

(2 + 2)D collective model based on a relativistic Boltzmann equation in the isotropization-time approximation: COMBOLT-ITAS. F. Taghavi¹, S. M. A. Tabatabaee Mehr², and F. Taghinavaz²¹*TUM School of Natural Sciences, Technische Universität München, Garching, Germany*²*School of Particles and Accelerators, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5531, Tehran, Iran*

(Received 20 May 2025; revised 12 December 2025; accepted 27 February 2026; published 26 March 2026)

A new model based on the relativistic Boltzmann equation in the isotropization time approximation is developed to investigate the collective behavior of the quark-gluon plasma produced in high-energy heavy-ion collisions. The equation is solved in (2 + 2)D (two spatial and two momentum-space dimensions). This framework couples preequilibrium dynamics with hydrodynamic evolution to simulate the dynamics of quasiparticle evolution. A numerical scheme based on the method of characteristics enables the evolution to begin from a specified initial Boltzmann distribution. In this work, the spatial structure of the initial distribution is modeled using the TRENTO framework. Our results show that a medium initialized at τ_0 on the order of 1 fm/c with a small shear viscosity to entropy density ratio ($\eta/s = 0.008$) evolves consistently with hydrodynamic simulations, such as those performed using the VISH2+1 code, while discrepancies arise for a medium with $\eta/s = 0.8$. Furthermore, when initialized with a highly anisotropic momentum distribution in the longitudinal direction at early times, the system exhibits spatially nonuniform thermalization in the transverse plane, leading to the emergence of a nontrivial hypersurface that marks the onset of hydrodynamic applicability. Finally, we compute the p_T -spectra for a nonfluctuating initial condition using the hybrid version of COMBOLT-ITA. In this hybrid setup, the description is switched from quasiparticles to hadrons, and URQMD is used to model the hadron gas dynamics. We compare these results with those obtained from the hybrid VISH2+1 initialized within the same setup. For a small shear viscosity, $\eta/s = 0.08$, the two results show a good level of consistency, whereas for a larger value, $\eta/s = 0.8$, a noticeable discrepancy emerges.

DOI: [10.1103/35t1-lcws](https://doi.org/10.1103/35t1-lcws)**I. INTRODUCTION**

Many experimental and theoretical studies have shown that a new state of matter, the Quark-Gluon Plasma (QGP), is created in high-energy heavy-ion collisions [1–8]. In this phase, quarks and gluons are deconfined and behave like an almost perfect fluid, characterized by a small η/s [9–13]. Relativistic hydrodynamics provides a robust framework for modeling the system's evolution during the intermediate, near-equilibrium stages [14,15]. This approach has proven highly effective in describing key phenomena in high-energy nuclear collisions, including collective flow in large systems, as well as the production of thermal photons and dileptons [16–19].

Experimental studies have established the presence of nonzero long-range correlations among final-state particles in small collision systems, including p -Pb and p - p collisions [20–23]. Harmonic flow coefficients can parametrize these correlations [24,25], thereby examining the hydrodynamic models. Evidence for analogous long-range near-side

correlations has also been reported in e^-e^+ collisions [26,27], as well as in e^-p and γp collisions and high-multiplicity jets [28–30]. For detailed discussions, we direct the reader to [31,32]. One interpretation of these correlations involves hydrodynamic models [33–35]. However, the validity of hydrodynamic descriptions for collective behaviors of these systems remains theoretically challenging.

In a perturbative manner, relativistic hydrodynamics is an effective theory description for the dynamics of many-body systems when the microscopic and macroscopic scales are well separated. This separation is quantified in terms of the Knudsen number, which is a ratio between microscopic and macroscopic length scales. Taking into account the studies of hydrodynamic description for small systems, the main question is the domain of validity of hydrodynamics. This domain is represented by the transport coefficients and other model parameters. There have been various studies to show that this domain is much bigger for larger systems than smaller systems [36,37]. On the other hand, smaller collisions have smaller volumes and survive for a very short time, rendering the applicability of fluid dynamics questionable. Likewise, the relativistic hydrodynamic formulation faces fundamental challenges regarding stability and causality in the Navier-Stokes equations [38,39]. Recent theoretical developments have proposed modified hydrodynamic approaches to address these limitations [40,41]. Nevertheless, numerical

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

investigations of necessary conditions to fulfill the causality criteria still encounter regions where these conditions are violated [42–44], highlighting ongoing challenges in hydrodynamic modeling.

Given these conceptual and practical limitations, a central question is how and when a many-body QCD system transitions from far-from-equilibrium dynamics to a regime where hydrodynamics becomes a reliable description. A dynamical approach to initializing the hydrodynamic energy density and flow fields has been proposed in [45] for high-energy nuclear collisions. Recent progress in QCD kinetic theory has provided a quantitative framework for studying this transition, often referred to as hydrodynamization (for recent reviews, see [46–48]). In this context, the inverse Reynolds number, Re^{-1} , has emerged as a useful measure of the relative importance of non-hydrodynamic to hydrodynamic modes and is therefore frequently used to characterize the onset of hydrodynamic behavior.

The collective behavior observed in small systems can be interpreted through various approaches. Prominent explanations include initial-state correlations [49], partonic scattering and the formation of color flux tubes [50], and hydrodynamic flow [51]. An alternative framework for describing such collective flow is kinetic theory. Kinetic theory provides a suitable framework for studying the evolution of the energy-momentum tensor, as the moments of the Boltzmann equation are directly related to its components and have proved to be remarkably successful in reproducing thermalization and collective behavior. Early work demonstrated that local thermal equilibrium in boost-invariant systems can be achieved using relativistic kinetic theory within the relaxation-time approximation [52]. Furthermore, recent studies have established that kinetic theory in the relaxation-time approximation can reproduce hydrodynamic modes for various scenarios through careful selection of momentum and relaxation time parameters [53–59]. Remarkably, numerical solutions of kinetic theory within the isotropization-time approximation (ITA) framework show strong potential to explain the large elliptic flow observed in proton-proton and proton-nucleus collisions [60]. Additionally, the ITA approach has proven to be a good method to investigate the relative weight of particlelike and fluidlike excitations to collective flow in small and large system sizes [57]. More importantly, kinetic theory offers new perspectives on hydrodynamic model limitations. Numerical solutions of the Boltzmann equation reveal that collective flow in high-energy collisions exhibits opacity dependence [57,61–63], directly testing relativistic hydrodynamics’ validity for interpreting flow observables across different system sizes.

Given the significant role of kinetic theory in explaining collective phenomena, this work explores its potential as a unified framework for studying the dynamical properties of nuclear matter in high-energy collisions. We numerically solve the Boltzmann equation within the isotropization-time approximation for a boost-invariant system of massless quasiparticles. Following established approaches, the collision kernel is chosen to probe both the near-free-streaming regime and the close-to-ideal hydrodynamic limit. The model can also be applied to systems with highly anisotropic momentum-

space distributions. Notably, the code reproduces the behavior of other hydrodynamic models within the limits of ideal hydrodynamics. Starting from a highly anisotropic momentum distribution at very early times, we demonstrate that different regions of the evolving matter reach local equilibrium at different times, resulting in a nontrivial geometry for the onset of hydrodynamic applicability.

In the present study, we introduce a Boltzmann equation solver in $(2+2)\text{D}$ called COMBOLT-ITA, *A Collective Model based on the relativistic Boltzmann equation in the Isotropization Time Approximation*. To apply the model in more realistic scenarios, we also introduce a hybrid version by switching the quasiparticle description to a hadronic gas, whose evolution is studied using URQMD [64,65]. This allows for a complete description of the system from the early-time dynamics to the final hadronic stage. The hybrid model, in which COMBOLT-ITA is coupled to additional components, is available in the public GitHub repository [66]. Using the hybrid model, we study the p_T spectrum of a nonfluctuating system and compare it with results from VISH2+1. In a followup study [67], the same setup has been used to describe recent measurements of OO collisions at the Large Hadron Collider (LHC) [68–70] on an event-by-event fluctuating basis.

The organization of this paper is as follows. In Sec. II, we elaborate on the solving procedure for the Boltzmann equation, including details of the initial-state construction, the method of characteristics, and other numerical inputs. In Sec. III, we present numerical results from COMBOLT-ITA and compare them with those obtained from VISH2+1 simulations for different values of η/s . In Sec. IV, we explain the results from momentum-anisotropic initial conditions and discuss the criteria for reaching hydrodynamic behavior. The p_T spectrum and the hybrid model are described in Sec. V. Finally, we summarize our findings and provide an outlook for future work in Sec. VI.

II. SOLVING THE $(2+2)\text{D}$ BOLTZMANN EQUATION

This section presents the framework used to solve the boost-invariant Boltzmann equation, including a description of the initial conditions and the numerical approach.

A. Boost-invariant Boltzmann equation

The Boltzmann distribution of massless quasiparticles, $f(x^\mu, p^\mu)$ with constraint $p^\mu p_\mu = 0$, satisfies the equation $p^\mu \partial_\mu f = -p^\mu u_\mu (f - f_{\text{eq}})/\tau_{\text{relax}}$, where τ_{relax} is the timescale in which the Boltzmann distribution tends to the equilibrium distribution. In this study, we frequently use Milne coordinates $(\tau, x_\perp, \eta_s)$ in which proper time and pseudorapidity are defined as $\tau = \sqrt{t^2 - z^2}$ and $\eta_s = \text{arctanh } z/t$. We use flat space-time metric $g_{\mu\nu} = \text{diag}(1, -1, -1, -\tau^2)$. The momentum components in the Milne coordinate are given as $p^\tau = p_\perp \cosh(y - \eta_s)$ and $p^\eta = p_\perp/\tau \sinh(y - \eta_s)$, where the rapidity is defined as $y = \text{arctanh } p^z/p^t$. We conventionally use the following boost-invariant quantities, in the momentum space:

$$w = p^\tau, \quad v_z = \tau p^\eta/p^\tau, \quad \phi_p = \arctan p^y/p^x. \quad (1)$$

The momentum of a massless quasiparticle moving with velocity z/t is given as $p^\mu = wv^\mu$ where four-velocity v^μ at $z = 0$ reads as

$$v^\mu = \left(1, \sqrt{1 - v_z^2} \cos \phi_p, \sqrt{1 - v_z^2} \sin \phi_p, v_z\right), \quad (2)$$

which means w and v_z are energy and longitudinal velocity in a comoving frame of a longitudinally expanding medium.

