

Analytic structure of stress-energy response functions and new Kubo formulas

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Determining the transport properties of quark-gluon plasma is one of the most important aspects of relativistic heavy ion collision studies. Field-theoretical calculations of transport coefficients such as the shear and bulk viscosities require Kubo formulas, which in turn require real-time correlation functions of stress-energy tensors. Consequently, knowing the analytic structure of these correlation functions is essential in any such studies. Using the energy-conservation laws and the results from gravity-hydrodynamics analysis, we determine the low-frequency and low-wavenumber analytic structures of all stress-energy correlation functions in the rest frame of the medium. By comparing with the diffusion and sound spectra from the second-order and third-order relativistic hydrodynamics, various new Kubo formulas are derived in the limit where the zero-frequency limit is taken first. We also show that the meaning of the Kubo formulas for relaxation times can change when higher-order terms are added to the hydrodynamics. A subtle issue of taking the zero frequency and zero wave number limits when using skeleton diagrams is addressed as well.

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I. INTRODUCTION

The main purpose of studying relativistic heavy ion collisions is to determine the properties of quark-gluon plasma (QGP). Among QGP properties, the equation of state and the transport coefficients are the most studied in relativistic heavy ion collisions. The QGP equation of state can be reliably calculated on the lattice up to the electroweak scale as it only involves the calculation of the stress-energy tensor average [1–4]. Field-theoretical calculations of transport coefficients, however, are more complex as they need to use the Kubo formulas involving real-time correlation functions.

Kubo formulas have been a staple of theoretical physics ever since they were first derived [5–7]. Relativistic field theory calculations of viscosities using the Kubo formulas have a correspondingly long history [8–48]. The standard Kubo

formulas for the shear and the bulk viscosities can be derived by considering the analytic structure of stress-energy tensor correlation functions in the the small-wave-number ($k = |\mathbf{k}|$) and small-frequency (ω) limit. Such derivations so far (for instance, see [11,32]) have used a limited amount of information on how the correlation functions depend on the transport coefficients.

Recently, more information on the structure of the correlation functions became available due to new developments in gravity-hydrodynamic analysis [49–58] as well as third-order relativistic hydrodynamics [59–69]. To take advantage of the new information, one needs to understand the analytic structure of the stress-energy tensor correlation functions more thoroughly. The main purpose of this work is to determine the general analytic structure of correlation functions that are compatible with the new information from gravity hydrodynamics and third-order hydrodynamics. By doing so, we are able to derive a number of new Kubo formulas for the viscosities, as we will show below. Interestingly, in most of the new formulas, the order of the limits is reversed. That is, the zero ω limit is taken first instead of the zero k limit.

Schematically, the usual Kubo formula takes the following form:

$$\gamma_A = iC_A \lim_{\omega \rightarrow 0} \lim_{k \rightarrow 0} \partial_\omega G_R^{\hat{A}\hat{A}}(\omega, \mathbf{k}), \quad (1)$$

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where $\hat{A}$ is an operator that is coupled to an external force that introduces a perturbation to the underlying equilibrium system. The retarded correlation function $G_R^{\hat{A}\hat{A}}(\omega, \mathbf{k})$ is calculated in equilibrium, and γ_A is the transport coefficient that controls the speed of dissipation of the perturbation. The coefficient C_A is an $O(1)$ constant.

The ω derivative is present because the Kubo formula above is actually about the small ω behavior of the spectral density, defined as twice the imaginary part of the retarded correlation function. The order of the limits in Eq. (1) corresponds to the thermodynamic limit where the zero k limit (equivalently, the infinite volume limit) is taken before the zero ω limit. The two limits usually do not commute.

There was, however, an exception. In Ref. [49] and many subsequent papers [53,54,56–58] one finds the following expansion of a retarded correlation function of the stress-energy tensor component $\hat{T}^{xy}$:

$$G_R^{xy,xy}(\omega, k_z) = -i\eta\omega + \eta\tau_\pi\omega^2 - \left(\frac{\kappa}{2} - \kappa^*\right)\omega^2 - \frac{\kappa}{2}k_z^2, \quad (2)$$

where η is the shear viscosity, τ_π is the corresponding relaxation time, κ and κ^* are additional transport coefficients, and we have omitted the higher order terms.¹ At first glance, this looks like a reasonable representation of the shear response function in the small ω and small k limit. However, if the two limits do not commute, one must specify whether this expansion assumes $k > \omega$ or $k < \omega$.

Careful inspection of the derivation of Eq. (2) reveals that the expansion above is perfectly valid, which seems to indicate that the two limits do commute, at least for this specific case. Hence, the alternate form of the Kubo formula,

$$\eta = i \lim_{k_z \rightarrow 0} \lim_{\omega \rightarrow 0} \partial_\omega G_R^{xy,xy}(\omega, k_z), \quad (3)$$

should work as well as the usual one with the $k_z \rightarrow 0$ limit taken first. Equation (3) cannot be valid if $G_R^{xy,xy}(\omega, k_z)$ possesses a diffusive hydrodynamic pole. Therefore, validating Eq. (3) is equivalent to showing that the correlation function $G_R^{xy,xy}(\omega, k_z)$ does not possess any hydrodynamic pole. This is indeed the case [51,54,70] ultimately because variation only in the z direction cannot induce diffusion in the transverse (x, y) directions.

This may be taken as a clue that if the full analytic structure of the retarded correlation functions of stress-energy tensors are revealed, there may be other similar formulas as yet unknown in the limit where $\omega \rightarrow 0$ is taken first. This is what we would like to explore in this work. Along the way, we also would like to address another puzzle that appears in the usual calculations of the viscosities using the skeleton expansion: it does not seem to matter which limit is taken first even when Eq. (2) is not directly used.

In recent gravity-hydrodynamics literature, there appeared a few Kubo-like formulas where some transport coefficients are obtained in the limit where $\omega \rightarrow 0$ is taken first

[50,52,53,55] (In this work, we will limit the analysis to only the two-point functions.). Most of these formulas concern thermodynamic transport coefficients that have more to do with the static properties of the system than the dissipative properties. This is understandable given that the zero-frequency limit of the retarded correlation function coincides with the static Euclidean correlation function. We will show below that these are faithfully reproduced within our analysis.

II. ANALYTIC STRUCTURE OF STRESS-ENERGY TENSOR CORRELATION FUNCTIONS

The purpose of this section is to explain briefly how to obtain the most general analytic structures of the stress-energy tensor correlation functions in the small ω and small k limit. The main tool we use is the energy-momentum conservation laws, supplemented by consistency with gravity-hydrodynamics results. Including additional conserved currents is left for future work.

A. Ward identity analysis

In an isotropic and static medium, the following Ward identities can be derived by taking two functional derivatives of the curved-space partition function with respect to the Riemannian metric, and then analytically continuing to the Minkowski space [34,71–73],

$$0 = k_\alpha (\tilde{G}^{\alpha\beta,\mu\nu}(\omega, \mathbf{k}) - \eta^{\beta\mu} \langle \hat{T}^{\alpha\nu} \rangle - \eta^{\beta\nu} \langle \hat{T}^{\alpha\mu} \rangle + \eta^{\alpha\beta} \langle \hat{T}^{\mu\nu} \rangle), \quad (4)$$

where $\hat{T}^{\mu\nu}$ is the stress-energy tensor. Because $\tilde{G}^{\alpha\beta,\mu\nu}(\omega, \mathbf{k})$ can contain possible contact terms, it is not exactly the same as the retarded correlation function, which is defined as

$$G_R^{\alpha\beta,\mu\nu}(\omega, \mathbf{k}) = -i \int d^4x e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}} \theta(t) \langle [\hat{T}^{\alpha\beta}(t, \mathbf{x}), \hat{T}^{\mu\nu}(0)] \rangle. \quad (5)$$

The flat-space metric we use is $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. Here, $\langle \hat{O} \rangle = (1/Z) \text{Tr}(e^{-\beta \hat{H}} \hat{O})$ is the thermal average of the operator $\hat{O}$. Using the CPT properties of the stress-energy tensor operators, one can show that $G_R^{\alpha\beta,\mu\nu}(\omega, \mathbf{k}) = G_R^{\mu\nu,\alpha\beta}(\omega, \mathbf{k})$.

