

Improved formula for Wigner function and spin polarization in a decoupling relativistic fluid at local thermodynamic equilibrium

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(Received 3 October 2025; accepted 17 August 2026; published 8 September 2026)

We present an upgraded formula for Wigner function and spin polarization of fermions emitted by a relativistic fluid at local thermodynamic equilibrium at the decoupling, which improves the one obtained by F. Becattini, M. Buzzegoli, and A. Palermo [Phys. Lett. B **820**, 136519 (2021)] and by S. Y. F. Liu and Y. Yin [J. High Energy Phys. **07** (2021) 188] that is used in numerical simulations of relativistic nuclear collisions. By using a new expansion method, applicable to decoupling hypersurfaces with arbitrary geometry, we reproduce the known term proportional to thermal vorticity and obtain an upgraded form of the spin-shear term which captures the dependence on the geometry. The new method also includes additional contributions whose physical nature is to be assessed. The new expression also naturally excludes contributions from space-time gradients in the normal direction of the hypersurface, providing a theoretical justification for the isothermal condition previously imposed *a priori*. This framework can be extended to particles with arbitrary spin.

DOI: 10.1103/5rpc-hg99

I. INTRODUCTION

Relativistic heavy-ion collisions provide a unique environment for studying strongly interacting matter, known as quark-gluon plasma (QGP), under extremely high temperature [1–3]. After a short preequilibrium evolution, the system is assumed to achieve local thermodynamic equilibrium (LTE) and its evolution is thereafter effectively described by relativistic hydrodynamics [4–8]. Physical quantities are then characterized by slowly varying macroscopic variables, such as flow velocity, temperature, and chemical potential. The nonequilibrium operator approach and its reformulation [9–13] offer a rigorous framework to obtain physical observables in a relativistic fluid based on quantum statistical field theory at local equilibrium, which is especially useful for spin [13–22], an inherently quantum observable [23–33].

If the QGP achieves LTE on some initial hypersurface Σ_0 , the density operator is obtained by maximizing the entropy $S = -\text{tr}(\hat{\rho} \ln \hat{\rho})$ with constrained energy-momentum and charge densities. With the Belinfante stress-energy operator $\hat{T}^{\mu\nu}$ [34] and the current operator $\hat{j}^\mu$, the density operator

takes the following form [9,12,13]:

$$\hat{\rho} = \frac{1}{Z} \exp \left[- \int_{\Sigma_0} d\Sigma_\mu(y) (\hat{T}^{\mu\nu}(y) \beta_\nu(y) - \hat{j}^\mu(y) \zeta(y)) \right], \quad (1)$$

where β is the four-temperature vector and ζ the reduced chemical potential μ/T . By means of Gauss's theorem, the exponent in the operator (1) can be transformed into the sum of a three-dimensional (3D) integral at the decoupling hypersurface Σ_D , i.e., when the fluid breaks up [35], and a 4D integral over the region encompassed by Σ_D and Σ_0 , which is essentially the region of existence of the QGP [13]:

$$\begin{aligned} & \int_{\Sigma_D} d\Sigma_\mu(y) (\hat{T}^{\mu\nu}(y) \beta_\nu(y) - \hat{j}^\mu(y) \zeta(y)) \\ & + \int_{\Omega} d^4y (\hat{T}^{\mu\nu}(y) \partial_\mu \beta_\nu(y) - \hat{j}^\mu(y) \partial_\mu \zeta(y)). \end{aligned} \quad (2)$$

While the first term is the LTE integral at the decoupling, the second term accounts for the dissipative terms of physical observables. In this work, we will confine ourselves to the calculation of observables, notably spin polarization, at LTE at the decoupling. The dissipative terms are assumed to be small compared to the LTE part, so that we will neglect the second integral in the Eq. (2) and focus on the zeroth-order approximation, i.e., the LTE density operator:

$$\hat{\rho}_{\text{LE}} = \frac{1}{Z} \exp \left[- \int_{\Sigma_D} d\Sigma_\mu(y) (\hat{T}^{\mu\nu}(y) \beta_\nu(y) - \hat{j}^\mu(y) \zeta(y)) \right]. \quad (3)$$

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The mean value of a local operator $\widehat{O}(x)$ on a point $x \in \Sigma_D$ calculated with the density operator (3), e.g.,

$$O_{LE}(x) \equiv \text{Tr}(\widehat{O}(x) \widehat{\rho}_{LE}), \quad (4)$$

should meet a definite requirement: it cannot depend on the values of the β and ζ fields outside the hypersurface Σ_D . The reason is clear: the density operator involves an integration over Σ_D , and, since x itself is a point on Σ_D , the functional form of these fields outside this hypersurface can be varied arbitrarily without affecting the result. This requirement has a remarkable implication: in a gradient expansion of the mean value at LTE, only the components along the tangent directions to Σ_D matter; the component perpendicular to the hypersurface should not appear in the final result.

In previous spin polarization calculations with (3), this requirement was not met, due to significant geometric approximations, tacitly or explicitly introduced [14,36]. For instance, in Ref. [14], for the purpose of calculating the LTE contribution, the region encompassed by the initial and the decoupling hypersurface was approximated with a rectangular region with infinitely distant timelike boundaries, while in Ref. [36] the LTE hypersurface was tacitly assumed to be perpendicular to the four-velocity field, which is not the case in general. In fact, according to analytical models [37–39] and numerical simulations [40–43], the decoupling hypersurface does not meet either case.

In this work we propose a new approach for evaluating quantities at LTE, which applies to hypersurfaces with arbitrary geometry and, as we will see, features the expected independence on the normal components of the gradients; technically, this method relies on an exchange of the order of integrations for the leading order correction of the Wigner function, which sheds light on the nature of the correction itself. Although our derivation is focused on the spin-1/2 case, the method can be readily extended to particles with arbitrary spin.

The paper is organized as follows. In the Sec. II, we introduce the local equilibrium density operator and define the linear response expansion in terms of thermodynamic gradients. In the Sec. III, we introduce the Wigner function for a free Dirac field and propose a new expansion scheme for its linear response expansion. This scheme allows the integration over the decoupling hypersurface to be performed exactly, at any given order, without resorting to geometrical approximations. In the Sec. IV, we apply the resulting formula for the linear response theory of the Wigner function to compute the first-order gradient correction to the spin polarization vector of hadrons. The new formula contains not only gradients of the thermodynamic fields, but also gradients of the normal vector, which are absent in the flat-hypersurface approximation. We finally discuss the physical implications of the new formula.

Notations

Throughout this paper we use the mostly minus signature convention for the flat metric $g^{\mu\nu} = \text{diag}(+, -, -, -)$. We adopt the natural system, $\hbar = k_B = c = 1$. The spacial part of a four-vector k^μ is denoted with boldface, $\mathbf{k}$; the scalar

product is denoted with a dot for both four-vectors $k \cdot p = k_\mu p^\mu$ and three-vectors $\mathbf{k} \cdot \mathbf{p} = k_j p^j$. Einstein index conventions are assumed, and contracted indices are summed all over the possible values, $\mu = 0, 1, 2, 3$; $j = 1, 2, 3$. The Levi-Civita pseudotensor $\epsilon^{\mu\nu\rho\sigma}$ is chosen so that $\epsilon^{0123} = +1$.

II. LOCAL THERMODYNAMIC EQUILIBRIUM

In the calculation of the LTE mean value (4), we can replace the four-temperature $\beta_\nu(y)$ and the reduced chemical potential $\zeta(y)$ in Eq. (3) with

$$\beta_\nu(y) = \beta_\nu(x) + [\beta_\nu(y) - \beta_\nu(x)] \equiv \beta_\nu(x) + \Delta\beta_\nu(y, x),$$

$$\zeta(y) = \zeta(x) + [\zeta(y) - \zeta(x)] \equiv \zeta(x) + \Delta\zeta(y, x),$$

and rewrite (4) as

$$O_{LE}(x) = \frac{1}{Z} \text{Tr} \left(\exp \left[-\beta(x) \cdot \widehat{P} + \zeta(x) \widehat{Q} - \int_{\Sigma_D} d\Sigma_\mu(y) (\widehat{T}^{\mu\nu}(y) \Delta\beta_\nu(y, x) - \widehat{j}^\mu(y) \Delta\zeta(y, x)) \right] \widehat{O}(x) \right), \quad (5)$$

where we have taken out $\beta_\nu(x)$ and $\zeta(x)$ from the integral and used the identity of the four-momentum operator $\widehat{P}^\nu = \int_{\Sigma_D} d\Sigma_\mu(y) \widehat{T}^{\mu\nu}(y)$ and the charge operator $\widehat{Q} = \int_{\Sigma_D} d\Sigma_\mu(y) \widehat{j}^\mu(y)$.

In the hydrodynamic limit, with a clear separation between the microscopic scale, associated with the correlation lengths of the operators, and the scale over which the β and ζ field significantly varies, the first two terms in the exponent of (5) dominate over the last term, which is expected to provide a correction. We can then use linear response theory to obtain a good approximation of the mean value of the operator $\widehat{O}(x)$ at LTE:

$$O_{LE}(x) = O_{GE}(x) + \Delta O_{LE}(x) + \Delta O_{LE}^{(\zeta)}(x),$$

where $O_{GE}(x)$ is the global equilibrium mean value calculated with a four-temperature equal to $\beta(x)$ and a reduced chemical potential equal to $\zeta(x)$:

$$O_{GE}(x) \equiv \langle \widehat{O}(x) \rangle_{GE} = \frac{\text{Tr}[\widehat{O}(x) e^{-\widehat{A}(x)}]}{\text{Tr}[e^{-\widehat{A}(x)}]}, \quad (6)$$

with $\widehat{A}(x) \equiv \beta(x) \cdot \widehat{P} - \zeta(x) \widehat{Q}$. The terms $\Delta O_{LE}(x)$ and $\Delta O_{LE}^{(\zeta)}(x)$ are leading corrections from the linear responses to $\Delta\beta$ and $\Delta\zeta$, respectively:

$$\begin{aligned} \Delta O_{LE}(x) &= - \int_{\Sigma_D} d\Sigma_\mu(y) \int_0^1 dz \Delta\beta_\nu(y, x) \\ &\quad \times \langle \widehat{O}(x), e^{-z\widehat{A}(x)} \widehat{T}^{\mu\nu}(y) e^{z\widehat{A}(x)} \rangle_{c,GE}, \\ \Delta O_{LE}^{(\zeta)}(x) &= \int_{\Sigma_D} d\Sigma_\mu(y) \int_0^1 dz \Delta\zeta(y, x) \\ &\quad \times \langle \widehat{O}(x), e^{-z\widehat{A}(x)} \widehat{j}^\mu(y) e^{z\widehat{A}(x)} \rangle_{c,GE}. \end{aligned} \quad (7)$$

