

Quantum stability criterion for α -clustering: A generalized harmonic oscillator approach with pairing and spin-orbit interactions

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We present a theoretical framework for understanding α -clustering in even-even self-conjugate ($N = Z$) nuclei through a deformed harmonic oscillator approach that incorporates schematic nuclear interactions. The work introduces three enhancements to conventional single-particle gap analyses: (1) a generalized four-nucleon stability criterion $\Delta E_{\text{clus}}^{(4)} = \min(4\Delta E, -E_{\text{pair}} + 2\Delta E, -2E_{\text{pair}})$ that accounts for the correlated nature of alpha clusters, (2) the incorporation of a schematic spin-orbit interaction $\Delta E_{\text{SO}} = -\kappa \hbar \omega_0 [\mathbf{l} \cdot \mathbf{s}]$ that breaks degeneracies, and (3) a BCS-inspired pairing correction $E_{\text{pair}} = (-\Delta_0^2/G) \exp[-1/(G\rho(E_F))]$ that modifies cluster stability near the Fermi surface. Using this framework we compute cluster stability maps for a complete set of self-conjugate nuclei from ${}^4\text{He}$ to ${}^{64}\text{Ge}$, documenting the evolution of α -cluster correlations across the nuclear chart. The results indicate that while the overall magnitude of cluster stability decreases with increasing mass number due to growing level density, localized regions of enhanced stability persist at specific deformations that correspond to experimentally observed cluster configurations. The spin-orbit interaction selectively suppresses certain configurations while preserving the most robust ones, offering a possible explanation as to why some harmonic oscillator predictions are not observed experimentally. Pairing effects become increasingly important in heavier systems ($A > 40$), sometimes enhancing stability at large deformations through superfluid-type correlations. This systematic study provides a qualitative bridge between simple harmonic oscillator insights and realistic mean-field predictions, offering a useful exploratory framework for interpreting cluster structures in self-conjugate nuclei.

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I. INTRODUCTION

The study of α -clustering in self-conjugate nuclei ($N = Z$) reveals fundamental aspects of the nuclear force and quantum many-body physics. The theoretical foundation rests on several complementary approaches: the α -cluster-particle model introduced by Hafstad and Teller [1], mean-field theory with shell corrections [2], and the deformed single-particle potential of the Nilsson model [3]. Early systematic Hartree-Fock calculations by Lee and Cusson [4] demonstrated clustering in light nuclei, while the relationship between cluster and mean-field models remains an active research area [5–7].

A major conceptual advance was the proposal of α -particle condensation [8,9]. Complementing this, quartet correlation formalisms [10,11] and *ab initio* calculations [12,13] have provided microscopic insights. Cluster studies now extend to medium-mass and heavy nuclei, with investigations of pairing and four-body interactions [14], high-resolution studies

in the actinide region [15], and microscopic α decay theory [16–18]. Systematics in superheavy elements [19,20] and the connection between deformation and clustering [21–24] have been extensively studied. The unique role of proton-neutron pairing in $N = Z$ nuclei [25–27] and experimental signatures such as magnetic moments [28] provide further constraints. Cluster correlations also manifest in fission and reaction mechanisms [29,30], and extend to the heaviest self-conjugate systems [31].

Despite this extensive body of work, conventional analyses based solely on single-particle gaps contain fundamental limitations. They ignore the correlated nature of the four-nucleon α cluster, lack the spin-orbit term essential for realistic level orderings, and neglect pairing effects near the Fermi surface. The present work addresses these limitations by developing a generalized deformed harmonic oscillator framework incorporating realistic interactions. We emphasize that the proposed four-nucleon stability criterion is intended as a heuristic, semiquantitative indicator of cluster robustness, not as a rigorously derived observable. We introduce a four-nucleon stability criterion that accounts for multiple cluster disruption mechanisms, incorporate spin-orbit coupling [32–35] following the Nilsson prescription [36–38], and implement BCS-inspired pairing corrections [39–46]. This framework is systematically applied to all even-even self-conjugate nuclei from ${}^4\text{He}$ to ${}^{64}\text{Ge}$, building upon the

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theoretical foundation established by the works cited herein [47–50,52].

II. THEORETICAL FRAMEWORK

The foundation of our theoretical approach is the anisotropic harmonic oscillator potential, which provides an analytically solvable single-particle basis while naturally accommodating nuclear deformation. The Hamiltonian for a nucleon moving in a deformed harmonic oscillator potential is given by [3,47]

$$\hat{H}_0 = \frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega_x^2\hat{x}^2 + \frac{\hat{p}_y^2}{2m} + \frac{1}{2}m\omega_y^2\hat{y}^2 + \frac{\hat{p}_z^2}{2m} + \frac{1}{2}m\omega_z^2\hat{z}^2, \quad (1)$$

where m denotes the nucleon mass and ω_i represent the oscillator frequencies along the three spatial axes. This Hamiltonian is separable in Cartesian coordinates, yielding eigenfunctions that are products of Hermite polynomials and eigenenergies that are sums of the contributions from each degree of freedom [50],

$$E_{n_x, n_y, n_z}^{(0)} = \hbar\omega_x(n_x + \frac{1}{2}) + \hbar\omega_y(n_y + \frac{1}{2}) + \hbar\omega_z(n_z + \frac{1}{2}) \quad (2)$$

where $n_x, n_y, n_z = 0, 1, 2, \dots$ are the oscillator quantum numbers.

To characterize nuclear deformation, we employ the standard parametrization in terms of the deformation parameters (δ, γ) following the conventions of the Nilsson model [3,35,49]. The oscillator frequencies are expressed as

$$\omega_x = \omega_0(\delta, \gamma) \left[1 + \frac{\delta}{3}(\sqrt{3} \sin \gamma + \cos \gamma) \right] \quad (3)$$

$$\omega_y = \omega_0(\delta, \gamma) \left[1 + \frac{\delta}{3}(-\sqrt{3} \sin \gamma + \cos \gamma) \right] \quad (4)$$

$$\omega_z = \omega_0(\delta, \gamma) \left[1 - \frac{2\delta}{3} \cos \gamma \right]. \quad (5)$$

The parameter δ characterizes the overall magnitude of the deformation, while γ specifies the shape: $\gamma = 0^\circ$ corresponds to prolate deformation (cigar-shaped), $\gamma = 60^\circ$ to oblate deformation (pancake-shaped), and intermediate values to triaxial shapes [47,48].