The equivalent Ward identities involving the retarded correlation functions can be obtained by requiring that the total energy-momentum is conserved, and the correlation functions are well behaved in the $k \rightarrow 0$ limit. The identities are [73,74]

$$\begin{aligned} \omega G_R^{00,0\mu} &= k_i (G_R^{0i,0\mu} + \eta^{i\mu} (\varepsilon_0 + P_0)), \\ \omega G_R^{0j,0\mu} &= k_i (G_R^{ij,0\mu} - \eta^{0\mu} \eta^{ij} (\varepsilon_0 + P_0)), \\ \omega G_R^{0j,lm} &= k_i (\tilde{G}^{ij,lm} - (\eta^{il} \eta^{jm} + \eta^{im} \eta^{jl} - \eta^{ij} \eta^{lm}) P_0), \\ \omega G_R^{00,ij} &= k_l G_R^{0l,ij}, \end{aligned} \quad (6)$$

where ε_0 is the equilibrium energy density and P_0 is the equilibrium pressure. From now on, we will use the enthalpy density $h_0 = \varepsilon_0 + P_0$ for this combination. The only exception

¹We follow the convention where only the nonzero components of the arguments are displayed in a function of a four-vector. Thus, $G_R^{xy,xy}(\omega, k_z)$ means that $k^\mu = (\omega, 0, 0, k_z)$.

here is $\tilde{G}^{ij,lm}$, for which we have not yet found the corresponding $G_R^{ij,lm}$. Fortunately, it is not strictly necessary to do so as this correlator does not directly involve the energy-momentum density. From this expression, it is clear that the imaginary part of the retarded correlators obeys the same identities but without the ε_0 and P_0 contact terms on the right-hand side.

Using the above Ward identities, all correlators involving the energy-momentum density operator $\hat{T}^{0\mu}$ can be expressed in terms of $\tilde{G}^{ij,lm}(\omega, \mathbf{k})$. In a spatially isotropic system, this correlator can be decomposed in terms of the five orthogonal rank-2 tensors involving $\delta_{ij} = \eta^{ij}$ and $k_i k_j$ [34] as follows:²

$$\begin{aligned} \tilde{G}^{ij,lm}(\omega, \mathbf{k}) = & P_0(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl} - \delta_{ij}\delta_{lm}) + G_1(\omega, k^2) \\ & \times (\hat{k}_i\hat{k}_l\hat{\delta}_{jm} + \hat{k}_i\hat{k}_m\hat{\delta}_{jl} + \hat{k}_j\hat{k}_m\hat{\delta}_{il} + \hat{k}_j\hat{k}_l\hat{\delta}_{im}) \\ & + G_2(\omega, k^2)(\delta_{il}\hat{\delta}_{jm} + \delta_{im}\hat{\delta}_{jl} - \delta_{ij}\hat{\delta}_{lm}) \\ & + G_L(\omega, k^2)\hat{k}_i\hat{k}_j\hat{k}_l\hat{k}_m \\ & + G_{LT}(\omega, k^2)(\hat{k}_i\hat{k}_j\hat{\delta}_{lm} + \hat{\delta}_{ij}\hat{k}_l\hat{k}_m) \\ & + G_T(\omega, k^2)\hat{\delta}_{ij}\hat{\delta}_{lm}, \end{aligned} \quad (7)$$

using $\hat{k}_i = k_i/k$ and $\hat{\delta}_{ij} = \delta_{ij} - \hat{k}_i\hat{k}_j$. From now on, we will refer to the stress-energy tensor correlation functions as the correlators and the functions appearing on the right-hand side of Eq. (7) as the response functions. From Eq. (7), one can see that only G_1 , G_L , and G_{LT} can appear in Eq. (6) because G_2 and G_T are transverse to $\mathbf{k}$ in all indices. See Appendix B for details.

To further investigate the behaviours of the response functions, consider first the small ω limit. From Eqs. (6) and (7), and using the fact that the left-hand side of Eq. (6) must all vanish in the $k \rightarrow 0$ limit due to the energy-momentum conservation, we get

$$G_L(\omega, k^2) = -h_0 \frac{\omega^2}{k^2} - \frac{\omega^4}{k^4} g_L(k^2) + O(\omega^5), \quad (8)$$

where we defined

$$g_L(k^2) = -G_R^{00,00}(0, \mathbf{k}) \quad (9)$$

with $g_L(0) = T c_v$. Here c_v is the heat capacity and T is the temperature [34]. Similarly, for G_{LT} and G_1 ,

$$G_{LT}(\omega, k^2) = -h_0 \frac{\omega^2}{k^2} - g_{LT}(k^2)\omega^2 + O(\omega^3), \quad (10)$$

$$G_1(\omega, k^2) = -h_0 \frac{\omega^2}{k^2} - g_1(k^2)\omega^2 + O(\omega^3), \quad (11)$$

where $g_{LT}(k^2)$ and $g_1(k^2)$ are yet to be examined.

B. Analytic structures of G_1 and G_L

We are now ready to write down the general form of the response functions G_1 , G_L , and G_{LT} . We must, of course,

be guided by the physics of the many-body system. Hence, we assume that in the small ω and k limit these response functions contain either a diffusion pole or a sound pole. The possible existence of a branch cut [75–77] is not explored in the present study. For G_1 , consider a flow velocity given by $\mathbf{u} = (u^x(y), 0, 0)$. Then $\partial_y u_x \neq 0$ but all other derivatives vanish.

Since nonequilibrium perturbations in the local rest frame can be expressed as $\sim \hat{T}^{ij} \partial_i u_j$ [8,78], the relevant response function is $\tilde{G}^{xy,xy}(\omega, k_y) = G_1(\omega, k_y^2)$. Since this is a response to the disturbance in the orthogonal direction of the hydrodynamic flow, G_1 represents the diffusive shear mode.

For the diffusive response function, the most general form that is compatible with Eq. (11) is [32,34,54,74,79]

$$G_1(\omega, k^2) = -\frac{\omega^2[h_0 + k^2 g_1(k^2) + i\omega F_1(\omega, k^2)]}{k^2 - i\omega/D(\omega, k^2) - \omega^2 B(\omega, k^2)}. \quad (12)$$

All functions of ω and k appearing in the analytic structures in this work are of the form

$$F_1(\omega, k^2) = F_{1R}(\omega^2, k^2) - i\omega F_{1I}(\omega^2, k^2), \quad (13)$$

where F_{1R} and F_{1I} are real functions. This is because the relevant spectral functions need to be odd in ω and even in k . Three special cases are $D(\omega, k^2) = D_R(k^2) - i\omega D_I(k^2)$ appearing above and $Z(\omega, k^2)$ and $F_{T1}(\omega)$ appearing in G_L , G_{LT} , and G_T below. The numerator in Eq. (12) can accommodate any polynomial of ω and k^2 that is consistent with Eq. (11). The properties of the functions appearing above will be explored in Sec. IV.

The longitudinal response function G_L can be obtained as $G_L(\omega, k_z^2) = \tilde{G}^{zz,zz}(\omega, k_z)$. This is relevant when the disturbance is in the same direction of the flow ($\partial_z u_z(z) \neq 0$) and one is interested in how that disturbance creates a pressure wave (sound) in the same direction. The most general form of the sound pole structure compatible with Eq. (8) is [32,34,54,74]

$$G_L(\omega, k^2) = -\frac{\omega^2[h_0 + \omega^2 F_L(\omega, k^2)]}{k^2 - \omega^2/Z(\omega, k^2) - i\omega^3 R(\omega, k^2)}, \quad (14)$$

where

$$Z(\omega, k^2) = Z_R(k^2) + \omega^2 Z_{R\omega}(k^2) - i\omega Z_I(k^2). \quad (15)$$

Taking the small ω limit of this expression and comparing with Eq. (8), we can see that

$$g_L(k^2) = \frac{h_0}{Z_R(k^2)} + k^2 F_{LR}(0, k^2), \quad (16)$$

which reveals that the functions $Z_R(k^2)$ and $F_{LR}(0, k^2)$ are thermodynamic.

The response function $G_{LT}(\omega, k_z^2) = \tilde{G}^{xx,zz}(\omega, k_z)$ is relevant when the disturbance is in the same direction of the flow and one is interested in how that disturbance creates a pressure wave (sound) in the orthogonal direction. Hence, we

²This expression first appeared in an unpublished note by L.G. Yaffe.

can express it as

$$G_{LT}(\omega, k^2) = -\frac{\omega^2[h_0 + k^2 g_{LT}(k^2) + i\omega F_{LT}(\omega, k)]}{k^2 - \omega^2/Z(\omega, k) - i\omega^3 R(\omega, k)}, \quad (17)$$

which is consistent with Eq. (10).

