

Density correlations under global and local charge conservation

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This work presents a formalism to compute spatial n -point correlations of a conserved charge density in a large thermal system in the canonical ensemble, with explicit results presented up to fourth order. The resulting correlators contain local and balancing terms expressed through the grand-canonical susceptibilities for any equation of state. The new formalism is used to introduce a Gaussian local baryon number conservation in (spatial) rapidity in heavy-ion collisions at CERN Large Hadron Collider conditions through a modulation of the balancing term. Connection to the conservation volume V_c approach is established, indicating that the latter is appropriate for observables within a limited range around midrapidity. Quantitative analysis of the experimental data of the ALICE Collaboration on net proton cumulants in central Pb-Pb collisions at $\sqrt{s_{NN}} = 5.02$ TeV shows significant evidence for local baryon conservation.

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Introduction. Fluctuations and correlations of particle numbers produced in heavy-ion collisions are key observables in studies of the quantum chromodynamics (QCD) phase structure, including the search for QCD critical point within energy scan programs at the BNL Relativistic Heavy Ion Collider (RHIC) [1–5], CERN Super Proton Synchrotron (SPS) [6], and FAIR/GSI [7–9], and chiral criticality at the CERN Large Hadron Collider (LHC) [10,11]. In the grand-canonical limit, fluctuations of conserved charges are directly proportional to the susceptibilities: $\kappa_n[B] = V\chi_n$, where $\chi_n = T^{n-1}\partial^n p/\partial\mu^n$ [12]. The susceptibilities also determine the local density fluctuations, namely [13],

$$\langle\delta\rho(\mathbf{r}_1)\delta\rho(\mathbf{r}_2)\rangle_{\text{local}} \propto \chi_2 \delta(\mathbf{r}_1 - \mathbf{r}_2), \quad (1)$$

where $\delta(\mathbf{r}_1 - \mathbf{r}_2)$ should be understood as a localized function of the order of correlation length, $|\mathbf{r}_1 - \mathbf{r}_2| \lesssim \xi$.

However, the total charge is conserved in heavy-ion collisions, $\langle\delta B_{\text{tot}}^2\rangle = \int d\mathbf{r}_1 d\mathbf{r}_2 \langle\delta\rho(\mathbf{r}_1)\delta\rho(\mathbf{r}_2)\rangle = 0$, therefore, a long-range contribution to $\langle\delta\rho(\mathbf{r}_1)\delta\rho(\mathbf{r}_2)\rangle$ that balances the local part must exist [14–17]. This is also the case for high-order, n -point charge density correlations. Their possible nonequilibrium evolution on a hydrodynamics background has been discussed recently in several works [18–25].

The importance of baryon conservation effects for controlling the noncritical baseline for proton number cumulants is now well established [26–28]. Canonical treatment of global baryon conservation provides the minimum baseline. At the same time, even stronger, *local* effects of exact charge

conservation [29–31] are likely to be relevant in heavy-ion collisions due to the presence of causally disconnected regions of the fireball [32]. Controlling these effects is crucial for extracting physics relevant to the QCD phase structure from experimental measurements of fluctuations. Currently, a simplified model of a truncated fireball in a conservation (correlation) volume V_c [30] is being used to constrain the local conservation of QCD charges at LHC conditions through measurements of various fluctuations and correlations of hadron numbers [33–35].

In Refs. [36–38] a subensemble acceptance method (SAM) was developed to correct cumulants measured inside a spatial subvolume for global charge conservation effects. Here, this method is generalized to compute the n -point spatial density correlator to the desired order. The resulting explicit form of the balancing contributions in equilibrium allows one to introduce the effect of local charge conservation, which can be constrained with experimental measurements. As a first application, it is demonstrated how the available measurements of net proton variance constrain the rapidity range of local baryon conservation at the LHC.