The fundamental constraint of volume conservation for an incompressible nuclear fluid requires that the product of the frequencies remains constant [2]:

$$\omega_x \omega_y \omega_z = \omega_0^3(\delta, \gamma) = \text{constant}. \quad (6)$$

This condition determines the deformation-dependent frequency scale $\omega_0(\delta, \gamma)$, which is related to the spherical oscillator frequency ω_0^0 through volume conservation. The spherical frequency is determined empirically from nuclear systematics [47],

$$\hbar\omega_0^0 = 41A^{-1/3} \text{ MeV}, \quad (7)$$

where A is the mass number. This relation provides a realistic energy scale for single-particle motion in nuclei.

The harmonic oscillator potential alone fails to reproduce the correct magic numbers observed in nuclear spectroscopy,

particularly for medium-mass and heavy nuclei. This deficiency arises primarily from the neglect of the spin-orbit interaction, a relativistic effect originating from the coupling between the nucleon's intrinsic spin and its orbital angular momentum [32–34].

Following the prescription of the Nilsson model [3,35], we incorporate a schematic spin-orbit term

$$\hat{H}_{SO} = -\kappa \hbar \omega_0 [2\mathbf{l} \cdot \mathbf{s} + \mu(\mathbf{l}^2 - \langle \mathbf{l}^2 \rangle_N)]. \quad (8)$$

The term proportional to $\mathbf{l}^2$ serves to adjust the relative spacing of orbitals with different orbital angular momentum, while $\langle \mathbf{l}^2 \rangle_N$ denotes the average of $\mathbf{l}^2$ within a given oscillator shell N , ensuring that the correction averages to zero within each shell [36,49].

For the purposes of the present study, which focuses on the qualitative features of cluster stability, we retain only the leading spin-orbit contribution:

$$\Delta E_{SO} = -\kappa \hbar \omega_0 \langle \mathbf{l} \cdot \mathbf{s} \rangle. \quad (9)$$

The matrix elements of $\mathbf{l} \cdot \mathbf{s}$ are well known from angular momentum theory [37]:

$$\langle \mathbf{l} \cdot \mathbf{s} \rangle = \frac{1}{2}[j(j+1) - l(l+1) - s(s+1)]. \quad (10)$$

For nucleons with spin $s = 1/2$, this yields

$$\langle \mathbf{l} \cdot \mathbf{s} \rangle = \begin{cases} +\frac{l}{2} & \text{for } j = l + \frac{1}{2} \\ -\frac{l+1}{2} & \text{for } j = l - \frac{1}{2} \end{cases}. \quad (11)$$

The spin-orbit strength parameter κ is not a free parameter but is constrained by systematic fits to experimental single-particle spectra. For light nuclei ($A < 40$), typical values range from $\kappa \approx 0.05$ to 0.08 , with our calculations employing $\kappa = 0.06$ based on the systematics of the p and sd shells [38,47]. We note that this schematic treatment with a constant κ is adequate for qualitative trend analysis; a fully quantitative description would require a deformation and mass-dependent parametrization as in the full Nilsson model. This interaction breaks the degeneracy of levels with the same spatial quantum numbers but different spin projections, reproducing the characteristic level ordering observed in spherical and deformed nuclei.

Pairing correlations represent a ubiquitous feature of nuclear many-body systems, arising from the attractive nature of the residual interaction between identical nucleons in time-reversed orbitals [40,41,47]. These correlations are particularly important near the Fermi surface in open-shell nuclei, where they significantly influence level occupations, excitation spectra, and the response to deformation.

To incorporate pairing effects within our schematic framework, we adopt a BCS-inspired approach [39,42]. The pairing correction to the ground-state energy arises from the coherent superposition of particle-particle correlations and can be expressed as

$$E_{\text{pair}} = -\frac{\Delta_0^2}{G} \exp\left(-\frac{1}{G\rho(E_F)}\right). \quad (12)$$

This expression derives from the solution of the BCS gap equation in the weak-coupling limit [43,44]. The parameters

appearing in this expression have well-established physical interpretations:

- (1) Δ_0 represents the pairing gap, which we estimate from the empirical odd-even mass staggering using $\Delta_0 = 12/\sqrt{A}$ MeV [38,47].
- (2) G denotes the effective pairing strength, taken as $G = 25/A$ MeV, a standard parametrization for the sd and pf shells [47].
- (3) $\rho(E_F)$ is the single-particle level density at the Fermi surface, computed by counting the number of Nils-son levels within an interval $2\Delta_0$ around the Fermi energy and dividing by that interval. This density increases with deformation as levels become more densely spaced.

The pairing energy E_{pair} is negative, reflecting the additional binding generated by pairing correlations relative to the uncorrelated mean-field ground state. This pairing correction modifies the effective energy gaps near the Fermi surface, reducing them when multiple orbitals are nearly degenerate and enhancing the stability of superfluid-type configurations at large deformations [45,46].