Since we are interested in the dynamics of the energy momentum tensor, one can integrate out the effect of w and define [57]

$$F(\tau, x_\perp, v_z, \phi_p) = \frac{1}{4\pi^2} \int_0^\infty w^3 dw f(\tau, x_\perp, w, v_z, \phi_p). \quad (3)$$

In this case, the energy-momentum is obtained via

$$T^{\mu\nu} = \frac{1}{2\pi} \int_0^{2\pi} d\phi_p \int_{-1}^1 dv_z v^\mu v^\nu F, \quad (4)$$

where $v^\tau = 1$ and $v^\eta = v_z/\tau$. For massless quasiparticles, one can write the Boltzmann equation in terms of the w integrated Boltzmann distribution F as follows [57]:

$$\begin{aligned} \partial_\tau F + \sqrt{1 - v_z^2} \cos \phi_p \partial_x F + \sqrt{1 - v_z^2} \sin \phi_p \partial_y F \\ - \frac{v_z(1 - v_z^2)}{\tau} \partial_{v_z} F = -\frac{4v_z^2}{\tau} F + C[F], \end{aligned} \quad (5)$$

where the collision kernel is given as

$$C[F] = -\gamma \left(u^\mu v_\mu \epsilon^{1/4} F - \frac{\epsilon^{5/4}}{(u^\mu v_\mu)^3} \right). \quad (6)$$

In the above, ϵ is the local energy density and u^μ is the fluid velocity, obtained from the Landau matching condition $T_v^\mu u^\mu = \epsilon u^\nu$. The parameter γ is written in terms of the relaxation time $\gamma = \tau_{\text{relax}}^{-1} \epsilon^{-1/4}$. As shown in Eq. (6), the collision kernel is independent of the specific form of the equilibrium distribution for a system exhibiting conformal symmetry. For this reason, the timescale τ_{relax} is often referred to as the *isotropization time* instead of the relaxation time. Consequently, the solution of the Boltzmann equation is obtained using the *isotropization time approximation* [60].

We can use γ as a free parameter. However, to capture the connection with hydrodynamic calculations more clearly, we follow Ref. [55] and assume $\tau_{\text{relax}} = 5(\eta/s)/T$ where η/s is the shear viscosity over the entropy density. Then assuming equation of state of a conformal medium, $\epsilon = C_0 T^4$, one finally finds

$$\gamma = \frac{1}{5} \frac{1}{C_0^{1/4} \eta/s}. \quad (7)$$

To be consistent with [62], we choose $C_0 = 13.9$, which can be estimated from a non-interacting gas of massless quarks and gluons (see [71]).

B. Initial state

Before explaining the numerical method used to solve Eq. (5), we will discuss how to prepare the initial state. Although the numerical method can be initiated from any

function $F_{\text{init}}(x_\perp, v_z, \phi_p)$, we will focus on a factorized initial state in the present study. In this approach, the spatial part of the distribution is initialized using a model such as MC-GLAUBER [72,73] or TRENTO [74]. The momentum part is incorporated using a smooth function that includes a free parameter, which controls the longitudinal momentum anisotropy.

We assume the initial Boltzmann distribution can be written as follows:

$$F_{\text{init}}(x_\perp, v_z, \phi_p) = 2\epsilon_0(x_\perp) \mathcal{P}_0(v_z, \phi_p), \quad (8)$$

where $\epsilon_0(x_\perp)$ is the initial energy density and is defined on a 2D Cartesian grid and can be obtained from an initial state model such as the TRENTO event generator. Here, we assume $\epsilon_0(x_\perp)$ is normalized to ϵ_{tot} and $\mathcal{P}_0(v_z, \phi_p)$ is normalized to unity. The factor 2 appears to set $T^{\tau\tau}$ in Eq. (4) equal to ϵ_0 at the initial time.

We require that the initial momentum distribution be isotropic in the transverse direction; however, it may be anisotropic in the longitudinal direction. To satisfy our requirement, we define the momentum distribution as follows:

$$\mathcal{P}_0(v_z, \phi_p) = (2\lambda \tanh^{-1} [\sinh(1/\lambda)] \cosh(v_z/\lambda))^{-1}. \quad (9)$$

The parameter λ is defined to control the degree of isotropy. For $\lambda \rightarrow \infty$, the distribution is uniform in the range $|v_z| \leq 1$ while for $\lambda \rightarrow 0$, the distribution approaches $\delta(v_z)$. Using Eq. (4), the initial energy-momentum tensor for $\lambda \rightarrow \infty$ is isotropic,

$$T_0^{\mu\nu}(x_\perp) = \epsilon(x_\perp) \text{diag}(1, 1/3, 1/3, 1/3\tau^2), \quad (10)$$

while for $\lambda \rightarrow 0$ one finds zero pressure in the longitudinal direction,

$$T_0^{\mu\nu}(x_\perp) = \epsilon(x_\perp) \text{diag}(1, 1/2, 1/2, 0). \quad (11)$$

The impact of the initial state anisotropy on the dynamics of the energy-momentum tensor will be discussed later in this study.

In the TRENTO event generator, the so-called reduced thickness function is given as

$$T_R(x_\perp) = \mathcal{N} \left(\frac{T_A^p(x_\perp) + T_B^p(x_\perp)}{2} \right)^{1/p}, \quad (12)$$

where p is a real valued free parameter, and $\mathcal{N}$ is the overall normalization. The participant thickness functions $T_{A(B)}$ are the sum of the participant nucleons, which are assumed to have a Gaussian profile with width w . Here, we set $p = 0$ and $w = 0.7$ fm, in agreement with Bayesian analyses estimations [75–78]. We conventionally consider

$$\epsilon_0(x_\perp) = \frac{1}{\tau_0} T_R(x_\perp), \quad (13)$$

meaning, we assumed $T_R(x_\perp)$ has dimension energy⁻³. The overall normalization is set to $\mathcal{N} = 15$, with the nucleon-nucleon inelastic cross section chosen for $\sqrt{s_{NN}} = 2.76$ TeV. This normalization leads to a reasonable estimate of the total initial energy density. However, the actual value of the normalization needs to be carefully determined by comparing the results with experimental data in the future.

C. Numerical method of solving the Boltzmann equation

We use the method of characteristics to solve Eq. (5). The method is the extended version of that in Ref. [57] in which the moment expansion in the spatial direction of the initial state was used.

1. Characteristic curves

Equation (5) represents an integrodifferential equation. The collision kernel is dependent on energy density and fluid velocity, which are derived from integration over F . We still, however, can write this equation formally in the following way:

$$\begin{cases} \frac{d\mathbf{y}(s)}{ds} = \mathbf{Q}(\mathbf{y}(s)), \\ \mathbf{y}(0) = \mathbf{y}_0, \end{cases} \quad (14)$$

where

$$\mathbf{y}(s) = \begin{pmatrix} \tau(s) \\ x(s) \\ y(s) \\ v_z(s) \\ \phi_p(s) \\ F(s) \end{pmatrix}, \quad \mathbf{y}_0 = \begin{pmatrix} \tau_0 \\ x_0 \\ y_0 \\ v_{z,0} \\ \phi_{p,0} \\ F_{\text{init}} \end{pmatrix},$$

$$\mathbf{Q}(\mathbf{y}) = \begin{pmatrix} 1 \\ \sqrt{1-v_z^2} \cos \phi_p \\ \sqrt{1-v_z^2} \sin \phi_p \\ -\frac{v_z(1-v_z^2)}{\tau} \\ 0 \\ -\frac{4v_z^2}{\tau} F + C[F] \end{pmatrix}. \quad (15)$$

Vector $\mathbf{y}(s)$ represents a characteristic curve traversing the $(\tau, x, y, v_z, \phi_p, F)$ space, each curve is labeled by $(x_0, y_0, v_{z,0}, \phi_{p,0})$. The collection of all characteristic curves forms the final solution, each parametrized by s . The final result is achieved by expressing $(s, x_0, y_0, v_{z,0}, \phi_{p,0})$ in terms of the variables $(\tau, x, y, v_z, \phi_p)$.

2. Free-streaming coordinates and free-streaming solution

In the absence of collision, Eq. (14) reduces to linear differential equation and can be easily solved. The first five components of Eq. (14) can be solved independently of F , which leads to $\tau = s + \tau_0$ and the free-streaming coordinates introduced in Ref. [57],

$$x_0 = x - \frac{\cos \phi_p}{\sqrt{1-v_z^2}} [\tau - \tau_0 V(\tau, \tau_0, v_z)], \quad (16a)$$

$$y_0 = y - \frac{\sin \phi_p}{\sqrt{1-v_z^2}} [\tau - \tau_0 V(\tau, \tau_0, v_z)], \quad (16b)$$

$$v_{z,0} = \tau v_z / [\tau_0 V(\tau, \tau_0, v_z)], \quad (16c)$$

$$\phi_{p,0} = \phi_p, \quad (16d)$$

where

$$V(\tau, \tau_0, v_z) = \sqrt{1 - (1 - \tau^2/\tau_0^2)v_z^2}. \quad (17)$$

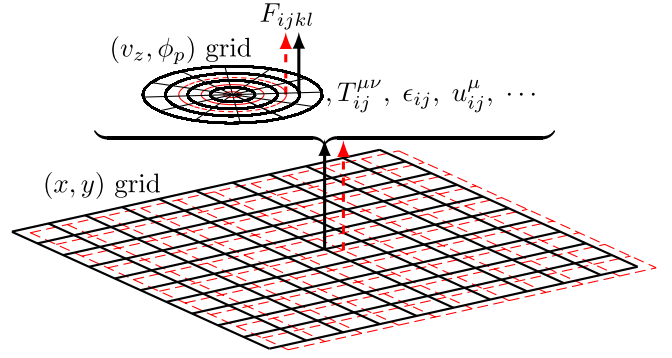


FIG. 1. The grid structure of discretized variables. At each time step, F is defined on a 4D grid, a spatial 2D Cartesian grid times a 2D polar grid. Energy-momentum tensor and hydrodynamic variables are defined on the Cartesian spatial part. The new grid after one time step is shown by red dashed lines.