To go further in figuring out the analytic forms of G_2 and G_T , one needs some more physics guidance. Physically, these functions have the following interpretations.

In purely hydrodynamic studies, $G_2(\omega, k_z^2) = \tilde{G}^{xy,xy}(\omega, k_z)$ does not appear as a response function since $\partial_x u_y(z) = \partial_y u_x(z) = 0$. It describes the evolution of the system in the (x, y) direction when variation is only in the z direction. As such, it cannot have a hydrodynamic pole. The response function $G_T(\omega, k^2) = (1/4)([\delta_{ij}\delta_{lm}\tilde{G}^{ij,lm}(\omega, \mathbf{k})])$ is the correlation function of $\hat{\delta}_{ij}\hat{T}^{ij} = 3\hat{P} - \hat{k}_i\hat{k}_j\hat{T}^{ij}$, where $\hat{P} = (1/3)\hat{T}^{ii}$ is the pressure. Hence, G_T is expected to have the sound pole.

Since G_2 and G_T are not accessible in purely hydrodynamic approach, we need to seek more information from gravity-hydrodynamics calculations where the perturbation is $\sim \hat{T}^{ij}h_{ij}$. Here h_{ij} is the metric perturbation.

C. Gravity-hydrodynamics results

In this section, we aim to clarify the pole structures of all five response functions in Eq. (7) by using the solutions of the linearized equations of motion worked out in [51–54]. We do not assume the conformal limit.

In this analysis, the metric perturbation is given by

$$g_{\mu\nu}(x) \approx \eta_{\mu\nu} + h_{\mu\nu}(x). \quad (18)$$

The linearized energy-momentum conservation laws in the rest frame of a fluid cell are given by

$$\partial_t \delta\epsilon + h_0 \partial_i u^i = -\frac{1}{2} h_0 \partial_i h_i^i \quad (19)$$

and

$$h_0 \partial_i u^i + \partial_j T^{ji} = -P_0 \partial_j h^{ji}, \quad (20)$$

where the spatial stress tensor is given by

$$T^{ji} = (P_0 + \delta P + \Pi)\delta^{ij} + \pi^{ji} - P_0 h^{ji}. \quad (21)$$

Indices are raised or lowered by the flat metric $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$. In particular, $h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$ and $h_\nu^\mu = \eta^{\mu\alpha}h_{\alpha\nu}$.³ Here, u^i is the flow velocity component, $\delta\epsilon$ is the deviation of the energy density from its equilibrium value, and $\delta P = v_s^2 \delta\epsilon$ is the pressure deviation with the speed of sound defined by $v_s^2 = \partial P / \partial \epsilon$.

³One should note that the linear piece in

$$g^{\mu\nu} = \eta^{\mu\nu} + \bar{h}^{\mu\nu} \quad (22)$$

is not $h^{\mu\nu}$. Because $g^{\mu\alpha}g_{\alpha\nu} = \delta_\nu^\mu$, we actually have

$$\bar{h}^{\mu\nu} = -h^{\mu\nu} \quad (23)$$

In this work, we will always use $h_{\mu\nu}$ and raise or lower the indices with the flat space metric.

The shear-tensor equation of motion is

$$\begin{aligned} \pi^{ij} = & -\eta[\partial^i u^j + \partial^j u^i - (2\delta^{ij}/3)\partial_l u^l] - \tau_\pi \partial_t \pi^{ij} - \eta \partial_t h^{(ij)} \\ & + \kappa(R^{(ij)} - 2R^{0(ij)0}) + 2\kappa^* R^{0(ij)0}, \end{aligned} \quad (24)$$

where η is the shear viscosity, τ_π is the shear relaxation time, and κ and κ^* are thermodynamic transport coefficients [49]. The angular brackets around indices indicate the symmetric and traceless part. The bulk-pressure evolution is governed by

$$\Pi = -\zeta \partial_l u^l - \tau_\Pi \partial_t \Pi + \xi_5 R + \xi_6 R^{00} - \frac{\zeta}{2} \partial_t h_l^l, \quad (25)$$

where ζ is the bulk viscosity, τ_Π is the bulk relaxation time, and ξ_5 and ξ_6 are additional thermodynamic constants. The linearized Riemann tensor is given by

$$R_{\alpha\mu\beta\nu} = \frac{1}{2}(\partial_\mu \partial_\beta h_{\alpha\nu} + \partial_\alpha \partial_\nu h_{\mu\beta} - \partial_\mu \partial_\nu h_{\alpha\beta} - \partial_\alpha \partial_\beta h_{\mu\nu}). \quad (26)$$

In the linear response analysis, the energy-momentum tensor averages are given by [49,51–54]

$$\begin{aligned} \langle \hat{T}^{\mu\nu}(\omega, k) \rangle_h = & \frac{\partial \langle T^{\mu\nu} \rangle_h}{\partial h_{\alpha\beta}} \Big|_{h_{\mu\nu}=0} h_{\alpha\beta}(\omega, k) \\ & - \frac{1}{2} G_R^{\mu\nu, \alpha\beta}(\omega, k) h_{\alpha\beta}(\omega, k) + O(h^2). \end{aligned} \quad (27)$$

The first term is there because the equilibrium in a curved space is different from the equilibrium in the flat Minkowski space. This term is the same as the counterterms in Eq. (4).

The goal here is to choose the metric perturbation and its spatial dependence judiciously so that only a particular response function is singled out. The details of the solutions will be given elsewhere. In this section, we will just quote relevant results. In all analyses below, the metric perturbation has the following form:

$$h_{ij}(t, x_l) = h_{ji}(t, x_l) = h_{ij}(\omega, k_l) e^{-i\omega t + ik_l x_l}, \quad (28)$$

where l is not summed.

We start with the one we already know something about. From Eq. (7), we know that $G^{xy,xy}(\omega, k_y) = G_1(\omega, k_y^2)$. Therefore, to get G_1 , we should use $h_{xy}(t, y) = h_{yx}(t, y)$ with all other metric components being zero. Solving for

$$\pi^{xy}(\omega, k_y) = -\bar{G}_1(\omega, k_y^2) h_{xy}(\omega, k_y), \quad (29)$$

we can obtain

$$\bar{G}_1(\omega, k^2) = \frac{-\omega^2(h_0 - i\frac{\kappa-2\kappa^*}{2D_T}\omega)}{k^2 - i\omega/D_T - (\tau_\pi/D_T)\omega^2}, \quad (30)$$

confirming that $G_1(\omega, k^2)$ has the diffusion structure. Here $D_T = \eta/h_0$ is the shear diffusion constant. The overbar on the response function indicates that it is obtained by solving the gravity-hydrodynamic equations up to the second order, and hence not as general as G_1 in Eq. (12).

From Eq. (7), we also know that $G_R^{xy,xy}(\omega, k_z) = G_2(\omega, k_z^2)$. Therefore, to get G_2 , we should use $h_{xy}(t, z) = h_{yx}(t, z)$ with

all other metric components set to zero. Solving for

$$\pi^{xy}(\omega, k_z) = -\bar{G}_2(\omega, k_z^2)h_{xy}(\omega, k_z), \quad (31)$$

we can obtain

$$\bar{G}_2(\omega, k^2) = -\frac{i\omega\eta + \frac{\kappa}{2}(\omega^2 + k^2) - \kappa^*\omega^2}{1 - i\omega\tau_\pi}. \quad (32)$$

This analytic form of $\bar{G}_2$ indeed justifies Eq. (2) as a well-defined expansion since $\bar{G}_2(\omega, k^2)$ has no hydrodynamic pole.

Comparing $\bar{G}_2$ with $\bar{G}_1$, we see that

$$\bar{G}_2(\omega, k^2) = \bar{G}_1(\omega, 0) - \frac{\frac{\kappa}{2}k^2}{1 - i\omega\tau_\pi}. \quad (33)$$

The most general form of G_2 (not $\bar{G}_2$) can then be written as

$$G_2(\omega, k^2) = G_1(\omega, 0) - k^2 F_2(\omega, k), \quad (34)$$

where F_2 is analytic around $\omega = 0$ and $k = 0$ but may contain poles on the imaginary ω axis away from $\omega = 0$.