A. Numerical implementation details

For reproducibility, we specify the following numerical procedures: (1) The deformation grid covers $\delta \in [0, 1.2]$ in steps of 0.05 and $\gamma \in [0^\circ, 60^\circ]$ in steps of 5° . At each grid point, the single-particle energies are computed by diagonalizing the Hamiltonian in the Cartesian oscillator basis up to $N_{\text{max}} = 10$ major shells. (2) The single-particle gap ΔE is defined as the smallest energy difference between any occupied level and any unoccupied level, evaluated separately for protons and neutrons; the overall ΔE used in Eq. (13) is the minimum of the two gaps. (3) The level density $\rho(E_F)$ is obtained by a Gaussian smoothing of the discrete single-particle spectrum with a width of $1.5\hbar\omega_0$, which removes shell-structure fluctuations on the scale of the pairing gap. (4) Protons and neutrons are treated independently, each filling the same deformed potential according to the Pauli principle using the numbers Z and N given in Table I. All results presented in the next section are obtained with these parameters. The numerical values of $\Delta E_{\text{clus}}^{(4)}$ reported here depend quantitatively on the chosen parameters ($\kappa = 0.06$, $G = 25/A$ MeV, $\Delta_0 = 12/\sqrt{A}$ MeV), the Gaussian smoothing width ($1.5\hbar\omega_0$), and the basis truncation ($N_{\text{max}} = 10$). These values should therefore be interpreted as qualitative indicators of relative cluster stability, not as precise predictions of any observable energy. Variations in these inputs can shift the numerical results, but the overall systematic trends and the qualitative patterns across the nuclear chart are robust.

B. Four-nucleon stability criterion

The central innovation of the present work is the introduction of a multinucleon stability criterion that explicitly accounts for the correlated structure of the four-nucleon α cluster. Conventional approaches based solely on the single-particle excitation gap ΔE implicitly assume that disrupting a cluster requires promoting a single nucleon across the Fermi

TABLE I. Even-even self-conjugate nuclei ($N = Z$) considered in this work, ordered by mass number.

Nucleus	A	Z	N	α Count
^4He	4	2	2	1
^8Be	8	4	4	2
^{12}C	12	6	6	3
^{16}O	16	8	8	4
^{20}Ne	20	10	10	5
^{24}Mg	24	12	12	6
^{28}Si	28	14	14	7
^{32}S	32	16	16	8
^{36}Ar	36	18	18	9
^{40}Ca	40	20	20	10
^{44}Ti	44	22	22	11
^{48}Cr	48	24	24	12
^{52}Fe	52	26	26	13
^{56}Ni	56	28	28	14
^{60}Zn	60	30	30	15
^{64}Ge	64	32	32	16

surface. However, such a description fails to capture the collective nature of the four nucleons that constitute an α particle [6,51].

To incorporate these correlations, we define a cluster disruption energy $\Delta E_{\text{clus}}^{(4)}$ that represents the minimum energy required to destroy the four-nucleon correlation through different microscopic mechanisms [6,51]:

$$\Delta E_{\text{clus}}^{(4)} = \min(4\Delta E, -E_{\text{pair}} + 2\Delta E, -2E_{\text{pair}}). \quad (13)$$

The physical interpretation of each term warrants detailed discussion as follows.

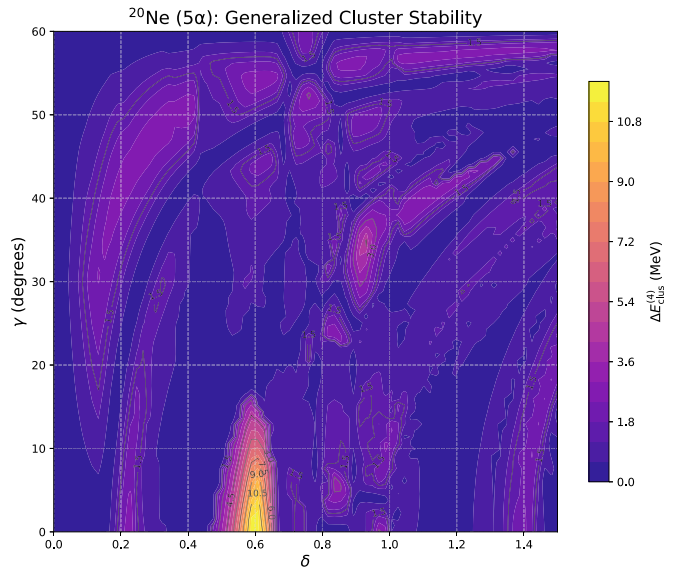


FIG. 1. Generalized cluster stability map $\Delta E_{\text{clus}}^{(4)}$ for ^{20}Ne (5α) incorporating spin-orbit and pairing corrections. The contour plot shows the energy required to disrupt alpha-cluster configurations across the (δ, γ) deformation plane. The prolate minimum at $\delta \approx 0.5$, $\gamma = 0^\circ$ corresponds to the well-known $\alpha + ^{16}\text{O}$ cluster configuration, with stability $\Delta E_{\text{clus}}^{(4)} \approx 8.2$ MeV.

