a function of Bjorken shear  $\alpha_B$ . At  $\alpha_B = 0$  all five eigenvalues equal  $2/15$  [ $SO(3)$  symmetry]. Finite Bjorken shear breaks  $SO(3) \rightarrow SO(2)$ : two eigenvalues decrease (longitudinal and  $v_2$  modes) while three remain near  $2/15$  (transverse modes).

### F. Christoffel symbols and geodesic equation

The Christoffel symbols of the five-dimensional shear sector at equilibrium are determined by the third cumulant [24]:

$$\Gamma_{bc}^a|_{\text{eq}} = \frac{1}{2} g^{ad} (\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc})|_{\text{eq}} = \frac{T_{abc}}{2C_2}. \quad (44)$$

The geodesic equation on the shear manifold is

$$\ddot{\alpha}^a + \frac{T_{abc}}{2C_2} \dot{\alpha}^b \dot{\alpha}^c = 0. \quad (45)$$

For Bjorken flow,  $\alpha^a = \alpha_B(\tau) \delta^{a2}$ , and the geodesic equation reduces to  $\ddot{\alpha}_B + [T_{222}/(2C_2)] \dot{\alpha}_B^2 = 0$ , with  $T_{222}/(2C_2) = 2/(7\sqrt{6}) \approx 0.117$ . The Israel-Stewart equations do not satisfy this: the collision and expansion terms provide  $O(1)$  forcing at all times.

### G. Scope of the curvature results

The explicit curvature statements obtained in this section should be interpreted at two distinct levels. First, there is a structural claim: once a finite-dimensional moment family is equipped with the Fisher-Rao metric, one can ask meaningful coordinate-invariant questions (coordinate invariant under smooth reparametrizations of that family) about the intrinsic curvature of that family. The word “intrinsic” is used throughout in this Riemannian-geometry sense—intrinsic to the truncated exponential family under the Fisher-Rao metric—and not in the sense of a physical property of the full kinetic theory. That statement is independent of the particular truncation used here, and follows from Sec. III D. Second, there are model-dependent claims: the numerical value  $R|_{\text{eq}} = -5/21$ , the signs of the ten sectional curvatures, and the associated hierarchy of Jacobi-growth rates derived in Sec. X. These results depend on the conformal, massless, RTA, low-dimensional setting adopted in the present paper.

For that reason, the curvature values reported here should not be read as universal signatures of QCD kinetic theory. They are explicit realizations within a tractable model. Whether analogous sign patterns, magnitudes, or sectional-curvature hierarchies survive on higher-dimensional manifolds or for more realistic collision kernels remains an open problem to which Sec. XI returns.

## VII. NONCONFORMAL CASE: THE TEMPERATURE-DEPENDENT DISSIPATIVE METRIC

For particles of mass  $m$ , the shear moment function becomes

$$H_\pi^{(m)}(\mathbf{p}) = \frac{p^{(i} p^{j)} \hat{\sigma}_{ij}}{E_p^2}, \quad E_p = \sqrt{p^2 + m^2}, \quad (46)$$

so the conformal radial-angular factorization no longer holds. As a result, the cumulant-generating function  $\Psi(T, \alpha) = \ln \int d\mu_p f_{\text{eq}}(p; T) e^{\alpha H_\pi^{(m)}(p)}$  depends nontrivially on both  $T$  and  $\alpha$ , and the dissipative metric component  $g_{\alpha\alpha}(T, \alpha) = \partial_\alpha^2 \Psi(T, \alpha)$  acquires explicit temperature dependence.

In the Bjorken truncation, the relevant manifold is the two-dimensional sector with coordinates  $(T, \alpha_\pi)$ . Its metric takes the form  $ds^2 = g_{TT}(T, \alpha) dT^2 + 2g_{T\alpha}(T, \alpha) dT d\alpha +$

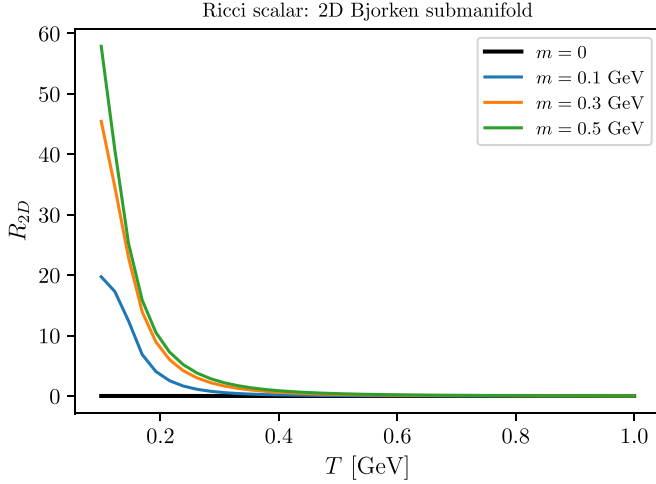


FIG. 5. Ricci scalar  $R_{2D}$  of the two-dimensional Bjorken submanifold  $(T, \alpha_\pi)$  at  $\alpha_\pi = 0$  vs temperature for different particle masses.  $R_{2D} = 0$  for  $m = 0$  because  $g_{\alpha\alpha}$  is  $T$  independent in the conformal limit. For  $m > 0$  the curvature peaks near  $T \sim m$ . This is the mass-induced curvature of the 2D submanifold, distinct from the  $R = -5/21$  of the 5D shear sector (Theorem 4).

$g_{\alpha\alpha}(T, \alpha) d\alpha^2$ . At  $\alpha = 0$ , rotational symmetry implies  $g_{T\alpha}(T, 0) = 0$ , but  $\partial_T g_{\alpha\alpha}(T, 0) \neq 0$  for  $m \neq 0$ . Therefore the two-dimensional  $(T, \alpha_\pi)$  geometry is generically curved in the massive case, whereas in the conformal massless limit the dissipative metric becomes temperature independent and this source of curvature disappears.

Figure 5 presents numerical evidence for a nonzero Ricci scalar in the two-dimensional  $(T, \alpha_\pi)$  sector of the massive theory. Figure 6 shows the underlying temperature dependence of  $g_{\alpha\alpha}|_{\text{eq}}(T, m)$ . For  $m = 0$ , the plotted curvature vanishes identically; for  $m > 0$ , it develops a peak near  $T \sim m$  and tends to zero in both the  $T \ll m$  and  $T \gg m$  limits. For

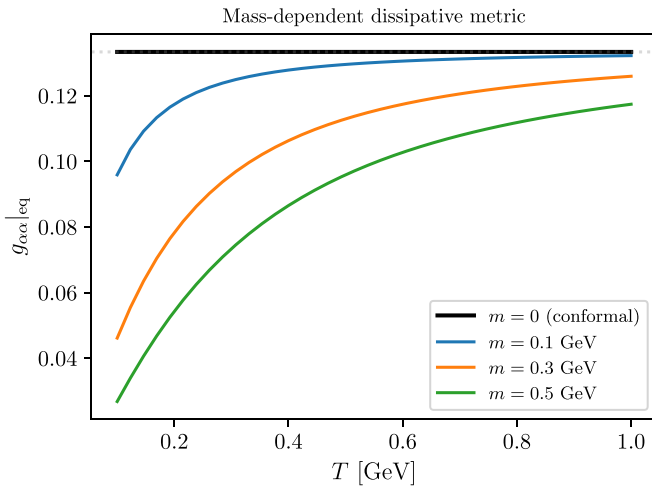


FIG. 6. Temperature dependence of the equilibrium dissipative metric  $g_{\alpha\alpha}|_{\text{eq}}(T, m)$  for different particle masses. For  $m = 0$  (conformal), the metric is the constant  $2/15$  (gray dotted). For  $m > 0$ ,  $g_{\alpha\alpha}|_{\text{eq}}$  decreases below the conformal value at low  $T$  and approaches it from below as  $T/m \rightarrow \infty$ .

TABLE V. Nonconformal dissipative metric  $g_{\alpha\alpha}|_{\text{eq}}(T, m)$  at selected temperatures for  $m = 0.3$  GeV (pionlike) and  $m = 0.5$  GeV. The conformal value is  $2/15 \approx 0.1333$ .

| $T$ (GeV) | $g_{\alpha\alpha} _{\text{eq}}(m = 0.3)$ | $g_{\alpha\alpha} _{\text{eq}}(m = 0.5)$ |
|-----------|------------------------------------------|------------------------------------------|
| 0.10      | 0.046                                    | 0.027                                    |
| 0.15      | 0.065                                    | 0.042                                    |
| 0.20      | 0.078                                    | 0.054                                    |
| 0.30      | 0.096                                    | 0.073                                    |
| 0.40      | 0.106                                    | 0.086                                    |
| 0.50      | 0.113                                    | 0.096                                    |
| 0.80      | 0.123                                    | 0.112                                    |
| 1.00      | 0.126                                    | 0.117                                    |

the representative  $m = 0.3$ -GeV curve,  $g_{\alpha\alpha}|_{\text{eq}}$  increases from approximately 0.065 at  $T = 150$  MeV to approximately 0.123 at  $T = 800$  MeV, approaching the conformal value  $2/15$  from below. Table V gives representative values.

The physical content of Table V is as follows. At low temperature ( $T \ll m$ ), the momentum distribution is concentrated near  $|\mathbf{p}| \sim T \ll m$ , so  $E_p \approx m$  and the shear moment function  $H_\pi^{(m)} \sim p^2/m^2$  is suppressed relative to its massless value  $H_\pi \approx 1$ . Consequently the variance  $g_{\alpha\alpha} = \langle H_\pi^2 \rangle$  is small. As  $T$  increases past  $m$ , the distribution populates relativistic momenta  $|\mathbf{p}| \gg m$ ,  $H_\pi^{(m)}$  approaches its conformal form, and  $g_{\alpha\alpha} \rightarrow 2/15$ . The crossover occurs at  $T \sim m/3 - m$ , precisely the regime where the trace anomaly  $(\epsilon - 3P)/T^4$  is maximal.

*Caveat: Illustrative massive gas, not QCD.* The analysis in this section uses a single-species classical massive gas with Boltzmann statistics. It does not include quantum statistics, flavor mixing, gluon contributions, or the temperature-dependent effective masses that characterize the QCD crossover. The results are therefore illustrative of the *mechanism* by which mass generates curvature, not a controlled prediction for the QGP.