The purely longitudinal response function G_L can be obtained from Eq. (7), by choosing $\tilde{G}_R^{zz,zz}(\omega, k_z)$. In terms of the metric perturbation, this corresponds to having $h_{zz}(t, z)$ as the sole nonzero component. By solving for

$$\delta T^{zz}(\omega, k_z) = -\frac{1}{2}\bar{G}_L(\omega, k_z^2)h_{zz}(\omega, k_z) \quad (35)$$

we obtain

$$\bar{G}_L(\omega, k^2) = \frac{N_L(\omega, k^2)}{S_{\text{sound}}(\omega, k^2)}. \quad (36)$$

The explicit expressions of the numerator N_L and the denominator S_{sound} can be found in Appendix A. Here, we just note that the numerator indeed has a sound pole and the whole expression reduces to that found in [51] in the conformal limit where $\zeta = 0$ and $\tau_\Pi = 0$.

The response function G_{LT} is given by

$$G^{xx,zz}(\omega, k_z) = G_{LT}(\omega, k_z^2), \quad (37)$$

which can be also obtained using $h_{zz}(t, z)$ but solving for

$$\delta T^{xx}(\omega, k_z) = -\frac{1}{2}\bar{G}_{LT}(\omega, k_z^2)h_{zz}(\omega, k_z). \quad (38)$$

The solution is

$$\bar{G}_{LT}(\omega, k^2) = \frac{N_{LT}(\omega, k^2)}{S_{\text{sound}}(\omega, k^2)}. \quad (39)$$

The numerator N_{LT} is given in Appendix A. Equation (39) is fully consistent with Eq. (17).

The last one to consider is G_T . From Eq. (7), one can get

$$G^{yy,zz}(\omega, k_x) + G^{zz,zz}(\omega, k_x) = 2G_T(\omega, k_x^2). \quad (40)$$

Hence, the perturbation to use is $h_{zz}(t, x)$ and we need to solve for

$$\delta T^{yy}(\omega, k_x) + \delta T^{zz}(\omega, k_x) = -\bar{G}_T(\omega, k_x^2)h_{zz}(\omega, k_x). \quad (41)$$

The response function is of the sound-pole type,

$$\bar{G}_T(\omega, k^2) = \frac{N_T(\omega, k^2)}{S_{\text{sound}}(\omega, k^2)}. \quad (42)$$

The numerator N_T can be found in Appendix A. The most general form of G_T turns out to be

$$G_T(\omega, k) = -\frac{\omega^2[h_0 + i\omega F_{T1}(\omega)]}{k^2 - \omega^2/Z(\omega, k) - i\omega^3 R(\omega, k)} - \frac{k^2[k^2 g_{T2}(k^2) + i\omega F_{T2}(\omega, k)]}{k^2 - \omega^2/Z(\omega, k) - i\omega^3 R(\omega, k)}. \quad (43)$$

Note that the function $F_{T1}(\omega) = F_{T1R}(\omega^2) - i\omega F_{T1I}(\omega^2)$ does not depend on k because any possible k dependence can be absorbed by $F_{T2}(\omega, k)$. The response functions $\bar{G}_{LT}$ and $\bar{G}_T$ were not calculated in [51].

What we would like to do next is to relate the $\omega = k = 0$ values of functions appearing in the expressions for the response functions to physical transport coefficients.

III. CONSTRAINTS FROM THE SMALL k CONDITIONS

By requiring that the coefficients of the various $\hat{k}\hat{k}\delta$ terms in Eq. (7) are $O(k^2)$ and the coefficient of the $\hat{k}_i\hat{k}_j\hat{k}_l\hat{k}_m$ term is $O(k^4)$ in the small k limit with $\omega \neq 0$, one can get three independent relationships between the response functions in the small k limit. One of the $O(k^2)$ relationships is

$$2G_1(\omega, k) - G_L(\omega, k) + G_{LT}(\omega, k) = O(k^2). \quad (44)$$

In the $k \rightarrow 0$ limit, this can be thought of as an equation for $F_{LTR}(\omega^2, 0)$ and $F_{LTI}(\omega^2, 0)$ in G_{LT} in terms of the functions appearing in G_1 and G_L . The equations are linear, but the solutions are fairly long and not very illuminating. The $\omega \rightarrow 0$ limits are, however, relatively simple:

$$F_{LTR} = \frac{2h_0 D_R}{Z_R} \quad (45)$$

and

$$F_{LTI} = -\frac{2(D_R^2 h_0 B_R + D_R F_{1R} - D_I h_0)}{Z_R} + 2D_R h_0 R_R + F_{LR} - \frac{2D_R h_0 Z_I}{Z_R^2}, \quad (46)$$

where we introduced a shorthand notation for the $\omega = k = 0$ values of the functions as the name of the functions themselves. That is, $D_R = D_R(0)$, $B_R = B_R(0, 0)$, etc., except for the response functions.

The second $O(k^2)$ relationship is

$$2G_T(\omega, k) - G_{LT}(\omega, k) - G_L(\omega, k) = O(k^2). \quad (47)$$

Taking $k \rightarrow 0$, we can use this condition to put conditions on $F_{T1}(\omega)$ in G_T . In the zero ω and k limit, we find

$$F_{T1R} = \frac{1}{2}F_{LTR}, \quad F_{T1I} = \frac{1}{2}(F_{LTI} + F_{LR}). \quad (48)$$

With $F_{LT}(\omega, 0)$ and $F_{T1}(\omega)$ thus determined, the remaining small k condition,

$$G_2 - 4G_1 + G_T + G_L - 2G_{LT} = O(k^4), \quad (49)$$

is fulfilled up to $O(k^2)$ with $\omega \neq 0$. $O(k^4)$ can then be achieved by solving for $F_{T2R}(\omega^2, 0)$ and $F_{T2I}(\omega^2, 0)$ in G_T to remove the $O(k^2)$ terms. The solutions are again long and not very

illuminating, but the $\omega = k = 0$ pieces are simple to write down:

$$F_{T2R} = 3h_0 D_R \quad (50)$$

and

$$F_{T2I} = \frac{F_{2R} - 4D_R^2 h_0}{Z_R} + 2g_{LT} + 3(h_0(D_I - D_R^2 B_R) - D_R F_{1R}). \quad (51)$$

Similarly, if one considers the small k behavior of $\tilde{G}^{ij,lm}(0, \mathbf{k})$, one can also deduce $G_2(0, k^2) = O(k^2)$, $G_T(0, k^2) = O(k^2)$ and

$$G_2(0, k^2) + G_T(0, k^2) = O(k^4), \quad (52)$$

from which one can get

$$g_{T2} + F_{2R} = 0. \quad (53)$$

IV. FIXING THE CONSTANTS WITH SECOND-ORDER HYDRODYNAMICS

We are now ready to relate the constants, namely the $\omega = 0$ and $k = 0$ values of the functions appearing in the response functions, to physical transport coefficients. First, consider the denominators of the response functions that give the dispersion relationships for diffusion and sound. By comparing the denominator of $G_1(\omega, k^2)$ with the dispersion relationship of the momentum diffusion from the second order hydrodynamics [34], one can show

$$D_R = D_T = \frac{\eta}{h_0}, \quad B_R D_R = \tau_\pi, \quad (54)$$

where D_R can be recognized as the shear diffusion constant D_T . By comparing the denominator of $G_L(\omega, k^2)$ and the sound dispersion relationship from second order hydrodynamics, one can get

$$\begin{aligned} Z_R &= v_s^2, \quad Z_{R\omega} = -\left(\tau_\pi \tau_\Pi v_s^2 + \tau_\Pi \frac{4\eta}{3h_0} + \tau_\pi \frac{\zeta}{h_0}\right), \\ Z_I &= \Gamma + v_s^2(\tau_\pi + \tau_\Pi), \quad R_R = \frac{\tau_\pi + \tau_\Pi}{v_s^2}, \\ R_I &= \frac{\tau_\pi \tau_\Pi - Z_I R_R}{v_s^2}. \end{aligned} \quad (55)$$

Here v_s is the speed of sound and $\Gamma = (4\eta/3 + \zeta)/h_0$ is the sound attenuation constant.

Using the above, we can further determine

$$\begin{aligned} F_{LTR} &= \frac{2\eta}{v_s^2}, \quad F_{LTI} = -\frac{2\eta\Gamma}{v_s^2} + \frac{2(K - \eta\tau_\pi)}{v_s^2} + F_{LR}, \\ F_{T1R} &= \frac{\eta}{v_s^2}, \quad F_{T1I} = -\frac{\eta\Gamma}{v_s^4} + \frac{(K - \eta\tau_\pi)}{v_s^2} + F_{LR}, \\ F_{T2R} &= 3\eta, \quad F_{T2I} = \frac{F_{2R} - \frac{4\eta^2}{h_0}}{v_s^2} + 3(K - \eta\tau_\pi) + 2g_{LT}, \end{aligned} \quad (56)$$

where $K = D_I h_0 - D_R F_{1R}$, F_{LR} , F_{2R} , and g_{LT} are still unknown.