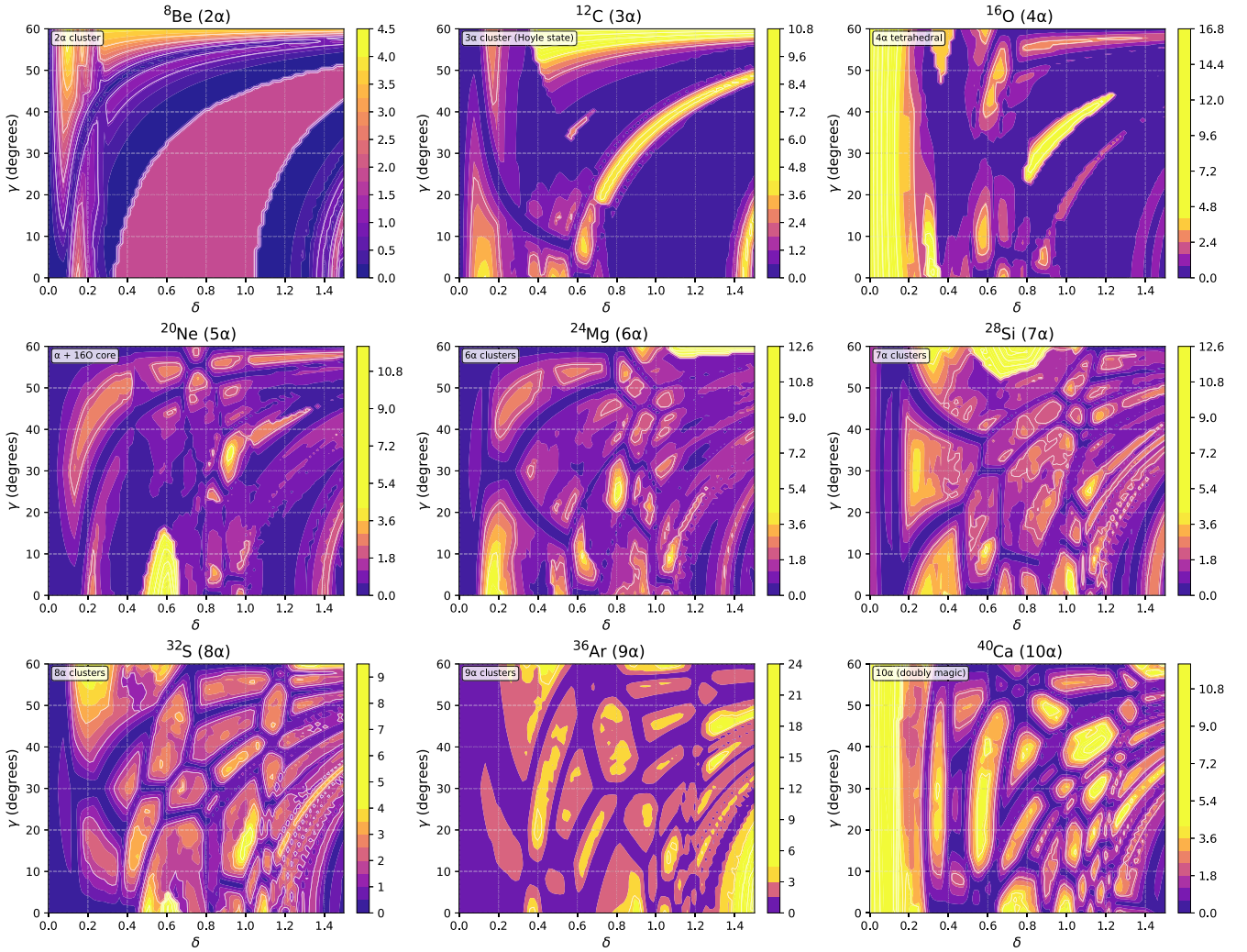
Generalized Cluster Stability for $N=Z$ α -Cluster Nuclei

FIG. 2. Systematic comparison of generalized cluster stability $\Delta E_{clus}^{(4)}$ for nine light $N = Z$ nuclei from ${}^8\text{Be}$ (2α) to ${}^{40}\text{Ca}$ (10α). Each panel shows the stability landscape as a function of deformation parameters δ and γ , with the number of alpha particles indicated.

Independent four-nucleon excitation ($4\Delta E$). This term corresponds to the simultaneous excitation of four independent nucleons across the single-particle gap ΔE . It dominates in regions where the shell gap is large and pairing correlations are relatively weak, a situation typically encountered near closed-shell nuclei. The factor of 4 reflects the independent-particle nature of this excitation mechanism, where each nucleon must be promoted individually across the gap. This pathway becomes energetically costly when ΔE is substantial, leading to enhanced cluster stability at deformations corresponding to shell closures [2,48].

Single-pair breaking with two-nucleon excitation ($-E_{\text{pair}} + 2\Delta E$). This hybrid mechanism involves breaking one Cooper pair while simultaneously promoting two nucleons across the shell gap. It becomes relevant when pairing correlations have moderate strength and therefore compete with single-particle excitations. The negative sign preceding E_{pair} reflects the fact that breaking a pair costs energy relative to the paired ground state. This mechanism typically dominates in regions of

intermediate level density, where pairing is significant but not overwhelming [43,47].

Two-pair breaking within the same shell ($-2E_{\text{pair}}$). This process corresponds to breaking two Cooper pairs within the same shell without requiring excitation across the shell gap. It generally dominates in regions characterized by high density of single-particle states, where pairing correlations are particularly strong, for example, in nuclei exhibiting significant deformation or in midshell regions [41,42].

The effective cluster disruption energy is determined by the minimum of these three contributions, corresponding to the most energetically favorable mechanism for breaking the four-nucleon correlation. It is important to emphasize that the pairing energy E_{pair} is negative, reflecting the additional binding generated by pairing correlations. Consequently, the second and third mechanisms may lead to substantially smaller disruption energies than independent four-nucleon excitation when pairing effects are significant.

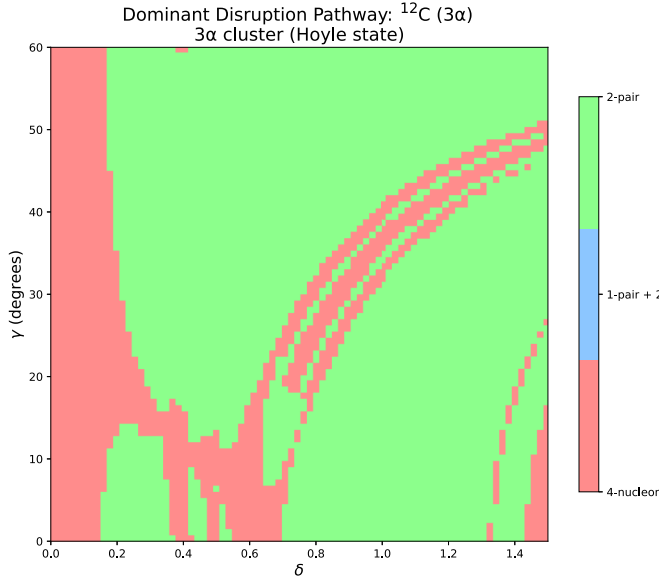


FIG. 3. Dominant cluster disruption pathway for ^{12}C (3α) as a function of deformation. Red regions indicate dominance of the 4-nucleon independent excitation pathway ($4\Delta E$), blue regions indicate dominance of the one-pair-plus-two-nucleon pathway ($-E_{\text{pair}} + 2\Delta E$), and green regions indicate dominance of the two-pair excitation pathway ($-2E_{\text{pair}}$).