Definitions. The primary quantity of interest here is the n -point spatial density correlator C_n , defined as

$$C_n(\mathbf{r}_1, \dots, \mathbf{r}_n) \equiv \left\langle \prod_{i=1}^n \delta\rho(\mathbf{r}_i) \right\rangle_c, \quad n \geq 2, \quad (2)$$

and $C_1(\mathbf{r}_1) = \rho(\mathbf{r}_1)$. Here, $\delta\rho(\mathbf{r}_i) = \rho(\mathbf{r}_i) - \langle\rho(\mathbf{r}_i)\rangle$ is the density fluctuation at coordinate $\mathbf{r}_i$, and the notation $\langle\cdots\rangle_c$ indicates cumulants, which differ from central moments starting from fourth order. Up to order four, the correlators read

$$C_2(\mathbf{r}_1, \mathbf{r}_2) = \langle\delta\rho_1\delta\rho_2\rangle, \quad C_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \langle\delta\rho_1\delta\rho_2\delta\rho_3\rangle, \quad (3)$$

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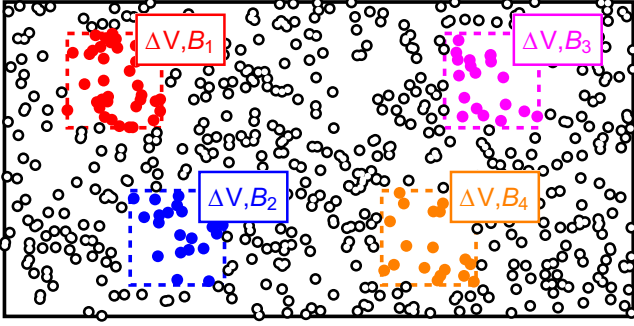


FIG. 1. Schematic illustration of the partition of the system volume V into distinct subvolumes of size ΔV . The points illustrate the distribution of the conserved charge B in a single microstate.

and [21]

$$\begin{aligned} \mathcal{C}_4(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) = & \langle \delta \rho_1 \delta \rho_2 \delta \rho_3 \delta \rho_4 \rangle - \langle \delta \rho_1 \delta \rho_2 \rangle \langle \delta \rho_3 \delta \rho_4 \rangle \\ & - \langle \delta \rho_1 \delta \rho_3 \rangle \langle \delta \rho_2 \delta \rho_4 \rangle - \langle \delta \rho_1 \delta \rho_4 \rangle \langle \delta \rho_2 \delta \rho_3 \rangle. \end{aligned} \quad (4)$$

Here, $\delta \rho_i \equiv \delta \rho(\mathbf{r}_i)$. Integrating $\mathcal{C}_n$ over any subvolume V_s therefore yields the value of the n th-order cumulant of the charge distribution inside V_s , i.e.,

$$\kappa_n[B_{V_s}] = \int_{\mathbf{r}_1 \in V_s} d\mathbf{r}_1 \dots \int_{\mathbf{r}_n \in V_s} d\mathbf{r}_n \mathcal{C}_n(\{\mathbf{r}_i\}). \quad (5)$$

Integrating over distinct volumes yields joint cumulants, which quantify the correlations between charges across different spatial domains.

Two-point correlator. The two-point correlator can be written as a sum of local and balancing parts,

$$\mathcal{C}_2(\mathbf{r}_1, \mathbf{r}_2) = \chi_2 \delta(\mathbf{r}_1 - \mathbf{r}_2) + \mathcal{C}'_2(\mathbf{r}_1, \mathbf{r}_2). \quad (6)$$

The local part corresponds to the susceptibility χ_2 . The balancing part satisfies $\int \mathcal{C}'_2(\mathbf{r}_1, \mathbf{r}_2) d\mathbf{r}_{1,2} = -\chi_2$ since the charge variance $\kappa_2[B]$ vanishes in the full volume. The correlation from global charge conservation is expected to be long range [39], entailing that $\mathcal{C}'_2(\mathbf{r}_1, \mathbf{r}_2)$ is uniform. One can thus empirically infer $\mathcal{C}'_2(\mathbf{r}_1, \mathbf{r}_2) = -\chi_2/V$ and

$$\mathcal{C}_2(\mathbf{r}_1, \mathbf{r}_2) = \chi_2 \delta(\mathbf{r}_1 - \mathbf{r}_2) - \frac{\chi_2}{V}. \quad (7)$$