With that caveat, the model suggests a qualitative picture. For the QGP near the deconfinement crossover ( $T_c \approx 155$  MeV), the effective strange quark mass  $m_s \approx 100$  MeV would place the strange sector in the crossover regime of the toy model. The light quark sector ( $m_{u,d} \approx 5$  MeV) would be deep in the conformal regime at  $T_c$ . If this qualitative pattern survives in a realistic multiflavor treatment with the lattice-QCD equation of state, the Fisher-Rao metric on the QGP moment manifold would have a strongly temperature-dependent dissipative sector driven primarily by the strange quark contribution. Verifying this requires a calculation with the QCD equation of state, which is deferred to future work.

The physical interpretation is straightforward: mass couples the thermodynamic and dissipative sectors through the temperature dependence of the momentum moments, and the resulting curvature provides a geometric diagnostic of conformal-symmetry breaking. In the present manuscript this statement is supported numerically rather than by a closed analytic formula for the Ricci scalar.

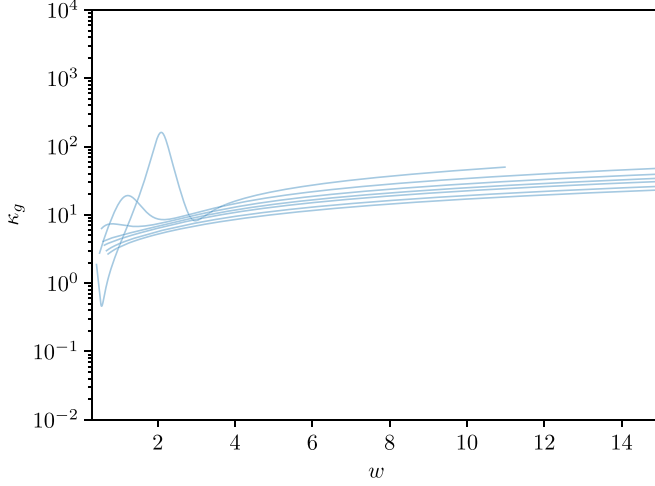


FIG. 7. Geodesic curvature  $\kappa_g$  of IS trajectories on the moment manifold. All trajectories have  $\kappa_g = O(1) - O(10)$  at all times. Different initial conditions converge to a common  $\kappa_g$  curve at late times, reflecting attractor convergence.

### VIII. GEODESIC CURVATURE OF ISRAEL-STEWART TRAJECTORIES

The geodesic curvature  $\kappa_g$  of a parametrized curve  $\gamma(\tau) = (T(\tau), \alpha(\tau))$  on  $(\mathcal{M}, g)$  is the magnitude of the covariant acceleration divided by the squared speed:

$$\kappa_g = \frac{\sqrt{A(a^T)^2 + B(a^\alpha)^2}}{A\dot{T}^2 + B\dot{\alpha}^2}, \quad (47)$$

where  $a^T = \ddot{T} + \Gamma_{TT}^T \dot{T}^2$  and  $a^\alpha = \ddot{\alpha} + \Gamma_{\alpha\alpha}^\alpha \dot{\alpha}^2$  are the components of the covariant acceleration, with Christoffel symbols from Eq. (44). A geodesic has  $\kappa_g = 0$ .

Figure 7 shows  $\kappa_g(\tau)$  along seven IS trajectories with initial shear  $\pi_0/\pi_{NS} \in \{-1, -0.5, 0, 0.5, 1, 2, 3\}$ .

The geodesic curvature is  $O(1) - O(10)$  everywhere, for all initial conditions, and does not decay to zero at late times. The IS attractor is not a geodesic, not approximately a geodesic, and does not asymptotically approach a geodesic.

#### A. Physical reason

Geodesics on  $\mathcal{M}$  are force-free trajectories—curves of minimum statistical distinguishability per unit parameter. The IS evolution is driven by two competing forces: collisional relaxation ( $-\pi/\tau_R$ , Sec. V) and longitudinal expansion ( $4\beta_\pi/(3\tau)$ ). These forces are external to the Riemannian geometry of  $\mathcal{M}$ ; they persist at all times and at all points on the attractor. The attractor is the trajectory that optimally balances these forces, not one that minimizes geometric length.

This rules out, at least in the 2D conformal Bjorken truncation, any variational characterization of the attractor as a geodesic or minimum-distance curve on  $\mathcal{M}$ . The attractor is a *dynamical* object—determined by the collision-expansion balance—not a *geometric* object—determined by the metric.

Figure 8 makes this visually explicit: IS trajectories (colored curves with arrows indicating the direction of increasing  $\tau$ ) follow paths dictated by the collision and expansion terms, while geodesics of the Fisher-Rao metric (black dashed) fol-

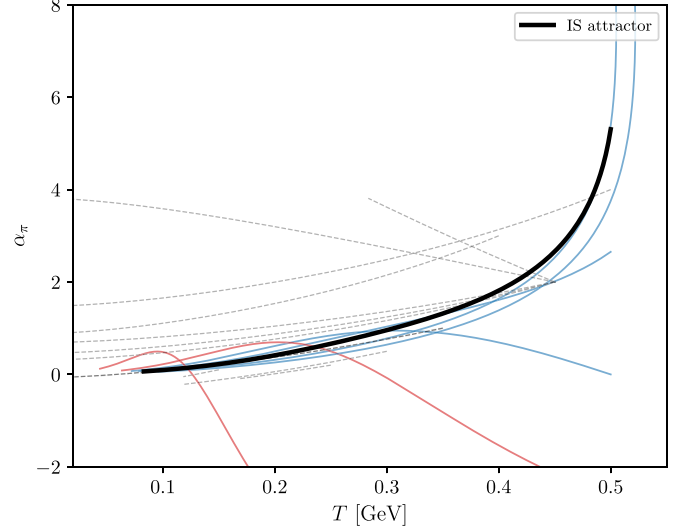


FIG. 8. Phase portrait on the  $(T, \alpha_\pi)$  moment manifold. Colored curves: IS trajectories for seven initial conditions (blue,  $\pi_0 \geq 0$ ; orange,  $\pi_0 < 0$ ). Black solid: exact RTA. Black dashed: geodesics of the Fisher-Rao metric. The IS trajectories and geodesics follow qualitatively different paths, confirming that the attractor is dynamical, not geometric.

low the intrinsic geometry. The two families of curves have qualitatively different behavior.

### IX. NUMERICAL RESULTS

I evaluate the Fisher-Rao framework numerically for a conformal massless gas with Boltzmann statistics and degeneracy  $g = 16$  (gluon proxy), at the Kovtun-Son-Starinets value [32]  $\eta/s = 1/(4\pi)$ . The IS Eqs. (12) and (13) are solved with initial temperature  $T_0 = 0.5$  GeV at  $\tau_0 = 0.2$  fm/cycle, for seven initial shear values  $\pi_0/\pi_{NS} \in \{0, 0.5, 1.0, 2.0, -0.5, 3.0, -1.0\}$ , where  $\pi_{NS} = 4\eta/(3\tau_0)$  is the Navier-Stokes value. The dimensionless time is  $w = \tau/\tau_R = \tau T/(5\eta/s)$ .

The exact RTA Boltzmann solution is obtained by solving the integral equation for the effective temperature  $T(\tau)$ , following the methods of Refs. [2,33,34], using the free-streaming functions  $R_\epsilon(y) = (1/2y^2) + \arctan(\sqrt{y^2 - 1})/(2\sqrt{y^2 - 1})$  and the corresponding  $R_\pi(y)$  derived from the zeroth and second angular moments of the free-streaming distribution [2]. The integral equation is iterated with relaxation factor 0.5 until the maximum fractional temperature change is below  $10^{-9}$  (typically  $\approx 60$  iterations on an  $N = 800$  grid). IS initial conditions are matched at  $\tau_0$  by specifying  $T(\tau_0) = T_0$  and  $\pi(\tau_0) = \pi_0$ ; the exact RTA is initialized with  $f(\tau_0) = f_{eq}(T_0)$  (equilibrium start). All computations are performed with the code `compute_v16.py`, available as Supplemental Material [35].

#### A. Classification of numerical quantities

The following quantities are *exact analytical*:  $g_{\alpha\alpha}|_{eq} = 2/15$ ,  $R = -5/21$ , all ten sectional curvatures, all cumulants ( $\kappa_2 - \kappa_4$ ,  $c_1$ ,  $c_2$ ), the Jacobi divergence rates, and the IS equa-

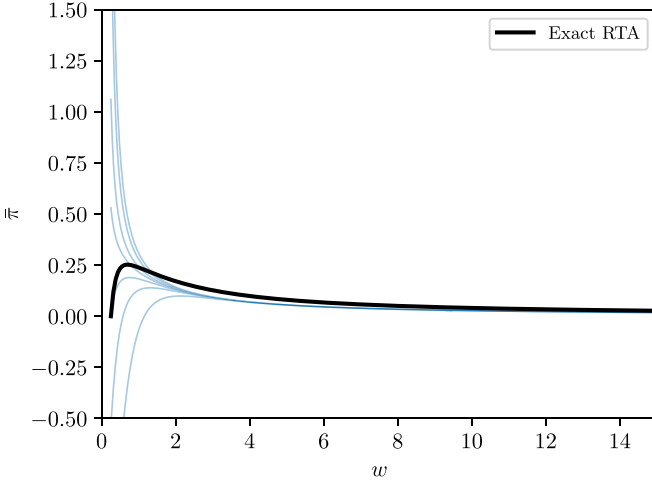


FIG. 9. Normalized pressure anisotropy  $\bar{\pi} = \pi/(\epsilon + P)$  vs dimensionless time  $w = \tau/\tau_R$  for conformal Bjorken flow at  $\eta/s = 1/(4\pi)$ . Blue curves: IS solutions with seven initial conditions. Red dashed: Navier-Stokes. Green dotted: free-streaming. All IS solutions converge to a common attractor by  $w \approx 3-5$ .

tion coefficients  $\delta_{\pi\pi} = 4/3$  and  $\tau_{\pi\pi} = 10/7$ . The following are computed *numerically from the cumulant-generating function*:  $g_{\alpha\alpha}(\alpha)$  at finite shear,  $\mathcal{D}_{\text{full}}(\alpha)$ , and the nonconformal table entries  $g_{\alpha\alpha}(T, m)$ . The following are computed *numerically from IS/RTA evolution*: all  $w$ -dependent quantities in this section (pressure anisotropy, Fisher-Rao distance along trajectories, transport recovery ratio, metric evolution, geodesic curvature, distance rate decomposition).