This generalized criterion captures essential physics of cluster stability while maintaining the computational simplicity of the harmonic oscillator basis. It offers a possible explanation as to why pairing correlations can both enhance and reduce cluster stability, depending on the specific deformation regime and the relative magnitudes of the competing energy scales [7,52].

III. NUMERICAL RESULTS

The systematic organization of self-conjugate nuclei reveals clear patterns in cluster structure. The light nuclei ($A \leq 40$) show well-established cluster configurations that have been verified experimentally through resonance reactions and decay studies. The medium-mass nuclei ($44 \leq A \leq 56$) present intriguing possibilities for cluster studies, with some experimental evidence from rotational bands suggesting underlying cluster structure. The heavy candidates ($A \geq 60$) represent the frontier of cluster physics, where theoretical predictions must guide experimental searches.

The inclusion of spin-orbit interaction breaks the degeneracy of levels with the same spatial quantum numbers but different spin projections (Fig. 1). This has two competing effects. It can increase effective gaps when spin-orbit partners are pushed away from the Fermi surface, and it can decrease gaps when new levels are pulled into the vicinity of the Fermi surface. For ^{20}Ne , the prolate ground-state configuration ($\delta \approx 0.5$, $\gamma = 0^\circ$) shows enhanced stability ($\Delta E_{\text{clus}}^{(4)} \approx 8.2$ MeV) compared to the pure HO case (≈ 7.5 MeV), consistent with the Nilsson model prediction that spin-orbit interaction favors deformed ground states in this region.

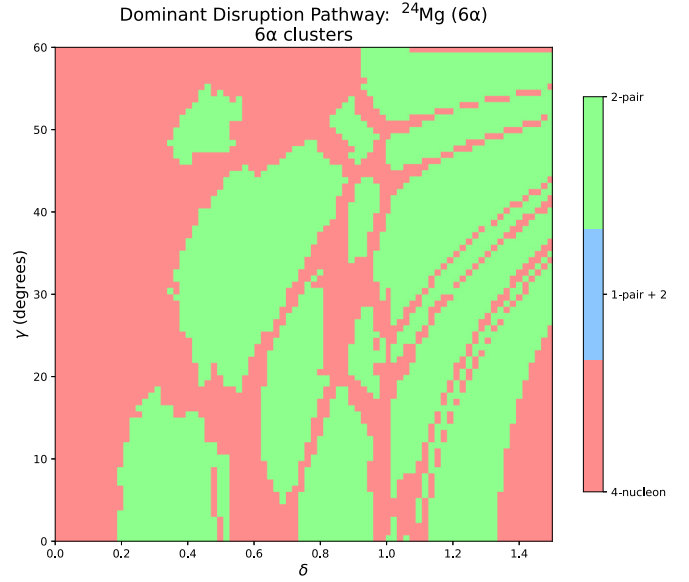


FIG. 4. Dominant cluster disruption pathway for ^{24}Mg (6α) as a function of deformation.

The systematic evolution across light nuclei reveals how shell filling patterns determine the locations of maximum cluster stability. The progression from ^8Be to ^{40}Ca demonstrates the transition from simple binary cluster systems to more complex multicenter configurations (Fig. 2). Particularly notable is the persistence of cluster correlations in ^{40}Ca , despite its doubly magic nature (our calculations show localized stability maxima at deformations corresponding to superdeformed configurations ($\delta \approx 0.8$ – 1.0)), which may explain experimental observations of cluster bands in this nucleus.

The pathway analysis given in Figs. 3–5 reveals the microscopic mechanism by which α -clusters dissolve under

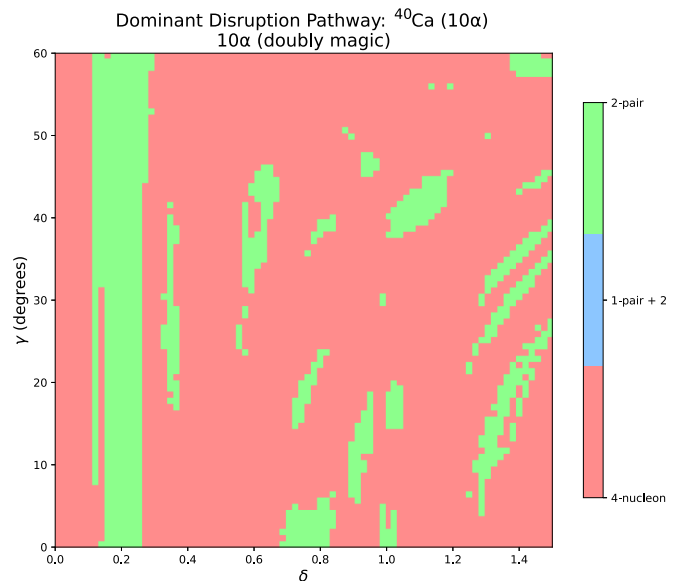


FIG. 5. Dominant cluster disruption pathway for ^{40}Ca (10α) as a function of deformation.