To obtain a rigorous derivation, extendable to high-order correlators, one can utilize the structure of the partition function in the thermodynamic limit. First, split the total volume into n distinct subvolumes of size ΔV plus the remainder volume equal to $V - n\Delta V$. This is schematically illustrated in Fig. 1, where different colored boxes show the subvolumes and the charge carriers inside them. The subvolumes can exchange the conserved charge with the remainder volume, hence the dashed lines at their boundaries. The subvolumes are assumed to be macroscopically large such as to capture the physics of correlation length, $\Delta V \gg \xi^3$. Denoting the baryon numbers in each subvolume by $\{B_1, \dots, B_n\}$, their

joint probability is then proportional to the product of the canonical partition functions $Z(V, B)$ [40] of the subvolumes [13]:

$$\begin{aligned} P(\{B_i\}) & \propto \left[\prod_{j=1}^n Z(\Delta V, B_j) \right] Z\left(V - n\Delta V, B - \sum_{j=1}^n B_j\right) \\ & \propto \left[\prod_{j=1}^n e^{-\Delta V f(\rho_j)} \right] e^{-(V - n\Delta V)f(\rho_{n+1})}. \end{aligned} \quad (8)$$

Here, $f(\rho)$ is the free energy density, $\rho_j = B_j/\Delta V$ for $j = 1, \dots, n$ and $\rho_{n+1} = (B - \sum_{j=1}^n B_j)/(V - n\Delta V)$. The joint probability (8) can be used to calculate the joint cumulants $\langle \delta B_1^{k_1}, \dots, \delta B_n^{k_n} \rangle_c$ and express these in terms of the susceptibilities χ_n that are encoded in the free energy density $f(\rho)$. The algorithm has been presented in Refs. [36,37] where a partition into two subvolumes was performed and the maximum term method applicable in the thermodynamic limit was used. The generalization to $n+1$ subvolumes is straightforward and the technical details are available in the Supplemental Material [41].

For the second-order cumulants, one obtains

$$\langle \delta B_i \delta B_j \rangle = \chi_2 \Delta V \delta_{ij} - \chi_2 \frac{(\Delta V)^2}{V}, \quad i, j = 1, \dots, n. \quad (9)$$

The first term corresponds to coinciding subvolumes and can be identified as the local term proportional to $\delta(\mathbf{r}_1 - \mathbf{r}_2)$ in Eq. (6). The second term corresponds to two distinct subvolumes and thus the nonlocal second term in Eq. (6). The two-point correlator, therefore, is

$$\mathcal{C}_2(\mathbf{r}_1, \mathbf{r}_2) = \chi_2 \delta(\mathbf{r}_1 - \mathbf{r}_2) - \frac{\chi_2}{V}, \quad (10)$$

in agreement with the empirical derivation [Eq. (7)].

Three-point correlator. The three-point correlator, $\mathcal{C}_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \langle \delta \rho_1 \delta \rho_2 \delta \rho_3 \rangle$, has three kinds of terms: (i) when all three coordinates coincide, $\mathbf{r}_1 = \mathbf{r}_2 = \mathbf{r}_3$, (ii) when two of the coordinates coincide, e.g., $\mathbf{r}_1 = \mathbf{r}_2 \neq \mathbf{r}_3$, and (iii) when all three coordinates are distinct $\mathbf{r}_1 \neq \mathbf{r}_2 \neq \mathbf{r}_3$. Thus, in addition to local and global terms, there are terms with a mixture of local correlations and non-local balancing due to charge conservation. The third-order joint cumulants read

$$\begin{aligned} \langle \delta B_i \delta B_j \delta B_k \rangle = & \delta_{ijk} \chi_3 \Delta V - (\delta_{ij} + \delta_{ik} + \delta_{jk}) \chi_3 (\Delta V)^2 \\ & + 2\chi_3 (\Delta V)^3, \quad i, j, k = 1, \dots, n. \end{aligned} \quad (11)$$