### B. Attractor convergence of pressure anisotropy

Figure 9 shows the normalized pressure anisotropy  $\bar{\pi} = \pi/(\epsilon + P)$  as a function of  $w$  for all seven IS initial conditions, together with the Navier-Stokes (first-order) and free-streaming solutions. The IS solutions converge to a common attractor curve by  $w \approx 3-5$ , confirming the standard attractor phenomenology [6]. The NS solution approaches the attractor from above (it overestimates the shear at early times), while the free-streaming solution approaches from below.

### C. Fisher-Rao distance and attractor convergence

Figure 10 shows the Fisher-Rao distance  $\mathcal{D}_{\text{full}}(\tau)$  for the same IS solutions. Three features are noteworthy.

- (i) All IS solutions converge to a common  $\mathcal{D}$  curve, confirming that the attractor is represented by a distinguished dynamical curve on the moment manifold.
- (ii) The convergence rate in  $\mathcal{D}$  is exponential with time constant  $\sim \tau_R$ , consistent with the collisional relaxation term  $-\mathcal{D}/\tau_R$  in Theorem 3.
- (iii) At late times ( $w \gg 1$ ), the attractor value of  $\mathcal{D}$  decreases as  $\mathcal{D}_* \sim 1/w$ , tracking the Navier-Stokes expectation  $\pi_{\text{NS}} \propto 1/\tau$ .

### D. Comparison with Knudsen and inverse Reynolds numbers

Figure 11 compares  $\mathcal{D}_{\text{full}}(\tau)$  with the Knudsen number  $\text{Kn} = \tau_R/\tau$  and inverse Reynolds number  $\text{Re}^{-1} = |\pi|/P$

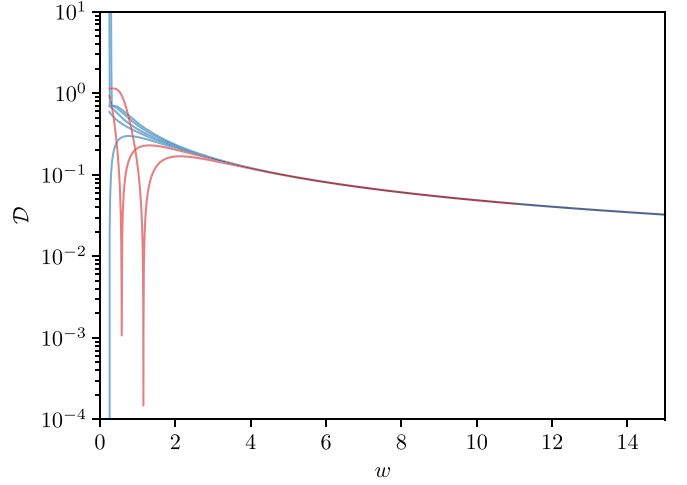


FIG. 10. Fisher-Rao distance within the projected 14-moment manifold,  $\mathcal{D}_{\text{full}}(\tau)$ , vs  $w = \tau/\tau_R$  for seven IS initial conditions (blue,  $\pi_0 \geq 0$ ; orange,  $\pi_0 < 0$ ). All solutions converge to a common attractor curve by  $w \approx 3$ . Green dotted: Navier-Stokes reference. Full range to  $w = 15$ .

along the NS-initialized IS solution. The three measures track each other at late times (all decrease as  $\sim 1/\tau$ ) but differ at early times:  $\text{Kn}$  decreases monotonically (it is purely geometric),  $\text{Re}^{-1}$  can increase transiently when viscous heating boosts  $|\pi|$  faster than  $P$  grows, and  $\mathcal{D}$  captures both effects through the metric. By Theorem 3, the linearized distance  $\mathcal{D}_{\text{lin}}$  is monotonically decreasing at leading order for  $\tau \gg \tau_R$ ; the numerical results confirm that  $\mathcal{D}_{\text{full}}$  exhibits similar late-time behavior.

### E. Metric component evolution

Figure 12 shows the ratio  $g_{\alpha\alpha}(\tau)/g_{\alpha\alpha}|_{\text{eq}}$  along the IS attractor (NS initialized) over the full range  $0 < w < 15$ . At early times ( $w \lesssim 1$ ), the metric drops to  $\approx 30\%$  of its equi-

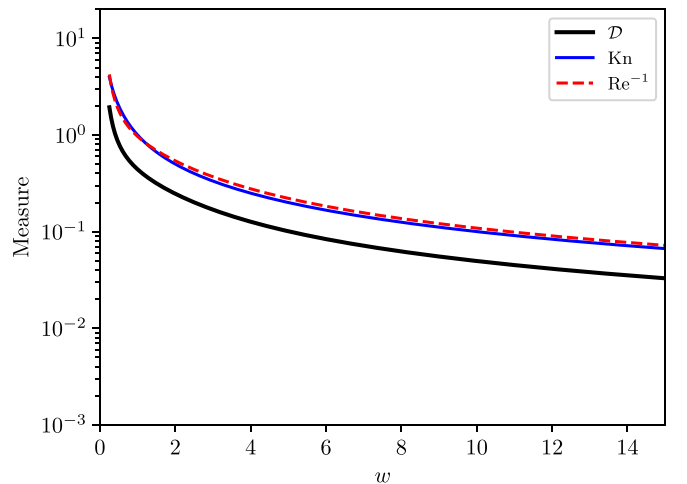


FIG. 11. Comparison of hydrodynamization measures: Fisher-Rao distance  $\mathcal{D}$  (black, solid), Knudsen number  $\text{Kn} = \tau_R/\tau$  (blue, solid), and inverse Reynolds number  $\text{Re}^{-1} = |\pi|/P$  (red, dashed), for the NS-initialized IS solution.

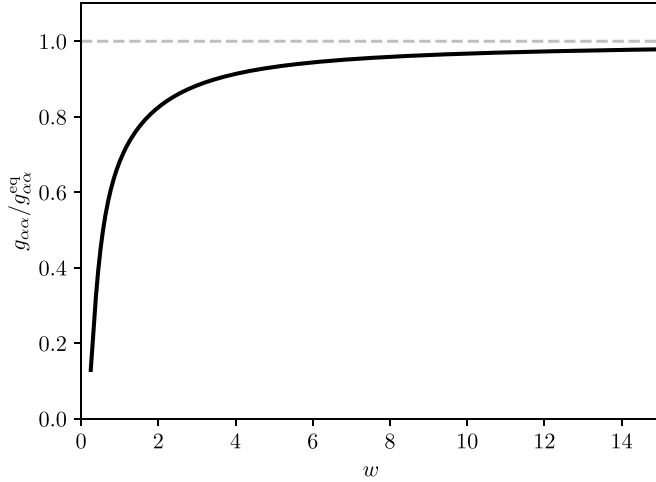


FIG. 12. Ratio of the dissipative Fisher-Rao metric component  $g_{\alpha\alpha}$  to its equilibrium value  $2/15$  along the IS attractor (NS-initialized). The departure from unity at early times ( $w \lesssim 3$ ) reflects the finite-shear corrections from Eq. (34). By  $w \gtrsim 4$ , the metric settles to within  $\approx 10\%$  of its equilibrium value.

librium value, reflecting the large finite-shear corrections from Eq. (34). By  $w \approx 5$  it recovers to within  $\approx 5\%$ , and by  $w \approx 10$  it is within  $\approx 1\%$ .

### F. Transport coefficient recovery

Figure 13 illustrates the geometric identity of Theorem 2 in dynamical evolution:  $\eta = 6P\tau_R g_{\alpha\alpha}$  in the conformal shear channel. At equilibrium this identity is exact. Away from equilibrium, the ratio  $(\eta/s)_{\text{geom}}/(\eta/s)_{\text{input}}$  deviates from unity because the local metric receives finite-shear corrections. The plot covers the full range  $0 < w < 15$ : the geometric extraction approaches the input value monotonically, reaching  $\approx 90\%$  accuracy by  $w \approx 4$  and  $>99\%$  by  $w \approx 10$ .

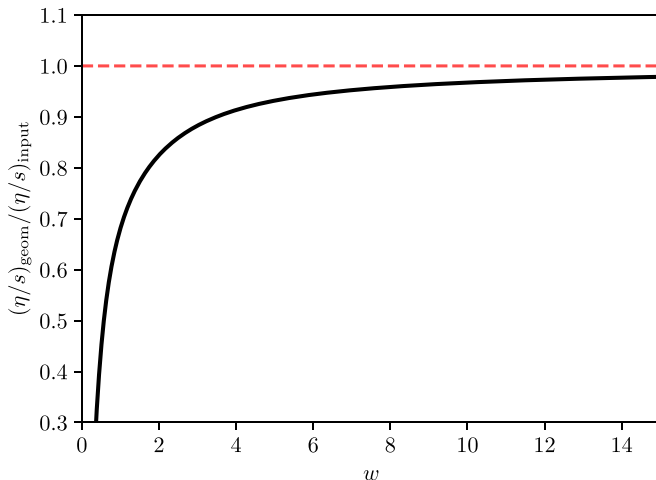


FIG. 13. Recovery of the shear viscosity from the Fisher-Rao metric:  $(\eta/s)_{\text{geom}}/(\eta/s)_{\text{input}}$  vs  $w$ , where  $(\eta/s)_{\text{geom}} = 6P\tau_R g_{\alpha\alpha}/s$ . The geometric extraction converges to the input value (red dashed) for  $w \gtrsim 3$ .

