Charge sum rules for quark fragmentation functions

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Charge sum rules for quark fragmentation functions are studied. The simultaneous implementation of the conservation of electric and baryon charges, strangeness, and isospin symmetry is achieved when the fragmentation to both mesons and baryons is considered. The results are compatible to Gell-Mann–Nishijima formulas and may be the new manifestation of superconformal symmetry between mesons and baryons. The numerical estimates are performed and compared with phenomenological models. The recently suggested violations of sum rules due to Wilson line contributions are discussed.

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I. INTRODUCTION

The basic concept underlying the theoretical analysis of most high-energy interactions in quantum chromodynamics (QCD) is factorization. As a result of the factorization, the physical quantities (e.g., cross sections) may be given as convolution of the two separated parts: the long-distance part that contains information on the structure of the nucleon in terms of its parton distribution functions (PDFs) and fragmentation functions (FFs), and the short-distance part which describes the hard interactions of the partons. The long-distance contributions are universal; i.e., they are the same in any inelastic reaction, and the short-distance parts depend only on the large scales related to the large momentum transfer and, therefore, can be evaluated using perturbative methods of QCD. PDFs and FFs controlled by the nonperturbative dynamics of QCD and determined from one process can be used for other processes.

The longitudinal fragmentation functions $D_i^h(z, Q^2)$ $(i = q, \bar{q}, g)$ are the final-state analogs of the PDFs indicating the probability density that an outgoing parton i produces a hadron h with the momentum fraction z. While the PDFs are fairly well known, on the contrary, the flavor-separated quark and gluon FFs, being relatively new objects, required for a quantitative description of hard scattering processes involving identified light hadrons in the final state, are not so well constrained. There are different sources to extract FFs from experimental data: semi-inclusive e^+e^- annihilation (SIA), single-inclusive production of a hadron h at a high transverse momentum p_T in hadron-hadron collisions, and unpolarized semi-inclusive deep inelastic lepton-nucleon scattering (SIDIS). Only the latter is crucial for a reliable determination of FFs, because only then one can separate $D_q^h(z, Q^2)$ from $D_{\bar{q}}^h(z, Q^2)$ (from the other processes only the sum of them can be determined).

In analogy to PDFs, FFs obey the various sum rules reflecting the conservation laws in QCD. These sum rules, among which the best known example is the momentum sum rule, provide useful phenomenological constraints for the practical extraction of the fragmentation functions. At present there are several sets of FFs that well describe data, but nevertheless differ quite a lot in different kinematic regions, particularly at very small z, where the next-to-leading order (NLO) perturbative QCD evolution may lead to the unphysical negative $D_q^h(z, Q^2)$. This make the energy and other sum rules for FFs a delicate concept which cannot be resolved unless the $z \to 0$ behavior of FFs is better understood.

In this work, we study the sum rule for the conserved flavor electric charge for FFs and demonstrate that it is upheld, implying simultaneously the conservation of the strangeness and baryon number. Based on this approach, we obtain the constraints for $D_{u\text{val}}^h(z, Q^2)$, where h denotes mesons π and K and baryons p, n, and Λ. We find these constraints confirmed by recent parametrizations of FFs.

We also discuss the possible cancellations of Wilson line contributions between quarks and antiquarks for valence fragmentation functions to resolve the problem of validation of the charge sum rules coming from the small $z \to 0$ region.

In the next section, we give some introduction to the fragmentation functions and their sum rules. Section III contains our main results on the charge sum rule. In Sec. IV, we compare the results with experimental data. Finally, we summarize our results in Sec. V.

II. FRAGMENTATION FUNCTIONS

Parton fragmentation functions $D_i^h(z, Q^2)$ carry information on the hadronization process and are related to a

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nonperturbative aspect of QCD. The fragmentation process of the hadron h at some hard scale Q^2 occurs from a parton i with the probability density $D_i^h(z, Q^2)$, where z is a fraction of the parton energy carrying by the hadron.

The fragmentation functions, similarly to the PDFs, obey the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) Q^2 evolution, and the initial $D_i^h(z, Q^2 = Q_0^2)$ can be parametrized in in different ways, usually in the form

$$D_i^h\big(z, Q_0^2\big) = \frac{M_i^h\, z^{\alpha_i}(1-z)^{\beta_i}[1+\gamma_i(1-z)^{\delta_i}]}{B[2+\alpha_i, \beta_i+1]+\gamma_i B[2+\alpha_i, \beta_i+\delta_i+1]}, \tag{1}$$

where M_i^h,

$$M_i^h < 1\,, \tag{2}$$

is the energy fraction for the hadron h, which is created from the parton i, and $B[a, b]$ denotes the Euler Beta function.