To go further, we need results from the gravity-hydrodynamics studies. To compare with Eq. (2), one can expand G_2 to get

$$\begin{aligned} G_2(\omega, k^2) &= -iD_R h_0 \omega + B_R D_R^2 h_0 \omega^2 - K \omega^2 - F_{2R} k^2 \\ &+ O(\omega k^2). \end{aligned} \quad (57)$$

The first two terms match the corresponding terms in Eq. (2) upon using Eq. (54). Matching the other two terms, we have

$$F_{2R} = \frac{\kappa}{2} = -g_{T2}, \quad (58)$$

$$K = \frac{\kappa}{2} - \kappa^*. \quad (59)$$

For the sound modes, we can compare our G_L with $\tilde{G}_L$ in Eq. (36). By requiring $G_L - \tilde{G}_L$ to vanish up to and including $O(\omega^6)$ with $k \sim \omega$, we get

$$v_s^2 F_{LR} = -\frac{2\kappa}{3} + \frac{4\kappa^*}{3} + 2\xi_5 - \xi_6 + \frac{h_0 Z'_R}{v_s^4}. \quad (60)$$

As argued before, F_{LR} is indeed thermodynamic in nature.

Similarly, comparing the gravity-hydro result $\tilde{G}_{LT}$ in Eq. (39) with G_{LT} in Eq. (17) gives a further condition:

$$g_{LT} = -\kappa + 2\kappa^*. \quad (61)$$

Within the context of the second-order hydrodynamics, it is consistent to set $Z'_R = \lim_{k \rightarrow 0} \partial Z_R(k^2)/\partial k^2 = 0$ (but not within the context of the third-order hydrodynamics, as can be seen below). This fixes all constants on the right-hand sides of Eq. (56) in terms of transport coefficients.

One can compare terms that are of higher order in ω and k to determine F_{1I} , B_I , and F_{2I} and the derivatives of other functions. But since the results are not very illuminating and do not yield useful Kubo formulas, we will not do so here.

V. THIRD-ORDER CONFORMAL HYDRODYNAMICS AND THE RELAXATION TIMES

In the previous section, we analyzed response functions within the context of the second-order hydrodynamics. Recently, third-order hydrodynamic results have been obtained by several groups [59–69]. In this section, we repeat some of the analysis so far to compare with the third-order results. For simplicity, we will restrict ourselves to the massless case, which may be referred to as the classical conformal theory. Within the kinetic theory, $\Pi = 0$ and $\tau_\Pi = 0$, and the relevant equations from [64] in the fluid cell rest frame are

$$\partial_t \pi^{ij} = -\frac{\pi^{ij}}{\tau_\pi} - \varphi_{-1|0} \sigma^{ij} - \partial_l \xi^{lij} \quad (62)$$

$$\partial_t \xi^{lij} = -\frac{1}{\tau_\xi} \xi^{lij} - \frac{3}{7} \partial^{(l} \pi^{ij)} - \partial_k \zeta^{klj}, \quad (63)$$

$$\partial_t \zeta^{ijkl} = -\frac{1}{\tau_\zeta} \zeta^{ijkl} - \frac{4}{9} \partial^{(i} \xi^{jkl)}, \quad (64)$$

where $\varphi_{-1|0} = 8T^4/5\pi^2$, and ξ^{lij} and ζ^{ijkl} are the new spin-3 and spin-4 degrees of freedom, respectively. These are still conformal because $\partial_l \xi^{lij} \sim \partial_l \partial^{(l} \pi^{ij)}$ can be decomposed as a linear combination of $\nabla^2 \pi^{ij}$ and $\partial^{(i} \partial_l \pi^{lj)}$. Both are Weyl invariant [69]. One can also get the same set of linearized

equations from [62] (up to ξ^{lij}), and a slightly different set from [59] (up to ζ^{ijkl}) that contains slightly different coefficients. The differences come from three different methods employed in these studies to get ξ^{lij} and ζ^{ijkl} . As this difference does not change the qualitative conclusion of this section, we will only consider the above equations.

By analyzing the resulting linearized equations of motion for the diffusive mode and comparing with G_1 , we get

$$\begin{aligned} D_R &= D_T, \quad B_R D_R = \tau_\pi + \tau_\xi + \tau_\zeta, \\ D_I &= \frac{8}{35} \tau_\pi \tau_\xi + \frac{5}{21} \tau_\xi \tau_\zeta + D_R (\tau_\xi + \tau_\zeta), \\ B_I D_R &= \tau_\pi \tau_\xi + \tau_\pi \tau_\zeta + \tau_\xi \tau_\zeta - D_I B_R. \end{aligned} \quad (65)$$

Compared to the second-order hydrodynamics results in Eq. (54), we see that the meaning of B_R has changed and D_I and B_I are now determined.

Previously, $B_R D_R$ was just τ_π in the conformal limit. Now it is the sum of all relaxation times. If one sets $\zeta^{ijkl} = 0$, one can show that $B_R D_R = \tau_\pi + \tau_\xi$. Hence, depending on how many additional modes are added, $B_R D_R$ can be τ_π , $\tau_\pi + \tau_\xi$, or $\tau_\pi + \tau_\xi + \tau_\zeta$. Therefore, the meaning of the Kubo formula

$$\frac{1}{2} \lim_{\omega \rightarrow 0} \lim_{k_z \rightarrow 0} \partial_\omega^2 G^{\text{xy,xy}}(\omega, k_z) = B_R D_R \eta - K \quad (66)$$

obtained from Eq. (57) does not seem to have a definite meaning, as the meaning of B_R changes as the order of hydrodynamics changes. One can certainly evaluate the left-hand side in a given theory, as was done in [34]. However, how it is related to the shear relaxation time is not so certain anymore. There does exist a possibility that the changes in K may compensate the changes in B_R . However, since K is purely thermodynamic in the second-order hydrodynamics, it does not seem likely that it will develop a τ_ζ or τ_ξ dependence in the third-order hydrodynamics.

Analysis of the longitudinal mode gives a similar conclusion. For instance,

$$v_s^2 R_R = \tau_\pi + \tau_\xi + \tau_\zeta \quad (67)$$

behaves the same way as B_R , and Z'_R , which vanished within the second-order hydrodynamics, is now

$$Z'_R = \frac{1}{945} \tau_\xi (81 \tau_\pi + 80 \tau_\zeta). \quad (68)$$

The results so far clearly indicate that the Kubo formulas involving the relaxation times are *not* stable against adding more relaxation equations for higher-order hydrodynamics. Hence, it is not so clear whether meaningful Kubo formulas for the relaxation times can actually exist, at least from the two-point correlation functions.⁴ For this reason, we will not consider Kubo formulas involving any relaxation times in the next section.

⁴References [52,53] contain a Kubo formula for τ_π in terms of a three-point correlation function of the stress-energy tensor components, which we have not yet analyzed in this context.

VI. NEW KUBO FORMULAS

With most of the constants now known, we can derive new Kubo formulas. The novel feature of many of these formulas is that the $\omega \rightarrow 0$ limit is taken first and then the $k \rightarrow 0$ limit is taken, in contrast to the usual requirement for the thermodynamic limit. Known examples of such Kubo formulas are the chiral magnetic conductivity and chiral vortical conductivity [80–83]. It is tempting to call this the static limit, but at this point it is not clear whether these new formulas can be actually calculated purely within the Matsubara formalism with $\omega_n = 0$ due to the presence of the ω derivatives.

First, consider

$$\begin{aligned} \Delta_{ij,lm} \tilde{G}^{ij,lm}(\omega, k) &= 20P + 8G_1(\omega, k^2) + 8G_2(\omega, k^2) \\ &\quad + \frac{4}{3}[G_L(\omega, k^2) + G_T(\omega, k^2) \\ &\quad - 2G_{LT}(\omega, k^2)], \end{aligned} \quad (69)$$

which is what is usually used to calculate the shear viscosity η . Here,

$$\Delta_{ij,lm} = (\delta_{il} \delta_{jm} + \delta_{im} \delta_{jl} - \frac{2}{3} \delta_{ij} \delta_{lm}) \quad (70)$$

isolates the spin-2 part of a rank-2 tensor. The usual Kubo formula for the shear viscosity is

$$\eta = \frac{i}{20} \lim_{\omega \rightarrow 0} \lim_{k \rightarrow 0} \Delta_{ij,lm} \partial_\omega \tilde{G}^{ij,lm}(\omega, k). \quad (71)$$

All five response functions contribute to this Kubo formula.