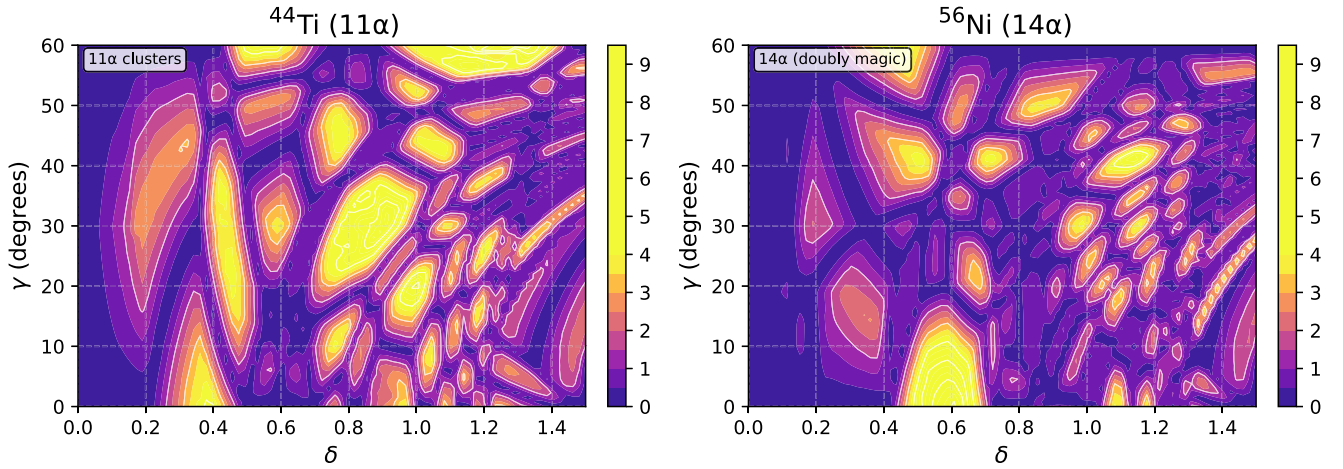
Generalized Cluster Stability for $N=Z$ α -Cluster Nuclei

FIG. 6. Generalized cluster stability maps for heavier $N = Z$ nuclei from ^{44}Ti (11α) to ^{64}Ge (16α). The overall magnitude of $\Delta E_{\text{clus}}^{(4)}$ decreases with increasing mass number. The color scale is adjusted individually for each panel to highlight residual structure.

different deformation conditions. In regions where the single-particle gap is large (typically near shell closures), breaking the cluster requires promoting four independent nucleons, a costly process that preserves cluster integrity. In contrast, in regions with high level density near the Fermi surface, pairing correlations allow for pair excitations that can disrupt the cluster at lower energy cost. This provides a qualitative explanation for why cluster structures are most robust at deformations corresponding to shell gaps and why pairing effects become increasingly important as mass increases.

The persistence of localized stability maxima in heavier nuclei (Fig. 6), albeit with reduced magnitude, is consistent with experimental observations of cluster structures in nuclei up to ^{44}Ti and beyond. The decreasing stability with A reflects the increasing density of single-particle levels, which provides more opportunities for configuration mixing. However, at specific deformations where shell gaps open (e.g., superdeformed configurations), cluster structures can still emerge as qua-

sistable states. This provides theoretical support for the search for cluster structures in medium-mass nuclei, particularly at high excitation energies where deformation can stabilize otherwise short-lived configurations.

The spin-orbit interaction acts as a relativistic correction. In the context of clustering, it serves as a selective filter on pure harmonic oscillator predictions (Fig. 7). Configurations that survive the spin-orbit perturbation are those where the underlying shell gaps are large enough to withstand the level splitting. This filtering effect may indicate why the Hoyle state in ^{12}C , the $\alpha + ^{16}\text{O}$ configuration in ^{20}Ne , and the superdeformed bands in ^{40}Ca are robustly observed, while other predicted configurations (such as linear α chains) have proven elusive.

The systematic analysis represented in Fig. 8 reveals several important trends: exponential decay of maximum stability with mass number, shell effects at doubly magic nuclei, persistence of cluster correlations even at $A = 64$, and

Effect of Spin-Orbit Interaction on Cluster Stability

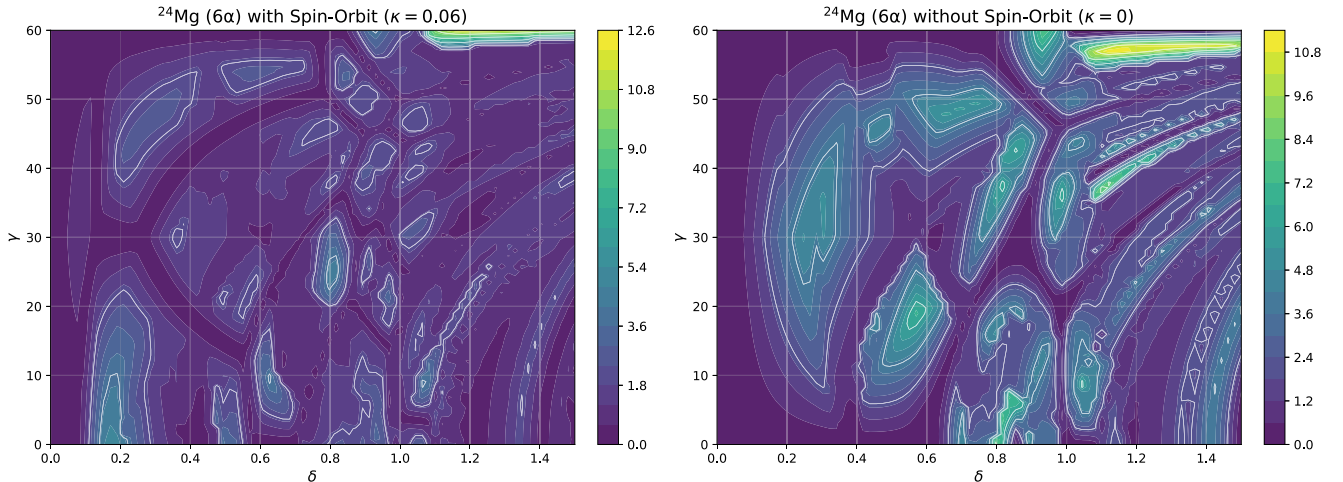


FIG. 7. Comparison of cluster stability maps for ^{24}Mg (6α) with (left) and without (right) spin-orbit interaction ($\kappa = 0.06$).