These are the terms contributing to the three-point correlator, $\mathcal{C}_3$. In particular, the middle term in Eq. (11) corresponds to the cases where two of the three subvolumes coincide. The correlator $\mathcal{C}_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ has three such terms: (i) $\mathbf{r}_1 = \mathbf{r}_2 \neq \mathbf{r}_3$, (ii) $\mathbf{r}_1 = \mathbf{r}_3 \neq \mathbf{r}_2$, (iii) $\mathbf{r}_2 = \mathbf{r}_3 \neq \mathbf{r}_1$. Therefore,

$$\mathcal{C}_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \chi_3 \delta_{1,2,3} - \frac{\chi_3}{V} [\delta_{1,2} + \delta_{1,3} + \delta_{2,3}] + 2 \frac{\chi_3}{V^2}. \quad (12)$$

Here, $\delta_{1,\dots,n} = \prod_{i=2}^n \delta(\mathbf{r}_1 - \mathbf{r}_i)$.

Four-point correlator. Following the iterative procedure to compute fourth-order joint cumulants one obtains

$$\begin{aligned} \mathcal{C}_4(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) = & \chi_4 \delta_{1,2,3,4} - \frac{\chi_4}{V} [\delta_{1,2,3} + \delta_{1,2,4} + \delta_{1,3,4} + \delta_{2,3,4}] - \frac{(\chi_3)^2}{\chi_2 V} [\delta_{1,2}\delta_{3,4} + \delta_{1,3}\delta_{2,4} + \delta_{1,4}\delta_{2,3}] \\ & + \frac{1}{V^2} \left[\chi_4 + \frac{(\chi_3)^2}{\chi_2} \right] [\delta_{1,2} + \delta_{1,3} + \delta_{1,4} + \delta_{2,3} + \delta_{2,4} + \delta_{3,4}] - \frac{3}{V^3} \left[\chi_4 + \frac{(\chi_3)^2}{\chi_2} \right]. \end{aligned} \quad (13)$$

Local conservation. Having the explicit expressions for the balancing terms entering the n -point density correlators allows one to introduce the local charge conservation effect by modulating the corresponding balancing contribution. Here, the considerations will be restricted to the two-point correlator, which reads

$$\mathcal{C}_2(\mathbf{r}_1, \mathbf{r}_2) = \chi_2 \left[\delta(\mathbf{r}_1 - \mathbf{r}_2) - \frac{\varkappa(\mathbf{r}_1, \mathbf{r}_2)}{V} \right]. \quad (14)$$

Here, $\varkappa(\mathbf{r}_1, \mathbf{r}_2)$ is the function that implements local charge conservation. Ideally, the function $\varkappa(\mathbf{r}_1, \mathbf{r}_2)$ should be computed by solving a formidable task of propagating density fluctuations on a hydrodynamic background. One would generally expect $\varkappa(\mathbf{r}_1, \mathbf{r}_2)$ to be a localized function of the difference $|\mathbf{r}_1 - \mathbf{r}_2|$, such as a Gaussian. It should be symmetric under exchange of its arguments, $\varkappa(\mathbf{r}_1, \mathbf{r}_2) = \varkappa(\mathbf{r}_2, \mathbf{r}_1)$, and satisfy the sum rules, $\int d\mathbf{r}_1 \varkappa(\mathbf{r}_1, \mathbf{r}_2) = \int d\mathbf{r}_2 \varkappa(\mathbf{r}_1, \mathbf{r}_2) = V$. The case $\varkappa(\mathbf{r}_1, \mathbf{r}_2) = 1$ corresponds to the equilibrium limit of global conservation. The grand-canonical limit is obtained by setting $\varkappa(\mathbf{r}_1, \mathbf{r}_2) = 0$.