In the above equations, $\langle \cdots \rangle_{c,GE}$ denotes the connected mean value at GE, i.e.,

$$\begin{aligned} \langle \widehat{O}_1(x), \widehat{O}_2(y) \rangle_{c,GE} &= \langle \widehat{O}_1(x) \widehat{O}_2(y) \rangle_{c,GE} \\ &- \langle \widehat{O}_1(x) \rangle_{GE} \langle \widehat{O}_2(y) \rangle_{GE}. \end{aligned}$$

III. WIGNER FUNCTION FOR SPIN-1/2 PARTICLES

A. Wigner operator

The covariant Wigner function $W(x, p)$ is a quantum generalization of the classical phase space distribution [47–50]; all physical observables can be expressed with suitable integrals of the Wigner function. For free spin-1/2 particles, it is defined as the mean value of the Wigner operator:

$$\widehat{W}_{ab}^+(x, p) \equiv \theta(p^0) \int \frac{d^4 y}{(2\pi)^4} e^{-ip \cdot y} \overline{\Psi}_b\left(x + \frac{y}{2}\right) \Psi_a\left(x - \frac{y}{2}\right), \quad (8)$$

where Ψ denotes the Dirac field and a, b are spinor indices. The step function $\theta(p^0)$ ensures that only particle contributions are included (for antiparticles the procedure is similar). The Dirac field can be expanded in terms of plane waves:

$$\begin{aligned} \Psi(x) &= \frac{1}{\sqrt{(2\pi)^3}} \sum_s \int \frac{d^3 p}{2\varepsilon} (u_s(p) \widehat{a}_s(p) e^{-ip \cdot x} \\ &+ v_s(p) \widehat{b}_s(p) e^{ip \cdot x}), \end{aligned} \quad (9)$$

where $\varepsilon = \sqrt{\mathbf{p}^2 + m^2}$ denotes the on-shell energy and $\widehat{a}_s(p_-)$ is the annihilation operator for a particle with momentum p_- (note that $p_{\pm}$ is on-shell) and spin component s , and the u 's are the familiar spinors solution of the free Dirac equation. The creation and annihilation operators obey the covariant anticommutation relation,

$$\{\widehat{a}_s(p), \widehat{a}_{s'}^\dagger(p')\} = 2\varepsilon_p \delta^3(\mathbf{p} - \mathbf{p}') \delta_{ss'},$$

and likewise for antiparticles. Replacing the integral measure in Eq. (9) with

$$\int \frac{d^3 p}{2\varepsilon} \rightarrow \int d^4 p \delta(p^2 - m^2) \theta(p^0),$$

and substituting the Dirac field into (8), the Wigner operator for the particle part turns out to be

$$\begin{aligned} \widehat{W}^+(x, p) &= \frac{1}{(2\pi)^3} \sum_{r,s} \int d^4 p_1 d^4 p_2 \delta(p_1^2 - m^2) \theta(p_1^0) \\ &\times \delta(p_2^2 - m^2) \theta(p_2^0) \delta\left(p - \frac{p_1 + p_2}{2}\right) \\ &\times e^{i(p_1 - p_2) \cdot x} \widehat{a}_r^\dagger(p_1) \widehat{a}_s(p_2) u_s(p_2) \bar{u}_r(p_1). \end{aligned} \quad (10)$$

One can now change the integration variables:

$$\begin{aligned} P &\equiv \frac{p_1 + p_2}{2}, \quad q \equiv p_1 - p_2, \quad \Rightarrow \quad p_1 = P + q/2, \\ p_2 &= P - q/2, \end{aligned} \quad (11)$$

so that

$$d^4 p_1 d^4 p_2 = d^4 P d^4 q$$

and

$$\delta(p_1^2 - m^2) \delta(p_2^2 - m^2) = \frac{1}{2} \delta\left(P^2 + \frac{q^2}{4} - m^2\right) \delta(P \cdot q).$$

Integrating in $d^4 P$ is now straightforward because of the $\delta^4(p - P)$ in Eq. (10), and, after replacing all variables according to (11), one readily gets the following integral expression:

$$\begin{aligned} \widehat{W}^+(x, p) &= \frac{1}{2(2\pi)^3} \sum_{r,s} \int d^4 q e^{iq \cdot x} \theta(p_+^0) \theta(p_-^0) \delta(p \cdot q) \\ &\times \delta\left(p^2 + \frac{q^2}{4} - m^2\right) \widehat{a}_r^\dagger(p_+) \widehat{a}_s(p_-) \\ &\times u_s(p_-) \bar{u}_r(p_+), \end{aligned} \quad (12)$$

where $p_{\pm} \equiv p \pm q/2$ are restricted on the mass shell.

B. Leading corrections to Wigner function

We now focus on the leading corrections to the Wigner function in the linear response theory, which can be derived by replacing $\widehat{O}$ in Eq. (7) by the Wigner operator $\widehat{W}^+(x, p)$. Plugging Eq. (12) into the first line in (7), we obtain

$$\begin{aligned} \Delta W_{LE}^+(x, p) &= -\frac{1}{2(2\pi)^3} \sum_{r,s} \int d^4 q \theta(p_+^0) \theta(p_-^0) e^{iq \cdot x} \delta(p \cdot q) \\ &\times \int_{\Sigma_D} d\Sigma_\mu(y) \int_0^1 dz \Delta\beta_\nu(y, x) \\ &\times \delta\left(p^2 + \frac{q^2}{4} - m^2\right) (\widehat{a}_r^\dagger(p_+) \widehat{a}_s(p_-)) \\ &\times e^{-z\widehat{A}(x)} \widehat{T}^{\mu\nu}(y) e^{z\widehat{A}(x)} \rangle_{c,GE} u_s(p_-) \bar{u}_r(p_+). \end{aligned} \quad (13)$$

Since the four-momentum operator $\widehat{P}^\mu$ is the generator of space-time translation, we have

$$\widehat{T}^{\mu\nu}(y) = e^{i\widehat{P} \cdot y} \widehat{T}^{\mu\nu}(0) e^{-i\widehat{P} \cdot y},$$

and, taking into account that $\widehat{A}(x) = \beta(x) \cdot \widehat{P} - \zeta(x) \widehat{Q}$ commutes with $\widehat{P}^\mu$,

$$\begin{aligned} \langle \widehat{a}_r^\dagger(p_+) \widehat{a}_s(p_-), e^{-z\widehat{A}(x)} \widehat{T}^{\mu\nu}(y) e^{z\widehat{A}(x)} \rangle_{c,GE} \\ = \langle e^{-\widehat{C}(x)} \widehat{a}_r^\dagger(p_+) \widehat{a}_s(p_-) e^{\widehat{C}(x)}, \widehat{T}^{\mu\nu}(0) \rangle_{c,GE}, \end{aligned} \quad (14)$$

where $\widehat{C}(x) = -z[\beta(x) \cdot \widehat{P} - \zeta(x) \widehat{Q}] + iy \cdot \widehat{P}$. Since $\widehat{a}^\dagger(p_{\pm}), \widehat{a}(p_{\pm})$ are the creation/annihilation operators of eigen-momentum states, they transform under translation as

$$\begin{aligned} e^{i\widehat{P} \cdot y} \widehat{a}^\dagger(p) e^{-i\widehat{P} \cdot y} &= e^{ip \cdot y} \widehat{a}^\dagger(p), \\ e^{i\widehat{P} \cdot y} \widehat{a}(p') e^{-i\widehat{P} \cdot y} &= e^{-ip' \cdot y} \widehat{a}(p'), \end{aligned}$$

so that

$$e^{-\widehat{C}(x)} \widehat{a}^\dagger(k_+) \widehat{a}(k_-) e^{\widehat{C}(x)} = e^{z\beta(x) \cdot q - iy \cdot q} \widehat{a}^\dagger(k_+) \widehat{a}(k_-).$$

Plugging Eq. (14) into Eq. (13) and completing the integral over z leads to

$$\begin{aligned} \Delta W_{\text{LE}}^+(x, p) = & \frac{1}{2(2\pi)^3} \sum_{r,s} \int_{\Sigma_D} d\Sigma_\mu(y) \Delta\beta_\nu(y, x) \\ & \times \int d^4q e^{iq \cdot (x-y)} \delta(p \cdot q) \frac{1 - e^{\beta(x) \cdot q}}{\beta(x) \cdot q} \\ & \times \delta\left(p^2 + \frac{q^2}{4} - m^2\right) \theta(p_+^0) \theta(p_-^0) \\ & \times u_s(p_-) \bar{u}_r(p_+) \langle \hat{a}_r^\dagger(p_+) \hat{a}_s(p_-), \hat{T}^{\mu\nu}(0) \rangle_{c,\text{GE}}. \end{aligned} \quad (15)$$

The linear response to $\Delta\zeta$, $\Delta W_{\text{LE}}^{+(\zeta)}$, is obtained by replacing $\Delta\beta_\nu(y, x)$ with $\Delta\zeta(y, x)$ and $\hat{T}^{\mu\nu}(0)$ with $-\hat{j}^\mu(0)$.

C. Integration over the decoupling hypersurface

In Ref. [14], the integration over Σ_D in Eq. (15) was carried out first, introducing a nontrivial geometric assumption, followed by the integration in q . In fact, it is more convenient to carry out the integration in q first because it does not require any special assumption other than the usual hydrodynamic limit. This reversed order of integration eventually does not imply special geometric assumptions, as mentioned in the Introduction, and it generates a nice expression, as we will see.

