

Attractor for (1 + 1)D viscous hydrodynamics with general rapidity distribution

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The attractor, which is supposed to be one of the possible answers for the early applicability of hydrodynamics in the evolution with different initial conditions, has attracted great attention to the fast decreasing of degrees of freedom in heavy ion collisions. We find attractor behaviors for 1 + 1D viscous hydrodynamics with general rapidity distribution based on Müller-Isreal-Stewart theory. Meanwhile, we also observe that a rapid expansion in the fluid velocity is essential for a rapid early time attractor.

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Introduction. Relativistic heavy-ion collisions, pursued at the BNL Relativistic Heavy-Ion Collider (RHIC) and at the CERN Large Hadron Collider (LHC), allow systematic studies of a color-deconfined phase of matter—the quark-gluon plasma (QGP). One of the breakthroughs in theoretical relativistic heavy-ion physics has been the realization of the great success of numerical hydrodynamic simulations in understanding and predicting measurements of the collective behavior of the observed hadrons (see, e.g., Refs. [1–3]). Such numerical simulations show the applicability of hydrodynamics after an unexpected short time after the initial collision. The hydrodynamics theory is generally believed to apply for systems near local equilibrium, whereas the initial condition of the relativistic heavy-ion collisions is far from equilibrium. The reason of fast hydrodynamization in relativistic heavy-ion collisions remains a puzzle, and the hydroattractor [4] is plausibly the answer.

The Müller-Isreal-Stewart (MIS) theory of hydrodynamics is commonly adopted in the numerical simulations. It contains conservation of energy-moment, equation of state, and relaxation of the shear stress tensor to its Navier-Stokes limit,

$$\mathcal{D}_\mu T^{\mu\nu} = 0, \quad (1)$$

$$\tau_\pi \Delta_{\alpha\beta}^{\mu\nu} u^\lambda \mathcal{D}_\lambda \pi^{\alpha\beta} = -\pi^{\mu\nu} + \eta \sigma^{\mu\nu}, \quad (2)$$

where $\mathcal{D}_\mu$ is the covariant derivative, $\sigma^{\mu\nu} = 2\Delta_{\alpha\beta}^{\mu\nu} \mathcal{D}^\alpha u^\beta$ the shear tensor, $\eta = C_\eta \frac{\varepsilon + P}{T}$ the shear viscosity, and $\tau_\pi = \frac{C_\tau}{T}$ the relaxation time. The spatial projection operators are defined as $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$, $\Delta_{\alpha\beta}^{\mu\nu} = \frac{1}{2} \Delta_\alpha^\mu \Delta_\beta^\nu + \frac{1}{2} \Delta_\beta^\mu \Delta_\alpha^\nu - \frac{1}{3} \Delta^{\mu\nu} \Delta_{\alpha\beta}$.

The shear stress tensor is orthogonal to the fluid velocity, $u_\mu \pi^{\mu\nu} = 0$, and traceless $\Delta_{\mu\nu} \pi^{\mu\nu} = 0$. By taking the conformal limit, the energy density is given by $\varepsilon = \frac{3}{\pi^2} T^4$, the pressure is $P = \varepsilon/3$, and the bulk pressure vanishes, so that the decomposition of the stress tensor is $T^{\mu\nu} = (\varepsilon + P)u^\mu u^\nu - P g^{\mu\nu} + \pi^{\mu\nu}$. The $\mathcal{N}$ super Yang-Mills theory [5] gives that $C_\eta = \frac{1}{4\pi}$ and $C_\tau = \frac{2-\ln 2}{2\pi}$.

Heller and Spalinski [4] found the first attractor solution for viscous hydrodynamics in (0 + 1) dimensions, which applies for systems that are boost-invariant in the longitudinal direction and homogeneous in transverse direction. In (0 + 1) D, there are only two independent hydrodynamic variables, the energy density and one of the diagonal shear viscous tensor, and they depend only on the proper time, but not the longitudinal or transverse coordinates. It was found, in [4], that by introducing the dimensionless variables $w \equiv \tau T$ and $f \equiv \tau \frac{\partial_\tau w}{w} = 1 + \tau \frac{\partial_\tau T}{T}$, Eqs. (1) and (2) can be combined into a single-variable equation that contains up to first-order derivatives,¹

$$w f \frac{df}{dw} + 4f^2 + \left(\frac{w}{C_\tau} - \frac{20}{3} \right) f = \frac{4C_\eta}{9C_\tau} - \frac{8}{3} + \frac{2w}{3C_\tau}. \quad (3)$$

Reference [4] found that solutions of Eq. (3) with different initial conditions converge to a universal curve, called *attractor*, within $w \sim C_\tau$. Noting that $w/C_\tau = \tau/\tau_\pi$ is the ratio between proper time and the temperature-dependent relaxation time, the attractor behavior implies rapid convergence of the hydrodynamic modes.