Net baryon number fluctuations at LHC. As a first application of the method, fluctuations of the net baryon number in Pb-Pb collisions at $\sqrt{s_{NN}} = 5.02$ GeV are considered here. Boost invariance is a good approximation of the midrapidity region where experimental measurements are performed at this energy. Hence, one can apply the devel-

oped formalism for uniform systems most straightforwardly to heavy-ion collisions at the LHC. Only the longitudinal (spatial) rapidity coordinate η is used, $\mathbf{r} = \eta$, and the longitudinal boost invariance with a cut-off, $\eta \in [-\eta_{\max}, \eta_{\max}]$, is imposed. The value of $\eta_{\max}$ is fixed such that the total fireball volume $V_{\text{tot}} = 2\eta_{\max} dV/dy$ is in line with the measurements of rapidity density of charged multiplicity [42]. The latter imply a Gaussian distribution of the volume with a width $\sigma_V = 4.12 \pm 0.10$, giving $V_{\text{tot}} = dV/dy \int dy e^{-\frac{y^2}{2\sigma_V^2}} = dV/dy \sqrt{2\pi} \sigma_V \simeq 10.2 dV/dy$, corresponding to $\eta_{\max} = 5.1$.

Local conservation is taken to be the only source of correlations, thus, the (grand-canonical) susceptibilities follow the Skellam distribution, $\chi_2 \propto d\langle B + \bar{B} \rangle/d\eta$. The local baryon conservation in rapidity is modeled by a Gaussian,

$$\varkappa(\eta_1, \eta_2) = 2\eta_{\max} \frac{\exp\left[-\frac{(\eta_1 - \eta_2)^2}{2\sigma_y^2}\right]}{\sqrt{2\pi} \sigma_y \text{erf}\left(\frac{\eta_{\max}}{\sqrt{2}\sigma_y}\right)}. \quad (15)$$

To preserve the symmetry $\varkappa(\eta_1, \eta_2) = \varkappa(\eta_2, \eta_1)$ and the sum rule $\int d\eta_1 \varkappa(\eta_1, \eta_2) = 2\eta_{\max}$, periodic boundary conditions in η are implemented such that $(\eta_1 - \eta_2)$ in Eq. (15) ranges from $-\eta_{\max}$ to $\eta_{\max}$ [43]. For $\sigma_y \rightarrow \infty$, one recovers the global baryon conservation limit.

The variance of net baryon number inside a (spatial) rapidity cut $|\eta| < \eta_{\text{cut}}$ is obtained by integrating the two-point density correlator, $\kappa_2[B - \bar{B}]|_{|\eta| < \eta_{\text{cut}}} = \int_{-\eta_{\text{cut}}}^{\eta_{\text{cut}}} d\eta_1 \int_{-\eta_{\text{cut}}}^{\eta_{\text{cut}}} d\eta_2 \mathcal{C}_2(\eta_1, \eta_2)$. One obtains

$$\begin{aligned} \kappa_2[B - \bar{B}]|_{|\eta| < \eta_{\text{cut}}} = & \int_{-\eta_{\text{cut}}}^{\eta_{\text{cut}}} d\eta_1 \int_{-\eta_{\text{cut}}}^{\eta_{\text{cut}}} d\eta_2 \frac{d\langle B + \bar{B} \rangle}{d\eta} \left[\delta(\eta_1 - \eta_2) - \frac{\exp\left[-\frac{(\eta_1 - \eta_2)^2}{2\sigma_y^2}\right]}{\sqrt{2\pi} \sigma_y \text{erf}\left(\frac{\eta_{\max}}{\sqrt{2}\sigma_y}\right)} \right] \\ = & \langle B + \bar{B} \rangle \left[1 - \frac{\left(1 - e^{-\frac{2\eta_{\text{cut}}^2}{\sigma_y^2}}\right) \sigma_y + \eta_{\text{cut}} \sqrt{2\pi} \text{erf}\left(\frac{\sqrt{2}\eta_{\text{cut}}}{\sigma_y}\right)}{\eta_{\text{cut}} \sqrt{2\pi} \text{erf}\left(\frac{\eta_{\max}}{\sqrt{2}\sigma_y}\right)} \right], \quad \eta_{\text{cut}} < \eta_{\max}/2 \\ = & \langle B + \bar{B} \rangle \left[1 - \frac{\sqrt{2}\eta_{\max}}{\sqrt{\pi} \sigma_y \text{erf}\left(\frac{\eta_{\max}}{\sqrt{2}\sigma_y}\right)} \alpha + O(\alpha^3) \right]. \end{aligned} \quad (16)$$

Here, $\alpha = \eta_{\text{cut}}/\eta_{\max}$ is the fraction of the system inside the rapidity cut, and the expressions are obtained for $\eta_{\text{cut}} < \eta_{\max}/2$ where periodic boundary conditions are irrelevant.