We use $\mathbf{x}$ as the shorthand notation for spatial and momentum coordinates, which we call laboratory coordinates in contrast to free-streaming coordinates $\mathbf{x}_0$. Using this notation, we show Eqs. (16) as $\mathbf{x}_0(\tau, \mathbf{x})$. The inverse function theorem ensures that $\mathbf{x}(\tau, \mathbf{x}_0)$ uniquely exists. By interchanging $\mathbf{x} \leftrightarrow \mathbf{x}_0$ and $\tau \leftrightarrow \tau_0$, one finds $\mathbf{x}(\tau, \mathbf{x}_0)$.

Using the last component of Eq.(14), we find the time evolution of the Boltzmann distribution,

$$F(\tau, \mathbf{x}) = \frac{F_{\text{init}}(\mathbf{x}_0(\tau, \mathbf{x}))}{V^4(\tau, \tau_0, v_z)}. \quad (18)$$

This is a wave-like solution in which, given the initial state, the Boltzmann distribution is known at any time.

3. 4D Grid and the grid points time dependence

We use a Cartesian grid in (x, y) space and a polar grid in (v_z, ϕ_p) space (see Fig. 1). We only show the $v_z > 0$ grids in the figure for simplicity. The discretized Boltzmann distribution F_{ijkl} is set up on this four-dimensional grid. The hydrodynamic variables are obtained after integration along (v_z, ϕ_p) ; therefore, they are defined only on the Cartesian grid. We assume $-c_{\text{max}} < (x, y) < c_{\text{max}}$, meaning that space is divided into N_c parts. We also start an equally spaced grid in $\phi_p \in [0, 2\pi)$ and $v_z \in [-1, 1]$ directions. The distance between two successive points in v_z grid, however, is not fixed, as we discuss in the following.

As time advances, the coordinates at $\mathbf{x}_0$ modify along the characteristic curves and reach the point $\mathbf{x}(\tau, \mathbf{x}_0)$. The coordinate ϕ_p remains unchanged but $v_{z,0}$ turns into $v_z = \tau v_{z,0} / [\tau_0 V(\tau, \tau_0, v_{z,0})]$ [see Eq. (16c)]. For $\tau > \tau_0$, we have $v_z < v_{z,0}$ except for $v_{z,0} = 0$ and $|v_{z,0}| = 1$. The red dashed circles in Fig. 1 indicate the locations of new v_z at time τ compared to their initial values at τ_0 shown by black circles. The grid points $|v_{z,0}|$ close to unity move faster inwards. This means that after some time the initial equally spaced v_z points are more and more concentrated around $v_z = 0$. In numerical calculations, it is important to consider this shift to avoid losing information about the distribution dynamics at large v_z values.

Depending on the value of (v_z, ϕ_p) , located at the tail of the black arrow pointing to F_{ijkl} in Fig. 1, all points on the spatial grid are shifted in the same ϕ_p direction and with same magnitude $[\tau_0 - \tau V(\tau_0, \tau, v_{z,0})]/\sqrt{1 - v_{z,0}^2}$. This is shown by a red dashed grid in the figure. As a result, the distance between the spatial grid points remains unchanged while the whole 2D grid moves, depending on the value of v_z .

4. Numerical solution in the presence of interaction

In the presence of interaction, we are still able to solve the first five components in Eq. (14), which leads to the same results as for the free-streaming coordinate. The last equation is no longer trivial and is written as

$$\begin{cases} \frac{dF(s, \mathbf{x}_0)}{ds} = -\frac{4v_z^2(s, \mathbf{x}_0)F(s, \mathbf{x}_0)}{s + \tau_0} + C[F(s, \mathbf{x}_0)] \\ F(0, \mathbf{x}_0) = F_{\text{init}}(\mathbf{x}_0). \end{cases} \bigg|_{\substack{\mathbf{x}_0 = \mathbf{x}_0(\tau, \mathbf{x}) \\ s = \tau - \tau_0}} \quad (19)$$

We aim to solve the above integrodifferential equation numerically. It is important to note that the equation is formulated in free-streaming coordinates. Once the solution F is obtained in this frame, it must be transformed into laboratory coordinates. To proceed, we approximate and rewrite Eq. (19) as

$$\begin{aligned} F(s + h, \mathbf{x}) \\ = \left[F(s, \mathbf{x}_0) - \frac{4v_z^2(s, \mathbf{x}_0)F(s, \mathbf{x}_0)}{s + \tau_0} + C[F(s, \mathbf{x}_0)] \right]_{\substack{\mathbf{x}_0 = \mathbf{x}_0(\tau, \mathbf{x}) \\ s = \tau - \tau_0}}. \end{aligned} \quad (20)$$

As discussed, the initial grid points at s move to the new location at $s + h$ along the characteristic curves. Therefore, we need to interpolate the distribution back to the original grid at each step to find hydrodynamic variables. Given that the free-streaming and laboratory coordinates are equivalent at initial time $\tau = \tau_0$, we perform the following recursive calculations:

- (1) At time step n , consider $F(\tau_n, \mathbf{x}_0)$ as the initial state $s = 0$, defined on the original grid $\mathbf{x}_0$.
- (2) Find $T^{\mu\nu}$ from Eq. (4).
- (3) Find ϵ and u^μ from the Landau matching condition $T_v^\mu u^\mu = \epsilon u^\nu$.
- (4) Find the right-hand side of Eq. (19) over all points $\mathbf{x}_0$ on the 4D grid and assign the result to distribution $F(\tau_{n+1}, \mathbf{x})$, where $\mathbf{x} = \mathbf{x}(\tau, \mathbf{x}_0)$.
- (5) Find $F(\tau_{n+1}, \mathbf{x}_0)$ by determining the value of the distribution in step 4 on the original grid (x_0, y_0) through interpolation.

The starting state of this recursive process is $F(\tau_0, \mathbf{x}_0) = F_{\text{init}}(\mathbf{x}_0)$.

For step number 2, we perform the integral (4) via trapezoidal approximation in v_z and ϕ_p directions. We note that the Landau matching in step number 3 is equivalent to an eigenvalue problem. The fluid velocity is the only timelike eigenvector of T_v^μ and the corresponding eigenvalue is the positive energy density. We use the eigenvalue solver provided

by the GNU Scientific Library (GSL) [79] for computing the eigenvalues and eigenvectors of T_v^μ . For the interpolation in step 5, we use a two-dimensional bicubic spline provided by GSL.

In addition to the above general remarks, we should add the following points. As discussed, the grid point length is modified in the v_z direction. The segments close to $|v_z| \lesssim 1$ grow over time, and we lose accuracy in this part of the distribution. To keep track of the dynamics of the Boltzmann distribution, we insert two new grid points at the center of the last segments in the positive and negative directions of the v_z grid when the segment becomes larger than 0.15. The numerical value is fixed by optimizing the numerical solution's performance and accuracy. One can choose larger values for systems closer to the free-streaming. The value of the Boltzmann distribution at the new points is obtained by cubic spline interpolation.

Steps 1 to 5 can be regarded as the Euler method, which advances a distribution from time τ_n to τ_{n+1} for one time step. However, it is possible to go beyond the Euler method by employing multistep Runge-Kutta methods with adaptive step sizes, such as the Dormand-Prince method, with only slight modifications to the five steps. Our preliminary investigation indicates that using a more complex method does not significantly enhance performance. We will save a deeper exploration of this topic for another study. Instead of using an adaptive step size, we use the following strategy: at step n , we choose

$$h_n = h_\infty \left(\frac{\tau_n}{\tau_n + h_\infty / \alpha_h} \right), \quad (21)$$

where α_h controls the growth rate of the time step size at the beginning. For small τ_n , the step size is proportional to τ_n as $h_n = \alpha_h \tau_n$. For large τ_n the step size approaches h_∞ . In the present study, we mostly use $\alpha_h = 0.02$ and $h_\infty = 0.005 \text{ fm}/c$.

III. EVOLUTION OF A SYSTEM WITH COMBOLT-ITA AND VISH2+1

In heavy-ion physics, the hydrodynamic stage typically begins after a short preequilibrium stage, around $\tau_0 \approx 1 \text{ fm}/c$, during which the medium is expected to become isotropized. Although the dynamics of the pre-equilibrium phase generate a complex energy-momentum tensor, we can start hydrodynamics at approximately $1 \text{ fm}/c$ with the assumption that the initial state, characterized by $\lambda \gg 1$, resembles an approximate initial condition. In this context, we anticipate that solving the Boltzmann equation reproduces results obtained using hydrodynamic equations, especially for near-ideal hydrodynamic evolution. To demonstrate this, we compare the results of the VISH2+1 model [71] with our model. We begin by considering the energy density illustrated in Fig. 2 (left and middle panels) as the starting point at $\tau_0 = 1.0 \text{ fm}/c$ for the evolution of an isotropic medium. In the VISH2+1 model, we apply the ideal equation of state, given by $\epsilon = C_0 T^4$, where $C_0 = 13.9$, consistent with the value used in COMBOLT-ITA. For initializing COMBOLT-ITA, we set $\lambda = 1000$, resulting in an isotropic initial momentum distribution. We then simulate three different values of $\eta/s = 0.008, 0.08$, and 0.8 .

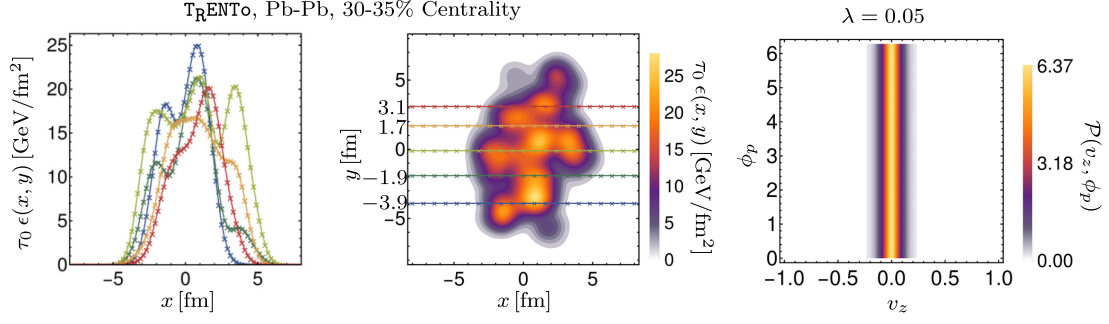


FIG. 2. The initial state of a Pb-Pb collision in 30–35% centrality obtained from the T_RENTo event generator (middle panel), and five slices along x direction (left panel). The initial momentum distribution for a highly anisotropic initial state (right panel).