By using our analytic forms and reversing the order of the limits, we have an alternative Kubo formula for η :

$$\eta = \frac{i}{12} \lim_{k \rightarrow 0} \lim_{\omega \rightarrow 0} \Delta_{ij,lm} \partial_\omega \tilde{G}^{ij,lm}(\omega, k). \quad (72)$$

Only G_2 and G_T contribute to this Kubo formula since all others vanish in the $\omega \rightarrow 0$ limit with $k \neq 0$. Note that deriving Eq. (72) is possible only because we have the analytic structures of all five response functions. Note also that even though both Eqs. (71) and (72) can be used to calculate η , the two orders of the limits do not commute.

Next, consider

$$G_R^{0x,0x}(\omega, k_y) = \frac{k_y^2}{\omega^2} G_1(\omega, k_y^2). \quad (73)$$

Taking a derivative with respect to ω , we obtain

$$\begin{aligned} \partial_\omega G_R^{0x,0x}(\omega, k_y) &= -i F_{1R}(0, k_y^2) - i \frac{g_1(k_y^2)}{D_R(k_y^2)} - i \frac{h_0}{k_y^2 D_R(k_y^2)} \\ &\quad + \frac{2h_0}{k_y^4 D_R^2(k_y^2)} \omega + O(\omega/k_y^2) + O(\omega^2) \end{aligned} \quad (74)$$

in the small $\omega < k_y$ limit. From Eq. (74), an interesting Kubo formula for η can be obtained:

$$\frac{\eta}{h_0} = -i \lim_{k_y \rightarrow 0} \lim_{\omega \rightarrow 0} \frac{h_0}{k_y^2 \partial_\omega G_R^{0x,0x}(\omega, k_y)}. \quad (75)$$

Taking one more ω derivative, one similarly gets

$$\frac{\eta}{h_0} = \left(\lim_{k_y \rightarrow 0} \lim_{\omega \rightarrow 0} \frac{2h_0}{k_y^4 \partial_\omega^2 G_R^{0x,0x}(\omega, k_y)} \right)^{1/2}. \quad (76)$$

One can get isotropic forms by substituting $k_y \rightarrow k$ and $G_R^{0x,0x} \rightarrow G_R^{0i,0i}/2$ on the right-hand sides.

These formulas are interesting because the denominator can be associated with the transport cross section σ_{tr} through $\eta/h_0 \sim l_{mfp} \sim 1/(n_{sc}\sigma_{tr})$, where l_{mfp} is the mean free path and n_{sc} denotes the density of the scatterers. In other words, putting a lower bound on η/s [47,84] is roughly equivalent to putting an upper bound on the in-medium transport cross-section. Thus, these formulas may provide intriguing possibilities to explore the lower bound of η/s in the context of finite temperature field theory. However, the exact relationship between the denominators of Eqs. (75) and (76) and the transport cross section is yet to be clarified.

We can derive a few more Kubo formulas for η and ζ all with the $\omega \rightarrow 0$ limit taken first. Noting $\tilde{G}^{xx,xx}(\omega, k_z) = P + G_2 + G_T$, we get

$$\lim_{k_z \rightarrow 0} \lim_{\omega \rightarrow 0} \partial_\omega \tilde{G}^{xx,xx}(\omega, k_z) = -4i\eta. \quad (77)$$

Reversing the order of the limits, we have

$$\lim_{\omega \rightarrow 0} \lim_{k_z \rightarrow 0} \partial_\omega \tilde{G}^{xx,xx}(\omega, k_z) = -i\left(\zeta + \frac{4\eta}{3}\right). \quad (78)$$

For the energy density and the pressure correlators, we have

$$\lim_{k \rightarrow 0} \lim_{\omega \rightarrow 0} \partial_\omega G_R^{00,00}(\omega, \mathbf{k}) = -\frac{i}{v_s^4} \left(\zeta + \frac{4\eta}{3} \right), \quad (79)$$

$$\lim_{k \rightarrow 0} \lim_{\omega \rightarrow 0} \partial_\omega G_R^{00,ii}(\omega, \mathbf{k}) = -4\frac{i\eta}{v_s^2}, \quad (80)$$

$$\lim_{k \rightarrow 0} \lim_{\omega \rightarrow 0} \partial_\omega \tilde{G}^{ii,jj}(\omega, \mathbf{k}) = -12i\eta. \quad (81)$$

We can combine them to get

$$\lim_{k \rightarrow 0} \lim_{\omega \rightarrow 0} \partial_\omega G^{\Delta P \Delta P}(\omega, \mathbf{k}) = -i\zeta, \quad (82)$$

where $\Delta \hat{P} = \hat{P} - v_s^2 \hat{\epsilon}$ with $\hat{P} = \hat{T}^{ii}/3$ being the pressure operator and $\hat{\epsilon} = \hat{T}^{00}$ the energy density operator. If one reverses the order of the limits, $\partial_\omega G_R^{00,00} \rightarrow 0$, $\partial_\omega G_R^{00,ii} \rightarrow 0$, and $\partial_\omega \tilde{G}^{ii,jj} \rightarrow -9i\zeta$. Hence we also have

$$\lim_{\omega \rightarrow 0} \lim_{k \rightarrow 0} \partial_\omega G^{\Delta P \Delta P}(\omega, \mathbf{k}) = -i\zeta. \quad (83)$$

Thus, like G_2 , it does not matter which order the limits are taken. This is because in this combination, the numerator behaves like $\sim (\omega^2 - v_s^2 k^2)$ for small ω and k which compensates the sound pole in the denominator. It is also interesting that both η and ζ can be obtained from the imaginary part (i.e., the spectral density) of the pressure autocorrelation function by taking different orders of the limits.

The following Kubo formulas for thermodynamic transport coefficients are equivalent to similar formulas in [51–53], and

hence serve as a good consistency check:

$$\begin{aligned} \lim_{k \rightarrow 0} \frac{v_s^2}{2} \partial_k^2 G_R^{00,00}(0, \mathbf{k}) &= -v_s^2 F_{LR} \\ &= +\frac{2\kappa}{3} - \frac{4\kappa^*}{3} - 2\xi_5 + \xi_6 - \frac{h_0 Z'_R}{v_s^4} \end{aligned} \quad (84)$$

and

$$\lim_{k \rightarrow 0} \partial_k^2 G_R^{00,ii}(0, \mathbf{k}) = -4g_{LT} = +4(\kappa - 2\kappa^*) \quad (85)$$

and

$$\lim_{k \rightarrow 0} \partial_k^2 G_R^{ii,jj}(0, \mathbf{k}) = 8g_{T2} = -4\kappa. \quad (86)$$

VII. ON DIAGRAMMATIC CALCULATIONS OF VISCOSITIES

Standard diagrammatic discussion of shear viscosity via the Kubo formula Eq. (71) starts with the one-loop diagram with bare propagators. Upon realizing that this diagram does not produce a sensible answer and also that the viscosity should inversely depend on the transport cross section, one then introduces resummed propagators to see that $\eta_{1\text{-loop}} \propto 1/\Gamma_{th}$, where Γ_{th} is the thermal width of the propagator (equivalently, the imaginary part of the self-energy). This inverse dependence then necessitates resummation of the ladder diagrams via a Schwinger-Dyson equation for the dressed vertex.

What gets often overlooked in this process is that, as long as $\Gamma_{th} \neq 0$, it does not matter whether the limit $k \rightarrow 0$ is taken first or the limit $\omega \rightarrow 0$ is taken first. As we have shown in this paper, only $\partial_\omega G_R^{xy,xy}(\omega, k_z)$ and $\partial_\omega G^{\Delta P \Delta P}(\omega, \mathbf{k})$ are independent of the order of the limits. All other response functions give different answers when different orders of the limits are taken. For instance, Eqs. (72) and (81) are different from the usual Kubo formula for η and ζ .