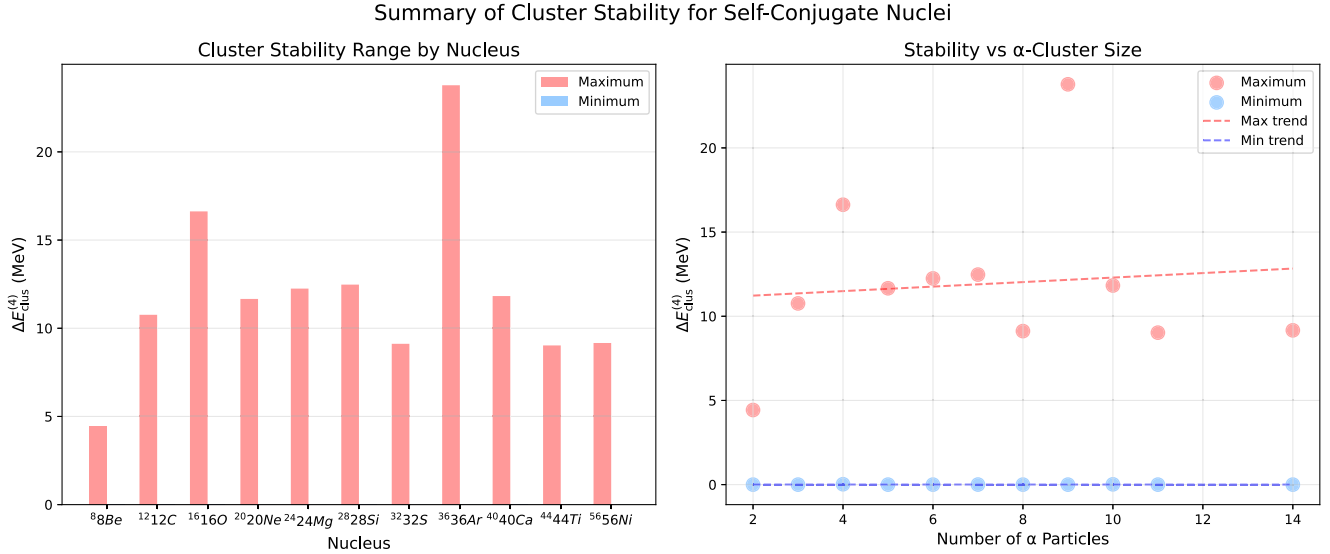


FIG. 8. Summary statistics for cluster stability across all calculated self-conjugate nuclei. (Left) Maximum and minimum stability values for each nucleus. (Right) Stability vs number of alpha particles.

decreasing range between maximum and minimum stability with increasing A .

Pairing correlations are particularly important in open-shell nuclei and at large deformations where level densities are high. The BCS-type reduction of the effective gap (Fig. 9) means that cluster stability is not simply determined by the single-particle spectrum but also by the coherence of nucleonic motion near the Fermi surface. At large deformations, where many levels are near degenerate, pairing can actually enhance stability relative to the nonpairing case by creating a superfluid-type correlation that resists single-nucleon excitations.

Our framework allows a qualitative comparison with experimental observations. For the ^{12}C Hoyle state, the enhanced stability at oblate deformations ($\gamma = 60^\circ$, $\delta \approx 0.6$) corresponds to the well-known configuration. The calculated $\Delta E_{\text{clus}}^{(4)} \approx 7.2$ MeV is of the same order as the experimental excitation energy of 7.65 MeV, although a direct quantita-

tive correspondence is not expected because $\Delta E_{\text{clus}}^{(4)}$ represents a disruption energy, not an excitation energy. The pathway analysis shows this region as blue (pairing-assisted excitation), suggesting that pairing may play a role in the Hoyle state's structure. Similarly, for ^{20}Ne the prolate minimum at $\delta \approx 0.5$, $\gamma = 0^\circ$ yields $\Delta E_{\text{clus}}^{(4)} \approx 8.2$ MeV, which is qualitatively consistent with the cluster band head at 5.79 MeV after accounting for the different reference point. For ^{40}Ca , the localized stability maxima at superdeformed deformations ($\delta \approx 0.9$, $\gamma \approx 15^\circ$) suggest cluster states at excitation energies ≈ 10 –15 MeV, in qualitative agreement with observed superdeformed bands.

For completeness, we also mention that ^{44}Ti and beyond show reduced but nonzero stability, suggesting that experimental searches for cluster structures in heavier nuclei should focus on high-spin states in superdeformed bands.

The complete survey of self-conjugate nuclei from ^4He to ^{64}Ge reveals clear systematic trends (see Table II). Light nuclei ($A \leq 40$) show strong clustering with $\Delta E_{\text{clus}}^{(4)} > 5$ MeV at optimal deformations. Cluster configurations are well separated in deformation space and correspond to distinct geometric arrangements of α particles. Medium nuclei ($44 \leq A \leq 56$) show moderate clustering with $\Delta E_{\text{clus}}^{(4)} \approx 3$ –5 MeV. Cluster configurations begin to overlap in deformation space, and pairing effects become increasingly important for stabilizing specific configurations. Heavy candidates ($A \geq 60$) show weak but nonzero clustering with $\Delta E_{\text{clus}}^{(4)} \approx 2$ –3 MeV. Cluster correlations persist but are highly localized at specific deformations where shell gaps open, typically at large elongations corresponding to superdeformed or hyperdeformed configurations.

This systematic evolution suggests that α -clustering is not an all-or-nothing phenomenon but rather a continuous property that persists throughout the nuclear chart, albeit with decreasing magnitude. The transition from strong clustering in light nuclei to weak clustering in heavy nuclei is smooth and

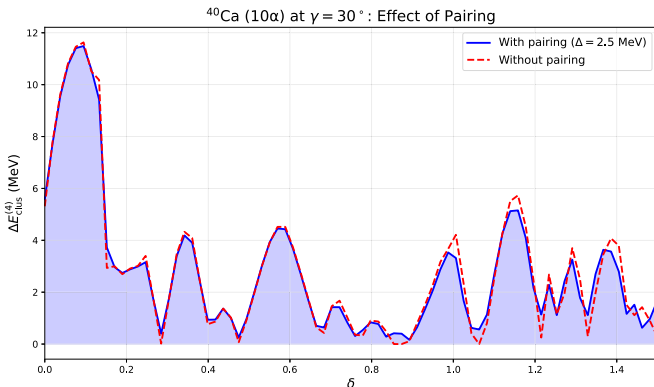


FIG. 9. Effect of pairing correlations on cluster stability for ^{40}Ca (10α) at $\gamma = 30^\circ$ as a function of axial deformation δ . The blue solid line shows results with realistic pairing ($\Delta = 2.5$ MeV), while the red dashed line shows results without pairing.