The momentum sum rule,

$$\sum_h \int_0^1 dz\, z D_i^h(z, Q^2) = 1, \tag{3}$$

is valid separately for each flavor i and involves a sum over all possible produced hadrons. The momentum sum rule is true at all scales Q^2, what is guaranteed by DGLAP evolution, and reflects the energy conservation. This is a rigorous assumption used in most phenomenological extractions of FFs (e.g., Refs. [1–6]).

However, in some recent fits of the fragmentation functions, the momentum sum rule is not imposed *a priori* but rather is used as an *a posteriori* check (e.g., Refs. [7–9]). There are two important reasons for such a procedure. First, the momentum sum rule, Eq. (3), requires the knowledge of the FFs of all produced hadrons, and second, it requires integration over z down to $z = 0$, while FFs can be usually determined from the experimental data only to $0.01 < z_{\min} < 0.2$. Hence, extrapolation of the FFs parametrizations to the small-z region may lead to their unphysical behavior in this region. The lack of knowledge of the real small-z behavior of FFs causes the verification of the sum rules to be problematic. This is not so crucial in the case of the momentum sum rule due to the suppression of small-z contributions, but may become essential in the case of the charge and particle number sum rules. Also, the low-z problem with the formulation of the charge and number sum rules for FFs from the theoretical point of view has been recently raised in Ref. [10].

III. CHARGE SUM RULE

The charge sum rule for FFs under study expresses charge conservation:

$$Q_i = \sum_h Q_h \int_0^1 dz D_i^h(z, Q^2), \tag{4}$$

where Q_i is the conserved (in particular, electric) charge of the parent quark of flavor i and the sum runs over all produced hadrons of charge Q_h. Unlike the case of the momentum sum rule, Eq. (3), where the suppression of small-z contributions occurs, the charge sum rule can be invalidated due to lack of the experimental data and also an adequate theoretical interpretation of the fragmentation functions in the range of $0 < z < z_{\min}$.

For a given process of charged-hadron production the small-z range depends on Q, $z_{\min} \sim 1/Q$ for large Q, and it is reasonable to consider instead of the charge sum rule, Eq. (4), its truncated contribution, which depends on the scale Q^2:

$$Q_i(Q^2) = \sum_h Q_h \int_{z_{\min}(Q^2)}^1 dz D_i^h(z, Q^2)\,. \tag{5}$$

In this way, one can test consistency of data with a theoretical model.

Note also that the region of small $z \leqslant \frac{\mu^2 x_B}{Q^2}$ (where μ is a scale of an order of typical hadron mass) in SIDIS may correspond [11] to the breaking of factorization of independent distribution and fragmentation functions and to the appearance of (extended) fracture [12–15] functions. The simultaneous consideration of sum rules for fragmentation and fracture functions may be, therefore, interesting.

In much of the data for the charged hadron production, the observed hadrons are identified as one of the three lightest ones: pions ($\pi^{\pm}$), kaons ($K^{\pm}$), and protons ($p/\bar{p}$). Here, in a simple approach to the charge sum rule, we consider these particles by adding step by step the subsequent components. Using only isospin SU(2) symmetry for the favored and unfavored fragmentation functions, and also the charge conjugation invariance of the strong interactions, we arrive at the generalized conservation law including charge, strangeness, and baryon number. On this basis, we obtain the constraints for the valence fragmentation functions of quark u, $D_{u\text{val}}^h(z, Q^2)$, into mesons π^+ and K^+ and baryons p, n, and Λ, and we compare the results with data.

Let us consider the charge sum rule for *up* and *down* quarks. Henceforth, we skip for simplicity the argument Q^2 in functions $D_q^h(z, Q^2)$. Writing out Eq. (4) explicitly, we have

$$Q_u = \sum_h \int_0^1 dz \left[D_u^{h^+}(z) - D_u^{h^-}(z)\right] = \sum_h \int_0^1 dz D_{u\text{val}}^{h^+}(z) \tag{6}$$

and

$$Q_d = \sum_h \int_0^1 dz \left[D_d^{h^+}(z) - D_d^{h^-}(z)\right] = \sum_h \int_0^1 dz\, D_{d\text{val}}^{h^+}(z)\,, \tag{7}$$

respectively, where

$$D_{q\text{val}}^{h^+}(z) = D_q^{h^+}(z) - D_{\bar{q}}^{h^+}(z) \equiv D_q^{h^+-h^-}(z). \tag{8}$$