The formula (15) can be rewritten in a way which makes apparent the effect of the hydrodynamic limit:

$$\Delta W_{\text{LE}}^+(x, p) = \frac{1}{(2\pi)^3} \int d^4q \delta(p \cdot q) G^{\mu\nu}(q) F_{\mu\nu}(q), \quad (16)$$

where

$$\begin{aligned} G^{\mu\nu}(q) = & \frac{1}{2} \delta\left(p^2 + \frac{q^2}{4} - m^2\right) \frac{1 - e^{\beta(x) \cdot q}}{\beta(x) \cdot q} \theta(p_+^0) \theta(p_-^0) \\ & \times \sum_{r,s} u_s(p_-) \bar{u}_r(p_+) \langle \hat{a}_r^\dagger(p_+) \hat{a}_s(p_-), \hat{T}^{\mu\nu}(0) \rangle_{c,\text{GE}} \end{aligned} \quad (17)$$

and

$$F_{\mu\nu}(q) = \int_{\Sigma_D} d\Sigma_\mu(y) e^{-iq \cdot (y-x)} \Delta\beta_\nu(y, x).$$

In the hydrodynamic limit, $\Delta\beta_\nu$ is a slowly varying function in space and time implying that the function $F_{\mu\nu}$, which is a Fourier transform in the variable q integrated in the variable y , is a peaked function about $q^\mu = 0$ (also taking into account that $q \cdot p = 0$). A good approximation of the integral (16) should be then obtained by a small- q expansion of the function $G^{\mu\nu}(q)$ in the integral (16):

$$G^{\mu\nu}(q) = \sum_{N=0}^{\infty} \frac{1}{N!} \left[\partial_{v_1}^q \partial_{v_2}^q \cdots \partial_{v_N}^q G^{\mu\nu}(q) \right] \Big|_{q^\mu=0} q^{v_1} q^{v_2} \cdots q^{v_N}. \quad (18)$$

Indeed, as will become clear later in Eq. (30), this expansion in q about $q = 0$ generates an expansion of the Wigner function

in the gradients of the thermohydrodynamic field β . Because of the presence of the δ and θ distributions in the function (17), the expansion (18) should be interpreted in a distributional sense, i.e., meaningful only upon eventual integration in dk^0 as specified in Eq. (37) later on.

Plugging (18) into (16), the leading order correction to the Wigner function becomes

$$\begin{aligned} \Delta W_{\text{LE}}^+(x, p) = & \frac{1}{(2\pi)^3} \sum_{N=0}^{\infty} \frac{1}{N!} \int_{\Sigma_D} d\Sigma_\mu(y) I_n^{v_1 v_2 \cdots v_N}(y-x) \\ & \times \Delta\beta_\nu(y, x) \left[\partial_{v_1}^q \partial_{v_2}^q \cdots \partial_{v_N}^q G^{\mu\nu}(q) \right] \Big|_{q=0}, \end{aligned} \quad (19)$$

where integrals over q are encoded in the rank- N function

$$\begin{aligned} I_N^{v_1 v_2 \cdots v_N}(y-x) & \equiv \int d^4q \delta(p \cdot q) e^{-iq \cdot (y-x)} q^{v_1} \cdots q^{v_N} \\ & = (2\pi)^3 \frac{(-i)^N}{|p^0|} \partial_x^{v_1} \cdots \partial_x^{v_N} \delta^3 \\ & \times \left(\mathbf{y} - \mathbf{x} - \frac{\mathbf{p}}{p^0} (y^0 - x^0) \right). \end{aligned} \quad (20)$$

Plugging Eq. (20) into (19) and repeatedly using the Leibniz rule (as shown in Appendix A), Eq. (19) can be transformed into

$$\begin{aligned} \Delta W_{\text{LE}}^+(x, p) = & \frac{1}{|p^0|} \sum_{N=0}^{\infty} \frac{(-i)^N}{N!} \left[\partial_{v_1}^q \cdots \partial_{v_N}^q G^{\mu\nu}(q) \right] \Big|_{q=0} \\ & \times \sum_{M=0}^N \frac{N! (-1)^M}{M! (N-M)!} \partial_x^{v_{M+1}} \cdots \partial_x^{v_N} \int_{\Sigma_D} d\Sigma_\mu(y) \\ & \times \left[\partial_x^{v_1} \cdots \partial_x^{v_M} \Delta\beta_\nu(y, x) \right] \\ & \times \delta^3 \left(\mathbf{y} - \mathbf{x} - \frac{\mathbf{p}}{p^0} (y^0 - x^0) \right). \end{aligned} \quad (21)$$

In general, Σ_D can be a topological complex hypersurface, with both spacelike and timelike parts. Notably, for a given spatial position $\mathbf{x}$ there might be multiple x^0 times belonging to Σ_D (see Fig. 1). Nevertheless, the hypersurface can be split into single-valued branches $x^0 = f_k(\mathbf{x})$ and the integration in (21) can be decomposed into the sum of integrals over those branches. For any arbitrary function $\Theta(x, y)$ the integration in y in the Eq. (21) yields (assuming that $p \cdot \sigma \neq 0$)

$$\begin{aligned} & \int_{\Sigma_D} d\Sigma_\mu(y) \delta^3 \left(\mathbf{y} - \mathbf{x} - \frac{\mathbf{p}}{p^0} (y^0 - x^0) \right) \Theta(y, x) \\ & = \sum_k s_k \int_{\Sigma_k} d^3y \sigma_\mu(y) \delta^3 \left(\mathbf{y} - \mathbf{x} - \frac{\mathbf{p}}{p^0} (y^0 - x^0) \right) \Theta(y, x) \\ & = \sum_{k,i} s_k \sigma_\mu(\bar{y}_{k,i}) \frac{|p^0|}{|p \cdot \sigma(\bar{y}_{k,i})|} \Theta(\bar{y}_{k,i}, x), \end{aligned} \quad (22)$$

where σ is the vector perpendicular to the hypersurface:

$$\sigma_\mu(x) = (1, -\partial f_k / \partial \mathbf{x}), \quad (23)$$

and $\bar{y}_{k,i}$ is the i th solution of the equation $\mathbf{y} = \mathbf{x} + (\mathbf{p}/p^0)[f_k(\mathbf{y}) - x^0(\mathbf{x})]$; s_k in Eq. (22) is a sign: $+1$ if σ_μ has the same direction of the outward pointing vector $d\Sigma_\mu$ and -1

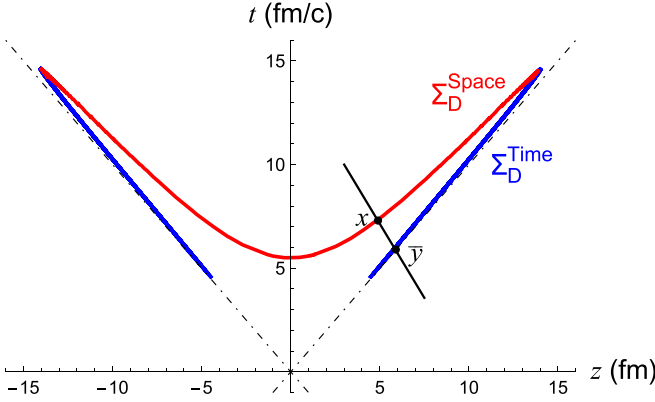


FIG. 1. A slice of the decoupling hypersurface Σ_D (red and blue lines) on the transverse plane $x = y = 0$, obtained by 3 + 1D viscous hydrodynamic simulations with the CLVisc model at $\sqrt{s_{NN}} = 27$ GeV [44–46]. The black solid line represents the world-line of a particle, which intersects Σ_D at two points: one on the spacelike branch and the other one on the timelike branch.

otherwise. For any given $\mathbf{x}$ and x^0 lying on Σ_D , the solutions are found by intersecting the world line of a particle off the mass shell (meaning $p^2 \neq m^2$) moving with velocity $\mathbf{p}/p^0$:

$$\mathbf{y} = \mathbf{x} + \frac{\mathbf{p}}{p^0}(y^0 - x^0), \quad (24)$$

with all branches of the hypersurface Σ_D . This is depicted in Fig. 1; there is at least one solution, which is $y = x$, but there might be more if the topology of the hypersurface makes it possible. We can then replace the sum over k, i in Eq. (22) with the sum over the p -dependent intersections $\bar{y}(x, p)$ for a given x .

Plugging Eq. (22) into Eq. (21), one obtains

$$\begin{aligned} \Delta W_{LE}^+(x, p) &= \sum_{N=0}^{\infty} \sum_{\bar{y}(x, p)} \frac{(-1)^N}{N!} [\partial_{v_1}^q \cdots \partial_{v_N}^q G^{\mu\nu}(q)]|_{q=0} \\ &\times \sum_{M=0}^N \frac{N!(-1)^M}{M!(N-M)!} d_x^{v_{M+1}} \cdots d_x^{v_N} \\ &\times \left[\frac{s_k \sigma_\mu(y)}{|p \cdot \sigma(y)|} \partial_x^{v_1} \cdots \partial_x^{v_M} \Delta \beta_\nu(y, x) \right]_{y=\bar{y}}, \end{aligned} \quad (25)$$

where the sum over $\bar{y}(x, p)$ runs over all the intersections of the particle world line with the hypersurface for a chosen space-time point x , including the trivial solution $y = x$. In Eq. (25) we have introduced the total derivative,

$$d_x^\mu = \frac{d}{dx_\mu},$$

to emphasize the difference between the derivative acting on the function *before* setting $y = \bar{y}(x, p)$ (that is ∂_x) and the derivative acting on the function *after* setting $y = \bar{y}(x, p)$. In general, using the chain rule for an arbitrary function

$g(x, \bar{y}(x, p))$, we have

$$\begin{aligned} \frac{d}{dx^\mu} g(x, \bar{y}(x, p)) &= \frac{\partial}{\partial x^\mu} g(x, \bar{y}(x, p)) \\ &+ \frac{\partial \bar{y}^\nu(x, p)}{\partial x^\mu} \frac{\partial g(x, y)}{\partial y^\nu} \bigg|_{y=\bar{y}(x, p)}. \end{aligned} \quad (26)$$

It can be shown (see Appendix A for the detailed calculation) that

$$\frac{\partial \bar{y}^\nu(x, p)}{\partial x^\mu} = \delta_\mu^\nu - \frac{p^\nu \sigma_\mu(\bar{y})}{p \cdot \sigma(\bar{y})} \equiv \Delta_\mu^\nu(\bar{y}). \quad (27)$$

Since $n_\mu = s_k \sigma_\mu / \sqrt{|\sigma \cdot \sigma|}$ is the outward pointing unit vector normal to the hypersurface Σ_D , the tensor $\Delta^{\nu\rho}$ can be also expressed as

$$\Delta^{\nu\rho}(\bar{y}) = g^{\nu\rho} - \frac{n^\nu(\bar{y})p^\rho}{p \cdot n(\bar{y})}. \quad (28)$$