It is then worth asking what is really being calculated with the usual skeleton diagram expansion technique using the resummed propagators. For simplicity, let us consider the one-loop diagram for

$$\eta = i \lim_{\omega \rightarrow 0} \lim_{k_z \rightarrow 0} \partial_\omega G_R^{xy,xy}(\omega, k_z) \quad (87)$$

in the real scalar $\lambda\phi^4$ theory. Since

$$\text{Im } G_R^{xy,xy}(\omega, k_z) = \frac{1}{2}(1 - e^{-\beta\omega})\sigma_{>}^{xy,xy}(\omega, k_z), \quad (88)$$

where $\sigma_{>}^{xy,xy}(\omega, k_z)$ is the greater Wightman function, we only need to examine

$$\begin{aligned} \sigma_{>, 1\text{-loop}}^{xy,xy}(\omega, k_z) &= 2 \int \frac{d^4 p}{(2\pi)^4} p_x^2 p_y^2 n(p^0 + \omega) [1 + n(p^0)] \\ &\quad \times \rho_\phi(p^0 + \omega, \mathbf{p} + k_z \mathbf{e}_z) \rho_\phi(p^0, \mathbf{p}), \end{aligned} \quad (89)$$

where

$$\rho_\phi(p) = -i \left(\frac{1}{(p^0 - i\Gamma_{th}/2)^2 - E_p^2} - \frac{1}{(p^0 + i\Gamma_{th}/2)^2 - E_p^2} \right) \quad (90)$$

is the spectral density for the resummed propagator with $E_p = \sqrt{\mathbf{p}^2 + m^2}$. The four poles of $\rho_\phi(p^0 + \omega, \mathbf{p} + k_z \mathbf{e}_z)$ are at $p^0 \approx \pm i\Gamma_{\text{th}}/2 - \omega \pm (E_p + p_z k_z/E_p)$. Since $\Gamma_{\text{th}}/2 \neq 0$, it does not matter to the pinching pole contribution which limit is taken first. We can just set $\omega = k = 0$. This is precisely what we expected from G_2 .

Now let us consider Eq. (71). We now have

$$\begin{aligned} \frac{1}{20} \Delta_{ij,lm} \sigma_{1\text{-loop}}^{ij,lm}(\omega, \mathbf{k}) &= \int \frac{d^4 p}{(2\pi)^4} n(p^0 + \omega) [1 + n(p^0)] \\ &\times \frac{1}{30} [4\mathbf{p}^4 + 8\mathbf{p}^2(\mathbf{p} \cdot \mathbf{k}) + (\mathbf{p} \cdot \mathbf{k})^2 \\ &+ 3\mathbf{k}^2 \mathbf{p}^2] \rho_\phi(p^0 + \omega, \mathbf{p} + \mathbf{k}) \\ &\times \rho_\phi(p^0, \mathbf{p}). \end{aligned} \quad (91)$$

The pole positions are at $p^0 \approx \pm i\Gamma_{\text{th}}/2 - \omega \pm (E_p + \mathbf{p} \cdot \mathbf{k}/E_p)$ and again there is no obstacle in just setting $\omega = k = 0$. But that violates what we found in Eqs. (71) and (72), that the two orders of the limits do not commute. Therefore, it is not clear whether Eq. (91) is evaluating Eq. (71) or Eq. (72). In fact, when $\omega = k = 0$, the right-hand-side of Eq. (91) should be interpreted as just the angle averaged version of Eq. (89). Hence all calculations that started with Eq. (71) but could be able to set $\omega = k = 0$ without any obstacles may have all been actually using G_2 via Eq. (87).

For the bulk viscosity calculation, the function $\partial_\omega G^{PP}$ does depend on the order of the limits while ladder diagrams do not. Fortunately, from Eqs. (82) and (83), we see that it is permissible to just set $\omega = k = 0$ as long as the autocorrelation function of $\Delta\hat{P} = \hat{P} - v_s^2 \hat{\epsilon}$ is used. This may imply that the ladder diagram resummation is only valid for this particular correlation function.

A question then arises: How do we use Kubo formulas that depend on the order of the limits?⁵ The answer seems to be that they can be used to calculate viscosities only if some nonperturbative approximation results in hydrodynamic poles for the correlation functions. For instance, Eq. (75) makes sense only if an approximation of $G_R^{0x,0x}(\omega, k_y) = (k_y^2/\omega^2)G_1(\omega, k_y^2)$ contains a diffusion pole. For instance, the usual skeleton diagram resummation does not accomplish this, but the Anderson localization schemes can yield a diffusion pole by assuming that the interaction is pointlike [85–87]. Thus, further development of nonperturbative approximation schemes is necessary to fully utilize the new Kubo formulas.

VIII. CONCLUSIONS

The main purpose of this work is to elucidate the analytic structure of the stress-energy tensor correlation functions in the small ω and small k limit. We succeeded in doing so by combining the energy-momentum conservation laws and the results from gravity-hydrodynamics calculations and the recently developed third-order hydrodynamics.

From this endeavor, we have drawn the following conclusions. First, the response function often used in the calculation of the shear viscosity, $G_R^{xy,xy}(\omega, k_z)$, does not in fact possess the shear diffusion pole. In retrospect, it is almost obvious that this particular correlator cannot possess the shear diffusion pole since variation only in the z direction cannot induce diffusion in the transverse (x, y) directions. On the other hand, $G_R^{xy,xy}(\omega, k_y)$ is the full hydrodynamics response function that possesses a proper shear diffusion pole. The usual calculations via skeleton expansion in fact correspond to using $G_R^{xy,xy}(\omega, k_z)$ even though the isotropic version of the Kubo formula is used. Similarly, skeleton expansion for the bulk viscosity makes sense only if the autocorrelation function of $\Delta\hat{P} = \hat{P} - v_s^2 \hat{\epsilon}$ is used.

Second, the Kubo formulas for the relaxation times are unreliable because the results of the second-order hydrodynamic analysis and the third-order one do not match. One should perhaps go back to the kinetic theory analysis to establish what the Kubo formulas for the relaxation times actually mean.

Third, we have shown that the viscosities can also be calculated with the $\omega \rightarrow 0$ limit taken first. For instance, the pressure-pressure spectral density can give the bulk viscosity or the shear viscosity depending on whether the $k \rightarrow 0$ limit is taken first or the $\omega \rightarrow 0$ limit is taken first. We have also derived some interesting Kubo formulas where the shear viscosity to enthalpy ratio is inversely proportional to some power of the correlation functions. These formulas may offer an intriguing possibility to further study the η/s bound in terms of the bound on the transport cross section.

Even though the $\omega \rightarrow 0$ limit is taken first, the new Kubo formulas are not really thermodynamical or static because usually an ω derivative is involved. Equivalently, these formulas use the imaginary part of the retarded correlation functions. In this regard, the Kubo formula in Eq. (76) is special because it uses the *real* part of a retarded correlation function. Although the second-order frequency derivative makes it nontrivial to connect it to a purely Euclidean formulation (thus to simpler lattice calculations), it may provide an intriguing possibility. We should note that, to use the new Kubo formulas fully, one needs an approximation scheme where the resummed correlation functions actually exhibit hydrodynamic poles. In this sense, investigation of new techniques to calculate the explicit pole structure of the correlation functions would be important.

Finally, we would like to note that, for the sake of brevity, we have omitted many of the details in the derivations in this work. We will report them in a separate publication.

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⁵Except the thermodynamic ones such as Eqs. (84)–(86).