The comparison between energy density obtained from COMBOLT-ITA and VISH2+1 is illustrated in Fig. 3. We present three time snapshots, at $\tau = 4.0, 5.0$, and 5.5 fm/c, where the differences between the two models become more evident. Curves in various colors represent different slices of energy density in the y direction, as shown in Fig. 2. It is noteworthy that the energy density dynamics of both models align closely, with the best agreement observed for the lowest value of η/s , as expected. However, even for $\eta/s = 0.008$, we still notice that the initial structures of the energy density become smoother in the VISH2+1 calculations, whereas the details persist in COMBOLT-ITA. This raises the question of whether numerical viscosity in the VISH2+1 calculations, other numerical artifacts, or a different relaxation process in the two approaches accounts for this observed difference. The difference is more evident for the largest value of $\eta/s = 0.8$, as expected [61]. In particular, we notice the most significant difference between the two models at a later time, $\tau = 5.5$ fm/c. In Fig. 4, we present a comparison between the COMBOLT-ITA and VISH2+1 calculations for the average transverse velocity $\{u_\perp\} = \{\sqrt{u_x^2(\tau, x_\perp) + u_y^2(\tau, x_\perp)}\}$ and the momentum anisotropy ϵ_p . Here, we use the notation

$$\{\mathcal{O}\} = \frac{\int dx_\perp \mathcal{O}(\tau, x_\perp) \epsilon(\tau, x_\perp)}{\int dx_\perp \epsilon(\tau, x_\perp)}.$$

We also define the local momentum anisotropy as

$$\begin{aligned} \hat{\epsilon}_p(\tau, x_\perp) &= \epsilon_{p,x}(\tau, x_\perp) + i\epsilon_{p,y}(\tau, x_\perp) \\ &= \frac{T^{xx}(\tau, x_\perp) - T^{yy}(\tau, x_\perp) + 2iT_{xy}(\tau, x_\perp)}{\int dx_\perp [T^{xx} + T^{yy}]}. \end{aligned} \quad (22)$$

The momentum anisotropy is obtained by integrating local momentum anisotropy over the transverse space,

$$\epsilon_p = \left| \int dx_\perp \hat{\epsilon}_p(\tau, x_\perp) \right|. \quad (23)$$

The quantity ϵ_p serves as a proxy for the elliptic flow v_2 , which represents an anisotropy in the final particle distribution. In Ref. [57], the energy anisotropic flow was defined and calculated from a Fourier series in the azimuthal direction, obtained by integrating the Boltzmann distribution of massless quasiparticles over space and momentum at late times when the azimuthal distribution becomes stable. This is conceptually (though not exactly) similar to ϵ_p , in which

the spatial integration is performed at each time step to construct the energy-momentum tensor, followed by computing the anisotropy among its transverse components. In Ref. [67], an experimentally motivated version of the energy anisotropic flow was defined based on the transverse energy of the particles. In the latter case, this quantity can be measured experimentally or calculated using realistic simulations, such as those used in Ref. [67], where the final states are massive hadrons.

The medium evolution for small values of $\eta/s = 0.008$, as calculated by COMBOLT-ITA and VISH2+1, shows good agreement, which is illustrated by the blue curves in Fig. 4. However, we observe a discrepancy between the two calculations at higher values of η/s . Specifically, VISH2+1 significantly overestimates the momentum anisotropy, ϵ_p , by a factor of 2 for $\eta/s = 0.8$. The figure also displays the results obtained from a medium with a highly anisotropic momentum distribution, as shown in Fig. 2 (right panel), where we set $\lambda = 0.05$. However, due to the rapid equilibration time in a medium close to ideal hydrodynamics, the influence of longitudinal anisotropy is not substantial, particularly on ϵ_p . We observe more differences for media with higher values of η/s .

The typical timescale for the onset of the preequilibrium stage, where an anisotropic momentum distribution is anticipated, is around 0.1 fm/c. We have analyzed a setup similar to that presented in Fig. 4, and the results were quite similar; thus, we will not present them here. Instead, in Fig. 5, we illustrate the evolution of the average inverse Reynolds number for systems with $\lambda = 0.05$ and 1000, initiated at $\tau_0 = 0.1$ and 1.0 fm/c. The effect of anisotropy is evident in this figure. The Re^{-1} parameter starts near one for anisotropic media and close to zero for isotropic media. The curves for both scenarios converge toward a common trend, with convergence occurring more quickly for media with a small η/s ratio. The curves from the two initiation times, although both having the same η/s , approach each other but do not exactly overlap.

IV. EARLY TIME DYNAMICS AND HYDRODYNAMIZATION SURFACE

In this section, we consider a system characterized by strong momentum-space anisotropy. Starting from $\tau_0 = 0^+$ fm/c, we evaluate when the medium becomes isotropized and when Re^{-1} is small. We assume the same initial energy density as shown in the left and middle panels of Fig. 2 and

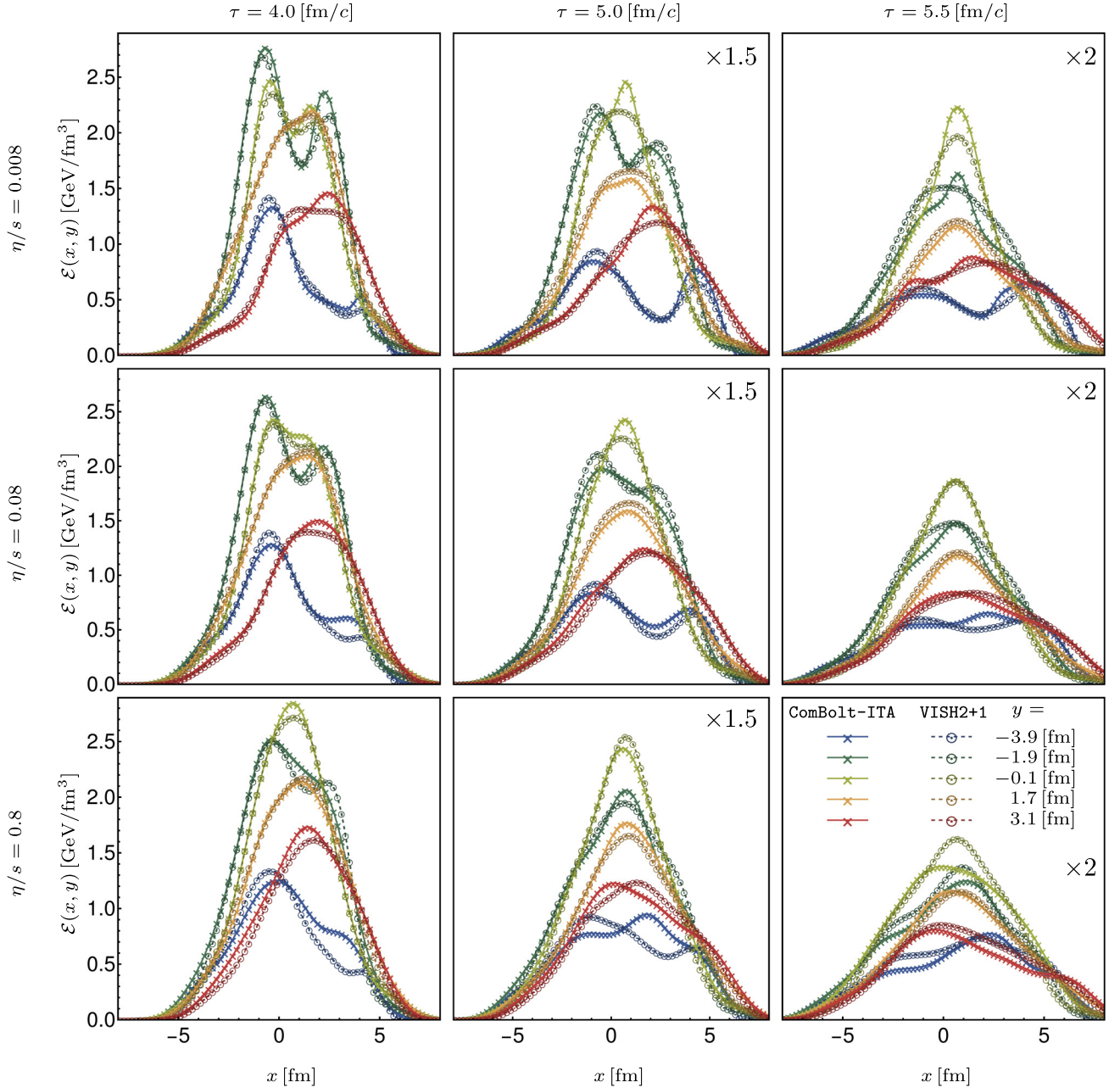


FIG. 3. Comparison of the energy density evolution from the same initial state, initiated at $\tau_0 = 1.0$ fm/c, as calculated by the CoMBolt-ITA and VISH2+1 models. The energy densities are in good agreement for $\eta/s = 0.008$ (top panels), while discrepancies between the two models become more pronounced for $\eta/s = 0.8$ (bottom panels).

a highly anisotropic momentum distribution ($\lambda = 0.05$). We initiate the evolution at $\tau_0 = 0.5 \times 10^{-4}$ fm/c. This extreme condition is outside the applicability of hydrodynamic models, such as VISH2+1.

Starting with a highly anisotropic medium, we quantitatively identify a space-time region within the expanding medium where hydrodynamics can be applied. We utilize the inverse Reynolds number, defined as

$$\text{Re}^{-1}(\tau, x_\perp) = \sqrt{\frac{6\pi^{\mu\nu}(\tau, x_\perp)\pi_{\mu\nu}(\tau, x_\perp)}{\epsilon^2(\tau, x_\perp)}}, \quad (24)$$

where the shear tensor $\pi^{\mu\nu}$ in a conformal system is derived from

$$T^{\mu\nu} = T_0^{\mu\nu} + \pi^{\mu\nu}. \quad (25)$$

Here, $T_0^{\mu\nu}$ represents the energy-momentum tensor in local equilibrium, given by $T_0^{\mu\nu} = (\epsilon + P)u^\mu u^\nu - P g^{\mu\nu}$, where $P = \epsilon/3$ is the pressure. Following the approach outlined in Ref. [62], we choose the normalization condition in Eq. (24) such that for a fully longitudinally anisotropic medium, we find $\text{Re}^{-1} = 1$. For the applicability of hydrodynamics, the medium should be close to the local equilibrium.