Hence, by using (26) and the above definitions, Eq. (25) can be rewritten as

$$\begin{aligned} \Delta W_{LE}^+(x, p) &= \sum_{N=0}^{\infty} \sum_{\bar{y}(x, p)} \frac{(-1)^N}{N!} [\partial_{v_1}^q \cdots \partial_{v_N}^q G^{\mu\nu}(q)]|_{q=0} \\ &\times \sum_{M=0}^N \frac{N!(-1)^M}{M!(N-M)!} \\ &\times [\partial_x^{v_{M+1}} + \Delta^{\alpha_{M+1}\rho_{M+1}}(\bar{y}) \partial_{\rho_{M+1}}^y] \\ &\cdots \times [\partial_x^{v_N} + \Delta^{\nu_N \rho_N}(\bar{y}) \partial_{\rho_N}^y] \partial_x^{v_1} \\ &\cdots \partial_x^{v_M} \frac{n_\mu(y)}{|p \cdot n(y)|} \Delta \beta_\nu(y, x) \bigg|_{y=\bar{y}}. \end{aligned} \quad (29)$$

Finally, using again the Binomial theorem, i.e.,

$$\begin{aligned} D^{(N)} &= (-\partial_x + \partial_x + D)^{(N)} \\ &= \sum_{M=0}^N \binom{N}{M} (-\partial_x)^{(M)} (\partial_x + D)^{(N-M)}, \end{aligned}$$

Eq. (29) can be recast into a compact form:

$$\begin{aligned} \Delta W_{LE}^+(x, p) &= \sum_{N=0}^{\infty} \sum_{\bar{y}(x, p)} \frac{1}{N!} D_y^N(\bar{y}) \\ &\times \left[G^{\mu\nu}(q) \frac{n_\mu(y)}{|p \cdot n(y)|} \Delta \beta_\nu(y, x) \right]_{q=0, y=\bar{y}(x, p)}. \end{aligned} \quad (30)$$

The operator D_y in Eq. (30) is defined as

$$D_y(\bar{y}) \equiv -i \Delta^{\nu\rho}(\bar{y}) \partial_\rho^y \partial_\nu^q \quad (31)$$

and the derivatives do not act on the operator in (28), which is evaluated over the intersections.

At any given N , the right-hand side of Eq. (30) contains derivatives of N th order in space and momentum q , so the q expansion is equivalent to a gradient expansion. The formula includes gradients of β_ν as well as gradients of the normal vector n to the hypersurface; the latter are obviously absent

if the hypersurface is a hyperplane, as is tacitly assumed in many calculations in thermal field theory, and yet for a curved hypersurface their contribution may not be negligible. Note that the N th term of the series contains mixed terms made by combinations of the derivative of the thermodynamic fields and the curvature term.

A crucial feature of Eq. (31) is the absence of the normal component of the spacial gradients. By decomposing the gradient into a normal component along n and a tangential component to the hypersurface perpendicular to n ,

$$\partial_\rho^y = n_\rho(n \cdot \partial^y) + \partial_{T\rho}^y \equiv n_\rho D + \partial_{T\rho}^y, \quad (32)$$

it turns out that only the tangential component survives:

$$\begin{aligned} D_y(\bar{y}) &= -i\Delta^{\nu\rho}(\bar{y})\partial_\rho^y\partial_v^q = \left(g^{\nu\rho} - \frac{n^\nu(\bar{y})p^\rho}{p \cdot n(\bar{y})}\right)(n_\rho D + \partial_{T\rho}^y)\partial_v^q \\ &= -i\Delta^{\nu\rho}(\bar{y})\partial_{T\rho}^y\partial_v^q. \end{aligned}$$

This result is precisely what one should have expected in view of the general conclusion about the independence of the LTE mean values of the thermodynamic fields outside the hypersurface Σ_D that was discussed in the Introduction.

The calculation of the correction $\Delta W_{LE}^{+(\zeta)}$ follows the same steps as the previous one, and one eventually gets

$$\begin{aligned} \Delta W_{LE}^{+(\zeta)}(x, p) &= \sum_{N=0}^{\infty} \sum_{\bar{y}(x,p)} \frac{1}{N!} D_y^N(\bar{y}) \\ &\times \left[H^\mu(q) \frac{n_\mu(y)}{|p \cdot n(y)|} \Delta\zeta(y, x) \right] \Big|_{q=0, y=\bar{y}(x,p)}, \end{aligned} \quad (33)$$

where

$$\begin{aligned} H^\mu(q) &= -\frac{1}{2}\delta\left(p^2 + \frac{q^2}{4} - m^2\right) \frac{1 - e^{\beta(x) \cdot q}}{\beta(x) \cdot q} \theta(p_+^0) \theta(p_-^0) \\ &\times \sum_{r,s} u_s(p_-) \bar{u}_r(p_+) \langle \hat{a}_r^\dagger(p_+) \hat{a}_s(p_-), \hat{j}^\mu(0) \rangle_{c,GE}. \end{aligned} \quad (34)$$

D. Discussion

It is somewhat surprising that the Wigner function at the point x receives contributions from the gradients of the fields β and ζ evaluated at points $\bar{y}(x, p)$, which can be relatively distant from x . These points lie on the world line (24), but this by no means implies that there is an actual particle traveling from $\bar{y}(x, p)$ to x ; its possible physical origin is discussed toward the end of the paper. The mathematical origin of this long-distance contribution can be understood by rewriting Eq. (15) as

$$\begin{aligned} \Delta W_{LE}^+(x, p) &= \frac{1}{(2\pi)^3} \int_{\Sigma_D} d\Sigma_\mu(y) \Delta\beta_\nu(y, x) \\ &\times \left[\int d^4q \delta(p \cdot q) e^{iq \cdot (x-y)} G^{\mu\nu}(q) \right]. \end{aligned} \quad (35)$$

In order for $y = x$ to be the only dominant contribution of the integral (35), the function of $y - x$ between square brackets

should be peaked around zero, but because of the presence of $\delta(p \cdot q)$ this function is constant through the whole world line (24), so that points other than x may give a large contribution to the integral.

An important question is about the number of terms that should be computed in the series (30) and (33) in order to obtain a satisfactory approximation. The speed of convergence of these series apparently depends on the ratio between two lengths; first, the correlation length l_c associated with the function $G^{\mu\nu}(H^\mu)$, i.e., $|\partial_q G^{\mu\nu}(q)|/|G^{\mu\nu}(q)|$ $[|\partial_q H^\mu(q)|/|H^\mu(q)|]$, which depends on the microscopic scales of the quantum field and the inverse temperature; second, the characteristic variation lengths of the thermohydrodynamic field β , $L_H = |\beta|/|\partial\beta|$ ($|\zeta|/|\partial\zeta|$), and that of the normal vector n , $L_G = |n|/|\partial n|$, which is a purely geometric quantity. So, the series will rapidly converge if

$$l_c/L_H \ll 1, \quad l_c/L_G \ll 1. \quad (36)$$

Such conditions are believed to be generally satisfied in high-energy heavy-ion collisions, yet the second one has not been carefully examined thus far. Here we provide a simple quantitative estimate for a hypersurface with constant proper time τ . The normal vector is approximated as $n^\mu = (\cosh \xi, 0, 0, \sinh \xi)$, with ξ being the space-time rapidity. The typical scale of its curvature is $L_G \sim \tau \cosh^{-1} \xi$. For most central Au + Au collisions at $\sqrt{s_{NN}} = 200$ GeV, $\tau \sim 10$ fm/c [51], corresponding to $6.5 \leq L_G \leq 10$ fm within the interval $|\xi| < 1$, which is much larger than the typical microscopic scale $l_c \sim 1/T \sim 1.2$ fm when $T = 160$ MeV. In the peripheral region with a larger ξ , the hypersurface becomes more curved with a larger l_c/L_G , and it is not obvious whether the contribution of derivatives of the normal vector at all orders with fixed order in the derivatives of $\Delta\beta$ may be important.

IV. SPIN POLARIZATION

With our formula (30) one can in principle calculate gradient corrections at LTE to all measurable quantities, notably particle spectra and spin polarization. We focus on the spin polarization of spin-1/2 fermions, which is one of the most studied cases and also plays a crucial role for studying vector meson's spin alignment [31–33], hyperon's spin correlation [52–54], and hypernuclei's spin polarization [55,56].

With the Wigner function, one can readily calculate the mean spin polarization using the formula [16]:

$$S^\mu(p) = \frac{1}{2} \frac{\int d^4p^0 \int_{\Sigma_D} d\Sigma \cdot p \operatorname{tr}[\gamma^\mu \gamma^5 W^+(x, p)]}{\int d^4p^0 \int_{\Sigma_D} d\Sigma \cdot p \operatorname{tr}[W^+(x, p)]}, \quad (37)$$

where the leading contribution to the denominator arises from W_{GE}^+ . With the definition (6), one can calculate the following mean value:

$$\langle \hat{a}_r^\dagger(p) a_s(p') \rangle_{GE} = 2\varepsilon \delta^3(\mathbf{p} - \mathbf{p}') n_F(x, p),$$

where $\varepsilon = \sqrt{\mathbf{p}^2 + m^2}$ is the on-shell energy and $n_F(x, p) = [1 + e^{\beta(x) \cdot p - \zeta}]^{-1}$ denotes the Fermi-Dirac distribution. Plugging this equation into the definition of the Wigner operator for the particles, Eq. (12), the following expression is

obtained:

$$W_{\text{GE}}^+(x, p) = \frac{1}{(2\pi)^3} \delta(p^2 - m^2) \theta(p^0) (\not{p} + m) n_F(x, p), \quad (38)$$

whose trace (in the four-dimensional spinor space) reads

$$\text{tr}(W_{\text{GE}}^+(x, p)) = \frac{4m}{(2\pi)^3} \delta(p^2 - m^2) \theta(p^0) n_F(x, p). \quad (39)$$

On the other hand, since $\text{tr}(\gamma^\mu \gamma^5 W_{\text{GE}}^+(x, p)) = 0$, the leading contribution to the numerator in Eq. (37) arises from ΔW_{LE}^+ .