The numerical calculations below are done for the full range, including $\eta_{\text{cut}} > \eta_{\max}/2$ where periodic boundary conditions are faithfully implemented.

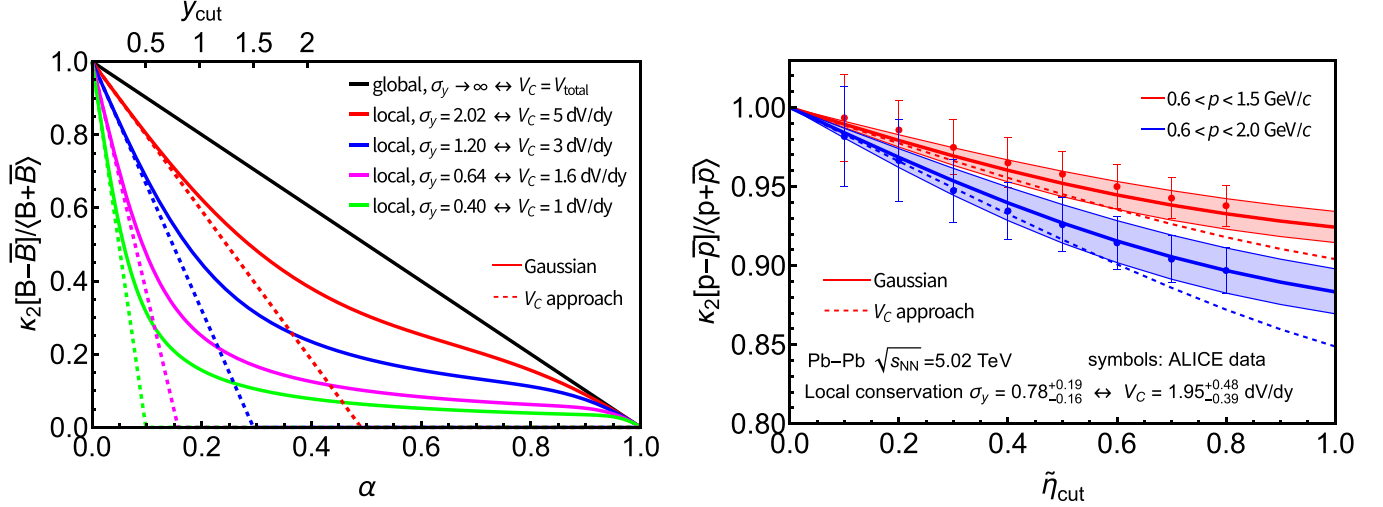


FIG. 2. Left panel: The normalized variance of net-baryon distribution as a function of acceptance fraction α in spatial rapidity, $|\eta| < \eta_{\text{cut}}$, computed for LHC conditions ($\sqrt{s_{NN}} = 5.02$ TeV). The solid lines correspond to calculations with different values of Gaussian width σ_y of local conservation in rapidity. Right panel: Dependence of the normalized variance of net-proton distribution in 0-5% central Pb-Pb collisions as a function of pseudorapidity cut. Black and red lines with a band depict the results in momentum acceptances of $0.6 < p < 1.5$ GeV/c and $0.6 < p < 2.0$ GeV/c, respectively. The symbols show the experimental data of the ALICE Collaboration [34]. The dashed lines in both panels correspond to the results from the equivalent V_C approach.