Other attractor solutions are found in viscous hydrodynamics that obeys Gubser symmetry [6–8], anisotropic hydrodynamics with Bjorken-like [9] expansions, and even for a Boltzmann equation [10,11]. Attractors in higher-order viscous hydro [12], Boltzmann equation for ultrarelativistic gas with hard-sphere interaction [13] are also found. Such attractor solutions correspond to systems with space-time

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¹Note that some factors are different from Eq. (9) of [4] since Eq. (2) only keeps up to first order terms.

dependence highly constrained by symmetry properties. Attractors for hydrodynamics with general spatial dependence is not found yet. Attractors for a hydrosolution that violates homogeneity in the transverse plane is also explored in Refs. [14–16], but a large-time convergence that is independent of the spacial coordinates was not observed. Meanwhile, it was speculated [16] that the longitudinal expansion, rather than interaction, is essential for an attractor. In the current Letter, we study the attractors for (1 + 1) D viscous hydrodynamics that remain homogeneous in the transverse plane but breaks the boost invariance. We refer the readers to the reviews [17–19] and the references therein for other developments.

One may generalize the Bjorken flow to a hydrosolution with nontrivial rapidity dependence by performing a “temporal shift” [20,21]. For the generalized solution, one may find that the attractor solution obtained in [4] remains applicable after redefining the quantities in a Lorentz-invariant manner, i.e., one should replace the proper-time derivative by the comoving derivative, $\mathcal{D}_\tau \rightarrow u^\mu \mathcal{D}_\mu$, and the proper time shall be replaced by the inverse of the expansion rate, $\tau \rightarrow 1/\theta \equiv 1/(\mathcal{D}_\mu u^\mu)$. That is, the dimensionless quantities in the attractor shall be replaced as

$$\tilde{w} \equiv T/\theta, \quad \varphi \equiv 1 + \frac{u^\mu \partial_\mu T}{\theta T}. \quad (4)$$

Such a generalization can be justified as follows. From the energy-momentum conservation equation (1), one may derive that $1 + 3 \frac{u^\mu \partial_\mu T}{T\theta} = \frac{\pi^{\mu\nu} \sigma_{\mu\nu}}{2sT\theta}$, which further leads to $\varphi_{\text{NS}}(\tilde{w}) = \frac{2}{3} + \frac{4C_\eta}{9\tilde{w}}$ when taking the Navier-Stokes theory that $\pi^{\mu\nu} = \eta \sigma^{\mu\nu}$. Therefore, quantities defined in Eq. (4) can at least converge to $\varphi_{\text{NS}}(\tilde{w})$. Nontrivially, by numerically solving the MIS viscous hydroequations with general initial rapidity distribution, we observe that Eq. (4) exhibits (early) attractor solution that follows Eq. (3) by replacing that $f \rightarrow \varphi$ and $w \rightarrow \tilde{w}$. We note that such quantities are similar to those defined in [14].

Numerical solution. In a system that is homogeneous in the transverse plane, one has $u^x = u^y = 0$ and $\partial_x = \partial_y = 0$. We further let $u^\tau \equiv \cosh \frac{\zeta}{2}$ and $u^\eta \equiv \frac{1}{\tau} \sinh \frac{\zeta}{2}$, so that the normalization condition is satisfied. The hydroequations (1) and (2) become

$$0 = \partial_\tau \left(\frac{2\varepsilon}{3} \cosh \zeta + \frac{\varepsilon}{3} + \pi^{\tau\tau} \right) + \frac{4\varepsilon}{3\tau} \cosh \zeta + \partial_\eta \left(\frac{2\varepsilon}{3\tau} \sinh \zeta + \pi^{\tau\eta} \right) + \tau \pi^{\eta\eta} + \frac{\pi^{\tau\tau}}{\tau}, \quad (5)$$

$$0 = \left(\frac{3}{\tau} + \partial_\tau \right) \left(\frac{2\varepsilon}{3\tau} \sinh \zeta + \pi^{\tau\eta} \right) + \partial_\eta \left(\frac{2\varepsilon}{3\tau^2} \cosh \zeta - \frac{\varepsilon}{3\tau^2} + \pi^{\eta\eta} \right), \quad (6)$$

$$0 = u^\lambda \partial_\lambda \pi^{\tau\tau} - \pi^{\tau\tau} \coth \frac{\zeta}{2} u^\lambda \partial_\lambda \zeta - \frac{\eta \sigma^{\tau\tau} - \pi^{\tau\tau}}{\tau_\pi}. \quad (7)$$