For a free Dirac field, the Belinfante symmetric energy-momentum tensor and the current operator are given by

$$\begin{aligned} \hat{T}^{\mu\nu}(x) &= \frac{i}{4} \bar{\Psi}(x) [\gamma^\mu (\overrightarrow{\partial}^\nu - \overleftarrow{\partial}^\nu) + \gamma^\nu (\overrightarrow{\partial}^\mu - \overleftarrow{\partial}^\mu)] \Psi(x), \\ \hat{j}^\mu(x) &= \bar{\Psi}(x) \gamma^\mu \Psi(x). \end{aligned} \quad (40)$$

Using the quantized field operator in Eq. (9), the operators $\hat{T}^{\mu\nu}(0)$ and $\hat{j}^\mu(0)$ turn out to be

$$\begin{aligned} \hat{T}^{\mu\nu}(0) &= \frac{1}{2(2\pi)^3} \sum_{r,s} \int d^4 p_+ d^4 p_- \delta(p_+^2 - m^2) \delta(p_-^2 - m^2) \bar{u}_s(p_-) (\gamma^\mu p_-^\nu + \gamma^\nu p_-^\mu) u_r(p_+) \hat{a}_s^\dagger(p_-) \hat{a}_r(p_+) \theta(p_+^0) \theta(p_-^0), \\ \hat{j}^\mu(0) &= \frac{1}{(2\pi)^3} \sum_{r,s} \int d^4 p_+ d^4 p_- \delta(p_+^2 - m^2) \delta(p_-^2 - m^2) \bar{u}_s(p_-) \gamma^\mu u_r(p_+) \hat{a}_s^\dagger(p_-) \hat{a}_r(p_+) \theta(p_+^0) \theta(p_-^0), \end{aligned} \quad (41)$$

for the particle terms only; the other terms, i.e., those involving antiparticles and the mixed ones do not contribute to the connected mean values:

$$\begin{aligned} \langle \hat{a}_r^\dagger(p_+) \hat{a}_s(p_-), \hat{T}^{\mu\nu}(0) \rangle_{c,\text{GE}} &= \frac{1}{2(2\pi)^3} \bar{u}_s(p_-) (\gamma^\mu p_-^\nu + \gamma^\nu p_-^\mu) u_r(p_+) n_F(x, p_+) [1 - n_F(x, p_-)], \\ \langle \hat{a}_r^\dagger(p_+) \hat{a}_s(p_-), \hat{j}^\mu(0) \rangle_{c,\text{GE}} &= \frac{1}{(2\pi)^3} \bar{u}_s(p_-) \gamma^\mu u_r(p_+) n_F(x, p_+) [1 - n_F(x, p_-)], \end{aligned} \quad (42)$$

where $p = (p_+ + p_-)/2$ and we have used

$$\langle \hat{a}_r^\dagger(p_+) \hat{a}_s(p_-), \hat{a}_{s'}^\dagger(p'_-) \hat{a}_{r'}(p'_+) \rangle_{c,\text{GE}} = 4\varepsilon_+ \varepsilon_- \delta^3(\mathbf{p}_- - \mathbf{p}'_-) \delta^3(\mathbf{p}_+ - \mathbf{p}'_+) \delta_{rr'} \delta_{ss'} n_F(x, p_+) [1 - n_F(x, p_-)],$$

with $\varepsilon_\pm = \sqrt{m^2 + \mathbf{p}_\pm^2}$ being the on-shell energies. Substituting Eq. (42) into (17) and (34) and using the relation

$$(1 - e^{\beta \cdot q}) n_F(x, p_+) (1 - n_F(x, p_-)) = n_F(x, p_+) - n_F(x, p_-),$$

one obtains

$$\begin{aligned} G^{\mu\nu}(q) &= \frac{n_F(x, p_+) - n_F(x, p_-)}{4(2\pi)^3 (\beta \cdot q)} \delta\left(p^2 + \frac{q^2}{4} - m^2\right) \theta(p_+^0) \theta(p_-^0) (\not{p}_- + m) (\gamma^\mu p_-^\nu + \gamma^\nu p_-^\mu) (\not{p}_+ + m), \\ H^\mu(q) &= - \frac{n_F(x, p_+) - n_F(x, p_-)}{2(2\pi)^3 (\beta \cdot q)} \delta\left(p^2 + \frac{q^2}{4} - m^2\right) \theta(p_+^0) \theta(p_-^0) (\not{p}_- + m) \gamma^\mu (\not{p}_+ + m). \end{aligned} \quad (43)$$

In order to calculate the spin polarization, we need to evaluate the axial-vector component of the Wigner function $\mathcal{A}^\mu(x, p) = \text{tr}(\gamma^\mu \gamma^5 W^+(x, k))$. Making use of the trace

$$\text{tr}[\gamma^\mu \gamma^5 (\not{p}_- + m) \gamma^\alpha (\not{p}_+ + m)] = -4i \epsilon^{\mu\alpha\lambda\tau} p_\lambda q_\tau$$

and Eqs. (43), (30), and (33), one obtains

$$\begin{aligned} \mathcal{A}_{\text{LE}}^\mu(x, p) + \mathcal{A}_{\text{LE}}^{\mu(\zeta)}(x, p) &= -i \sum_{N=0}^{\infty} \sum_{\bar{y}(x,p)} \frac{1}{N!} D_y^N \left\{ \frac{n_F(x, p_+) - n_F(x, p_-)}{(2\pi)^3 (\beta \cdot q)} \delta\left(p^2 + \frac{q^2}{4} - m^2\right) \theta(p_+^0) \theta(p_-^0) \right. \\ &\quad \times \epsilon^{\mu\rho\lambda\tau} p_\lambda q_\tau \frac{n_v(y)}{|p \cdot n(y)|} [(g_\rho^\nu p^\alpha + g_\rho^\alpha p^\nu) \Delta\beta_\alpha(y, x) - 2g_\rho^\nu \Delta\zeta(y, x)] \Big\} \Big|_{q=0, y=\bar{y}(x,p)}. \end{aligned} \quad (44)$$

As the function (in fact a distribution) to be derived in q is odd in that variable, only odd terms in the series contribute. The leading order contribution arises from the term of $N = 1$. After taking a derivative with respect to q , in $q = 0$, this reads

$$\begin{aligned} \mathcal{A}_{\text{LE}}^\mu(x, p) + \mathcal{A}_{\text{LE}}^{\mu(\zeta)}(x, p) &\simeq \frac{2}{(2\pi)^3} \epsilon^{\mu\rho\lambda\tau} p_\lambda n_F(x, p) [1 - n_F(x, p)] \delta(p^2 - m^2) \sum_{\bar{y}(x,p)} \Delta_{\tau\kappa}(\bar{y}) \\ &\quad \times \left\{ \frac{1}{2} (g_\rho^\nu p^\alpha + g_\rho^\alpha p^\nu) \left[\partial_y^\kappa \frac{n_v(y)}{|p \cdot n(y)|} \Delta\beta_\alpha(y, x) \right] - g_\rho^\nu \left[\partial_y^\kappa \frac{n_v(y)}{|p \cdot n(y)|} \Delta\zeta(y, x) \right] \right\} \Big|_{y=\bar{y}(x,p)}. \end{aligned} \quad (45)$$

The derivative with respect to y acts on $\Delta\beta_\alpha(y, x)$, $\Delta\zeta(y, x)$, and $n_v(y)$. However (see Appendix B) the curvature of Σ_D , which relates to the derivative of $n_v(y)$, does not contribute to (45) due to the presence of the antisymmetric Levi-Civita tensor. Plugging (39) and (45) into (37), the final expression of the spin polarization can be derived (see Appendix B for the details) at the leading order, which reads

$$S^\mu(p) \simeq -\frac{1}{8mN_p} \int_{\Sigma_D} d\Sigma(x) \cdot p n_F(1 - n_F) \epsilon^{\mu\nu\rho\lambda} p_\nu \times \sum_{\bar{y}(x,p)} \text{sgn}[p \cdot n(\bar{y})] \left\{ \varpi_{\rho\lambda}(\bar{y}) + \frac{2n_\rho(\bar{y})}{p \cdot n(\bar{y})} \times [\xi_{\lambda\alpha}(\bar{y})p^\alpha - \partial_\lambda^\zeta \zeta(\bar{y})] \right\}. \quad (46)$$

where $N_p = \int_{\Sigma_D} d\Sigma \cdot p \theta(p^0) n_F(x, p)$ is the total particle number. The three terms in Eq. (46) can be identified as contributions of the thermal vorticity $\varpi^{\mu\nu} \equiv (\partial^\nu \beta^\mu - \partial^\mu \beta^\nu)/2$, the thermal shear tensor $\xi^{\mu\nu} \equiv (\partial^\nu \beta^\mu + \partial^\mu \beta^\nu)/2$, and the spin Hall effect [57] proportional to $\partial\zeta$, respectively, while the curvature of Σ_D does not contribute at this order.

Equation (46) improves upon the formulas known in the literature in a twofold manner. First, it gives a new expression for the shear-induced polarization. In contrast to previous derivations, the factor $\hat{t}^\mu/p \cdot \hat{t}$ in Refs. [14,17] or $u^\mu/p \cdot u$ in Refs. [36,58] (where $\hat{t}^\mu$ is the unit time vector in the QGP frame and u^μ is the fluid velocity), obtained as the result of significant geometric approximations, is replaced by $n^\mu/|p \cdot n|$. The new formula comes down to that in Ref. [17] if Σ_D is a hyperplane, and to that in Ref. [36] if the velocity field is normal to Σ_D , hence it reproduces previously obtained theoretical results under specific geometric assumptions. This result solves the long-standing issue of the ambiguity of shear-induced polarization formula, which was addressed in several papers [18,59]. Remarkably, in Eq. (46), the vorticity and the shear induced terms both receive contributions from the derivative of the β field along n_μ , but in the combination these contributions cancel out according to the general features of the new formula (30). Indeed, by using Eq. (32),

$$\begin{aligned} & \epsilon^{\mu\nu\rho\lambda} p_\nu \left[\varpi_{\rho\lambda} + \frac{2n_\rho \xi_{\lambda\alpha} p^\alpha}{p \cdot n} \right] \\ &= \epsilon^{\mu\nu\rho\lambda} p_\nu \left[\frac{1}{2} \partial_\lambda \beta_\rho - \frac{1}{2} \partial_\rho \beta_\lambda + \frac{n_\rho (\partial_\lambda \beta_\alpha + \partial_\alpha \beta_\lambda) p^\alpha}{p \cdot n} \right] \\ &= \epsilon^{\mu\nu\rho\lambda} p_\nu \left[-\partial_\rho \beta_\lambda + \frac{n_\rho (\partial_\lambda \beta_\alpha + \partial_\alpha \beta_\lambda) p^\alpha}{p \cdot n} \right] \\ &= \epsilon^{\mu\nu\rho\lambda} p_\nu \left[-n_\rho D\beta_\lambda - \partial_{T\rho} \beta_\lambda \right. \\ & \quad \left. + \frac{n_\rho (n_\lambda D\beta_\alpha + n_\alpha D\beta_\lambda + \partial_{T\lambda} \beta_\alpha + \partial_{T\alpha} \beta_\lambda) p^\alpha}{p \cdot n} \right] \\ &= \epsilon^{\mu\nu\rho\lambda} p_\nu \left[-n_\rho D\beta_\lambda - \partial_{T\rho} \beta_\lambda + n_\rho D\beta_\lambda \right] \end{aligned}$$

$$+ \frac{n_\rho (\partial_{T\lambda} \beta_\alpha + \partial_{T\alpha} \beta_\lambda) p^\alpha}{p \cdot n} \Big] \\ = \epsilon^{\mu\nu\rho\lambda} p_\nu \left[-\partial_{T\rho} \beta_\lambda + \frac{n_\rho (\partial_{T\lambda} \beta_\alpha + \partial_{T\alpha} \beta_\lambda) p^\alpha}{p \cdot n} \right],$$

where we have used the antisymmetric properties of the Levi-Civita symbol. The last expression apparently shows that the precise combination of thermal vorticity and thermal shear in Eq. (46) implies the cancellation of the derivatives along the direction n_ρ perpendicular to the hypersurface. This occurs only for a definite value of the coefficient of thermal-shear-induced polarization in Eq. (46); if the coefficient multiplying ξ were different from 2, no cancellation would take place. Particularly, when the decoupling hypersurface is isothermal and the direction of the normal vector n_μ coincides with $\partial_\mu T$, derivatives of the temperature do not contribute. Therefore, in this upgraded formalism, the absence of temperature gradients [17,60,61] arises naturally. Second, the thermal vorticity contribution acquires a modification by the sign function $\text{sgn}(p \cdot n)$, which is apparently necessary when Σ_D is not entirely spacelike and future oriented.