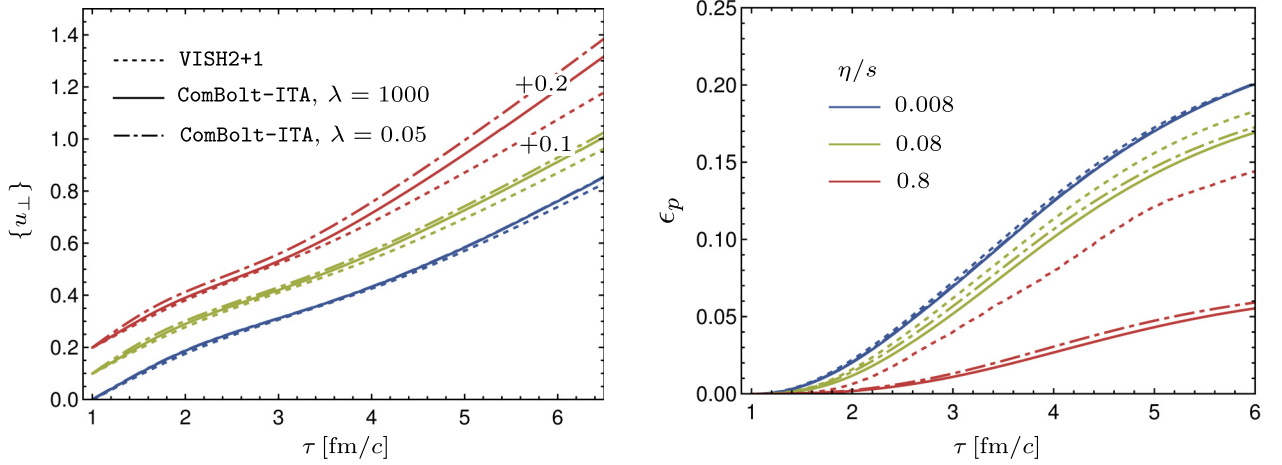


FIG. 4. A comparison of the average transverse velocity and momentum anisotropy evolution, initiated from the same isotropic initial state at $\tau_0 = 1.0$ fm/c, is presented for both COMBOLT-ITA and VISH2+1. There is good agreement between the two calculations for these quantities with $\eta/s = 0.008$. However, the discrepancy is more significant for larger η/s . VISH2+1 overestimates the momentum anisotropy by more than a factor of 2.

The evolution of Re^{-1} for $\eta/s = 0.04$ and 0.08 is depicted in Fig. 6 (upper panel). At the initial time, Re^{-1} reaches the highest value. As time progresses, this value decreases, and the rate of decrease depends on the η/s ratio. The medium with a smaller η/s demonstrates a faster decline toward lower average Re^{-1} , consistent with findings in Ref. [62]. We observe that the tail of the distribution, which corresponds to regions of lower energy density, exhibits a slower decrease. In contrast, the central regions with higher density approach local equilibrium more quickly. We also observe that for $\eta/s = 0.08$ almost all medium regions are more than 15% deviate from the local equilibrium until $\tau \approx 2.5$ fm/c. This time duration reduces to approximately 1 fm/c for medium with $\eta/s = 0.04$. The nontrivial behavior of longitudinal-

to-transverse pressure, as studied in anisotropic hydrodynamics [80], shows features consistent with the Re^{-1} behavior presented here.

In Fig. 6 (lower panel), we illustrate the evolution of the local momentum anisotropy for three values of η/s : 0.008, 0.08, and 0.8. The surface at which the description switches from quasiparticles to hadrons, referred to here as the freeze-out surface (red dashed contour), is defined as the region where the energy density reaches $\epsilon_{\text{sw}} = 0.3$ GeV/fm³ [81]. We also show the contours of $\text{Re}^{-1} = 0.1$ and 0.2 in the figure. Starting from an isotropic initial state in the transverse direction, with $\epsilon_p = 0$, we observe that the medium with a smaller value of η/s develops anisotropy more rapidly and achieves higher anisotropic values during the same time period compared to media with higher η/s . The results indicate that for $\eta/s = 0.008$ the medium is nearly in local equilibrium at the onset of momentum anisotropy development, at around $\tau \approx 0.5$ fm/c, at which the freeze-out surface lies within the region where $\text{Re}^{-1} = 0.2$. This suggests that hydrodynamic models can be employed to study the evolution of this system and accurately capture momentum anisotropy until the entire medium has frozen out. In contrast, for $\eta/s = 0.08$, the area close to local equilibrium moves inside the freeze-out surface by approximately $\tau \approx 2.5$ fm/c. Consequently, the medium remains outside the near-equilibrium domain for a considerable period, during which momentum anisotropy continues to develop. The last row of Fig. 6 (lower panel) indicates that the medium does not approach equilibrium until much later in time. This suggests that modeling the preequilibrium stage as a free-streaming medium may overlook aspects of momentum anisotropy development during that early phase.

We have demonstrated that the medium reaches local equilibrium on a nontrivial surface. To visualize this surface more clearly, we present contours of various values of Re^{-1} in the x - τ plane, enclosed by dashed curves representing the freeze-out surface in Fig. 7. This visualization highlights that the medium approaches local equilibrium on a nontrivial space-time surface rather than at a single characteristic time. It is

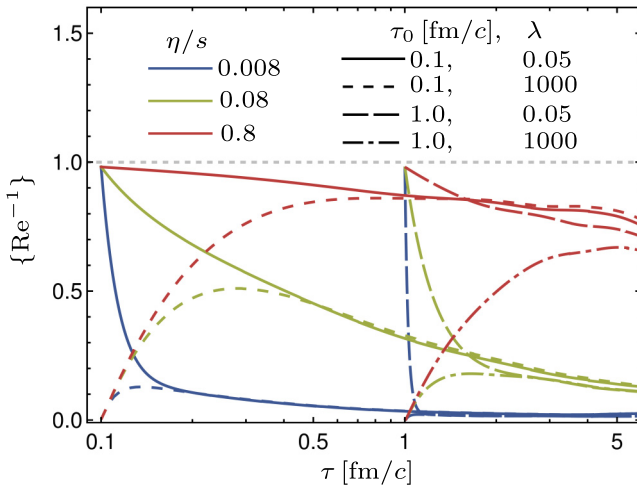


FIG. 5. Average inverse Reynolds number as a function of time for three different values of η/s and two initial times. Both the initial isotropic medium ($\lambda = 1000$) and the highly anisotropic medium ($\lambda = 0.05$) converge to a common curve, with the convergence occurring more rapidly for the medium with a smaller η/s .

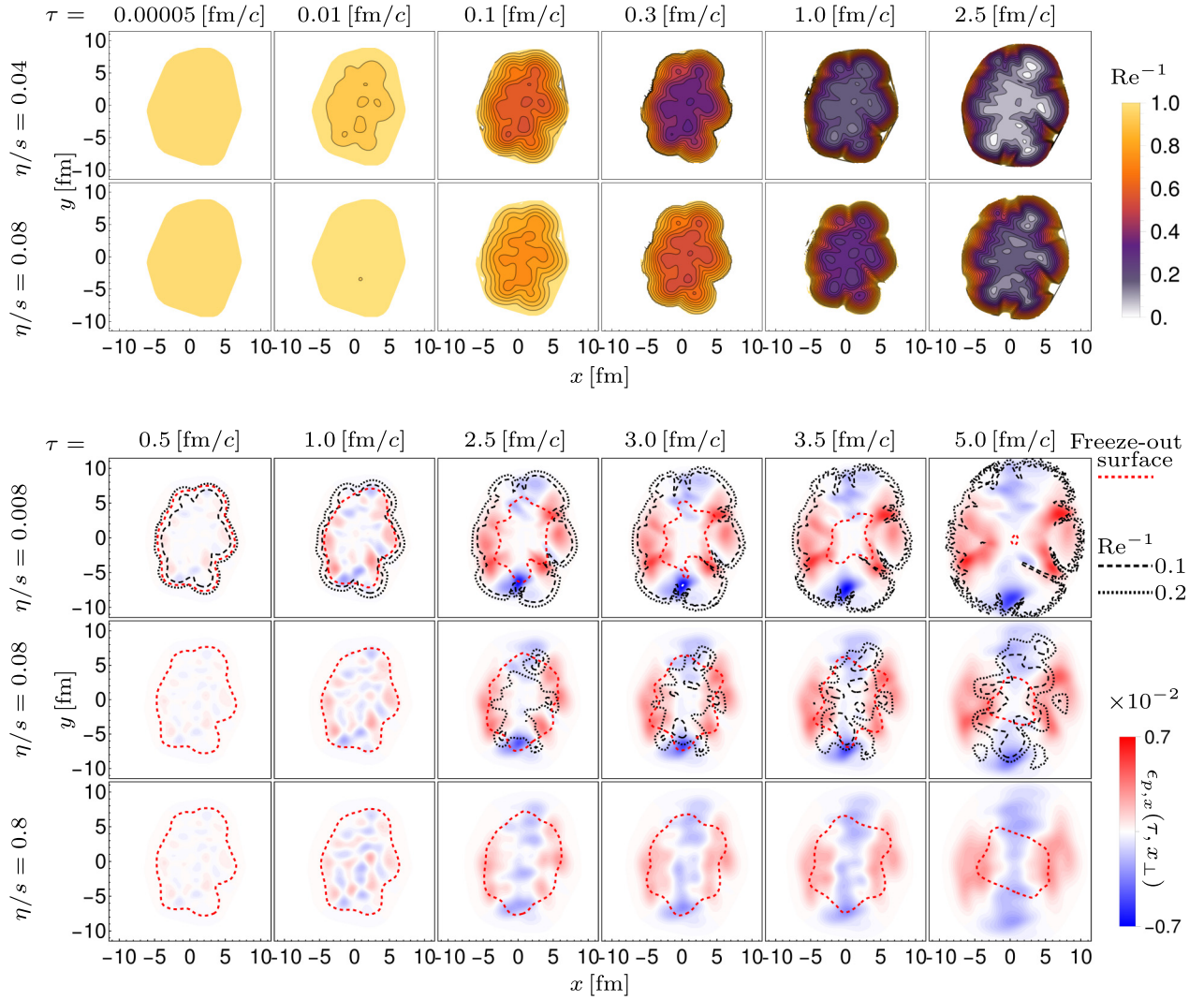


FIG. 6. The time evolution of Re^{-1} for media with $\eta/s = 0.04$ and 0.08 (upper panel), and the time evolution of the local momentum anisotropy for media with $\eta/s = 0.008$, 0.08 , and 0.8 (lower panel). The freeze-out surface and contours of $\text{Re}^{-1} = 0.1$ and 0.2 are shown with red dashed and black dashed curves in the lower panel.

important to emphasize that there is no universal threshold in Re^{-1} that uniquely defines this hydrodynamization surface; different values correspond to different degrees of hydrodynamization. In Ref. [62], the thresholds 0.8, 0.6, and 0.4 were examined for averaged values of Re^{-1} in the transverse plane, and hydrodynamics was assumed to be applicable below these values after a fixed time τ . While such thresholds work reasonably well for large opacities (small η/s), they allow only partial hydrodynamization and begin to fail for systems that remain farther from equilibrium.