In the above formulas, Eqs. (6)–(8), we have used charge symmetry of FFs:

$$D_q^{h^-}(z) = D_{\bar{q}}^{h^+}(z)\,. \tag{9}$$

The charged hadron production is dominated by charged pions, and this approximation was used long ago (see p. 16 of Ref. [16]) to perform the pioneering check of the charge sum rule and get the u-quark charge from semi-inclusive deep

inelastic neutrino data. Therefore, as a first step, we consider only the pion contributions to the sums over h in Eqs. (6) and (7):

$$Q_u = \frac{2}{3} = \sum_{h=\pi^\pm} Q_h \int_0^1 dz D_u^h(z) = \int_0^1 dz D_{u\mathrm{val}}^{\pi^+}(z) \tag{10}$$

and

$$Q_d = -\frac{1}{3} = \sum_{h=\pi^\pm} Q_h \int_0^1 dz D_d^h(z) = \int_0^1 dz D_{d\mathrm{val}}^{\pi^+}(z). \tag{11}$$

Assuming the isospin SU(2) symmetry for the favored and unfavored pion fragmentation functions [$\pi^+ = (u\bar{d})$, $\pi^- = (d\bar{u})$],

$$D_u^{\pi^+}(z) = D_{\bar{d}}^{\pi^+}(z), \quad D_{\bar{u}}^{\pi^+}(z) = D_d^{\pi^+}(z), \tag{12}$$

we have

$$D_{d\mathrm{val}}^{\pi^+}(z) = -D_{u\mathrm{val}}^{\pi^+}(z), \tag{13}$$

and hence, using Eqs. (6) and (7), one gets the incompatible relations:

$$Q_u = \frac{2}{3} = \int_0^1 dz D_{u\mathrm{val}}^{\pi^+}(z), \tag{14}$$

$$Q_d = -\frac{1}{3} = -\int_0^1 dz D_{u\mathrm{val}}^{\pi^+}(z). \tag{15}$$

Still, taking their difference divided by 2, one gets

$$\frac{Q_u - Q_d}{2} = \frac{1}{2} = \int_0^1 dz D_{u\mathrm{val}}^{\pi^+}(z), \tag{16}$$

which is nothing else than the sum rule relating the quark isospin projection to that of pions. This also shows, that the result obtained in the pioneering paper [16] should be more close to $1/2$ rather than to $2/3$, which is perfectly compatible within 30% accuracy of the data, explicitly mentioned in Ref. [16] just after the presented result.

At the same time, taking the sum (on which we will concentrate in what follows) instead of the difference, one has the obvious contradiction

$$Q_d = -Q_u \tag{17}$$

or, equivalently,

$$\tfrac{1}{3} = 0\,. \tag{18}$$

Now, let us check if adding to the charge sum rules for Q_u and Q_d also the kaon contributions will remove the discrepancy, Eq. (18). Thus, taking into account pions and kaons in the sum over h in Eqs. (6) and (7), we can write

$$Q_u + Q_d = \sum_{h=\pi^\pm, K^\pm} (\cdots) = \int_0^1 dz \big[D_{u\mathrm{val}}^{K^+}(z) + D_{d\mathrm{val}}^{K^+}(z)\big], \tag{19}$$

where, as we have shown above in Eq. (17), the pion contribution to $Q_u + Q_d$ is 0. By using the isospin SU(2) symmetry for the kaon fragmentation functions

$$D_{d\mathrm{val}}^{K^+}(z) = D_{u\mathrm{val}}^{K^0}(z) \equiv D_u^{K^0}(z) - D_u^{\bar{K}^0}(z), \tag{20}$$

Eq. (19) takes the following form:

$$Q_u + Q_d = \int_0^1 dz \left[D_{u\mathrm{val}}^{K^+}(z) + D_{u\mathrm{val}}^{K^0}(z)\right]. \tag{21}$$

Note that the appearing combination of kaon fragmentation functions should be zero due to the strangeness (equal zero for light quarks) conservation, so that the discrepancy persists. Note that the fragmentation functions of K^0 mesons are added here as a matter of principle. They correspond to the states of different quark contents, while experimentally the ones with definite CP symmetry are measured, for which the relevant valence contributions are zero, up to small CP-symmetry violation.