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: DETAILS OF $\bar{G}_L$, $\bar{G}_{LT}$ AND $\bar{G}_T$

The numerator for $\bar{G}_L$ is given by

$$N_L(\omega, k^2) = -h_0 \left(v_s^2 \omega^2 - i\tau_\pi v_s^2 \omega^3 - i\tau_\pi v_s^2 \omega^3 - \tau_\pi \tau_\pi v_s^2 \omega^4 - \frac{4i\eta \omega^3}{3h_0} - \frac{i\omega^3 \zeta}{h_0} - \frac{4\eta \tau_\pi \omega^4}{3h_0} - \frac{\tau_\pi \omega^4 \zeta}{h_0} - \frac{2\kappa \omega^4}{3h_0} \right. \\ \left. + \frac{4\kappa^* \omega^4}{3h_0} + \frac{2\omega^4 \xi_5}{h_0} - \frac{\omega^4 \xi_6}{h_0} + \frac{2i\kappa \tau_\pi \omega^5}{3h_0} - \frac{4i\kappa^* \tau_\pi \omega^5}{3h_0} - \frac{2i\tau_\pi \omega^5 \xi_5}{h_0} + \frac{i\tau_\pi \omega^5 \xi_6}{h_0} \right), \quad (\text{A1})$$

which is in fact independent of k at this order. The denominator exhibits the typical sound pole structure

$$S_{\text{sound}}(\omega, k^2) = k^2 v_s^2 - \omega^2 + i\tau_\pi \omega^3 + i\tau_\pi \omega^3 + \tau_\pi \tau_\pi \omega^4 - \frac{4i\eta k^2 \omega}{3h_0} - \frac{ik^2 \omega \zeta}{h_0} - ik^2 \tau_\pi v_s^2 \omega - ik^2 \tau_\pi v_s^2 \omega \\ - \frac{4\eta k^2 \tau_\pi \omega^2}{3h_0} - \frac{k^2 \tau_\pi \omega^2 \zeta}{h_0} - k^2 \tau_\pi \tau_\pi v_s^2 \omega^2. \quad (\text{A2})$$

This has exactly the same structure as G_L in Eq. (14). It also coincides with the sound dispersion relationship in the second-order hydrodynamics. The correlator $\bar{G}_L(\omega, k_z)$ coincides with the correlator $G_R^{zzzz}(0, k_z)$ calculated in [51] in the conformal limit where $\zeta = 0$ and $\tau_\pi = 0$. The sound pole denominator is common to all sound response functions G_L , G_{LT} , and G_T .

The numerator for $\bar{G}_{LT}$ is given by

$$N_{LT}(\omega, k^2) = h_0 \left(v_s^2 \omega^2 - i\tau_\pi v_s^2 \omega^3 - i\tau_\pi v_s^2 \omega^3 - \tau_\pi \tau_\pi v_s^2 \omega^4 + \frac{2i\eta \omega^3}{3h_0} - \frac{i\omega^3 \zeta}{h_0} + \frac{\kappa \omega^4}{3h_0} - \frac{2\kappa^* \omega^4}{3h_0} + \frac{2\eta \tau_\pi \omega^4}{3h_0} - \frac{\tau_\pi \omega^4 \zeta}{h_0} \right. \\ \left. + \frac{2\omega^4 \xi_5}{h_0} - \frac{\omega^4 \xi_6}{h_0} - \frac{ik \tau_\pi \omega^5}{3h_0} + \frac{2i\kappa^* \tau_\pi \omega^5}{3h_0} - \frac{2i\tau_\pi \omega^5 \xi_5}{h_0} + \frac{i\tau_\pi \omega^5 \xi_6}{h_0} - \frac{k^2 \kappa v_s^2 \omega^2}{h_0} + \frac{2k^2 \kappa^* v_s^2 \omega^2}{h_0} + \frac{ik^2 \kappa \tau_\pi v_s^2 \omega^3}{h_0} \right. \\ \left. - \frac{2ik^2 \kappa^* \tau_\pi v_s^2 \omega^3}{h_0} + \frac{4i\eta k^2 \omega^3 \xi_5}{h_0^2} - \frac{2i\eta k^2 \omega^3 \xi_6}{h_0^2} + \frac{ik^2 \kappa \omega^3 \zeta}{h_0^2} - \frac{2ik^2 \kappa^* \omega^3 \zeta}{h_0^2} \right), \quad (\text{A3})$$

which is consistent with G_{LT} in Eq. (17).

The numerator for $\bar{G}_T$ is given by

$$N_T(\omega, k^2) = -h_0 \left(v_s^2 \omega^2 - i\tau_\pi v_s^2 \omega^3 - i\tau_\pi v_s^2 \omega^3 - \frac{i\eta \omega^3}{3h_0} - \frac{i\omega^3 \zeta}{h_0} - \tau_\pi \tau_\pi v_s^2 \omega^4 - \frac{\eta \tau_\pi \omega^4}{3h_0} - \frac{\tau_\pi \omega^4 \zeta}{h_0} - \frac{2i\tau_\pi \omega^5 \xi_5}{h_0} \right. \\ \left. + \frac{i\tau_\pi \omega^5 \xi_6}{h_0} + \frac{2\omega^4 \xi_5}{h_0} - \frac{\omega^4 \xi_6}{h_0} - \frac{\kappa \omega^4}{6h_0} + \frac{\kappa^* \omega^4}{3h_0} + \frac{ik \tau_\pi \omega^5}{6h_0} - \frac{ik^* \tau_\pi \omega^5}{3h_0} + \frac{3i\eta k^2 v_s^2 \omega}{h_0} + \frac{3\eta k^2 \omega^2 \zeta}{h_0^2} \right. \\ \left. + \frac{3\eta k^2 \tau_\pi v_s^2 \omega^2}{h_0} - \frac{2k^2 \omega^2 \xi_5}{h_0} + \frac{\kappa k^2 v_s^2 \omega^2}{2h_0} + \frac{\kappa k^2 \omega^2}{6h_0} - \frac{\kappa^* k^2 v_s^2 \omega^2}{h_0} - \frac{\kappa k^4 v_s^2}{2h_0} - \frac{ik k^2 \tau_\pi v_s^2 \omega^3}{2h_0} - \frac{ik k^2 \tau_\pi \omega^3}{6h_0} \right. \\ \left. + \frac{2ik^2 \tau_\pi \omega^3 \xi_5}{h_0} + \frac{4i\eta k^2 \omega^3 \xi_5}{h_0^2} - \frac{2i\eta k^2 \omega^3 \xi_6}{h_0^2} - \frac{ik k^2 \omega^3 \zeta}{2h_0^2} + \frac{ik^* k^2 \omega^3 \zeta}{h_0^2} + \frac{ik^* k^2 \tau_\pi v_s^2 \omega^3}{h_0} + \frac{ik k^4 \tau_\pi v_s^2 \omega}{2h_0} + \frac{ik k^4 \omega \zeta}{2h_0^2} \right. \\ \left. - \frac{4i\eta k^4 \omega \xi_5}{h_0^2} \right), \quad (\text{A4})$$

which is consistent with G_T in Eq. (43).

APPENDIX B: CORRELATION FUNCTIONS IN TERMS OF THE RESPONSE FUNCTIONS

From Eqs. (6) and (7), we can deduce

$$G_R^{0j,lm}(\omega, k) = \frac{k}{\omega} (G_1(\omega, k^2) (\hat{k}_l \hat{\delta}_{jm} + \hat{k}_m \hat{\delta}_{jl}) + G_L(\omega, k^2) \\ \times \hat{k}_j \hat{k}_l \hat{k}_m + G_{LT}(\omega, k^2) \hat{k}_j \hat{\delta}_{lm}), \quad (\text{B1})$$

$$G_R^{0i,0j}(\omega, k) = \frac{k^2}{\omega^2} (G_1(\omega, k^2) \hat{\delta}_{ij} + G_L(\omega, k^2) \hat{k}_i \hat{k}_j), \quad (\text{B2})$$

$$G_R^{00,ij} = \frac{k^2}{\omega^2} (G_L(\omega, k^2) \hat{k}_i \hat{k}_j + G_{LT}(\omega, k^2) \hat{\delta}_{ij}), \quad (\text{B3})$$

$$\omega^3 G_R^{00,0i}(\omega, k) = k_i k^2 \left(G_L(\omega, k^2) - \frac{\omega^2}{k^2} h_0 \right), \quad (\text{B4})$$

$$\omega^4 G_R^{00,00}(\omega, k) = k^4 \left(G_L(\omega, k^2) - \frac{\omega^2}{k^2} h_0 \right), \quad (\text{B5})$$

$$\tilde{G}_R^{ii,jj}(\omega, k) = 3P + G_L(\omega, k^2) + 4G_{LT}(\omega, k^2) + 4G_T(\omega, k^2), \quad (\text{B6})$$

$$\tilde{G}_R^{xy,xy}(\omega, k_y) = -P + G_1(\omega, k_y^2), \quad (\text{B7})$$

$$\tilde{G}_R^{xy,xy}(\omega, k_z) = -P + G_2(\omega, k_z^2), \quad (\text{B8})$$

$$\tilde{G}_R^{yy,zz}(\omega, k_x) + \tilde{G}_R^{zz,zz}(\omega, k_x) = 2G_T(\omega, k_x^2), \quad (\text{B9})$$

$$G^{00,xx}(\omega, k_z) = \frac{k_z^2}{\omega^2} G_{LT}(\omega, k_z^2), \quad (\text{B10})$$

$$G^{0x,0x}(\omega, k_y) = \frac{k_y^2}{\omega^2} G_1(\omega, k_y^2). \quad (\text{B11})$$

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