The contour in Fig. 7 with the largest value of Re^{-1} for which hydrodynamics remains applicable can be considered the *hydrodynamization surface*. A similar concept was discussed using the Gubser solution in Ref. [35] and in Ref. [82] using KØMPØST. Since KØMPØST is applicable only for a short period of time (see Ref. [62]), our framework allows us to test this idea up to much later times. To identify the most accurate value of Re^{-1} , one would need to perform

an analysis comparing the result with a hydrodynamic code similar to that in Ref. [62], while retaining the full transverse dependence of Re^{-1} , an investigation we leave for future work.

As shown in the figure, the central region of the medium with $\eta/s = 0.04$ reaches small values of Re^{-1} very quickly. The outer region of the hydrodynamization surface intersects the freeze-out surface at $\tau \approx 1.0$ fm/c with $\text{Re}^{-1} \approx 0.5$. For a medium with $\eta/s = 0.08$, equilibration occurs later, and the difference in equilibration time between the central region and the region near the freeze-out surface becomes more gradual. This indicates that, although hydrodynamics becomes applicable to the core around $\tau \approx 1$ fm/c, a substantial portion of the tail of the distribution remains out of equilibrium. This underscores the importance of dynamically connecting the preequilibrium stage to the collective evolution, particularly in regions that approach, but do not fully attain, local equilibrium.

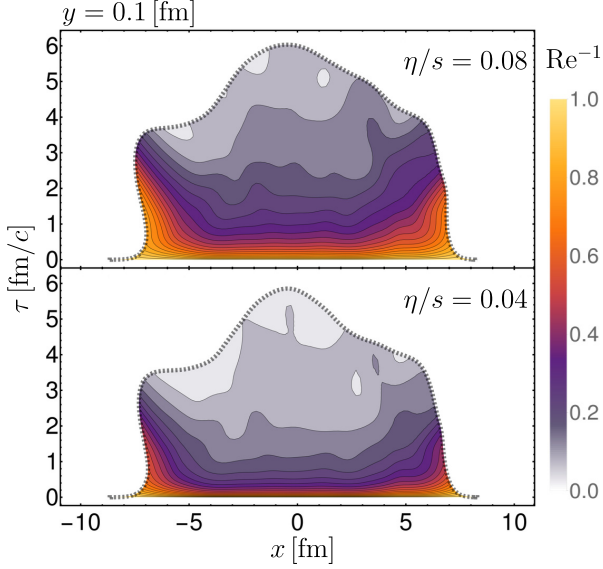


FIG. 7. Contour plot of Re^{-1} in the τ - x plane for $\eta/s = 0.08$ (top panel) and $\eta/s = 0.04$ (bottom panel). The contours are enclosed by the freeze-out surface, shown as a dashed curve.

V. TRANSVERSE MOMENTUM SPECTRUM VIA A HYBRID MODEL

In a more realistic scenario, the system's evolution must be analyzed in terms of the final-state degrees of freedom, namely hadrons. To this end, we need to model the transition from the quasiparticle description to hadronic degrees of freedom. The freeze-out surface is obtained using the CORNELIUS algorithm [83]. For massive quasiparticles, one can use the full Boltzmann distribution to construct the hadron distribution smoothly. In the massless case, however, we only have access to the w -integrated Boltzmann distribution [see Eq. (3)]. For this reason, we follow the standard approach and use $T^{\mu\nu}$ to reconstruct the hadronic Boltzmann distribution according to the prescription of Pratt and Torrieri [84], as implemented in the FRZOUT module provided by the Duke group [85]. The resulting particles are then propagated through the URQMD model [64,65] to account for hadronic interactions.

To assess the similarities and differences between the evolution in our model and that in a hydrodynamic setup, we consider two values of η/s , namely 0.08 and 0.8, with the isotropy parameter fixed at $\lambda = 1000$. As before, we use the VISH2+1 model for comparison. The initial energy density profile is taken to be Gaussian with a width of $R_0 = 2.0$ fm and $\epsilon_0 = 1000$ GeV. We begin both the COMBOLT-ITA hybrid and the VISH2+1 hybrid evolution at $\tau_0 = 1$ fm. For the VISH2+1 setup, the evolution assumes an ideal equation of state and subsequently follows the same procedure as in our model: particlization is performed using FRZOUT, followed by hadronic evolution with URQMD. In both cases, we adopt a switching energy of $\epsilon_{\text{sw}} = 0.3$ GeV for particlization. For small η/s , we have validated the freeze-out surface obtained from our code with that obtained from VISH2+1 to be similar. This means that one expects the

hadrons distribution from both models to be similar in this limit.

In Fig. 8, we present the transverse momentum spectra of pions, kaons, and protons obtained from the two frameworks, where the evolution is governed either by the hybrid VISH2+1 (empty markers) or by COMBOLT-ITA (filled markers) for $\eta/s = 0.08$ (left panel) and $\eta/s = 0.8$ (right panel). For $\eta/s = 0.08$, a reasonable level of agreement between the hydrodynamic and kinetic theory results persists up to $p_T \approx 2.5$ GeV, beyond which the ratio between the two approaches begins to show noticeable differences. As shown in Fig. 4, the hydrodynamic evolution produces a smaller transverse velocity than kinetic theory, thereby contributing to the observed deviations. To investigate this behavior more closely, we consider the case of $\eta/s = 0.8$. Here, the combined effect of enhanced viscosity and stronger transverse velocity in the kinetic theory leads to a more pronounced deviation between the two approaches. The spectrum obtained from hydrodynamics exhibits a steeper slope, while that from kinetic theory is noticeably flatter. As a consequence, the kinetic theory produces more particles with higher momentum in this viscous medium compared to the hybrid VISH2+1 framework.

VI. CONCLUSION AND OUTLOOK

In this work, we introduced a $(2+2)\text{D}$ model, COMBOLT-ITA, which solves the Boltzmann equation under the isotropization time approximation. The numerical method employs a characteristic approach, starting from an arbitrary initial Boltzmann distribution. In this study, the initial Boltzmann distribution was constructed using TRENTO for spatial energy distribution and connected to a nonfluctuating momentum distribution. The longitudinal anisotropy of the momentum distribution can be adjusted using a free parameter λ .

The model is expected to be compatible with hydrodynamic calculations for an initially isotropized medium that closely resembled ideal hydrodynamics, beginning at an initial time on the order of 1 fm/c. We showed that, by choosing a large value for λ and starting with the same initial spatial distribution at time $\tau_0 = 1.0$ fm/c, the outcomes from solving the Boltzmann equation closely resemble those obtained from hydrodynamic equations. In particular, we compared COMBOLT-ITA with VISH2+1 and observed good agreement for a medium approaching ideal hydrodynamics with $\eta/s = 0.008$. However, for media with larger values of η/s , the evolutions of the two models diverge, which aligns with observations made in Ref. [58] using a simpler spatial initial geometry.

We investigated the medium's evolution starting from a state of high initial momentum anisotropy at very early times. In particular, we studied the evolution of Re^{-1} and found that different regions of the medium reach local equilibrium at different times. For a medium with a small $\eta/s = 0.008$, equilibration occurs very early in regions with higher energy density, while in the tail of the distribution it happens later, around $\tau \approx 0.5$ fm/c. When η/s is larger, equilibration is delayed even further. Even in a “medium-size” system, such

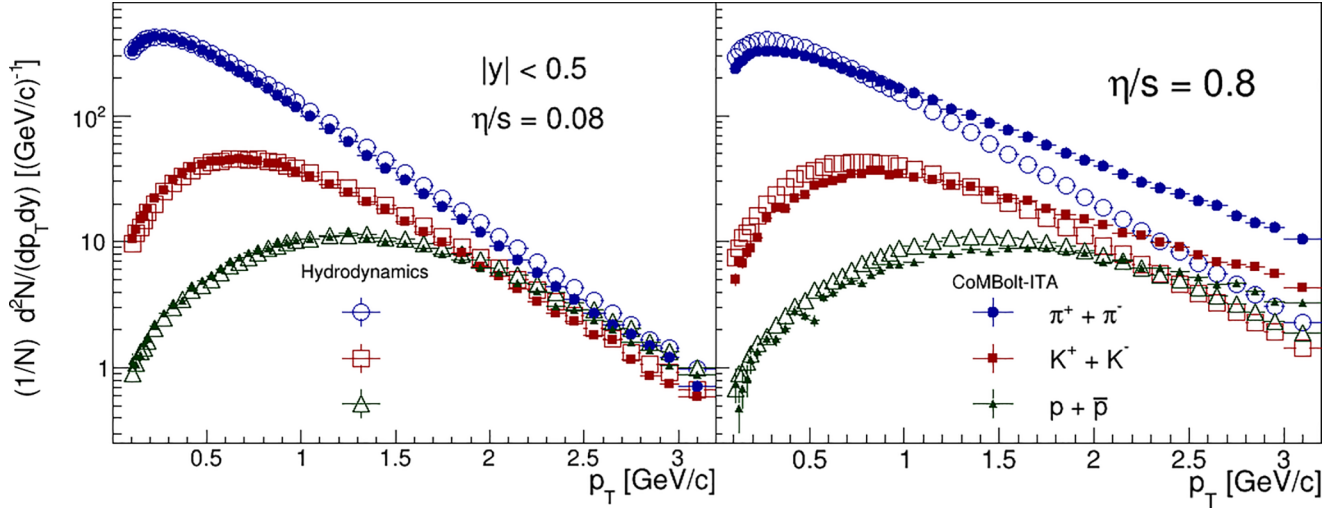


FIG. 8. Comparison of the transverse momentum spectra of pions (blue markers), kaons (red markers), and protons (green markers) obtained from hybrid VISH2+1 evolution (filled markers) and from CoMBolt-ITA (empty markers) for $\eta/s = 0.08$ (left panel) and $\eta/s = 0.8$ (right panel).

as Pb-Pb collisions at 30–35% centrality with $\eta/s = 0.08$, we find that $\text{Re}^{-1} \approx 0.5$ in the central region of the medium at $\tau \approx 1 \text{ fm}/c$, and it becomes even larger in the regions closer to the freeze-out surface.