Bearing in mind the above conclusion and also the motivation to find some general law in terms of the fragmentation functions, we continue our analysis by including also *baryons*. Thus, after taking into account the contributions to the charge sum rules, Eqs. (6) and (7), coming from pions, kaons, and also protons, we obtain

$$\begin{aligned} Q_u + Q_d = \int_0^1 dz \big[&D_u^{K^+ - K^-}(z) + D_u^{K^0 - \bar{K}^0}(z) \\ &+ D_u^{p-\bar{p}}(z) + D_d^{p-\bar{p}}(z)\big]. \end{aligned} \tag{22}$$

Assuming the charge and isospin SU(2) symmetry for the proton [$p = (uud)$] and neutron [$n = (ddu)$] FFs,

$$D_d^p(z) = D_u^n(z), \quad D_{\bar{d}}^p(z) \approx D_{\bar{u}}^p(z) \approx D_{\bar{u}}^n(z)\,, \tag{23}$$

we can write Eq. (22) as

$$\begin{aligned} \tfrac{1}{3} = &\left\langle D_u^{K^+}\right\rangle - \left\langle D_u^{K^-}\right\rangle + \left\langle D_u^{K^0}\right\rangle - \left\langle D_u^{\bar{K}^0}\right\rangle \\ &+ \left\langle D_u^{p}\right\rangle - \left\langle D_u^{\bar{p}}\right\rangle + \left\langle D_u^{n}\right\rangle - \left\langle D_u^{\bar{n}}\right\rangle, \end{aligned} \tag{24}$$

where we have used the hadron multiplicity in the form

$$\left\langle D_q^h\right\rangle \equiv \int_0^1 dz D_q^h(z). \tag{25}$$

Note that four last terms now express the *baryon* charge so that the current sum rule simultaneously provides conservation of strangeness, electric, and baryon charges, which is a sort of manifestation of the Gell-Mann–Nishijima formula.

As soon as in the "favored" approximation the fragmentation function of K^+ dominates (and the strangeness conservation is numerically questionable), it is instructive to add and subtract to the sum rule the same term (which is also theoretically attracting providing the baryonic strangeness contribution in addition to the mesonic one) $\langle D_u^\Lambda\rangle - \langle D_u^{\bar{\Lambda}}\rangle$, where Λ is a uds hyperon, and then we group all terms in *strange* and *nonstrange* parts:

$$\begin{aligned} \tfrac{1}{3} = &\overbrace{\left\langle D_u^{K^+}\right\rangle - \left\langle D_u^{K^-}\right\rangle + \left\langle D_u^{K^0}\right\rangle - \left\langle D_u^{\bar{K}^0}\right\rangle + \left\langle D_u^{\bar{\Lambda}}\right\rangle - \left\langle D_u^{\Lambda}\right\rangle}^{S_u = 0} \\ &+ \underbrace{\left\langle D_u^{p}\right\rangle - \left\langle D_u^{\bar{p}}\right\rangle + \left\langle D_u^{n}\right\rangle - \left\langle D_u^{\bar{n}}\right\rangle + \left\langle D_u^{\Lambda}\right\rangle - \left\langle D_u^{\bar{\Lambda}}\right\rangle}_{B_u = 1/3}. \end{aligned} \tag{26}$$

One can see that the right-hand side of Eq. (26) is a sum of the strangeness $S = 0$ and the baryon number $B = 1/3$ of the u quark (like in the Gell-Mann–Nishijima formula), and the

final result reads

$$\tfrac{1}{3} = \tfrac{1}{3}\,. \tag{27}$$

We have demonstrated that the charge sum rules for the quark fragmentation functions hold, including simultaneously the contributions of mesons and baryons providing the conservation of the strangeness, electric, and baryon charges.

Let us, for completeness, write the expression for the isospin conservation, $I_3(u)$, including the pions, kaons, and baryons:

$$\tfrac{1}{2} = \langle D_u^{\pi^+}\rangle - \langle D_u^{\pi^-}\rangle + \tfrac{1}{2}\big(\langle D_u^{K^+}\rangle - \langle D_u^{K^-}\rangle - \langle D_u^{K^0}\rangle + \langle D_u^{\bar{K}^0}\rangle + \langle D_u^{p}\rangle - \langle D_u^{\bar{p}}\rangle - \langle D_u^{n}\rangle + \langle D_u^{\bar{n}}\rangle\big)\,. \tag{28}$$

The last two equations form our main result. They are clearly compatible to Gell-Mann–Nishijima formulas for quarks and hadrons, and it is crucial that fragmentation to both mesons and baryons must be considered. This may provide a new manifestation of superconformal symmetry [17] between mesons and baryons.

Another interesting aspect of the obtained sum rules is provided by the recent investigation [10] of the sum rule violations due to Wilson line contributions implied by quark and baryon quantum numbers mismatch. Our result suggests that these effects are strongly constrained by Gell-Mann–Nishijima formulas. The simplest way of the realization of such constraints is probably the cancellations of Wilson line contributions between quarks and antiquarks for valence fragmentation functions. This possibility may be supported by the fact [18] that resummed evolution kernels for multiplicities in quark and gluon jets (that is, the first moments of quark and gluon fragmentation functions) obey the Casimir scaling. As the gluons' fragmentations to particles and antiparticles coincide, their quark multiplicities' evolutions are the same, due to Casimir scaling. One may therefore expect that the Wilson line contribution [10] is also the same, although additional investigations are required. It is interesting to mention, that Casimir scaling in Ref. [18] is related to supersymmetric properties of evolution, which may be compared with superconformal symmetry in Ref. [17].