Figure 2 shows the dependence of $\kappa_2[B - \bar{B}]/(B + \bar{B})$ on the acceptance fraction α for different values of σ_y . In the limit $\sigma_y \rightarrow \infty$ (global baryon conservation) one obtains the well-known result, $\kappa_2[B - \bar{B}]/(B + \bar{B}) = 1 - \alpha$ [27,44,45]. With a decreasing value of σ_y , the drop with α is steeper. At small α , this is evident from Eq. (16) by a steeper negative slope. These results are qualitatively consistent with prior estimates of local baryon conservation [29,31].

Connection to the V_C approach. Analysis of the small α behavior allows one to establish the connection between Gaussian local conservation and the V_C approach of Refs. [30,33–35]. In the V_C approach, the fireball is truncated to k units around midrapidity, $|\eta| < k/2$, which is treated in the canonical ensemble. The V_C approach is thus equivalent to a global conservation model with a reduced rapidity cut-off, $\eta_{\text{cut}} \rightarrow k/2$, and thus entails a linear dependence on α with a modified slope

$$\left(\frac{\kappa_2[B - \bar{B}]}{B + \bar{B}} \right)_{V_C = k dV/d\eta} = 1 - \tilde{\alpha} = 1 - \frac{2\eta_{\text{max}}}{k} \alpha. \quad (17)$$

Comparing it with Eq. (16) one obtains the connection between k and σ_y :

$$k(\sigma_y) = \sqrt{2\pi} \sigma_y \text{erf} \left(\frac{\eta_{\text{max}}}{\sqrt{2}\sigma_y} \right) \approx \sqrt{2\pi} \sigma_y, \quad (18)$$

where the latter approximation applies for $\sigma_y \ll \eta_{\text{max}}$. The dashed lines in Fig. 2 depict the V_C approach results using k from Eq. (18). The comparison with the full lines allows one to establish the validity range of the V_C approach. The V_C approach is accurate for $|y_{\text{cut}}| \lesssim k/4$ and breaks down at higher rapidities. For measurements within one unit at midrapidity one thus has to restrict $k \gtrsim 2$ when applying the V_C approach.

Proton fluctuations and kinematical cuts. Momentum cuts replace the coordinate cuts in experimental measurements. In particular, net proton cumulants have been measured by the ALICE Collaboration in Pb-Pb collisions at the LHC for utilizing cuts in 3-momentum, $0.6 < p < 1.5(2.0)$ GeV/c, and kinematical pseudorapidity, $|\tilde{\eta}| < \tilde{\eta}_{\text{cut}}$ [34,46]. The present formalism can be used to incorporate such kinematical cuts if the acceptance probabilities for all (anti)baryons are independent. Namely, one obtains the net proton variance by integrating $C_2(\eta_1, \eta_2)$ over all spatial rapidities and applying a Bernoulli filter for each (η_1, η_2) pair to account for momentum cuts.

Let us denote by $p(\eta)$ the probability that an (anti)baryon emitted from spatial rapidity η ends up as an (anti)proton inside the acceptance, taken to be identical for protons and antiprotons. The first term in C_2 (14) is the self-correlation term [47], thus the binomial acceptance dilutes it by a single power of $p(\eta)$. The second term corresponds to the balancing contribution and thus the two-baryon correlations. It is multiplied by $p(\eta_1)p(\eta_2)$ [48,49]. The normalized variance of the net proton number inside momentum acceptance thus reads

$$\frac{\kappa_2[p - \bar{p}]_{\text{acc}}}{\langle p + \bar{p} \rangle} = 1 - \frac{\langle p(\eta_1)p(\eta_2) \rangle}{\langle p(\eta) \rangle} \quad (19)$$

with $\langle p(\eta) \rangle = \frac{1}{2\eta_{\text{max}}} \int d\eta p(\eta)$ and $\langle p(\eta_1)p(\eta_2) \rangle = \frac{1}{(2\eta_{\text{max}})^2} \int d\eta_1 d\eta_2 p(\eta_1)p(\eta_2) \chi(\eta_1, \eta_2)$.