Different components of the π tensor are connected with each other when implementing the orthogonal relation.

In this work, we initialize the hydroprofile by a parametrization inspired by an asymmetric analytical solution [20]

$$\frac{\varepsilon}{\varepsilon_i} = (t_0 + a \tau_i e^\eta)^{\frac{2}{3}(a^2-2)} (t_0 + \tau_i e^{-\eta}/a)^{\frac{2}{3}(a^2-2)}, \quad (8)$$

$$\zeta = f_\zeta \ln \frac{t_0 e^{-\eta} + a \tau_i}{t_0 e^\eta + \tau_i/a}, \quad (9)$$

$$\pi^{\eta\eta} = f_\pi \frac{\varepsilon}{\tau_i^2}. \quad (10)$$

τ_i is the initial proper time, t_0 is of the time unit and controls the rapidity plateau, $1/\sqrt{2} < a < \sqrt{2}$ controls the level of asymmetry, and f_ζ and f_π are additional parameters alters the initial profile from the analytical solution. Taking $f_\zeta = 1$ and $f_\pi = 0$, as well as turning off shear viscosity $C_\eta = 0$, the solution would return to the analytical solution found in [20]. We fix $t_0 = 0.1\tau_i$ and $\varepsilon_i = \tau_i^{-4}$ and vary a , f_ζ , and f_π and check attractor behavior. We note that such initialization parametrization is for the convenience of controlling different components, and the conclusion does not rely on it.

We begin by exploring the influence of initial flow velocity. We take $a = 1.2$, $f_\pi = 0.2$, and $f_\zeta = \{0, 0.5, 1\}$, and numerically solve Eqs. (5)–(7) with the corresponding initial profile. After obtaining the space-time dependent temperature and velocity, we construct $\tilde{w}$ and φ according to Eq. (4), as well as the pressure anisotropy $\frac{P_L}{P_T} = \frac{\varepsilon + 3\pi_x^x}{\varepsilon - 3\pi_x^x/2}$. We plot them

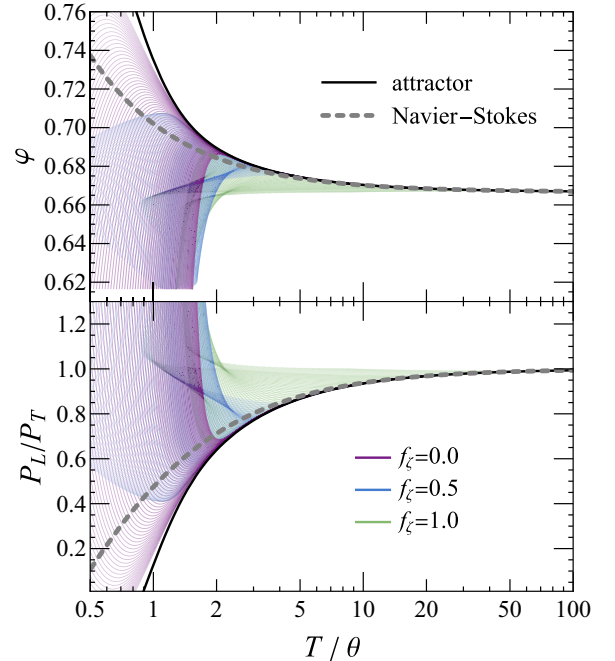


FIG. 1. Novel attractor quantity (upper) and pressure anisotropy (lower) versus the scaled comoving time. In all four panels, thin curves in purple, blue, and green correspond to three different hydroprofiles with initial flow parameters $f_\zeta = 0$, $f_\zeta = 0.5$, and $f_\zeta = 1$, respectively, whereas different curves in the same color are from different rapidity slices in the same hydroprofile. For comparison, black thick solid curves correspond to attractor solutions whereas gray dotted curves are the 0 + 1D solution of Navier-Stokes hydro.