The formula (46) features additional contributions from the nontrivial solution of Eq. (24) with $\bar{y}(x, p) \neq x$. They are geometrically related to particles moving inward across the hypersurface and traversing the fluid region [62–65]. This appears unphysical and it may be, in our formalism, a spurious effect generated by not having used interacting fields in the Wigner operator when dealing with in-plasma region in the formula (8). Such additional terms need careful studies in the future but here we propose that they can be discarded as follows: for the classical Cooper-Frye formula, the usual way to drop the inward-moving particles is to introduce a cutoff $\theta(p \cdot n)$ [63–66]. Likewise, introducing $\theta(p \cdot n(x))\theta(p \cdot n(\bar{y}))$ in Eq. (46) will remove the additional intersections in Eq. (46).

It should also be stressed that the formula (46) only contains the leading order derivatives of the normal vector. In fact, the full formula at the leading order in the gradients of β and ζ must include all odd-order derivatives of the geometry-dependent factor. In this improved approximation, Eq. (44) yields

$$\begin{aligned} & \mathcal{A}_{\text{LE}}^\mu(x, p) + \mathcal{A}_{\text{LE}}^{\mu(\zeta)}(x, p) \\ & \simeq -i\epsilon^{\mu\rho\lambda\gamma} p_\lambda \sum_{\bar{y}(x,p)} [(g_\rho^\nu p^\alpha + g_\rho^\alpha p^\nu) \partial_\tau \beta_\alpha(y) - 2g_\rho^\nu \partial_\tau \zeta(y)] \\ & \times \sum_{K=0}^{\infty} \frac{1}{K!} \Delta^{\sigma\tau} \Delta^{\sigma_1\tau_1} \dots \Delta^{\sigma_K\tau_K} \partial_\sigma^q \partial_{\sigma_1}^q \dots \partial_{\sigma_K}^q \\ & \times \left[\frac{n_F(x, p_+) - n_F(x, p_-)}{(2\pi)^3 (\beta \cdot q)} \delta\left(p^2 + \frac{q^2}{4} - m^2\right) \theta(p_+^0) \right. \\ & \times \theta(p_-^0) q_\gamma \Big] \Bigg|_{q=0} \frac{\partial_{\tau_1}^y \dots \partial_{\tau_K}^y \left[\frac{n_v(y)}{|p \cdot n(y)|} \right]}{\Big|_{y=\bar{y}(x,p)}}. \quad (47) \end{aligned}$$

The leading correction to the formula (46) arises from the $K=3$ term in Eq. (47). At this order, the spin polarization receives contributions from $(\partial_\mu \partial_\nu n_\alpha)(\partial_\lambda \beta_\kappa)$, $(\partial_\mu n_\alpha)(\partial_\nu n_\rho)(\partial_\lambda \beta_\kappa)$, $(\partial_\mu \partial_\nu n_\alpha)(\partial_\lambda \zeta)$, and $(\partial_\mu n_\alpha)(\partial_\nu n_\rho)(\partial_\lambda \zeta)$.

Explicit expressions for these effects are lengthy and they are not shown in this paper. Their impacts on global and local spin polarization in heavy-ion collisions will be studied in future works.

V. CONCLUSIONS

In conclusion, we have developed a new method to calculate the Wigner function at LTE on the decoupling hypersurface in heavy-ion collisions. By inverting the momentum and the hypersurface integrals, we have been able to remove specific assumptions on the geometry of the hypersurface and to include in the gradient expansion those of the normal vector to the hypersurface. The space-time derivatives in the normal direction of the hypersurface are naturally excluded, which has a remarkable implication in the vanishing contributions from the temperature gradient if the hypersurface is

isothermal. We have derived upgraded expressions of the spin polarization of spin-1/2 fermions, which merits verifications in numerical simulations.

ACKNOWLEDGMENTS

The authors thank G.-L. Ma and Q. Wang for insightful discussions. This work is supported in part by the Italian Ministry of University and Research, Project No. PRIN2022, “Advanced probes of the Quark Gluon Plasma,” Next Generation EU, Mission 4 Component 1.

DATA AVAILABILITY

The data that support the findings of this article are openly available [67].

APPENDIX A: COMPACT FORM OF WIGNER FUNCTION

Plugging Eq. (20) into Eq. (19), we obtain

$$\Delta W_{\text{LE}}^+(x, p) = \frac{1}{|p^0|} \sum_{N=0}^{\infty} \frac{(-i)^N}{N!} [\partial_{v_1}^q \partial_{v_2}^q \cdots \partial_{v_N}^q G^{\mu\nu}(q)]|_{q=0} \int_{\Sigma_D} d\Sigma_\mu(y) \Delta\beta_\nu(y, x) \left[\partial_x^{v_1} \partial_x^{v_2} \cdots \partial_x^{v_N} \delta^3\left(\mathbf{y} - \mathbf{x} - \frac{\mathbf{p}}{p^0}(y^0 - x^0)\right) \right]. \quad (\text{A1})$$

The presence of the partial derivatives of the δ -function makes the integral difficult to evaluate. It is convenient to convert the above equation to the form in (21) through the following steps. First, we split the partial derivative into the sum of two different operators,

$$\partial_x = \partial_x^{(1)} + \partial_x^{(2)},$$

acting only on the δ function and $\Delta\beta_\nu(y, x)$, respectively. Thereby, the partial derivatives in Eq. (A1) can be recast as $[\partial^q \cdot \partial_x^{(1)}]^N$ and taken out of the integral. The latter can be expressed as follows, by means of the binomial theorem:

$$[\partial^q \cdot \partial_x^{(1)}]^N = [\partial^q \cdot (\partial_x^{(1)} + \partial_x^{(2)}) - \partial^q \cdot \partial_x^{(2)}]^N = \sum_{M=0}^N \frac{N!(-1)^M}{M!(N-M)!} [\partial^q \cdot \partial_x]^{N-M} [\partial^q \cdot \partial_x^{(2)}]^M.$$

The derivative $\partial_x^{(2)}$ only acts on the $\Delta\beta$ function and can be moved into the integral, yielding the expression in the Eq. (21).

In the Eq. (21), the space-time point y is located on Σ_D . As described in the main text, Σ_D can be split into several branches, where in each the decoupling time y^0 is a single-valued function of $\mathbf{y}$, i.e., $y^0 = f_k(\mathbf{y})$. Then Eq. (22) follows because, on each branch,

$$\delta^3\left(\mathbf{y} - \mathbf{x} - \frac{\mathbf{p}}{p^0}(y^0 - x^0)\right) = \sum_{k,i} \frac{|p^0|}{|p \cdot \sigma(\bar{\mathbf{y}}_{k,i})|} \delta^3(\mathbf{y} - \bar{\mathbf{y}}_{k,i}), \quad (\text{A2})$$

where $\bar{\mathbf{y}}_{k,i}^\mu(x)$ denotes the i th crossing point of the world line (24) with the k th branch of Σ_D . The prefactor on the right-hand side is the inverse of the determinant of the matrix:

$$\frac{\partial \bar{y}^j}{\partial y^i} \left[\mathbf{y} - \mathbf{x} - \frac{\mathbf{p}}{p^0}(f_k(\mathbf{y}) - x^0) \right]^i = \delta_j^i - \frac{p^j}{p^0} \frac{\partial f_k(\mathbf{y})}{\partial y^i} = \delta_j^i - \frac{p^j}{p^0} \sigma^j = \frac{p^0 \delta_j^i - p^j \sigma^j}{p^0}.$$

Carrying out the integration in d^3y , and using the relations (22), Eq. (21) is converted to Eq. (25).