IV. COMPARISON TO DATA

To test consistency of our results with a phenomenological model of the fragmentation functions, we compare the theoretical predictions on the charge and isospin sum rules with the numerical estimates of their truncated z contributions based on some recent parametrizations of FFs.

In addition to our main results on the charge and isospin sum rules, below we obtain some constraints on $\langle D^h_{u\mathrm{val}}\rangle$, where $h = \pi^+, K^+, p,$ and Λ, which can also be tested numerically. Namely, gathering together Eqs. (24), (26) and also the expression for the electric charge Q_u:

$$\langle D^{K^+}_{u\mathrm{val}}\rangle + \langle D^{p}_{u\mathrm{val}}\rangle + \langle D^{n}_{u\mathrm{val}}\rangle = \tfrac{1}{3}\,, \tag{29}$$

$$\langle D^{p}_{u\mathrm{val}}\rangle + \langle D^{n}_{u\mathrm{val}}\rangle + \langle D^{\Lambda}_{u\mathrm{val}}\rangle = \tfrac{1}{3}, \tag{30}$$

$$\langle D^{\pi^+}_{u\mathrm{val}}\rangle + \langle D^{K^+}_{u\mathrm{val}}\rangle + \langle D^{p}_{u\mathrm{val}}\rangle = \tfrac{2}{3}, \tag{31}$$

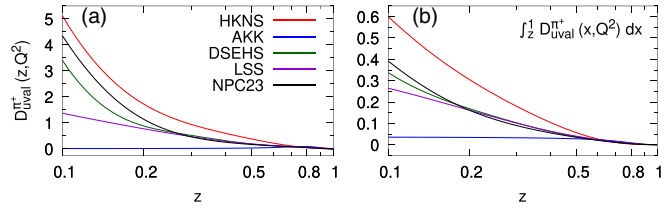


FIG. 1. The valence fragmentation function $D^{\pi^+}_{u\mathrm{val}}(z, Q^2)$ (a) and its contribution to the charge of the u quark, $\int_z^1 D^{\pi^+}_{u\mathrm{val}}(x, Q^2)\,dx$ (b), for different parametrization sets at $Q^2 = 25\ \mathrm{GeV}^2$.

and assuming (based on the concept of a common function for favored fragmentation functions from up and down quarks and on a flavor symmetry)

$$D^{p}_{u\mathrm{val}}(z) \approx 2D^{n}_{u\mathrm{val}}(z) \approx 2D^{\Lambda}_{u\mathrm{val}}(z)\,, \tag{32}$$

we arrive at

$$\langle D^{\pi^+}_{u\mathrm{val}}\rangle = \tfrac{5}{12}, \quad \langle D^{p}_{u\mathrm{val}}\rangle = \tfrac{1}{6}\,,$$
$$\langle D^{K^+}_{u\mathrm{val}}\rangle = \langle D^{n}_{u\mathrm{val}}\rangle = \langle D^{\Lambda}_{u\mathrm{val}}\rangle = \tfrac{1}{12}\,. \tag{33}$$

In our analysis we use the following various published NLO parametrization sets: HKNS-2007 [2], AKK-2008 [3], DSEHS-2014 [4], and LSS-2015 [6] for pions; HKNS-2007, AKK-2008, and DEHSS-2017 [5] for kaons; HKNS-2007, DSS-2007 [19], AKK-2008, and BS-2003 [20] for protons; and DSV-1998 [21], BS-2003, AKK-2008, and SAK-2020 [22] for Λ's. For the charged hadrons, we also employ a very recent set of FFs, NPC23 [9,23]. A multitude of processes have been used for the extraction of FFs. The fits are based on SIA (all sets but LSS), SIDIS (LSS, DSEHS, DEHSS, DSS, and NPC23), and $pp(\bar{p})$ (AKK, BS, DSEHS, DEHSS, DSS, and NPC23). DGLAP NLO evolution of FFs has been performed with the help of a correspondingly modified numerical code provided by HKNS [2].