To employ the code in more realistic scenarios and to gain further insight into the differences between the evolution described by CoMBolt-ITA and VISH2+1, we used the hybrid version of both frameworks and calculated the p_T spectra of pions, kaons, and protons. For a nonfluctuating Gaussian initial energy density, we observed that when $\eta/s = 0.08$ the two models' predictions agree well. In contrast, for a more viscous medium with $\eta/s = 0.8$, a substantial difference appears in the particle spectra. This discrepancy originates from differences in the transverse flow velocity and the shear-stress tensor developed during the evolution. As shown in Fig. 4, VISH2+1 generates a smaller transverse velocity compared to CoMBolt-ITA, leading to the differences in the particle spectra observed in Fig. 8.

One key advantage of CoMBolt-ITA and similar models over those based on anisotropic hydrodynamics [80,86–88] is their ability not only to initiate the evolution from a highly anisotropic medium but also to describe scenarios with minimal collective behavior, or even near-free-streaming evolution, within a single unified framework. Moreover, the dynamical nature of achieving local equilibrium makes these models more practical than approaches relying on separate preequilibrium descriptions, such as K ϕ MP ϕ ST [89] followed by hydrodynamics. Finally, a notable advantage of models based on solving the Boltzmann equation is that they generally avoid the causality issues that can arise in hydrodynamic simulations [42,44].

The model can be applied to study collectivity in both small and large systems on an event-by-event basis and to describe experimental flow measurements. In a followup study [67], the hybrid model was used to investigate the anisotropic flow of OO collisions measured at the LHC [68–70].

Looking ahead, the current implementation assumes an ideal equation of state. Incorporating a realistic equation of state is essential for improving the model's competitiveness with hydrodynamic approaches. This will require introducing an energy-dependent mass scale, as suggested in Refs. [90,91], and represents a key direction for future development.

Furthermore, a recent study has shown that a more realistic collision kernel based on underlying QCD dynamics can be computed efficiently using machine-learning techniques [92]. Implementing such a collision kernel in our framework is also planned for future work.

ACKNOWLEDGMENTS

S.F.T. would like to thank Harri Niemi, Victor Ambrus, and Sören Schlichting for valuable discussions. S.F.T. is supported by the Deutsche Forschungsgemeinschaft (DFG) through Grant No. 517518417.

DATA AVAILABILITY

The data that support the findings of this article are openly available [66].

[1] T. Matsui and H. Satz, J/ψ suppression by quark-gluon plasma formation, *Phys. Lett. B* **178**, 416 (1986).

[2] S. A. Bass, M. Gyulassy, H. Stoecker, and W. Greiner, Signatures of quark gluon plasma formation in high-energy

- heavy ion collisions: A critical review, *J. Phys. G* **25**, R1 (1999).
- [3] D. Teaney, J. Lauret, and E. V. Shuryak, Flow at the SPS and RHIC as a quark gluon plasma signature, *Phys. Rev. Lett.* **86**, 4783 (2001).
- [4] J. Adams, *et al.* (STAR Collaboration), Experimental and theoretical challenges in the search for the quark gluon plasma: The STAR Collaboration's critical assessment of the evidence from RHIC collisions, *Nucl. Phys. A* **757**, 102 (2005).
- [5] H. van Hees, V. Greco, and R. Rapp, Heavy-quark probes of the quark-gluon plasma at RHIC, *Phys. Rev. C* **73**, 034913 (2006).
- [6] E. Shuryak, Physics of strongly coupled quark-gluon plasma, *Prog. Part. Nucl. Phys.* **62**, 48 (2009).
- [7] G.-Y. Qin and X.-N. Wang, Jet quenching in high-energy heavy-ion collisions, *Int. J. Mod. Phys. E* **24**, 1530014 (2015).
- [8] W. Busza, K. Rajagopal, and W. van der Schee, Heavy ion collisions: The big picture, and the big questions, *Annu. Rev. Nucl. Part. Sci.* **68**, 339 (2018).
- [9] P. Kovtun, D. T. Son, and A. O. Starinets, Viscosity in strongly interacting quantum field theories from black hole physics, *Phys. Rev. Lett.* **94**, 111601 (2005).
- [10] P. Romatschke and U. Romatschke, Viscosity information from relativistic nuclear collisions: How perfect is the fluid observed at RHIC? *Phys. Rev. Lett.* **99**, 172301 (2007).
- [11] D. T. Son and A. O. Starinets, Viscosity, black holes, and quantum field theory, *Annu. Rev. Nucl. Part. Sci.* **57**, 95 (2007).
- [12] T. Schäfer and D. Teaney, Nearly perfect fluidity: From cold atomic gases to hot quark gluon plasmas, *Rep. Prog. Phys.* **72**, 126001 (2009).
- [13] H. Song, S. A. Bass, U. Heinz, T. Hirano, and C. Shen, 200 A GeV Au + Au collisions serve a nearly perfect quark-gluon liquid, *Phys. Rev. Lett.* **106**, 192301 (2011); **109**, 139904(E) (2012).
- [14] P. Kovtun, Lectures on hydrodynamic fluctuations in relativistic theories, *J. Phys. A* **45**, 473001 (2012).
- [15] P. Romatschke and U. Romatschke, *Relativistic Fluid Dynamics In and Out of Equilibrium*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, 2019).
- [16] K. Dusling and D. Teaney, Simulating elliptic flow with viscous hydrodynamics, *Phys. Rev. C* **77**, 034905 (2008).
- [17] U. Heinz and R. Snellings, Collective flow and viscosity in relativistic heavy-ion collisions, *Annu. Rev. Nucl. Part. Sci.* **63**, 123 (2013).
- [18] G. Vujanovic, C. Young, B. Schenke, R. Rapp, S. Jeon, and C. Gale, Dilepton emission in high-energy heavy-ion collisions with viscous hydrodynamics, *Phys. Rev. C* **89**, 034904 (2014).
- [19] J.-F. Paquet, C. Shen, G. S. Denicol, M. Luzum, B. Schenke, S. Jeon, and C. Gale, Production of photons in relativistic heavy-ion collisions, *Phys. Rev. C* **93**, 044906 (2016).
- [20] V. Khachatryan, *et al.* (CMS Collaboration), Observation of long-range near-side angular correlations in proton-proton collisions at the LHC, *J. High Energy Phys.* **09** (2010) 091.
- [21] B. Abelev, *et al.* (ALICE Collaboration), Long-range angular correlations on the near and away side in p-Pb collisions at $\sqrt{s_{NN}} = 5.02$ TeV, *Phys. Lett. B* **719**, 29 (2013).
- [22] G. Aad, *et al.* (ATLAS Collaboration), Observation of associated near-side and away-side long-range correlations in $\sqrt{s_{NN}}=5.02$ TeV proton-lead collisions with the ATLAS detector, *Phys. Rev. Lett.* **110**, 182302 (2013).
- [23] S. Acharya, *et al.* (ALICE Collaboration), Multiplicity and event-scale dependent flow and jet fragmentation in pp collisions at $\sqrt{s} = 13$ TeV and in p-Pb collisions at $\sqrt{s_{NN}} = 5.02$ TeV, *J. High Energy Phys.* **03** (2024) 092.
- [24] V. Khachatryan, *et al.* (CMS Collaboration), Evidence for collectivity in pp collisions at the LHC, *Phys. Lett. B* **765**, 193 (2017).
- [25] P. Božek, Collective flow in p-Pb and d-Pd collisions at TeV energies, *Phys. Rev. C* **85**, 014911 (2012).
- [26] Y. C. Chen, *et al.* (Belle Collaboration), Two-particle angular correlations in e^+e^- collisions to hadronic final states in two reference coordinates at Belle, *J. High Energy Phys.* **03** (2023) 171.
- [27] Y.-C. Chen, *et al.*, Long-range near-side correlation in e^+e^- collisions at 183-209 GeV with ALEPH archived data, *Phys. Lett. B* **856**, 138957 (2024).
- [28] I. Abt, *et al.* (ZEUS Collaboration), Two-particle azimuthal correlations as a probe of collective behaviour in deep inelastic ep scattering at HERA, *J. High Energy Phys.* **04** (2020) 070.
- [29] A. Tumasyan, *et al.* (CMS Collaboration), Two-particle azimuthal correlations in γp interactions using pPb collisions at $\sqrt{s_{NN}} = 8.16$ TeV, *Phys. Lett. B* **844**, 137905 (2023).
- [30] A. Hayrapetyan, *et al.* (CMS Collaboration), Observation of enhanced long-range elliptic anisotropies inside high-multiplicity jets in pp collisions at $\sqrt{s} = 13$ TeV, *Phys. Rev. Lett.* **133**, 142301 (2024).
- [31] J. L. Nagle and W. A. Zajc, Small system collectivity in relativistic hadronic and nuclear collisions, *Annu. Rev. Nucl. Part. Sci.* **68**, 211 (2018).
- [32] J. F. Grosse-Oetringhaus and U. A. Wiedemann, A decade of collectivity in small systems, [arXiv:2407.07484](https://arxiv.org/abs/2407.07484).
- [33] R. D. Weller and P. Romatschke, One fluid to rule them all: Viscous hydrodynamic description of event-by-event central p+p, p+Pb and Pb+Pb collisions at $\sqrt{s} = 5.02$ TeV, *Phys. Lett. B* **774**, 351 (2017).
- [34] W. Zhao, Y. Zhou, H. Xu, W. Deng, and H. Song, Hydrodynamic collectivity in proton-proton collisions at 13 TeV, *Phys. Lett. B* **780**, 495 (2018).
- [35] S. F. Taghavi, Smallest QCD droplet and multiparticle correlations in p-p collisions, *Phys. Rev. C* **104**, 054906 (2021).
- [36] H. Niemi and G. S. Denicol, How large is the Knudsen number reached in fluid dynamical simulations of ultrarelativistic heavy ion collisions? [arXiv:1404.7327](https://arxiv.org/abs/1404.7327).
- [37] C. Shen, J.-F. Paquet, G. S. Denicol, S. Jeon, and C. Gale, Collectivity and electromagnetic radiation in small systems, *Phys. Rev. C* **95**, 014906 (2017).
- [38] W. A. Hiscock and L. Lindblom, Generic instabilities in first-order dissipative relativistic fluid theories, *Phys. Rev. D* **31**, 725 (1985).
- [39] W. Israel, Nonstationary irreversible thermodynamics: A causal relativistic theory, *Ann. Phys.* **100**, 310 (1976).
- [40] P. Kovtun, First-order relativistic hydrodynamics is stable, *J. High Energy Phys.* **10** (2019) 034.
- [41] F. S. Bemfica, M. M. Disconzi, and J. Noronha, First-order general-relativistic viscous fluid dynamics, *Phys. Rev. X* **12**, 021044 (2022).
- [42] C. Plumberg, D. Almaalol, T. Dore, J. Noronha, and J. Noronha-Hostler, Causality violations in realistic simulations of heavy-ion collisions, *Phys. Rev. C* **105**, L061901 (2022).