In Eq. (26), the derivative of $\bar{y}(x, p)$ with respect to x is obtained by taking into account that $\bar{y}$ locates on the world line:

$$\bar{\mathbf{y}} = \mathbf{x} + \frac{\mathbf{p}}{p^0}(\bar{y}^0 - x^0). \quad (\text{A3})$$

Taking partial derivatives with respect to x^μ of the above equation,

$$\frac{\partial \bar{y}^j}{\partial x^\mu} - \frac{p^j}{p^0} \frac{\partial \bar{y}^0}{\partial \bar{y}^l} \frac{\partial \bar{y}^l}{\partial x^\mu} = \left[\delta_l^j + \frac{p^j}{p^0} \sigma_l(\bar{\mathbf{y}}) \right] \frac{\partial \bar{y}^l}{\partial x^\mu} = \delta_\mu^j - \frac{p^j}{p^0} \delta_\mu^0, \quad (\text{A4})$$

where $j, l = 1, 2, 3$ and where $\sigma_\mu(\bar{y})$ is the normal vector of Σ_D at the space-time point $\bar{y}$. For a point $\bar{y} = \bar{y}_{k,i}$, which denotes the i th solution of $\bar{y} = \mathbf{x} + \frac{\mathbf{p}}{p^0}(\bar{y}^0 - x^0)$ on the k th branch of the hypersurface, we obtain

$$\sigma_\mu(\bar{y}_{k,i}) = \left(1, -\frac{\partial f_k(\bar{y}_{k,i})}{\partial \bar{y}_{k,i}}\right). \quad (\text{A5})$$

The 3×3 matrix

$$A_l^j = \left[\delta_l^j + \frac{p^j}{p^0} \sigma_l(\bar{y}_{k,i}) \right]$$

can be inverted, and one obtains

$$\frac{\partial \bar{y}_{k,i}^j}{\partial x^\mu} = A^{-1j}_l \left(\delta_\mu^l - \frac{p^l}{p^0} \delta_\mu^0 \right) = \left[\delta_\mu^j - \frac{p^j \sigma_l(\bar{y}_{k,i})}{p \cdot \sigma(\bar{y}_{k,i})} \right] \left(\delta_\mu^l - \frac{p^l}{p^0} g_\mu^0 \right).$$

If $\mu = 0$ we then have

$$\frac{\partial \bar{y}_{k,i}^j}{\partial x^0} = -\frac{p^j}{p \cdot \sigma(\bar{y}_{k,i})},$$

while if $\mu = m$ spacial index

$$\frac{\partial \bar{y}_{k,i}^j}{\partial x^m} = \delta_m^j - \frac{p^j \sigma_m(\bar{y}_{k,i})}{p \cdot \sigma(\bar{y}_{k,i})},$$

so that, altogether,

$$\frac{\partial \bar{y}_{k,i}^j}{\partial x^\mu} = \delta_\mu^j - \frac{p^j \sigma_\mu(\bar{y}_{k,i})}{p \cdot \sigma(\bar{y}_{k,i})}. \quad (\text{A6})$$

Similarly, one can calculate the derivative of y^0 with respect to x :

$$\frac{\partial \bar{y}_{k,i}^0}{\partial x^\mu} = \frac{\partial \bar{y}_{k,i}^0}{\partial \bar{y}_{k,i}^j} \frac{\partial \bar{y}_{k,i}^j}{\partial x^\mu} = -\sigma_j(\bar{y}_{k,i}) \left[\delta_\mu^j - \frac{p^j \sigma_\mu(\bar{y}_{k,i})}{p \cdot \sigma(\bar{y}_{k,i})} \right],$$

whence we obtain

$$\frac{\partial \bar{y}_{k,i}^0}{\partial x^0} = \sigma_0(\bar{y}_{k,i}) - \frac{p^0}{p \cdot \sigma(\bar{y}_{k,i})}, \quad \frac{\partial \bar{y}_{k,i}^0}{\partial x^m} = -\frac{p^0 \sigma_m(\bar{y}_{k,i})}{p \cdot \sigma(\bar{y}_{k,i})}. \quad (\text{A7})$$

Equations (A6) and (A7) can be written in a compact form as

$$\frac{\partial \bar{y}_{k,i}^\nu}{\partial x^\mu} = \delta_\mu^\nu - \frac{p^\nu \sigma_\mu(\bar{y}_{k,i})}{p \cdot \sigma(\bar{y}_{k,i})} \equiv \Delta_\mu^\nu(\bar{y}_{k,i}), \quad (\text{A8})$$

so that Eq. (26) can be rewritten as

$$\frac{d}{dx^\mu} g(x, \bar{y}_{k,i}(x)) = \frac{\partial}{\partial x^\mu} g(x, \bar{y}_{k,i}(x)) + \Delta_\mu^\nu(\bar{y}_{k,i}) \frac{\partial g(x, y)}{\partial y^\nu} \Big|_{y=\bar{y}_{k,i}(x, p)}. \quad (\text{A9})$$

By using the last result, the Eq. (25) can be rewritten as:

$$\begin{aligned} \Delta W_{\text{LE}}^+(x, p) &= \sum_{N=0}^{\infty} \sum_{k,i} s_k \frac{(-1)^N}{N!} [\partial_{v_1}^q \cdots \partial_{v_N}^q G^{\mu\nu}(q)] \Big|_{q=0} \sum_{M=0}^N \frac{N!(-1)^M}{M!(N-M)!} [\partial_x^{v_{M+1}} + \Delta^{\alpha_{M+1} \rho_{M+1}}(\bar{y}_{k,i}) \partial_{\rho_{M+1}}^y] \\ &\quad \cdots \times [\partial_x^{v_N} + \Delta^{v_N \rho_N}(\bar{y}_{k,i}) \partial_{\rho_N}^y] \partial_x^{v_1} \cdots \partial_x^{v_M} \frac{\sigma_\mu(y)}{|p \cdot \sigma(y)|} \Delta \beta_\nu(y, x) \Big|_{y=\bar{y}_{k,i}}. \end{aligned} \quad (\text{A10})$$

Using again the Binomial theorem, the last expression can be recast as

$$\Delta W_{\text{LE}}^+(x, p) = \sum_{N=0}^{\infty} \sum_{k,i} s_k \frac{(-i)^N}{N!} [\Delta^{\alpha \rho}(\bar{y}_{k,i}) \partial_\rho^y \partial_\alpha^q]^N G^{\mu\nu}(p, q) \frac{\sigma_\mu(y)}{|p \cdot \sigma(y)|} \Delta \beta_\nu(y, x) \Big|_{q=0, y=\bar{y}_{k,i}}. \quad (\text{A11})$$

Now, because of its definition [see text below Eq. (23)], the sign s_k is such that

$$s_k \sigma_\mu = n_\mu \sqrt{|\sigma \cdot \sigma|},$$

where n_μ is the outward pointing unit vector normal to the hypersurface Σ_D . Besides,

$$\frac{n_\mu \sqrt{|\sigma \cdot \sigma|}}{|p \cdot \sigma|} = \frac{n_\mu}{|p \cdot n|}$$

and

$$\Delta_\mu^\nu = \delta_\mu^\nu - \frac{p^\nu \sigma_\mu}{p \cdot \sigma} = \delta_\mu^\nu - \frac{p^\nu n_\mu}{p \cdot n}.$$

By taking the above equalities into account and introducing the operator D_y in Eq. (31), we can express ΔW_{LE}^+ as shown in Eq. (30). Note the argument of the operator $\Delta^{\alpha\rho}$ is $\bar{y}$, such that the derivative with respect to y does not act on $\Delta^{\alpha\rho}$.

APPENDIX B: SPIN POLARIZATION FOR SPIN-1/2 FERMIONS

In Eq. (45), the contribution from $\Delta\beta$ read

$$\mathcal{A}_{\text{LE}}^\mu(x, p) \simeq \frac{2}{(2\pi)^3} n_F(x, p) [1 - n_F(x, p)] \delta(p^2 - m^2) \epsilon^{\mu\rho\lambda\tau} p_\lambda \frac{1}{2} (g_\rho^\nu p^\alpha + g_\rho^\alpha p^\nu) \sum_{\bar{y}(x, p)} \Delta_{\tau\kappa}(\bar{y}) \left[\partial_y^\kappa \frac{n_v(y)}{|p \cdot n(y)|} \Delta\beta_\alpha(y, x) \right] \Big|_{y=\bar{y}(x, p)}. \quad (\text{B1})$$

The derivative with respect to y acts on both $\Delta\beta_\alpha(y, x)$ and $n_v(y)$, giving rise to two terms. For the former, one obtains a factor that is proportional to

$$\begin{aligned} \frac{1}{2} \epsilon^{\mu\rho\lambda\tau} p_\lambda \Delta_{\tau\kappa}(\bar{y}) n_v(\bar{y}) (g_\rho^\nu p^\alpha + g_\rho^\alpha p^\nu) \left[\partial_y^\kappa \beta_\alpha(y) \Big|_{y=\bar{y}(x, p)} \right] &= \frac{1}{2} \epsilon^{\mu\nu\lambda\tau} p_\lambda \Delta_{\tau\kappa}(\bar{y}) n_v(\bar{y}) p^\alpha \left[\partial_y^\kappa \beta_\alpha(y) \Big|_{y=\bar{y}(x, p)} \right] + [p \cdot n(\bar{y})] \\ &\times \frac{1}{2} \epsilon^{\mu\alpha\lambda\tau} p_\lambda \Delta_{\tau\kappa}(\bar{y}) \left[\partial_y^\kappa \beta_\alpha(y) \Big|_{y=\bar{y}(x, p)} \right]. \end{aligned}$$

Plugging the explicit form of $\Delta_{\tau\kappa}(\bar{y})$ in Eq. (28), it becomes

$$\frac{1}{2} \epsilon^{\mu\nu\lambda\kappa} p_\lambda n_v(\bar{y}) p^\alpha \left[\partial_y^\nu \beta_\alpha(y) \Big|_{y=\bar{y}(x, p)} \right] + [p \cdot n(\bar{y})] \frac{1}{2} \epsilon^{\mu\alpha\lambda\kappa} p_\lambda \left[\partial_y^\nu \beta_\alpha(y) \Big|_{y=\bar{y}(x, p)} \right] - \frac{1}{2} \epsilon^{\mu\alpha\lambda\tau} p_\lambda p^\kappa n_\tau(\bar{y}) \left[\partial_y^\nu \beta_\alpha(y) \Big|_{y=\bar{y}(x, p)} \right],$$

whence, renaming repeated indices,

$$[p \cdot n(x)] \frac{1}{2} \epsilon^{\mu\nu\rho\lambda} p_\nu \left[\partial_y^\rho \beta_\lambda(y) \Big|_{y=\bar{y}(x, p)} \right] - \frac{1}{2} \epsilon^{\mu\nu\rho\lambda} p_\nu n_\rho(x) p^\alpha \left[\partial_y^\nu \beta_\alpha(y) + \partial_\alpha^\nu \beta_\lambda(y) \Big|_{y=\bar{y}(x, p)} \right]. \quad (\text{B2})$$

By using the definitions of thermal vorticity and thermal shear,

$$\varpi_{\rho\lambda} = -\frac{1}{2}(\partial_\rho \beta_\lambda - \partial_\lambda \beta_\rho), \quad \xi_{\rho\lambda} = \frac{1}{2}(\partial_\rho \beta_\lambda + \partial_\lambda \beta_\rho),$$

in Eq. (B2) and reinstating the missing factors, we obtain the first part of $\mathcal{A}_{\text{LE}}^\mu$:

$$\mathcal{A}_{\text{LE}-1}^\mu(x, p) = -\frac{\delta(p^2 - m^2)}{(2\pi)^3} n_F(x, p) (1 - n_F(x, p)) \sum_{\bar{y}(x, p)} \text{sgn}[p \cdot n(\bar{y})] \epsilon^{\mu\nu\rho\lambda} p_\nu \left[\varpi_{\rho\lambda}(\bar{y}) + \frac{2}{p \cdot n(\bar{y})} n_\rho(\bar{y}) \xi_{\lambda\alpha}(\bar{y}) p^\alpha \right]. \quad (\text{B3})$$