A comparison of the valence u-quark FF for the pion at the scale $Q^2 = 25\ \mathrm{GeV}^2$, $D^{\pi^+}_{u\mathrm{val}}(z, Q^2)$, for different parametrization sets is shown in Fig. 1(a). The initial scales Q_0^2 are $1\ \mathrm{GeV}^2$ for HKNS, LSS, and DSEHS, $2\ \mathrm{GeV}^2$ for AKK and $25\ \mathrm{GeV}^2$ for NPC23 fits, respectively. The AKK FF was extracted directly in the valence $(u - \bar{u})$ form from measurements in which particles were distinguished from their antiparticles and has a high uncertainty [3]. Large differences in the small-z behavior of $D^{\pi^+}_{u\mathrm{val}}$ between various fits reflect in the predictions for its truncated contributions to the charge sum rule, $\int_z^1 D^{\pi^+}_{u\mathrm{val}}(x, Q_0^2)\,dx$ [see Fig. 1 (right)].

Only in the LSS fit the conservation of the electric charge has been imposed *a priori* with the assumption $\langle D_{u\mathrm{val}^{\pi^+}}\rangle = 2/3$. However, the increase of the other FFs at small values of z is phenomenologically not relevant as their range of applicability is restricted by kinematics in SIA, SIDIS, and hadron-hadron collisions to $z > z_{\min}$. Thus, for the HKNS FFs, $z_{\min} = 0.05$ for the data at $Q = M_Z$, and $z_{\min} = 0.1$ for the data at $Q < M_Z$. In the case of the AKK and DSEHS parametrizations, $z_{\min} = 0.05$, while for the NPC23 fit $z_{\min} = 0.01$ for pion and $z_{\min} = 0.088$ for kaon and proton FFs. This is taken into account in the following figures illustrating our

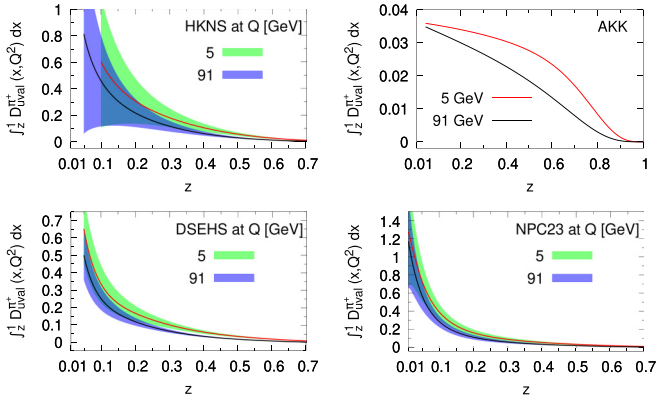


FIG. 2. Q^2 dependence of the truncated moment $\int_z^1 D^{\pi^+}_{u\text{val}}(x,Q^2)\,dx$ for HKNS, AKK, DSEHS, and NPC23 fits. The range of z is limited to $z_{\min} < z < 1$ for each fit, respectively. The uncertainty bands are shown.

results to avoid dealing with an unphysical extrapolation of FFs to the region $z < z_{\min}$.

Figure 2 shows the truncated pion contribution to the charge sum rule, $\int_z^1 D^{\pi^+}_{u\text{val}}(x,Q^2)\,dx$, for HKNS, AKK, DSEHS, and NPC23 fits at $Q = 5$ GeV and $Q = M_Z$ together with the uncertainty bands, for the restricted range of $z_{\min} < z < 1$, respectively. The charge sum rule, Eq. (4), which reflects the electric charge conservation holds at any scale Q^2 while its truncated at $z_{\min}$ version, Eq. (5), depends on Q^2. The HKNS results suffer from significantly larger uncertainties, estimated by the Hessian method, at small z in comparison to DSEHS and NPC23 fits. In turn, the AKK fit which at the initial scale $Q_0^2 = 2$ GeV^2 has a simple form, $D^{\pi^+}_{u\text{val}}(z,Q_0^2) \sim z^{13}(1-z)^{2.5}$, definitely underestimates the pion contribution to the charge sum rule.

The separate contributions of the pions, kaons, and protons and their sum to the electric charge of the u quark ($Q_u = 2/3$) is shown in Fig. 3. One can see that the predictions of the HKNS, NPC23 and, particularly, DSEHS (DEHSS, DSS) fits