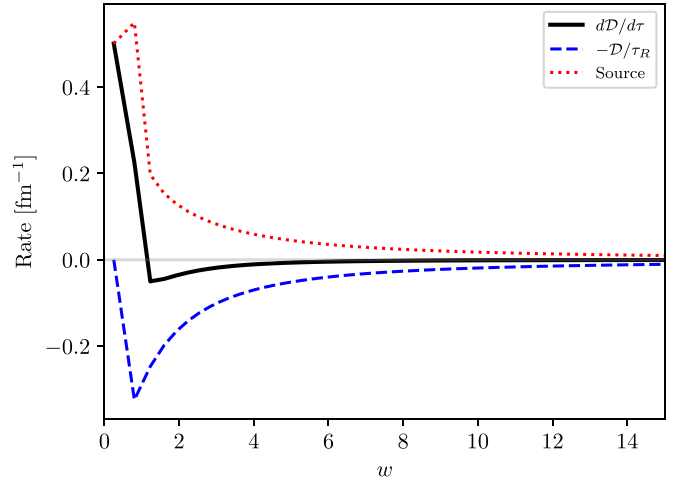


FIG. 14. Decomposition of  $d\mathcal{D}_{\text{full}}/d\tau$  into collisional relaxation ( $-\mathcal{D}/\tau_R$ , blue dashed) and expansion source (red dotted) along the exact RTA Boltzmann solution. Total rate: black solid. The crossing at  $w \approx 1$  defines the geometric hydrodynamization time. Full range to  $w = 15$ .

### G. Distance rate decomposition

Figure 14 decomposes  $d\mathcal{D}_{\text{full}}/d\tau$  numerically into a collisional relaxation contribution  $-\mathcal{D}_{\text{full}}/\tau_R$  and the remainder (expansion-driven source). This decomposition is motivated by the leading-order structure of Theorem 3 for  $\mathcal{D}_{\text{lin}}$ , but the plotted quantity is the full geodesic distance computed from the exact RTA solution. At early times ( $w \lesssim 1$ ), the expansion source dominates, driving the system away from equilibrium. At  $w \approx 1$ , the two rates cross: collisional relaxation begins to dominate, and  $\mathcal{D}_{\text{full}}$  enters an approximately monotonically decreasing regime. This crossing defines a geometric hydrodynamization time:  $\tau_{\text{hydro}} \approx \tau_R$ , or equivalently,  $w_{\text{hydro}} \approx 1$ .

### H. Summary of numerical results

Table VI collects the principal numerical quantities computed in this section.

TABLE VI. Summary of numerical results for conformal Bjorken flow at  $\eta/s = 1/(4\pi)$ ,  $g = 16$ ,  $T_0 = 0.5$  GeV,  $\tau_0 = 0.2$  fm/cycle. All IS results use NS-initialized trajectory.

| Quantity                                                     | Value              | Reference |
|--------------------------------------------------------------|--------------------|-----------|
| $g_{\alpha\alpha} _{\text{eq}}$                              | $2/15$ (exact)     | Theorem 2 |
| $\eta/s$                                                     | $1/(4\pi)$ (input) | Setup     |
| $w_{\text{hydro}}$ (rate crossing)                           | $\approx 1$        | Fig. 14   |
| $\mathcal{D}$ at $w = 1$ (NS-init)                           | $\approx 0.4$      | Fig. 10   |
| $\mathcal{D}$ at $w = 5$ (NS-init)                           | $\approx 0.05$     | Fig. 10   |
| $(\eta/s)_{\text{geom}}/(\eta/s)_{\text{input}}$ at $w = 4$  | $\approx 0.90$     | Fig. 13   |
| $(\eta/s)_{\text{geom}}/(\eta/s)_{\text{input}}$ at $w = 10$ | $> 0.99$           | Fig. 13   |
| $g_{\alpha\alpha}/g_{\alpha\alpha}^{\text{eq}}$ at $w = 5$   | $\approx 0.95$     | Fig. 12   |
| Attractor convergence                                        | $w \approx 2-3$    | Fig. 9    |

TABLE VII. Geodesic deviation rates on the five-dimensional conformal shear manifold at equilibrium. The divergence rate  $\lambda = \sqrt{|K|}$  is kernel independent.

| Plane        | Mode coupling                      | $K$       | $\lambda$ | Behavior |
|--------------|------------------------------------|-----------|-----------|----------|
| $(E^1, E^2)$ | $v_2 \leftrightarrow$ Bjorken      | $-10/49$  | 0.452     | Diverge  |
| $(E^2, E^3)$ | Bjorken $\leftrightarrow$ directed | $-10/49$  | 0.452     | Diverge  |
| $(E^1, E^4)$ | $v_2 \leftrightarrow xz$           | $-25/196$ | 0.357     | Diverge  |
| $(E^1, E^5)$ | $v_2 \leftrightarrow yz$           | $-25/196$ | 0.357     | Diverge  |
| $(E^3, E^4)$ | Directed $\leftrightarrow xz$      | $-25/196$ | 0.357     | Diverge  |
| $(E^3, E^5)$ | Directed $\leftrightarrow yz$      | $-25/196$ | 0.357     | Diverge  |
| $(E^4, E^5)$ | $xz \leftrightarrow yz$            | $-25/196$ | 0.357     | Diverge  |
| $(E^1, E^3)$ | $v_2 \leftrightarrow$ directed     | $+5/49$   | 0.319     | Focus    |
| $(E^2, E^4)$ | Bjorken $\leftrightarrow xz$       | $+5/196$  | 0.160     | Focus    |
| $(E^2, E^5)$ | Bjorken $\leftrightarrow yz$       | $+5/196$  | 0.160     | Focus    |

## X. GEODESIC DEVIATION HIERARCHY ON THE SHEAR MANIFOLD

The sectional curvatures computed in Theorem 4 are not merely geometric curiosities. Through the Jacobi equation for geodesic deviation, they determine the rate at which nearby distributions separate or converge on the Fisher-Rao manifold. This section derives the resulting hierarchy, connects it to the preequilibrium response functions of QCD kinetic theory [36,37], and leaves open the question of whether this equilibrium ordering has any dynamical consequence.

### A. Jacobi equation on the shear manifold

Let  $\gamma(s)$  be a geodesic on the five-dimensional shear manifold  $(\mathcal{M}_5, g_{ab})$  with tangent vector  $\dot{\gamma}^a$ . A nearby geodesic  $\gamma'(s)$  is described by the Jacobi field  $J^a(s)$ , satisfying

$$\frac{D^2 J^a}{ds^2} + R^a_{bcd} \dot{\gamma}^b J^c \dot{\gamma}^d = 0, \quad (48)$$

where  $D/ds$  denotes the covariant derivative along  $\gamma$ . For a geodesic in the direction of basis vector  $E^{(a)}$  and a deviation in the direction  $E^{(b)}$ , the transverse component satisfies

$$\ddot{J}^b + K(a, b) J^b = 0, \quad (49)$$

where  $K(a, b)$  is the sectional curvature of the  $(a, b)$  plane. The solutions are the following.

- (1)  $K < 0$ .  $J^b(s) \sim \exp(\lambda s)$  with  $\lambda = \sqrt{|K|}$ , exponential divergence.
- (2)  $K > 0$ .  $J^b(s) \sim \cos(\lambda s)$  with  $\lambda = \sqrt{K}$ , oscillatory focusing.
- (3)  $K = 0$ .  $J^b(s) \sim s$ , linear (flat space).

The divergence rate  $\lambda = \sqrt{|K|}$  is a purely geometric quantity: it depends only on the angular distribution of the moment function  $H_\pi$  on  $S^2$ , not on the collision kernel, the coupling strength, or the value of  $\eta/s$ .

### B. Hierarchy of divergence and focusing rates

Table VII lists the geodesic deviation rates for all ten independent two-planes.

The hierarchy has three tiers.

- (1) *Strongest divergence* ( $\lambda = \sqrt{10}/7 \approx 0.452$ ). The planes coupling the Bjorken direction  $E^{(2)}$  to the two dominant transverse harmonics ( $v_2$  and directed flow). Nearby geodesics in these planes separate exponentially, meaning that distributions differing in both longitudinal and transverse shear are more distinguishable on the Fisher-Rao manifold than a flat-geometry estimate would predict.
- (2) *Intermediate divergence* ( $\lambda = 5/14 \approx 0.357$ ). All planes involving at least one off-diagonal shear component ( $xz$  or  $yz$ ) that is not paired with the Bjorken direction.
- (3) *Focusing* ( $\lambda = \sqrt{5}/7 \approx 0.319$  for  $v_2$ -directed;  $\lambda = \sqrt{5}/14 \approx 0.160$  for Bjorken-off-diagonal). Nearby geodesics converge. Distributions differing in these channel pairs are *less* distinguishable than flat-geometry estimates predict.

Two structural features are noteworthy. First, the hierarchy is entirely determined by the representation theory of  $SO(3)$ : the angular moments of  $P_2(\mu)$ ,  $P_2(\mu)^2$ , and  $P_2(\mu)^3$  under the equilibrium measure on  $S^2$  fix the metric, the Amari-Chentsov tensor, and hence all sectional curvatures. Second, the sign pattern has a physical interpretation: the Bjorken direction  $E^{(2)}$  diverges from all other directions *except* the off-diagonal shear components  $E^{(4)}$  and  $E^{(5)}$ , with which it focuses. This reflects the kinematic fact that longitudinal expansion preferentially populates the longitudinal shear mode while leaving the off-diagonal modes at their equilibrium level.

### C. What the kernel does and does not determine

The deviation rates in Table VII are equilibrium quantities in the precise sense of Sec. III D: they are built from  $g_{ab}|_{\text{eq}}$  and  $T_{abc}|_{\text{eq}}$ , which are fixed by angular integrals against  $f_{\text{eq}}$  alone. The collision operator does not enter these integrals. Consequently, within any theory whose near-equilibrium moment manifold is the DNMR 14-moment expansion with the same shear basis, the equilibrium sectional curvatures and the resulting deviation rates have the same values: this is a Schur statement about the equilibrium distribution, not a prediction about the dynamics.

The distinction matters for interpretation. What is determined by  $f_{\text{eq}}$  alone is the *geometric* content of the equilibrium shear sector—the sectional-curvature numbers and the sign pattern that organizes them into Bjorken-vs-elliptic-flow and Bjorken-vs-off-diagonal pairs. What is *not* determined by  $f_{\text{eq}}$  alone is any *rate* that follows from placing an actual trajectory on the manifold: the relaxation rate  $1/\tau_R$ , the equilibration time  $\tau_{\text{hydro}}$ , and—crucially—whether the geometric deviation pattern translates into a dynamical hydrodynamization pattern all depend on the collision operator. Section VIII and the outlook below state this dependence explicitly.