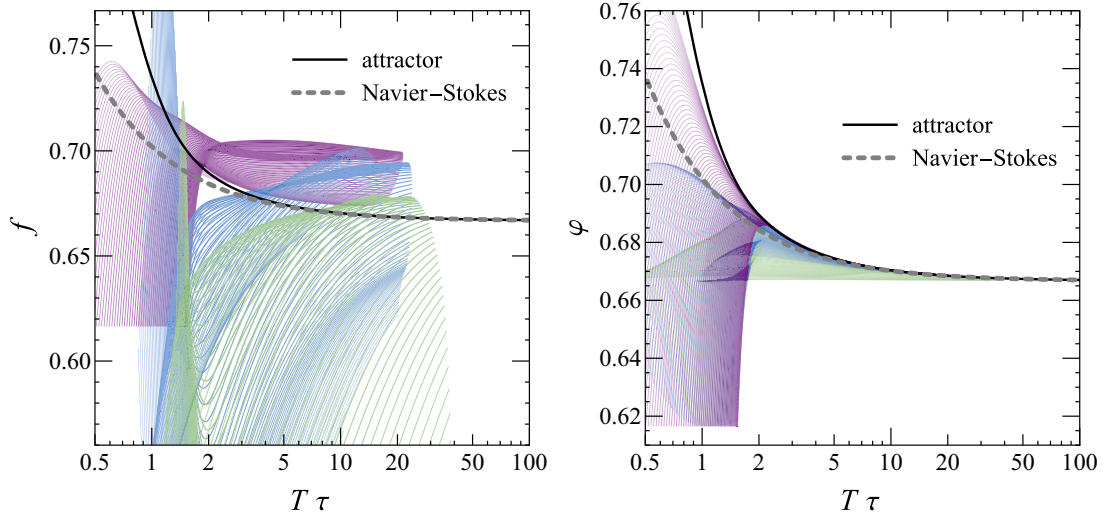


FIG. 2. Bjorken-based (left) and novel (right) attractor quantities versus scaled proper time. Parameter settings and color coding are the same as Fig. 1.

for different rapidity slides and present the results in Fig. 1. For all rapidity slides and for all values of f_ζ , all curves eventually approach the attractor solution, i.e., solution of Eq. (3) but with the replacement that $f \rightarrow \varphi$ and $w \rightarrow \tilde{w}$. Meanwhile, the attractor of pressure anisotropy is given by $A^P(\tilde{w}) = \frac{3-4\varphi(\tilde{w})}{2\varphi(\tilde{w})-1}$, see, e.g., [9]. For assessment of the attractor speed, we also show results from the Navier-Stokes theory, which takes $\pi^{\mu\nu} = \eta\sigma^{\mu\nu}$ rather than solving the relaxation time equation (2) and gives that $\varphi_{\text{NS}}(\tilde{w}) = \frac{2}{3} + \frac{4C_\eta}{9\tilde{w}}$ and $A^P_{\text{NS}}(\tilde{w}) = \frac{3-4\varphi_{\text{NS}}(\tilde{w})}{2\varphi_{\text{NS}}(\tilde{w})-1}$. It is apparent that the hydrosolution converges to the attractor solution faster than the convergency between attractor and Navier-Stokes solution when $f_\zeta = 0$ or

0.5, whereas the late time attractor is observed when $f_\zeta = 1$. From [20] one can learn that an analytical solution, corresponding to $f_\zeta = 1$, expands slower than the Bjorken flow, i.e., $f_\zeta = 0$. We conclude that the expansion with the speed of a Bjorken flow—or at least close enough—is essential for the early time attractor.