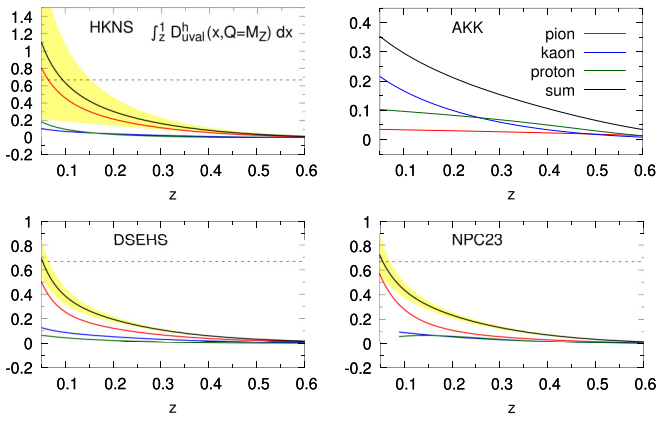


FIG. 3. Contributions to the charge of the u quark coming from the pions, kaons, and protons, $\int_z^1 D^h_{u\text{val}}(x,Q_0^2)\,dx$, and their sum, Eq. (31), for different FFs fits at $Q = M_Z$. The uncertainty band for the sum is shown.

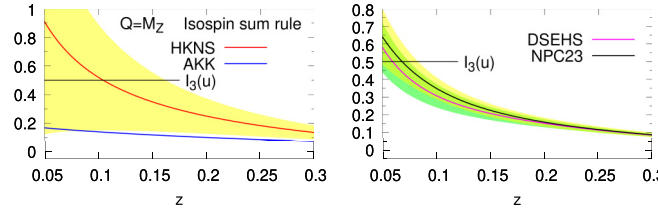


FIG. 4. Truncated contributions to the isospin sum rule for the u quark, Eq. (34).

are in good agreement with the theoretical charge sum rule, Eq. (4).

In Fig. 4 we present the truncated at $z_{\min}$ contributions to the isospin sum rule for the u quark utilizing the HKNS, AKK, DSEHS (DEHSS, DSS), and NPC23 FFs. Based on Eq. (28), which incorporates both the mesons and the baryons in the sum for $I_3(u)$, and using the relations Eqs. (20) and (32), where it is usually assumed that $D^{K^+}_{d\,\text{val}} = 0$, one obtains

$$I_3(u) \approx \langle D^{\pi^+}_{u\text{val}}\rangle + \tfrac{1}{2}\langle D^{K^+}_{u\text{val}}\rangle + \tfrac{1}{4}\langle D^{p}_{u\text{val}}\rangle. \tag{34}$$

The value $I_3(u) = 0.5$ is reached at $z \approx 0.1$ for the HKNS result and near $z_{\min}$ for the DSEHS and NPC23 predictions. The AKK fit of $I_3(u)$, similarly as in the case of the electric charge sum rule, lies significantly below the theoretical value.

Concluding, the data on the final-state hadronization encoded in the tested parton-to-hadron FFs are mostly in agreement with the theoretical charge and isospin sum rules.

It is also interesting to notice that the constraints on $\langle D^h_{u\text{val}}\rangle$ obtained in Eq. (33) are supported by the phenomenological estimates as well (within systematic uncertainties), with the exception of AKK parametrizations for the pions and kaons. This is shown in Fig. 5, where we present the truncated at z first moment of $D^h_{u\text{val}}$ for mesons: pions and kaons; and for baryons: protons and Λ's. However, the comparison for the Λ's should be treated with caution as the data and hence the obtained fits are accessible only for the sum $\Lambda + \bar{\Lambda}$. Therefore, our assumption that $D^{\bar{\Lambda}}_{\bar{u}} \ll D^{\Lambda}_{u}$, and also the

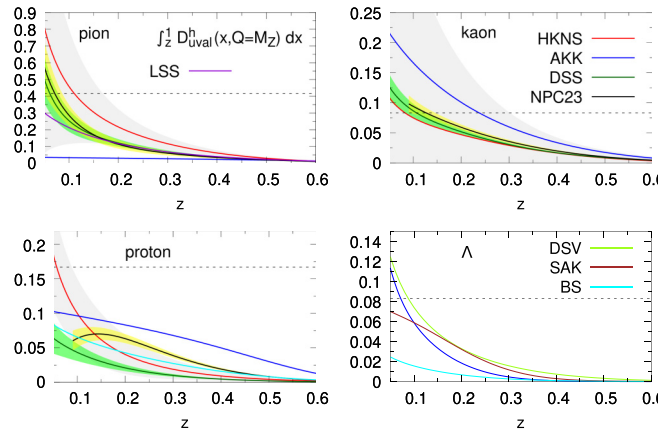


FIG. 5. The truncated moments, $\int_z^1 D^h_{u\text{val}}(x,Q^2)\,dx$, for mesons (the pions and kaons) and baryons (the protons and Λ's), calculated for different FFs sets, compared to the constraints for $\langle D^h_{u\text{val}}\rangle$, Eq. (33). The uncertainty bands are shown.