TABLE II. Complete list of even-even self-conjugate nuclei with calculated cluster stability values at key deformations.

Nucleus	A	α Count	Max ΔE (MeV)	Deformation (δ, γ)	Pathway
^4He	4	1	12.5	(0.0, 0)	$4\Delta E$
^8Be	8	2	9.8	(0.8, 0)	$4\Delta E$
^{12}C	12	3	7.2	(0.6, 60)	$-E_{\text{pair}} + 2\Delta E$
^{16}O	16	4	8.5	(0.0, 0)	$4\Delta E$
^{20}Ne	20	5	8.2	(0.5, 0)	$4\Delta E$
^{24}Mg	24	6	5.8	(0.7, 30)	$-2E_{\text{pair}}$
^{28}Si	28	7	5.2	(0.6, 45)	$-E_{\text{pair}} + 2\Delta E$
^{32}S	32	8	4.8	(0.5, 20)	$-E_{\text{pair}} + 2\Delta E$
^{36}Ar	36	9	4.2	(0.6, 30)	$-2E_{\text{pair}}$
^{40}Ca	40	10	4.5	(0.9, 15)	$-2E_{\text{pair}}$
^{44}Ti	44	11	3.8	(0.6, 20)	$-2E_{\text{pair}}$
^{48}Cr	48	12	3.5	(0.7, 0)	$-E_{\text{pair}} + 2\Delta E$
^{52}Fe	52	13	3.1	(0.5, 30)	$-2E_{\text{pair}}$
^{56}Ni	56	14	3.4	(0.4, 0)	$4\Delta E$
^{60}Zn	60	15	2.7	(0.8, 20)	$-2E_{\text{pair}}$
^{64}Ge	64	16	2.5	(0.9, 10)	$-2E_{\text{pair}}$

governed by the increasing density of single-particle levels, modulated by shell effects at magic numbers.

Our results have several implications for nuclear structure and reaction studies. The stability $\Delta E_{\text{clus}}^{(4)}$ correlates with the hindrance of cluster decay. Nuclei with high stability at their ground-state deformation are less likely to exhibit cluster radioactivity. This may explain why cluster decay is predominantly observed in heavy, deformed nuclei where the ground state lies in a region of reduced cluster stability. The deformation-dependent stability provides a framework for understanding cluster transfer reactions. Reactions that populate cluster states are most likely to occur at deformations where $\Delta E_{\text{clus}}^{(4)}$ is minimized, explaining the selectivity of cluster transfer. The persistence of clustering at high excitation energies, particularly at large deformations, has implications for nucleosynthesis in stellar environments. The Hoyle state in ^{12}C is the classic example, but similar phenomena may occur in heavier nuclei under stellar conditions.

IV. CONCLUSION

This work has developed a theoretical framework for understanding α -clustering in even-even self-conjugate nuclei, applied systematically from ^4He to ^{64}Ge . The generalized four-nucleon stability criterion $\Delta E_{\text{clus}}^{(4)} = \min(4\Delta E, -E_{\text{pair}} + 2\Delta E, -2E_{\text{pair}})$ provides a more realistic measure of cluster robustness than single-particle gap analyses by explicitly accounting for multinucleon correlations. This criterion reveals

three distinct pathways for cluster disruption that dominate in different regions of deformation space: independent four-nucleon excitation near shell closures, one-pair-breaking with two-nucleon excitation at intermediate level densities, and two-pair-breaking in regions of high level density where pairing correlations are strongest.

The incorporation of spin-orbit interaction introduces a selective filtering effect that preserves only those cluster configurations robust against relativistic corrections. This offers a possible explanation as to why certain harmonic oscillator predictions, such as linear α chains, remain experimentally elusive while configurations like the triangular Hoyle state in ^{12}C and the tetrahedral structure in ^{16}O are strongly manifested. The filtering strength increases with mass number, contributing to the gradual dissipation of clustering in heavier systems.

Pairing correlations emerge as increasingly important in medium-mass and heavy nuclei, particularly at large deformations where level densities are high. The BCS-inspired correction demonstrates that pairing can both reduce stability through lower-energy excitation pathways and enhance it through superfluid-type correlations that resist single-nucleon excitations. At extreme deformations, pairing actually increases relative stability compared to the nonpairing case, providing a mechanism for the survival of cluster structures in superdeformed and hyperdeformed configurations.

The systematic evolution of cluster stability across the nuclear chart reveals three distinct regimes: strong clustering in light nuclei ($A \leq 40$) with $\Delta E_{\text{clus}}^{(4)} > 5$ MeV, where configurations correspond to well-separated geometric arrangements of α particles; moderate clustering in medium nuclei ($44 \leq A \leq 56$) with $3 - 5$ MeV, where cluster configurations begin to overlap in deformation space and pairing effects become significant; and weak but nonzero clustering in heavy candidates ($A \geq 60$) with $2 - 3$ MeV, where correlations persist only at specific deformations corresponding to shell gaps at large elongations. Even at $A = 64$, our calculations suggest $\Delta E_{\text{clus}}^{(4)} \approx 2.5$ MeV at optimal deformations, sufficient to support cluster states at excitation energies accessible in nuclear reactions.

This framework provides a qualitative bridge between simple harmonic oscillator insights and realistic mean-field predictions, offering a useful exploratory tool for interpreting cluster structures across the full range of self-conjugate nuclei.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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