$p(\eta)$ is evaluated using the blast-wave model [50], which reasonably describes proton transverse momentum spectra [51]. The details are available in the Supplemental Material [41]. In addition to the kinematic cuts, $p(\eta)$ also contains efficiency factor $q \simeq 0.33$ to distinguish protons from all baryons. The value of q is based on the HRG model estimation at freeze-out [52].

The results are shown in the right panel of Fig. 2 and they exhibit sensitivity to the range σ_y of local baryon conservation. By fitting the largest acceptance point, $0.6 < p < 2.0$ GeV/c and $|\tilde{\eta}| < 0.8$, one obtains $\sigma_y = 0.78^{+0.19}_{-0.16}$. This corresponds to the conservation volume $V_c = (1.95^{+0.48}_{-0.39})dV/dy$. With the same parameters, one obtains a reasonable description of the data for smaller pseudorapidity cuts as well as for the momentum cut $0.6 < p < 1.5$ GeV/c. Note that this estimate neglects possible baryon interactions [53] and baryon annihilation [54]. Results of the equivalent V_c approach are also shown in Fig. 2, by the dashed lines. For the extracted values of σ_y one observes that the V_c approach overpredicts the suppression due to local baryon conservation. In particular, the same data analyzed in the V_c approach suggests $V_c \simeq 3dV/dy$ [35], which is considerably larger than given by the Gaussian charge correlation. Therefore, estimates of the correlation volume based on V_c should be regarded as an upper bound on the value of k .

The conservation volume $V_c = (1.95^{+0.48}_{-0.39})dV/dy$ is about five times smaller than the total fireball volume $V_{\text{tot}} \simeq 10.2dV/dy$. Thus, experimental measurements of the net proton variance at $\sqrt{s_{NN}} = 5.02$ TeV Pb-Pb collisions provide significant evidence for local baryon conservation, as for global baryon conservation only one would expect $V_c = V_{\text{tot}}$. This observation calls for a more detailed analysis of local conservation effects in other systems, such as Au-Au collisions at RHIC-BES, where so far only global baryon conservation has been considered. The analysis of RHIC-BES-I data in [28] indicates the additional suppression of proton number variance relative to global baryon conservation only at $\sqrt{s_{NN}} \gtrsim 20$ GeV, which was modeled by the excluded volume effect in [28]. This suppression may also be indicative of local baryon conservation effects at RHIC, similar to what is obtained for LHC in the present work. These questions will

be addressed in a forthcoming study utilizing the extension of the formalism to non-boost-invariant systems.

Summary and outlook. This work presented a novel formalism to compute the n -point correlator of a conserved charge density in the canonical ensemble. The main results are given by Eqs. (10), (12), and (13) expressing two-, three-, and four-point density correlations in terms of grand-canonical susceptibilities. The results are universal and applicable for any system in equilibrium in the thermodynamic limit, namely when $V \gg \xi^3$ holds. By modulating the balancing contribution term to the two-point baryon density correlator, a Gaussian local conservation of baryon charge was introduced. The analysis of experimental data on net proton fluctuations from 5.02 TeV Pb-Pb collisions indicates local baryon conservation with a Gaussian width $\sigma_y = 0.78^{+0.19}_{-0.16}$, corresponding to the conservation volume V_c spanning $k \simeq \sqrt{2\pi}\sigma_y = (1.95^{+0.48}_{-0.39})$ units of rapidity.

The formalism leaves plenty of room for extensions and applications. One possibility is to utilize the input for the susceptibilities from lattice QCD or effective theories, or incorporate local correlations among baryons such as the repulsive excluded volume effect, baryon clustering, and baryon annihilation. One can also extend it to multiple conserved charges, as well as nonconserved quantities correlated to conserved charges, such as the individual hadron numbers and balance functions, as well as factorial cumulants [55]. This will allow one to apply the formalism at lower energies, such as RHIC, SPS, HADES, and CBM, in particular by utilizing particleization hypersurfaces from hydrodynamic simulations. It is also of interest to implement the correlations induced by (local) charge conservation into a Monte Carlo sampler of the hadron resonance gas at particleization.

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