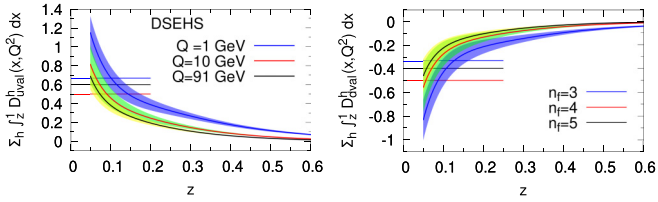


FIG. 6. Truncated contributions to the electric charge of the u quarks and the d-quarks for DSEHS (DEHSS, DSS) FFs at different scales Q^2. Comparison to the modified charge sum rule, Eq. (35) [10] (horizontal lines). Different n_f numbers of active flavors of the so-called "orphan quarks" at $z = 0$ correspond to the scales Q^2, respectively.

assumption from Eq. (32) which relates $D^{\Lambda}_{u\text{val}}$ to $D^{p}_{u\text{val}}$, may be too rough.

In our analysis we avoid dealing with the region of the small $z \to 0$, where the parton to hadron fragmentation functions are unknown, by using the truncated at $z_{\min} \sim 0.05$ contributions to the charge and isospin sum rules. We note a good agreement of the phenomenological results with the theoretical predictions near the smallest accessible experimental $z_{\min}$.

Our results seem to favor the possible cancellations of Wilson line contributions between quarks and antiquarks for valence fragmentation functions. This resolves the problem of violation of the charge sum rules for FFs coming from the final-state hadronization at $z \to 0$ raised in Ref. [10]. Nevertheless, let us finally present a possible modification of the charge sum rule proposed in Ref. [10] and its impact on our tests.

To make the charge sum rule valid despite the mismatch of final-state quark numbers, its modified form incorporates the so-called "orphan quarks" j at $z = 0$:

$$\sum_h Q_h \int_0^1 dz D_i^h(z, Q^2) \approx Q_i - \frac{1}{n_f} \sum_{j\ \text{active}} Q_j, \qquad (35)$$

where n_f is the number of all active j-quark flavors. In this way, the modified charge sum rule, Eq. (35), in a contrary to the standard one, Eq. (4), depends on the scale Q^2. This is shown in Fig. 6 where we compare the truncated contributions to the charge sum rules for the u- and d-quark for DSEHS (DEHSS, DSS) FFs at different scales Q^2 with the modified charge sum rule, Eq. (35), for different numbers of active flavors n_f of the so-called "orphan quarks" at $z = 0$. The scales $Q = 1$, 10, and 91 GeV are taken to be below the charm threshold, $Q^2 < m_c^2$, between the charm and bottom ones, $m_c^2 < Q^2 < m_b^2$, and above the bottom one, $Q^2 > m_b^2$, respectively. These correspond to $n_f = 3$, 4, and 5, respectively.

If the only active quark flavors are u, d, and s ($n_f = 3$), then the subtracted term in Eq. (35) vanishes and the charge sum rule, Eq. (4), is recovered. If $n_f = 4$ or 5, one has $Q_i \approx 0.5$ or 0.6 for the u quark, and $Q_i \approx -0.5$ or -0.4 for the d quark.

The valence FFs under study, being nonsinglets, do not mix with other FFs in their Q^2 evolution and this property holds in all orders in pQCD. Therefore, the thresholds for heavy quarks enter into evolution only via the running coupling constant $\alpha_s(Q^2)$. From Fig. 6, we observe that the modification of the charge rule, Eq. (35), does not have much impact on the results of our phenomenological analysis.

V. CONCLUSIONS

We demonstrated that the charge sum rules for the quark fragmentation functions hold, including simultaneously the contributions of mesons and baryons providing the conservation of the strangeness, electric, and baryon charges, Eq. (26). We also obtained the expression for the isospin conservation, Eq. (28). The results are compatible to Gell-Mann–Nishijima formulas for quarks and hadrons, manifesting a new aspect of quark-hadron duality.

Using our results, we formulated the constraints for $D^h_{u\text{val}}(z, Q^2)$, Eq. (33), where h denotes mesons π and K and baryons p, n, and Λ. The numerical estimates based on some recent parametrizations of FFs confirm these constraints and also are in agreement with the truncated contributions to the charge and isospin sum rules.

We also discussed the possible cancellations of Wilson line contributions between quarks and antiquarks for valence fragmentation functions to resolve the problem of violation of the charge sum rules for FFs coming from the small-z region.

Finally, we presented a possible modification of the charge sum rule incorporating the so-called "orphan quark" FFs and discussed its impact on our phenomenological tests.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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