On the other hand, the contribution from the derivative of the normal vector n is proportional to

$$\epsilon^{\mu\rho\lambda\tau} p_\lambda (g_\rho^\nu p^\alpha + g_\rho^\alpha p^\nu) \Delta_{\tau\kappa}(\bar{y}) \Delta\beta_\alpha(\bar{y}, x) \left[\partial_y^\kappa \frac{n_v(y)}{|p \cdot n(y)|} \right] \Big|_{y=\bar{y}}, \quad (\text{B4})$$

which in turn gives rise to two terms. The first, obtained by contracting with p^ν , is proportional to

$$p^\nu \left[\partial_y^\kappa \frac{n_v(y)}{|p \cdot n(y)|} \right] \Big|_{y=\bar{y}} = \left[\partial_y^\kappa \text{sgn}[p \cdot n(y)] \right] \Big|_{y=\bar{y}} \propto \delta(p \cdot n(\bar{y})),$$

which vanishes unless $p \cdot n(\bar{y}) = 0$. In our derivation, in order to ensure that the δ function can be solved in terms of the y variable, we had to assume that $p \cdot n(\bar{y}) = 0$, so we neglect the above term. Therefore, we can rewrite the above term, without changing it, as

$$\epsilon^{\mu\rho\lambda\tau} p_\lambda (g_\rho^\nu p^\alpha - g_\rho^\alpha p^\nu) \Delta_{\tau\kappa}(\bar{y}) \Delta\beta_\alpha(\bar{y}, x) \left[\partial_y^\kappa \frac{n_v(y)}{|p \cdot n(y)|} \right] \Big|_{y=\bar{y}}, \quad (\text{B5})$$

i.e., by changing the sign of the second term in the bracket. Using the Schouten identity,

$$\epsilon^{\mu\nu\lambda\tau} p^\alpha + \epsilon^{\nu\lambda\tau\alpha} p^\mu + \epsilon^{\lambda\tau\alpha\mu} p^\nu + \epsilon^{\tau\alpha\mu\nu} p^\lambda + \epsilon^{\alpha\mu\nu\lambda} p^\tau = 0, \quad (\text{B6})$$

we have

$$\epsilon^{\mu\rho\lambda\tau} p_\lambda (g_\rho^\nu p^\alpha - g_\rho^\alpha p^\nu) = (\epsilon^{\mu\nu\lambda\tau} p^\alpha + \epsilon^{\lambda\tau\alpha\mu} p^\nu) p_\lambda = -(\epsilon^{\nu\lambda\tau\alpha} p^\mu + \epsilon^{\tau\alpha\mu\nu} p^\lambda + \epsilon^{\alpha\mu\nu\lambda} p^\tau) p_\lambda.$$

Plugging this relation into Eq. (B5) and taking into account that $p^\tau \Delta_{\tau\kappa}(\bar{y}) = 0$, the expression in (B5) can be rewritten as

$$-(\epsilon^{\nu\lambda\tau\alpha} p^\mu + \epsilon^{\tau\alpha\mu\nu} p^\lambda) p_\lambda \Delta_{\tau\kappa}(\bar{y}) \Delta\beta_\alpha(\bar{y}, x) \left[\partial_y^\kappa \frac{n_\nu(y)}{|p \cdot n(y)|} \right] \Big|_{y=\bar{y}} = -\epsilon^{\nu\lambda\tau\alpha} (p^\mu p_\lambda - g_\lambda^\mu p^2) \Delta_{\tau\kappa}(\bar{y}) \Delta\beta_\alpha(\bar{y}, x) \left[\partial_y^\kappa \frac{n_\nu(y)}{|p \cdot n(y)|} \right] \Big|_{y=\bar{y}}. \quad (\text{B7})$$

With the help of the definition of $\Delta_{\tau\kappa}(\bar{y})$, one can prove that

$$\epsilon^{\nu\lambda\tau\alpha} \Delta_\tau^\kappa(\bar{y}) = \epsilon^{\nu\lambda\tau\alpha} \left[g_\tau^\kappa - \frac{p^\kappa n_\tau(\bar{y})}{p \cdot n(\bar{y})} \right] = (p^\tau \epsilon^{\nu\lambda\kappa\alpha} - p^\kappa \epsilon^{\nu\lambda\tau\alpha}) \frac{n_\tau(\bar{y})}{p \cdot n(\bar{y})} = -(p^\nu \epsilon^{\lambda\kappa\alpha\tau} + p^\lambda \epsilon^{\kappa\alpha\tau\nu} + p^\alpha \epsilon^{\tau\nu\lambda\kappa}) \frac{n_\tau(\bar{y})}{p \cdot n(\bar{y})}, \quad (\text{B8})$$

where we have again used the Schouten identity (B6) in the last step. Substituting (B8) into Eq. (B7) and noting that (see also above discussion)

$$p^\lambda (p^\mu p_\lambda - g_\lambda^\mu p^2) = 0, \quad p^\nu \left[\partial_y^\kappa \frac{n_\nu(y)}{|p \cdot n(y)|} \right] \Big|_{y=\bar{y}} = [\partial_y^\kappa \text{sgn}[p \cdot n(y)]] \Big|_{y=\bar{y}} = 0,$$

the expression in Eq. (B7) gets its final form:

$$(p^\mu p_\lambda - g_\lambda^\mu p^2) \epsilon^{\lambda\tau\nu\kappa} \frac{n_\tau(\bar{y})}{p \cdot n(\bar{y})} \left[\partial_y^\kappa \frac{n_\nu(y)}{|p \cdot n(y)|} \right] \Big|_{y=\bar{y}} p^\alpha \Delta\beta_\alpha(\bar{y}, x). \quad (\text{B9})$$

Defining the new vector:

$$\Omega^\lambda(\bar{y}) \equiv \epsilon^{\lambda\tau\nu\kappa} \frac{n_\tau(\bar{y})}{p \cdot n(\bar{y})} \left[\partial_y^\kappa \frac{n_\nu(y)}{|p \cdot n(y)|} \right] \Big|_{y=\bar{y}} = \frac{\text{sgn}[p \cdot n(\bar{y})]}{[p \cdot n(\bar{y})]^2} \epsilon^{\lambda\tau\nu\kappa} n_\tau(\bar{y}) [\partial_y^\kappa n_\nu(y)] \Big|_{y=\bar{y}}, \quad (\text{B10})$$

which depends on the curvature of the decoupling hypersurface at point $\bar{y}$, Eq. (B9) can be rewritten as

$$(p^\mu p_\lambda - g_\lambda^\mu p^2) p^\alpha \Delta\beta_\alpha(\bar{y}, x) \Omega^\lambda(\bar{y}). \quad (\text{B11})$$

However, according to the definition of the normal unit vector, we have

$$n_\mu = A \sigma_\mu,$$

with A a suitable scalar and σ as in Eq. (23). Hence

$$\partial_\kappa n_\nu = A \partial_\kappa \sigma_\nu + (\partial_\kappa A) n_\nu. \quad (\text{B12})$$

Since

$$\partial_\kappa^\nu \sigma_\nu(y) = \begin{cases} 0, & \kappa = 0 \quad \text{or} \quad \nu = 0, \\ -\frac{\partial f_\kappa(y)}{\partial y^\nu \partial y^\nu}, & \kappa = i, \quad \nu = j, \quad i, j = 1, 2, 3, \end{cases}$$

it turns out that the gradient of σ in Eq. (B12) is symmetric under the exchange of its indices. Therefore, the contraction with $\epsilon^{\lambda\tau\nu\kappa}$ in Eq. (B10) of both the terms in Eq. (B12) vanishes and so $\Omega^\lambda = 0$. In conclusion, the expression in (B9) vanishes, demonstrating that the curvature of Σ_D does not contribute to the spin polarization at the leading order.

On the other hand, the contribution from $\Delta\zeta$ is given by

$$\mathcal{A}_{\text{LE},\zeta}^\mu(x, p) = -\frac{2}{(2\pi)^3} \sum_{\bar{y}(x,p)} n_F(x, p) [1 - n_F(x, p)] \delta(p^2 - m^2) \epsilon^{\mu\nu\lambda\tau} p_\lambda \Delta_{\tau\kappa}(\bar{y}) \left[\partial_y^\kappa \frac{n_\nu(y)}{|p \cdot n(y)|} \Delta\zeta(y, x) \right] \Big|_{q=0, y=\bar{y}(x,p)}. \quad (\text{B13})$$

Using Eq. (B8), one can prove that

$$\epsilon^{\mu\nu\lambda\tau} p_\lambda \Delta_{\tau\kappa} \left[\partial_y^\kappa \frac{n_\nu(y)}{|p \cdot n(y)|} \right] \Big|_{y=\bar{y}} = (p^\mu p_\lambda - p^2 g_\lambda^\mu) \epsilon^{\lambda\nu\kappa\tau} \frac{n_\tau(\bar{y})}{p \cdot n(\bar{y})} \left[\partial_y^\kappa \frac{n_\nu(y)}{|p \cdot n(y)|} \right] \Big|_{y=\bar{y}} - \epsilon^{\lambda\kappa\tau\mu} p_\lambda \frac{n_\tau(\bar{y})}{p \cdot n(\bar{y})} p^\nu \left[\partial_y^\kappa \frac{n_\nu(y)}{|p \cdot n(y)|} \right] \Big|_{y=\bar{y}}. \quad (\text{B14})$$

As shown in the previous section, the terms in the above equation vanish, indicating that the curvature of Σ_D does not contribute to $\mathcal{A}_{\text{LE},\zeta}^\mu$ in Eq. (B13). On the other hand, the contribution from $\partial_y \zeta$ reads

$$\mathcal{A}_{\text{LE},\zeta}^\mu(x, p) = \frac{2}{(2\pi)^3} \sum_{\bar{y}(x,p)} n_F(x, p)[1 - n_F(x, p)] \delta(p^2 - m^2) \epsilon^{\mu\nu\rho\lambda} p_\nu \frac{n_\rho(\bar{y})}{|p \cdot n(\bar{y})|} \partial_\lambda^\bar{y} \zeta(\bar{y}). \quad (\text{B15})$$

Finally, plugging Eqs. (39), (B3), and (B15) into Eq. (37), the vorticity- and shear-induced polarization as well as the spin Hall effect in (46) are obtained.

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