For comparison, we also present the Bjorken-based and the novel attractor quantities versus the temperature scaled proper time, see Fig. 2. It is clear that no attractor behavior is observed for the Bjorken-based quantity. While there seems to be some attractor behavior in $\varphi(w)$, we emphasize that $\varphi(\tilde{w})$ is a better one. In addition to the theoretical advantage that $\tilde{w}$ is invariant under Lorentz boost along any

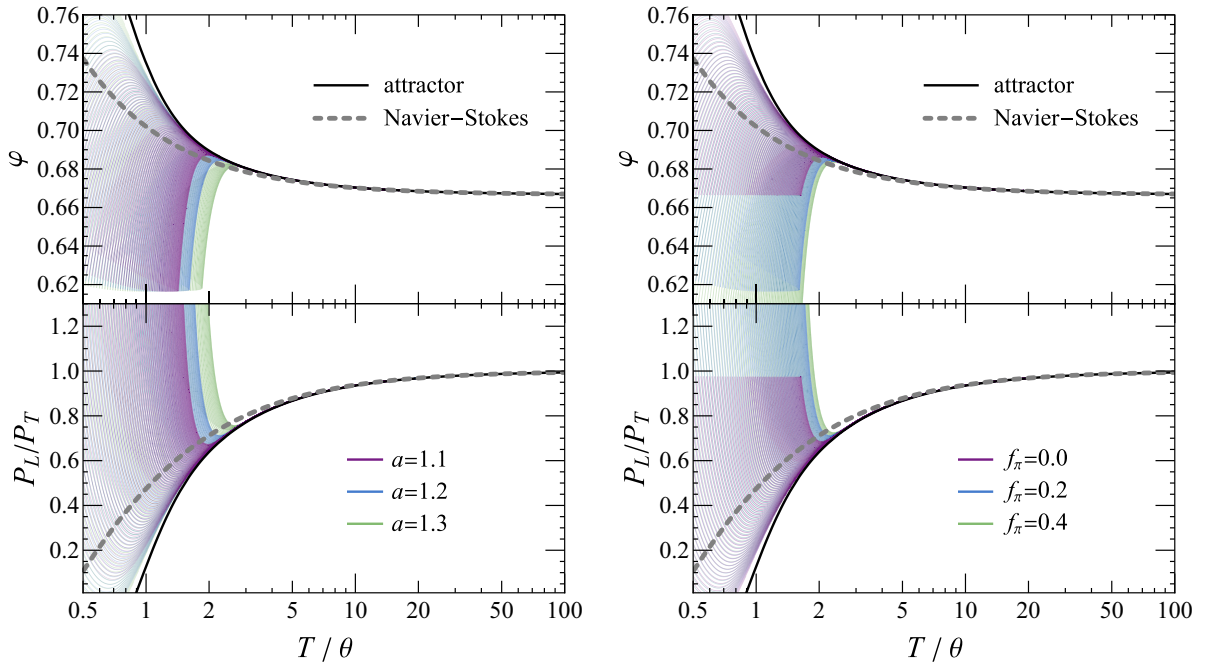


FIG. 3. Same as Fig. 1 but for the comparison of initial parameters $a \in \{1.1, 1.2, 1.3\}$ (left) and $f_\pi \in \{0.0, 0.2, 0.4\}$ (right).

direction, the phenomenological advantage is more apparent in the $\tilde{w} \in [1, 2]$ sector of the blue curves, and the narrower white region between the purple curves and the attractor at early time.

After exploring the effect of initial flow, we move on to study the asymmetry and initial shear. We fix $f_\zeta = 0.2$, and take $f_\pi = 0.2$ and $a \in \{1.1, 1.2, 1.3\}$ in the former, and $a = 1.2$ and $f_\pi \in \{0, 0.2, 0.4\}$ in the latter. Results are shown in Fig. 3. In general, we observe that the hydrosolution approaches to the attractor more rapidly in more symmetric system (with a closer to unity) or with less initial shear stress tensor (smaller f_π). Early time attractor behavior is observed in all these parameter sets.

Summary and outlooks. Attractor behavior in hydrodynamic evolution is important in understanding the rapid applicability of the theory of hydrodynamics in relativistic heavy-ion collisions. So far, large-time attractor behaviors are only observed in systems that are boost-invariant in the

longitudinal direction and homogeneous in the transverse direction. For realistic heavy-ion collisions, it is crucial to relieve the aforementioned assumptions and study the attractor behavior for general initial conditions. In the current Letter, we move a step forward by proposing a novel attractor that replaces the quantities of the attractor found by Heller and Spalinski [4] into the corresponding Lorentz-invariant version. We numerically solve the $1+1$ dimensional viscous hydrodynamic equation with general initialization and find that the hydrosolution approaches to the novel attractor solution. We also observe that the initial expansion rate that is close to Bjorken flow is essential for the emergence of an early-time attractor. The generalization of attractor observables sheds light on the study of attractors in hydrosystems with realistic $3+1$ dimensional expansion.

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