### D. Outlook

The geodesic deviation hierarchy is a geometric property of the DNMR moment manifold: it is a numerical property of the equilibrium Fisher-Rao manifold within the 14-moment truncation; because it is built from angular integrals against

TABLE VIII. Comparison of geometric frameworks for relativistic fluid dynamics. “Internal” refers to the momentum-space/distribution function geometry; “external” refers to spacetime geometry.

| Framework                  | Space               | Type     | Coupling            | This paper                       |
|----------------------------|---------------------|----------|---------------------|----------------------------------|
| Ruppeiner [30]             | Equilibrium states  | Internal | Any                 | Thermodynamic sector of $g_{AB}$ |
| Fluid-gravity [38]         | Bulk spacetime      | External | Strong, large $N_c$ | Complementary                    |
| Carter variational [39,40] | Fluid worldsheet    | External | Any                 | Independent                      |
| Geroch-Lindblom [41]       | State space         | Internal | Any                 | Different generating function    |
| BDNK gauge freedom [42]    | Frame bundle        | Internal | Any                 | Fisher metric is frame invariant |
| Sarbach-Zannias [43]       | Tangent bundle $TM$ | External | Any                 | Provides kinematic arena         |
| Fisher-Rao (this work)     | Moment manifold     | Internal | Any (RTA here)      |                                  |

$f_{\text{eq}}$ , the Schur argument of Sec. III D applies and the ordering does not depend on the collision kernel at the level of those integrals. Whether it has dynamical consequences for mode-resolved hydrodynamization rates is an open question that requires either mode-resolved kinetic-theory simulations [36,37] or a derivation of the DNMR-to-response-function mapping at the level of the linearized Boltzmann equation. I do not claim a correspondence here; I report the geometry.

## XI. DISCUSSION

### A. Relation to existing geometric frameworks

The Fisher-Rao metric constructed here stands in precise relation to several existing geometric frameworks in relativistic fluid dynamics. Table VIII summarizes the comparison.

The  $(T, T)$  component of the equilibrium Fisher-Rao metric reproduces the Ruppeiner metric [30,44] exactly,  $g_{TT}|_{\text{eq}} = C_V/T^2 = 4\epsilon/T^3$  [Eq. (24)]; the dissipative block  $g_{\alpha\alpha}|_{\text{eq}} = 2/15$  has no analog in equilibrium thermodynamic geometry. The gauge analogy of Dore *et al.* [42] identifies the choice of hydrodynamic variables with gauge transformations; the Fisher metric on the full distribution  $f$  is invariant under those frame transformations. The tangent-bundle formulation of Sarbach and Zannias [43] provides the mass-shell arena on which the present construction lives. The Geroch-Lindblom divergence-type formulation [41] parametrizes viscous fluids by a generating function on state space; here the cumulant-generating function  $\Psi(\alpha)$  plays that role, with  $g_{\alpha\alpha} = \Psi''$  and  $T_{abc} = \Psi'''$ . The fluid-gravity correspondence [38,45] operates on external spacetime rather than the internal momentum manifold considered here and does not overlap in content within the regime of this paper.

### B. Implications for the matching problem

Theorem 3 motivates a leading-order geometric matching criterion: where  $\mathcal{D}_{\text{lin}}$  decreases monotonically along the IS evolution, the system is moving on the moment manifold toward equilibrium under collisional contraction. At leading order this happens once the collisional relaxation term dominates the expansion source—the rate-crossing condition of Sec. IX. A possible geometric matching criterion within the present model is  $\mathcal{D} < \mathcal{D}_{\text{threshold}}$ , calibrated against the regime in which IS and the exact RTA solution already agree. Because  $\mathcal{D}$  is a Riemannian distance it is invariant under

reparametrization, so the criterion is frame independent; the standard switching time  $\tau_{\text{switch}} \approx 1$  fm/cycle based on  $\text{Kn} < 1$  is a ratio-of-scales proxy for the same condition.

At  $\eta/s = 1/(4\pi)$ , a provisional calibration against the 5%-tolerance convergence of IS to exact RTA (Figs. 9–11) places  $\tau^* \approx 2\tau_R$  ( $w \approx 2$ ) and  $\mathcal{D}_{\text{threshold}} \approx 0.15$ , i.e.  $|\alpha_\pi| \approx 0.4$ . The value is  $\eta/s$  dependent; a systematic calibration across coupling strengths is left to future work. The geometric rate crossing of Fig. 14 near  $w \approx 1$  and the tolerance-based agreement near  $w \approx 2$  are not inconsistent: the former is the onset of collisional dominance (change in the dominant mechanism on the manifold); the latter is quantitative IS/RTA agreement.

### C. Extension to realistic collision kernels

The RTA kernel is analytically tractable but physically crude; the full QCD kernel (AMY  $2 \rightarrow 2$  scattering [46],  $1 \rightarrow 2$  radiative processes, medium-induced radiation) modifies the relaxation structure. The construction splits naturally. Within the restricted class of kernels considered here—those that share the equilibrium distribution  $f_{\text{eq}}$ —the equilibrium metric and equilibrium geometric values  $g_{\alpha\alpha}|_{\text{eq}} = 2/15$  and  $R|_{\text{eq}} = -5/21$  are unchanged at the level of the Schur argument of Sec. III D—the angular integrals against  $f_{\text{eq}}$  that define these quantities are fixed by  $\text{SO}(3)$  irreducibility and are insensitive to the collision kernel so long as the kernel preserves  $f_{\text{eq}}$  as its equilibrium distribution. The kernel dependence of the *dynamics* on the manifold is a separate question addressed below. The distance *dynamics* is kernel dependent: the simple relaxation  $-\mathcal{D}/\tau_R$  is replaced by a contraction involving the spectrum of the linearized collision operator. A mode-by-mode Fisher-Rao analysis against the KØMPØST spectrum [36] is the natural next test of whether the equilibrium deviation pattern organizes the dynamical hydrodynamization pattern.

### D. Block structure within the 14-moment truncation

The DNMR 14-moment basis decomposes at equilibrium into scalar, vector, and rank-2 traceless symmetric  $\text{SO}(3)$ -irreducible blocks. By the Schur argument of Sec. III D, the equilibrium metric is block diagonal within this basis. The conformal Bjorken setting of the present paper activates only the shear block; extensions to nonconformal matter, to heat flux, or to baryon diffusion couple additional blocks and lie outside the derivations here.

### E. Labeling the regimes by $\mathcal{D}$

Within the present truncation, the standard regimes of the kinetic-to-hydrodynamic crossover can be labeled by the scalar  $\mathcal{D}$ : equilibrium at  $\alpha_\pi = 0$ ,  $\mathcal{D} = 0$ ; the tangent-space regime where  $\mathcal{D}_{\text{full}}/\mathcal{D}_{\text{lin}}$  is within 7–35% of unity (Table II); and the free-streaming limit  $\tau_R \rightarrow \infty$ , where  $\mathcal{D}$  is driven purely by the expanding momentum distribution. The attractor is a dynamical curve, not a geodesic of the metric (Sec. VIII). No correspondence between these labels and physical hydrodynamization rates is claimed.

### F. Cumulant structure and the Amari-Chentsov framework

The cumulant expansion of the Fisher-Rao metric,  $g_{\alpha\alpha}(\alpha) = \kappa_2 + \kappa_3\alpha + \frac{1}{2}\kappa_4\alpha^2 + \dots$ , connects the present construction to Amari's theory of statistical manifolds [24]. The key structural features are the following.

- (i) *The second cumulant  $\kappa_2 = 2/15$  is the metric.* This is the variance of  $H_\pi$  under the equilibrium distribution and determines the local curvature of the log-likelihood surface.
- (ii) *The third cumulant  $\kappa_3 = -4\sqrt{6}/315$  is the Amari-Chentsov tensor.* Its nonvanishing means the shear-sector manifold is not dually flat: the  $e$  connection and  $m$  connection have nonzero torsion. Physically,  $\kappa_3 \neq 0$  reflects the angular asymmetry of  $H_\pi$  on the unit sphere:  $P_2(\mu)$  is not a symmetric function about its mean.
- (iii) *The fourth cumulant  $\kappa_4 = -8/525$  controls the curvature.* The Riemann tensor of the five-dimensional shear manifold is determined algebraically by  $\kappa_2$ ,  $\kappa_3$ , and the isometry group [24]. The negative value of  $\kappa_4$  is responsible for the negative Ricci scalar  $R = -5/21$ : the manifold has more curvature than the  $\kappa_3$ -induced connection alone would produce.
- (iv) *Higher cumulants are computable but not needed at equilibrium.* At finite shear the full function  $g_{\alpha\alpha}(\alpha)$  is required; the  $O(\alpha^2)$  truncation of the cumulant expansion breaks down beyond  $|\alpha| \approx 2$  (Fig. 1).

### G. Open problems

Two concrete questions follow directly from the 14-moment closure and the IS dynamics studied here.

- (i) The  $O(\text{Re}^{-2})$  nonlinear collision coefficients  $\phi_1 - \phi_8$  in the DNMR notation [26] are absent from the IS equations used here; including them would modify the geodesic curvature of IS trajectories on the manifold but not the Fisher-Rao metric itself, which depends only on the distribution function.
- (ii) The  $O(1)$  geodesic curvature of IS trajectories on the 2D conformal submanifold establishes that, within the present setting, the dynamics and the geometry are distinct curves. Whether the magnitude and scaling of  $\kappa_g$  on the full 14-dimensional manifold retain this feature is a concrete test of how much of the dynamics-geometry distinction survives within the 14-moment truncation.

## XII. LIMITATIONS

The construction and results of this paper fail, or are not claimed to hold, in the following specific ways.

- (1) *Nonconformal coupling of shear to other blocks.* The Schur-lemma argument underlying the robustness of the equilibrium shear-sector geometry (Sec. III D) relies on orthogonality between  $\text{SO}(3)$  irreducible blocks of the DNMR basis at equilibrium. Once nonconformal effects are introduced—for instance bulk-shear mixing at finite particle mass (Sec. VII)—this orthogonality is broken at equilibrium and the Schur argument that holds the 5D shear Ricci scalar fixed under orthogonal-block extensions no longer applies. The nonconformal illustration of Sec. VII uses a *classical massive gas* as a solvable nonconformal setting; it is not QCD and should not be read as a direct prediction for the QGP.
- (2) *Truncation breakdown at  $\ell > 2$ .* The DNMR 14-moment truncation is known to fail when the distribution develops structure at angular momenta  $\ell > 2$ . The Fisher-Rao analysis inherits this limitation: the moment manifold resolves only what the truncation captures. The equilibrium geometric values ( $g_{\alpha\alpha}|_{\text{eq}} = 2/15$ ,  $R = -5/21$ , the Jacobi rates) are exact properties of the 14-moment manifold, not of the full Boltzmann phase space.
- (3) *Kernel dependence of all rates.* The equilibrium Fisher metric and Amari-Chentsov tensor are kinematic quantities computed from angular integrals against  $f_{\text{eq}}$ ; they are fixed by the  $\text{SO}(3)$  representation structure of the basis (Schur's lemma) and do not depend on the collision operator. The *dynamics* on the manifold—the relaxation rate  $1/\tau_R$ , the equilibration time, and any map from the equilibrium geodesic-deviation pattern to a dynamical hydrodynamization pattern—does depend on the collision operator. No claim of kernel independence for dynamical rates is made at any point in this paper.
- (4) *The attractor is not a geodesic.* The  $O(1)$  geodesic curvature of Israel-Stewart trajectories established in Sec. VIII is a *negative* result in the following precise sense: it rules out any description of the hydrodynamic attractor as a geodesic of the Fisher-Rao metric within the present truncation. It does not provide a geometric characterization of the attractor; that remains an open problem.
- (5) *Finite-shear geometry is kinematic, not dynamical.* The 15–35% gap between  $\mathcal{D}_{\text{full}}$  and  $\mathcal{D}_{\text{lin}}$  is a statement about the statistical manifold itself (Secs. V and IX): it is computed from the exponential-family structure and tested against the exact RTA Boltzmann solution projected onto the 14-moment basis. It is not a prediction for IS dynamics, because the IS equations are linearized at  $O(\alpha_\pi)$  and do not see the  $O(\alpha_\pi^2)$  corrections that produce the gap. More generally, the finite-distance Fisher-Rao geodesic quantities, the sectional curvatures, the Ricci scalar,

and the Jacobi-deviation rates depend on the  $O(\alpha_\pi^2)$  and higher structure of the exponential family; because the dynamics used in this paper does not probe that structure, these quantities are reported as diagnostics of the *chosen statistical family* (the DNMR-moment exponential-family embedding) and not as diagnostics of the dynamics or of the full kinetic theory. The geometry is kinematic; the dynamics is linearized; they are compatible but distinct, as stated in Sec. III D.

- (6) *The Fisher-Rao distance does not encode full quantum distinguishability.* The present framework operates on classical Boltzmann distributions in the moment truncation. It does not address quantum distinguishability, entanglement, or off-shell correlators; these require a different statistical structure and are outside the scope of this paper.
- (7) *Closure dependence of finite-shear and curvature quantities.* The DNMR 14-moment state does not uniquely determine a distribution function: it is compatible with a family of completions that share the same 14 moments but differ at higher orders. The exponential family (19) is the Jaynes maximum-entropy completion of that truncation and is used here as a modeling choice. Quantities evaluated at equilibrium (the dissipative metric component  $g_{\alpha_\pi\alpha_\pi}|_{\text{eq}} = 2/15$ , the Navier-Stokes relation  $\eta = 4P\tau_R/5$ ) are closure independent within the 14-moment truncation because all admissible completions coincide at  $\alpha_\pi = 0$  to leading order. Finite-shear quantities that depend on  $O(\alpha_\pi^2)$  and higher structure—the full geodesic distance  $\mathcal{D}_{\text{full}}$ , the finite-shear deviation, the sectional curvatures  $K$ , the Ricci scalar  $R$ , the Jacobi-deviation rates—are closure dependent: they are properties of the maximum-entropy completion, and alternative completions of the same 14 moments would in general yield different values. These closure-dependent results are reported in this paper as diagnostics of the maximum-entropy exponential-family embedding used throughout, not as diagnostics of the underlying physical state; the maximum-entropy closure is not derived here from the kinetic equations and is not argued to be dynamically selected.
- (8) *Operational status of  $\mathcal{D}$ .* The Fisher-Rao distance  $\mathcal{D}$  is not an experimentally measured quantity in this paper; it is reconstructed from the same exact RTA kinetic solution used to validate hydrodynamic truncations, projected onto the 14-moment basis and then inserted into the exponential family of Sec. III D. Different reconstruction schemes (different closures of the 14-moment state) would in general produce different values for the closure-dependent quantities of L7. The quantity is model-internal: it is invariant under reparametrizations within the chosen statistical family, but not under change of truncation, change of closure, or change of distribution class.

### XIII. CONCLUSIONS

I have constructed the Fisher-Rao geometry on the DNMR 14-moment manifold in the relaxation-time approximation and conformal Bjorken flow, and computed its consequences for the kinetic-to-hydrodynamic crossover in this specific setting. The seven results established in this paper are the following.

- (1) The equilibrium dissipative Fisher metric in the conformal shear channel is  $g_{\alpha\alpha}|_{\text{eq}} = 2/15$ , an exact dimensionless constant (Theorem 2, proved in the Appendix). Through a thermodynamic prefactor fixed by the equation of state, it determines the shear viscosity as  $\eta = (4s\tau_R/5) g_{\alpha\alpha}|_{\text{eq}} = 4P\tau_R/5$ . Near-equilibrium transport is an equilibrium statistical geometric quantity in this setting.
- (2) The full geodesic Fisher-Rao distance in conformal Bjorken flow, computed within the DNMR 14-moment exponential family, differs from its standard linearized approximation by 15–35% at moderate shear when compared against the exact RTA Boltzmann solution projected onto the same 14-moment basis (Table II). This is the model-internal observable used to test the geometry against exact RTA: the gap is reconstructible from that projection and quantifies the breakdown of tangent-space expansions as a geometric property of the projected manifold, not of the full Boltzmann phase space.
- (3) The five-dimensional conformal shear sector of the moment manifold at equilibrium is intrinsically curved, with Ricci scalar  $R|_{\text{eq}} = -5/21$  (Theorem 4). Any one-dimensional Bjorken restriction is flat; the curvature is an intrinsically higher-dimensional property of the shear sector. The value is fixed by the  $\text{SO}(3)$  representation structure of the basis and the angular structure of  $P_2(\mu)$  on  $S^2$  via Schur's lemma applied to the equilibrium integrals.
- (4) In a nonconformal classical massive gas (used only as a solvable nonconformal setting, not QCD), the dissipative metric becomes temperature dependent, and the two-dimensional  $(T, \alpha_\pi)$  sector has a nonzero Ricci scalar that peaks near  $T \sim m$ . This is illustrative of how nonconformal coupling breaks the Schur orthogonality on which (3) relies.
- (5) Israel-Stewart trajectories on the moment manifold have  $\kappa_g = \mathcal{O}(1)$  geodesic curvature throughout (Sec. VIII): the hydrodynamic attractor is a dynamical trajectory, not a geodesic of the Fisher-Rao metric. This establishes that, within the present setting, the geometry and the dynamics are distinct curves on the manifold.
- (6) The Fisher-Rao distance provides a coordinate-invariant, information-theoretic measure of deviation from local equilibrium *within the truncated exponential family of the DNMR 14-moment basis*; by Theorem 3, the linearized distance is monotone-decreasing once the collisional relaxation term dominates the expansion source, so the crossover from expansion-driven

growth to collision-driven contraction suggests a geometric matching criterion within the present model. This is a diagnostic of the projected *Ansatz* and its evolution under IS dynamics; it is not a diagnostic of the full kinetic theory.

- (7) The Jacobi equation for geodesic deviation on the five-dimensional shear manifold organizes the ten shear-channel pairs into a pattern of divergence and focusing rates (Table VII). The Bjorken–elliptic-flow and Bjorken–directed-flow planes carry the strongest deviation ( $\lambda = \sqrt{10}/7$ ); the Bjorken–off-diagonal planes focus ( $\lambda = \sqrt{5}/14$ ). These rates are equilibrium quantities—fixed by  $g_{ab}|_{\text{eq}}$  and  $T_{abc}|_{\text{eq}}$ , which are angular integrals against  $f_{\text{eq}}$  alone (Schur’s lemma). Whether the dynamics on the manifold realizes this pattern as a dynamical hierarchy is a separate, kernel-dependent question; the paper does not claim it does.

*Scope.* The results are obtained within one specific setting: the DNMR 14-moment truncation, the RTA collision kernel, and conformal Bjorken flow. *Definitional results* (Fisher-Rao metric, the Hessian identity with relative entropy, positive definiteness, the metric-to-transport relation, the geodesic framework, the cumulant expansion) are formulated independently of the specific collision kernel and truncation used here; whether each of them carries over to a given alternative setting is a case-by-case question that the present paper does not undertake to settle in general. *Equilibrium geometric values* (the numbers  $g_{\alpha\alpha}|_{\text{eq}} = 2/15$ ,  $R = -5/21$ , the sectional curvatures, the Jacobi-deviation rates, the 15–35% finite-shear correction) are features of the DNMR 14-moment shear basis and the equilibrium angular structure of  $P_2(\mu)$  on  $S^2$ . Within the restricted class of truncation extensions considered here—those that add only SO(3)-irreducible blocks orthogonal to the shear block at equilibrium—these values are unchanged at the level of the argument given, where “unchanged” means that the equilibrium angular integrals against  $f_{\text{eq}}$  coincide before and after the enlargement by Schur’s lemma. Within extensions that couple the shear block at equilibrium, the argument does not hold. The statement is about equilibrium integrals against  $f_{\text{eq}}$ , not about the kinetic dynamics of the extension. Under extensions to angular moments  $\ell > 2$  or away from the 14-moment truncation, no claim is made. Section XII enumerates these failure modes explicitly.

*The contribution is therefore not a general reformulation of relativistic kinetic theory, but a controlled geometric description of the conformal Bjorken QGP in the RTA 14-moment truncation, with one observable reconstructible from the exact kinetic solution within the model—the 15–35% finite-shear distance gap against exact RTA—and explicit failure modes.*

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## DATA AVAILABILITY

The data that support the findings of this article are generated by the code supplied in the Supplemental Material [35].

## APPENDIX: PROOF OF THEOREM 2: MOMENTUM-SPACE INTEGRATION

I compute  $g_{\alpha\pi\alpha\pi}|_{\text{eq}} = \langle H_\pi^2 \rangle_{\text{eq}}$  and the transport coefficient  $\beta_\pi$  from first principles for the conformal massless classical gas with degeneracy  $g$ .

### 1. Conventions

Work in the LRF,  $u^\mu = (1, \mathbf{0})$ . Following the conventions of Eq. (17), the probability measure is  $d\mu_p = (up) d\Gamma_p = d^3p / [(2\pi)^3]$  (since  $p^0 = E_p = |\mathbf{p}|$  for massless particles in the LRF). The equilibrium distribution is  $f_{\text{eq}} = g e^{-E_p/T}$  (dimensionless, Boltzmann statistics). The number density is

$$n_{\text{eq}} = \int \frac{d^3p}{(2\pi)^3} g e^{-p/T} = \frac{g}{2\pi^2} \int_0^\infty p^2 e^{-p/T} dp = \frac{g T^3}{\pi^2}. \quad (\text{A1})$$

The energy density is  $\epsilon = 3g T^4/\pi^2$ , the pressure  $P = g T^4/\pi^2$ , and the entropy density  $s = 4g T^3/\pi^2$ .

### 2. The shear moment function

The Bjorken shear tensor in the LRF is  $\sigma^{ij} = \text{diag}(-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3})/\tau$  (for longitudinal expansion along  $z$ ), with magnitude  $|\sigma| = (\sigma_{ij}\sigma^{ij})^{1/2} = \sqrt{2/3}/\tau$ . The unit shear tensor is  $\hat{\sigma}^{ij} = \sigma^{ij}/|\sigma|$ .

The shear moment function (10) contracts the traceless momentum tensor with the unit shear direction:

$$H_\pi(\mathbf{p}) = \frac{p^{(i} p^{j)} \hat{\sigma}_{ij}}{E_p^2}, \quad (\text{A2})$$

where  $p^{(i} p^{j)} = p^i p^j - \frac{1}{3} p^2 \delta^{ij}$ . Since  $\hat{\sigma}$  is traceless, the  $\delta^{ij}$  term does not contribute, giving  $p^{(i} p^{j)} \hat{\sigma}_{ij} = p^i p^j \hat{\sigma}_{ij}$ .

Introduce spherical coordinates  $\mathbf{p} = p(\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$ . Then

$$p^i p^j \hat{\sigma}_{ij} = \sqrt{\frac{3}{2}} \frac{p^2}{3} (1 - 3 \cos^2 \theta) = -\sqrt{\frac{2}{3}} p^2 P_2(\cos \theta), \quad (\text{A3})$$

where  $P_2(\mu) = (3\mu^2 - 1)/2$  is the second Legendre polynomial. Since  $E_p = p$  for massless particles,

$$H_\pi = -\sqrt{\frac{2}{3}} P_2(\cos \theta). \quad (\text{A4})$$

Crucially,  $H_\pi$  is a *pure angular function* for massless particles—it depends on  $\hat{\mathbf{p}}$  but not on  $|\mathbf{p}|$ . This decoupling is specific to the conformal (massless) case.

### 3. Angular integral

$H_\pi^2 = \frac{2}{3} [P_2(\cos \theta)]^2$ . The angular integral is evaluated using the orthogonality relation for Legendre polynomials,

$$\int_{-1}^1 [P_\ell(\mu)]^2 d\mu = 2/(\ell + 1):$$

$$\begin{aligned} \int d\Omega H_\pi^2 &= \frac{2}{3} \int_0^{2\pi} d\phi \int_{-1}^1 d\mu [P_2(\mu)]^2 \\ &= \frac{2}{3} \times 2\pi \times \frac{2}{5} = \frac{8\pi}{15}. \end{aligned} \quad (\text{A5})$$

#### 4. Fisher metric component: $g_{\alpha\alpha}$

The Fisher metric in the  $\alpha_\pi$  direction at equilibrium is

$$g_{\alpha\alpha}|_{\text{eq}} = \frac{1}{n_{\text{eq}}} \int d\Gamma_p (up) f_{\text{eq}} H_\pi^2, \quad (\text{A6})$$

where the factor  $(up) = E_p$  arises from the probability measure  $d\mu_p = (up) d\Gamma_p$  defined in Eq. (17). Writing out the integration,

$$\begin{aligned} g_{\alpha\alpha}|_{\text{eq}} &= \frac{g}{n_{\text{eq}}} \int_0^\infty \frac{p^2 dp}{(2\pi)^3 p} p e^{-p/T} \int d\Omega H_\pi^2 \\ &= \frac{g}{n_{\text{eq}}} \frac{1}{(2\pi)^3} \frac{8\pi}{15} \int_0^\infty p^2 e^{-p/T} dp. \end{aligned} \quad (\text{A7})$$

The radial integral is  $\int_0^\infty p^2 e^{-p/T} dp = 2 T^3$ . Substituting,

$$\begin{aligned} g_{\alpha\alpha}|_{\text{eq}} &= \frac{g \times 8\pi \times 2T^3}{n_{\text{eq}} \times 8\pi^3 \times 15} = \frac{g \times 2T^3}{n_{\text{eq}} \times 15\pi^2} \\ &= \frac{g \times 2T^3 \times \pi^2}{gT^3 \times 15\pi^2} = \frac{2}{15}. \end{aligned} \quad (\text{A8})$$

This is an exact, dimensionless constant, independent of  $T$ .

#### 5. Transport coefficient: $\beta_\pi$

The shear viscosity in the Anderson-Witting RTA is obtained from the Navier-Stokes correction  $\delta f = (\tau_R f_{\text{eq}}/E_p T) p^i p^j \sigma_{ij}$  (derived by linearizing the Boltzmann equation (6)). The resulting shear stress is

$$\pi^{ij} = \frac{\tau_R}{T} \int d\Gamma_p \frac{f_{\text{eq}}}{E_p} p^i p^j p^k p^l \sigma_{kl}. \quad (\text{A9})$$

Evaluating the rank-4 angular integral via  $\int d\Omega \hat{p}^i \hat{p}^j \hat{p}^k \hat{p}^l = (4\pi/15)(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})$  and contracting with the traceless  $\sigma_{kl}$ ,

$$\int d\Omega \hat{p}^i \hat{p}^j \hat{p}^k \hat{p}^l \sigma_{kl} = \frac{8\pi}{15} \sigma^{ij}. \quad (\text{A10})$$

Hence,

$$\begin{aligned} \pi^{ij} &= \frac{\tau_R}{T} \frac{g}{(2\pi)^3} \frac{8\pi}{15} \int_0^\infty p^4 e^{-p/T} dp \sigma^{ij} \\ &= \frac{\tau_R}{T} \frac{g \times 8\pi \times 24T^5}{8\pi^3 \times 15} \sigma^{ij} = \frac{8g \tau_R T^4}{5\pi^2} \sigma^{ij}. \end{aligned} \quad (\text{A11})$$

Comparing with the constitutive relation  $\pi^{ij} = 2\eta \sigma^{ij}$  gives

$$\eta = \frac{4g \tau_R T^4}{5\pi^2} = \frac{4P \tau_R}{5} = \tau_R \beta_\pi, \quad (\text{A12})$$

confirming  $\beta_\pi = 4P/5$ . The consistency check  $\eta/s = \tau_R T/5$  reproduces the standard result [2,26].

#### 6. Relating $g_{\alpha\alpha}$ to $\beta_\pi$

The Fisher metric involves the integral  $\mathcal{I}_1 = \int d\Gamma_p E_p f_{\text{eq}} H_\pi^2 = n_{\text{eq}} \times 2/15$ , while the transport coefficient involves  $\mathcal{I}_3 = \int d\Gamma_p E_p^3 f_{\text{eq}} H_\pi^2$ . Since  $H_\pi$  is purely angular for massless particles, the ratio is

$$\frac{\mathcal{I}_3}{\mathcal{I}_1} = \frac{\int_0^\infty p^4 e^{-p/T} dp}{\int_0^\infty p^2 e^{-p/T} dp} = \frac{24T^5}{2T^3} = 12 T^2. \quad (\text{A13})$$

Therefore,

$$\begin{aligned} \beta_\pi &= \frac{\mathcal{I}_3}{15T} = \frac{12T^2 \mathcal{I}_1}{15T} = \frac{12T^2 n_{\text{eq}}}{15T} g_{\alpha\alpha}|_{\text{eq}} \\ &= \frac{12T n_{\text{eq}}}{15} \times \frac{2}{15} = \frac{4P}{5}, \end{aligned} \quad (\text{A14})$$

where the last step uses  $n_{\text{eq}} = gT^3/\pi^2$  and  $P = gT^4/\pi^2$ . This establishes the geometric identity

$$\eta = \tau_R \beta_\pi = \frac{12T n_{\text{eq}}}{15} \tau_R g_{\alpha\alpha}|_{\text{eq}}. \quad (\text{A15})$$

The prefactor  $12T n_{\text{eq}}/15 = 4(\epsilon + P)/(5T) = 4s/5$  is a thermodynamic quantity depending only on the equation of state. For an arbitrary equation of state, the prefactor generalizes to  $\mathcal{I}_3/(\mathcal{I}_1 \times 15T)$ , which is computable given the equilibrium distribution and remains a property of the equation of state alone, not of the nonequilibrium dynamics. ■

#### 7. Away-from-equilibrium corrections

For finite  $\alpha_\pi$ , the Fisher metric is expanded through the cumulant generating function  $\Psi(\alpha) = \ln \langle e^{\alpha H_\pi} \rangle_{\text{eq}}$ , whose second derivative is  $g_{\alpha\alpha}(\alpha) = \Psi''(\alpha) = \kappa_2 + \kappa_3 \alpha + \frac{1}{2} \kappa_4 \alpha^2 + \dots$ , where  $\kappa_n$  are the equilibrium cumulants of  $H_\pi$ .

The third cumulant ( $\kappa_3 = \langle H_\pi^3 \rangle_{\text{eq}}$  since  $\langle H_\pi \rangle_{\text{eq}} = 0$ ) is nonzero because  $[P_2(\mu)]^3$  has a nonvanishing integral:

$$\kappa_3 = \frac{1}{2} \left( \frac{2}{3} \right)^{3/2} \int_{-1}^1 [-P_2(\mu)]^3 d\mu = -\frac{4\sqrt{6}}{315}. \quad (\text{A16})$$

The fourth cumulant is  $\kappa_4 = \langle H_\pi^4 \rangle_{\text{eq}} - 3\kappa_2^2$ :

$$\langle H_\pi^4 \rangle_{\text{eq}} = \frac{4}{105}, \quad \kappa_4 = \frac{4}{105} - \frac{12}{225} = -\frac{8}{525}. \quad (\text{A17})$$

The correction coefficients in Eq. (34) are therefore

$$c_1 = \frac{\kappa_3}{\kappa_2} = -\frac{3}{7} \left( \frac{2}{3} \right)^{3/2}, \quad c_2 = \frac{\kappa_4}{2\kappa_2} = -\frac{2}{35}. \quad (\text{A18})$$

Both are negative, indicating that the metric *decreases* with increasing shear: the distribution becomes less statistically distinguishable from neighboring states as it moves away from equilibrium. The linear term  $c_1 \neq 0$  reflects the asymmetry between positive and negative shear for the Bjorken geometry.

- [1] P. Romatschke and U. Romatschke, *Relativistic Fluid Dynamics in and Out of Equilibrium* (Cambridge University, Cambridge, England, 2019).
- [2] W. Florkowski, M. P. Heller, and M. Spaliński, New theories of relativistic hydrodynamics in the LHC era, *Rep. Prog. Phys.* **81**, 046001 (2018).
- [3] M. P. Heller and M. Spaliński, Hydrodynamics beyond the gradient expansion: Resurgence and resummation, *Phys. Rev. Lett.* **115**, 072501 (2015).
- [4] J. Berges, M. P. Heller, A. Mazeliauskas, and R. Venugopalan, QCD thermalization: *Ab initio* approaches and interdisciplinary connections, *Rev. Mod. Phys.* **93**, 035003 (2021).
- [5] U. Heinz, Thermalization at RHIC, *AIP Conf. Proc.* **739**, 163 (2005).
- [6] P. Romatschke, Relativistic fluid dynamics far from local equilibrium, *Phys. Rev. Lett.* **120**, 012301 (2018).
- [7] M. P. Heller, R. A. Janik, and P. Witaszczyk, Hydrodynamic gradient expansion in gauge theory plasmas, *Phys. Rev. Lett.* **110**, 211602 (2013).
- [8] M. P. Heller, A. Serantes, M. Spaliński, V. Svensson, and B. Withers, Relativistic hydrodynamics: A singular perspective, *Phys. Rev. X* **12**, 041010 (2022).
- [9] J. Brewer, L. Yan, and Y. Yin, Adiabatic hydrodynamization in rapidly-expanding quark–gluon plasma, *Phys. Lett. B* **816**, 136189 (2021).
- [10] K. Rajagopal, B. Scheiing-Hitschfeld, and R. Steinhorst, Adiabatic hydrodynamization and the emergence of attractors: A unified description of hydrodynamization in kinetic theory, *J. High Energy Phys.* **04** (2025) 028.
- [11] M. P. Heller, A. Serantes, M. Spaliński, and B. Withers, The space of transport coefficients allowed by causality, *Nat. Phys.* **20**, 1948 (2024).
- [12] F. S. Bemfica, M. M. Disconzi, and J. Noronha, Causality and existence of solutions of relativistic viscous fluid dynamics with gravity, *Phys. Rev. D* **98**, 104064 (2018).
- [13] F. S. Bemfica, M. M. Disconzi, and J. Noronha, First-order general-relativistic viscous fluid dynamics, *Phys. Rev. X* **12**, 021044 (2022).
- [14] P. Kovtun, First-order relativistic hydrodynamics is stable, *J. High Energy Phys.* **10** (2019) 034.
- [15] P. Romatschke, Relativistic viscous fluid dynamics and non-equilibrium entropy, *Class. Quantum Grav.* **27**, 025006 (2010).
- [16] R. Khan *et al.*, Uncertainty quantification of holographic transport and energy loss for the hot and baryon-dense QGP, *arXiv:2603.20482*.
- [17] S. I. Finazzo, R. Rougemont, H. Marrochio, and J. Noronha, Hydrodynamic transport coefficients for the non-conformal quark–gluon plasma from holography, *J. High Energy Phys.* **02** (2015) 051.
- [18] H.-T. Ding, H.-T. Shu, and C. Zhang, Shear and bulk viscosities of the gluon plasma across the transition temperature from lattice QCD, *Phys. Rev. D* **113**, 074503 (2026).
- [19] P. Pavan *et al.*, Shear viscosity from quenched to full lattice QCD, *PoS LATTICE2024*, 199 (2025).
- [20] J. E. Bernhard, J. S. Moreland, and S. A. Bass, Bayesian estimation of the specific shear and bulk viscosity of quark–gluon plasma, *Nat. Phys.* **15**, 1113 (2019).
- [21] D. Everett *et al.* (JETSCAPE Collaboration), Multisystem Bayesian constraints on the transport coefficients of QCD matter, *Phys. Rev. C* **103**, 054904 (2021).
- [22] C. Shen, B. Schenke, and W. Zhao, Viscosities of the baryon-rich quark–gluon plasma from beam energy scan data, *Phys. Rev. Lett.* **132**, 072301 (2024).
- [23] N. N. Čencov, *Statistical Decision Rules and Optimal Inference* (American Mathematical Society, Providence, 1982).
- [24] S.-I. Amari and H. Nagaoka, *Methods of Information Geometry* (American Mathematical Society, Providence, 2000).
- [25] J. D. Bjorken, Highly relativistic nucleus-nucleus collisions: The central rapidity region, *Phys. Rev. D* **27**, 140 (1983).
- [26] G. S. Denicol, H. Niemi, E. Molnár, and D. H. Rischke, Derivation of transient relativistic fluid dynamics from the Boltzmann equation, *Phys. Rev. D* **85**, 114047 (2012); Erratum: **91**, 039902(E) (2015).
- [27] W. Israel and J. M. Stewart, Transient relativistic thermodynamics and kinetic theory, *Ann. Phys. (NY)* **118**, 341 (1979).
- [28] I. Müller, Zum Paradoxon der Wärmeleitungstheorie, *Z. Phys.* **198**, 329 (1967).
- [29] H. Grad, On the kinetic theory of rarefied gases, *Commun. Pure Appl. Math.* **2**, 331 (1949).
- [30] G. Ruppeiner, Riemannian geometry in thermodynamic fluctuation theory, *Rev. Mod. Phys.* **67**, 605 (1995); Erratum: **68**, 313(E) (1996).
- [31] R. Jordan, D. Kinderlehrer, and F. Otto, The variational formulation of the Fokker–Planck equation, *SIAM J. Math. Anal.* **29**, 1 (1998).
- [32] P. K. Kovtun, D. T. Son, and A. O. Starinets, Viscosity in strongly interacting quantum field theories from black hole physics, *Phys. Rev. Lett.* **94**, 111601 (2005).
- [33] M. Strickland, J. Noronha, and G. Denicol, Anisotropic nonequilibrium hydrodynamic attractor, *Phys. Rev. D* **97**, 036020 (2018).
- [34] C. Chattopadhyay, U. Heinz, S. Pal, and G. Vujanovic, Higher order and anisotropic hydrodynamics for Bjorken and Gubser flows, *Phys. Rev. C* **97**, 064909 (2018).
- [35] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/by8d-d9z2> for the Python analysis code `compute_v16.py` used to generate the numerical results reported in the figures and numerical tables, including the Fisher-Rao metric evaluations, finite-shear distance comparison, IS/RTA trajectory calculations, geodesic-curvature calculation, curvature checks, and Jacobi-deviation calculations.
- [36] A. Kurkela, A. Mazeliauskas, J.-F. Paquet, S. Schlichting, and D. Teaney, Matching the nonequilibrium initial stage of heavy ion collisions to hydrodynamics with QCD kinetic theory, *Phys. Rev. Lett.* **122**, 122302 (2019).
- [37] A. Kurkela, A. Mazeliauskas, J.-F. Paquet, S. Schlichting, and D. Teaney, Effective kinetic description of event-by-event pre-equilibrium dynamics in high-energy heavy-ion collisions, *Phys. Rev. C* **99**, 034910 (2019).
- [38] S. Bhattacharyya, V. E. Hubeny, S. Minwalla, and M. Rangamani, Nonlinear fluid dynamics from gravity, *J. High Energy Phys.* **02** (2008) 045.
- [39] B. Carter, Convective variational approach to relativistic thermodynamics of dissipative fluids, *Proc. R. Soc. A* **433**, 45 (1991).
- [40] N. Andersson and G. L. Comer, Relativistic fluid dynamics: Physics for many different scales, *Living Rev. Relativity* **10**, 1 (2007).
- [41] R. Geroch and L. Lindblom, Dissipative relativistic fluid theories of divergence type, *Phys. Rev. D* **41**, 1855 (1990).

- [42] T. Dore, J. Noronha, H. Niemi, and G. Torrieri, Fluctuating relativistic dissipative hydrodynamics as a gauge theory, [Ann. Phys. \(NY\) \*\*442\*\*, 168902 \(2022\)](#).
- [43] O. Sarbach and T. Zannias, The geometry of the tangent bundle and the relativistic kinetic theory of gases, [Class. Quantum Grav. \*\*31\*\*, 085013 \(2014\)](#).
- [44] F. Weinhold, Metric geometry of equilibrium thermodynamics, [J. Chem. Phys. \*\*63\*\*, 2479 \(1975\)](#).
- [45] V. E. Hubeny, S. Minwalla, and M. Rangamani, The fluid/gravity correspondence, [arXiv:1107.5780](#).
- [46] P. B. Arnold, G. D. Moore, and L. G. Yaffe, Effective kinetic theory for high temperature gauge theories, [J. High Energy Phys. \*\*01\*\* \(2003\) 030](#).