- [43] F. S. Bemfica, M. M. Disconzi, V. Hoang, J. Noronha, and M. Radosz, Nonlinear constraints on relativistic fluids far from equilibrium, *Phys. Rev. Lett.* **126**, 222301 (2021).
- [44] R. Krupczak, *et al.* (ExTrEMe Collaboration), Causality violations in simulations of large and small heavy-ion collisions, *Phys. Rev. C* **109**, 034908 (2024).
- [45] M. Okai, K. Kawaguchi, Y. Tachibana, and T. Hirano, New approach to initializing hydrodynamic fields and mini-jet propagation in quark-gluon fluids, *Phys. Rev. C* **95**, 054914 (2017).
- [46] S. Schlichting and D. Teaney, The first fm/c of heavy-ion collisions, *Annu. Rev. Nucl. Part. Sci.* **69**, 447 (2019).
- [47] J. Berges, M. P. Heller, A. Mazeliauskas, and R. Venugopalan, QCD thermalization: *Ab initio* approaches and interdisciplinary connections, *Rev. Mod. Phys.* **93**, 035003 (2021).
- [48] J. Jankowski and M. Spaliński, Hydrodynamic attractors in ultrarelativistic nuclear collisions, *Prog. Part. Nucl. Phys.* **132**, 104048 (2023).
- [49] S. Schlichting and P. Tribedy, Collectivity in small collision systems: An initial-state perspective, *Adv. High Energy Phys.* **2016**, 8460349 (2016).
- [50] C. Bierlich, G. Gustafson, and L. Lönnblad, A shoving model for collectivity in hadronic collisions, [arXiv:1612.05132](https://arxiv.org/abs/1612.05132).
- [51] U. W. Heinz and J. S. Moreland, Hydrodynamic flow in small systems or: “How the heck is it possible that a system emitting only a dozen particles can be described by fluid dynamics?” *J. Phys.: Conf. Ser.* **1271**, 012018 (2019).
- [52] G. Baym, Thermal equilibration in ultra-relativistic heavy-ion collision, *Phys. Lett. B* **138**, 18 (1984).
- [53] G. Ferini, M. Colonna, M. Di Toro, and V. Greco, Scalings of elliptic flow for a fluid at finite shear viscosity, *Phys. Lett. B* **670**, 325 (2009).
- [54] M. Ruggieri, F. Scardina, S. Plumari, and V. Greco, Thermalization, isotropization and elliptic flow from nonequilibrium initial conditions with a saturation scale, *Phys. Rev. C* **89**, 054914 (2014).
- [55] P. Romatschke, Retarded correlators in kinetic theory: Branch cuts, poles and hydrodynamic onset transitions, *Eur. Phys. J. C* **76**, 352 (2016).
- [56] A. Kurkela and U. A. Wiedemann, Analytic structure of non-hydrodynamic modes in kinetic theory, *Eur. Phys. J. C* **79**, 776 (2019).
- [57] A. Kurkela, U. A. Wiedemann, and B. Wu, Flow in AA and pA as an interplay of fluid-like and non-fluid like excitations, *Eur. Phys. J. C* **79**, 965 (2019).
- [58] V. E. Ambrus, S. Schlichting, and C. Werthmann, Development of transverse flow at small and large opacities in conformal kinetic theory, *Phys. Rev. D* **105**, 014031 (2022).
- [59] V. Nugara, V. Greco, and S. Plumari, Far-from-equilibrium attractors with full relativistic Boltzmann approach in 3+1D: Moments of distribution function and anisotropic flows v_n , *Eur. Phys. J. C* **85**, 311 (2025).
- [60] A. Kurkela, U. A. Wiedemann, and B. Wu, Opacity dependence of elliptic flow in kinetic theory, *Eur. Phys. J. C* **79**, 759 (2019).
- [61] A. Kurkela, S. F. Taghavi, U. A. Wiedemann, and B. Wu, Hydrodynamization in systems with detailed transverse profiles, *Phys. Lett. B* **811**, 135901 (2020).
- [62] V. E. Ambrus, S. Schlichting, and C. Werthmann, Opacity dependence of transverse flow, preequilibrium, and applicability of hydrodynamics in heavy-ion collisions, *Phys. Rev. D* **107**, 094013 (2023).
- [63] V. E. Ambrus, S. Schlichting, and C. Werthmann, Establishing the range of applicability of hydrodynamics in high-energy collisions, *Phys. Rev. Lett.* **130**, 152301 (2023).
- [64] S. A. Bass, *et al.*, Microscopic models for ultrarelativistic heavy ion collisions, *Prog. Part. Nucl. Phys.* **41**, 255 (1998).
- [65] M. Bleicher, *et al.*, Relativistic hadron hadron collisions in the ultrarelativistic quantum molecular dynamics model, *J. Phys. G* **25**, 1859 (1999).
- [66] S. F. Taghavi, CoMBolt-ITA hybrid: Kinetic-theory-based event generator, gitHub repository, version 1.0. (2025), <https://github.com/FaridTaghavi/ComBolt-Hybrid>.
- [67] S. F. Taghavi and S. M. A. Tabatabaee Mehr, Opacity estimation of OO collision from CoMBolt-ITA hybrid, [arXiv:2512.05009](https://arxiv.org/abs/2512.05009).
- [68] I. J. Abualrob, *et al.* (ALICE Collaboration), Evidence of nuclear geometry-driven anisotropic flow in OO and Ne-Ne collisions at $\sqrt{s_{NN}} = 5.36$ TeV, [arXiv:2509.06428](https://arxiv.org/abs/2509.06428).
- [69] G. Aad, *et al.* (ATLAS Collaboration), Measurement of the azimuthal anisotropy of charged particles in $\sqrt{s_{NN}} = 5.36$ TeV $^{16}\text{O} + ^{16}\text{O}$ and $^{20}\text{Ne} + ^{20}\text{Ne}$ collisions with the ATLAS detector, [arXiv:2509.05171](https://arxiv.org/abs/2509.05171).
- [70] A. Hayrapetyan, *et al.* (CMS Collaboration), Observation of long-range collective flow in OO and NeNe collisions and implications for nuclear structure studies, [arXiv:2510.02580](https://arxiv.org/abs/2510.02580).
- [71] H. Song and U. W. Heinz, Causal viscous hydrodynamics in 2+1 dimensions for relativistic heavy-ion collisions, *Phys. Rev. C* **77**, 064901 (2008).
- [72] M. L. Miller, K. Reygers, S. J. Sanders, and P. Steinberg, Glauber modeling in high energy nuclear collisions, *Annu. Rev. Nucl. Part. Sci.* **57**, 205 (2007).
- [73] B. Alver, M. Baker, C. Loizides, and P. Steinberg, The PHOBOS Glauber Monte Carlo, [arXiv:0805.4411](https://arxiv.org/abs/0805.4411).
- [74] J. S. Moreland, J. E. Bernhard, and S. A. Bass, Alternative ansatz to wounded nucleon and binary collision scaling in high-energy nuclear collisions, *Phys. Rev. C* **92**, 011901 (2015).
- [75] J. E. Bernhard, J. S. Moreland, and S. A. Bass, Bayesian estimation of the specific shear and bulk viscosity of quark-gluon plasma, *Nat. Phys.* **15**, 1113 (2019).
- [76] D. Everett, *et al.* (JETSCAPE Collaboration), Multisystem Bayesian constraints on the transport coefficients of QCD matter, *Phys. Rev. C* **103**, 054904 (2021).
- [77] J. E. Parkkila, A. Onnerstad, and D. J. Kim, Bayesian estimation of the specific shear and bulk viscosity of the quark-gluon plasma with additional flow harmonic observables, *Phys. Rev. C* **104**, 054904 (2021).
- [78] J. E. Parkkila, A. Onnerstad, S. F. Taghavi, C. Mordasini, A. Bilandzic, M. Virta, and D. J. Kim, New constraints for QCD matter from improved Bayesian parameter estimation in heavy-ion collisions at LHC, *Phys. Lett. B* **835**, 137485 (2022).
- [79] M. Galassi, *et al.*, *GNU Scientific Library Reference Manual*, 3rd ed. (Network Theory Ltd., 2009), <http://www.gnu.org/software/gsl/>.
- [80] M. Martinez, R. Ryblewski, and M. Strickland, Boost-invariant (2+1)-dimensional anisotropic hydrodynamics, *Phys. Rev. C* **85**, 064913 (2012).
- [81] A. Bazavov, *et al.* (HotQCD Collaboration), Equation of state in (2+1)-flavor QCD, *Phys. Rev. D* **90**, 094503 (2014).
- [82] A. Kurkela, A. Mazeliauskas, J.-F. Paquet, S. Schlichting, and D. Teaney, Effective kinetic description of event-by-event pre-equilibrium dynamics in high-energy heavy-ion collisions, *Phys. Rev. C* **99**, 034910 (2019).

- [83] P. Huovinen and H. Petersen, Particlization in hybrid models, *Eur. Phys. J. A* **48**, 171 (2012).
- [84] S. Pratt and G. Torrieri, Coupling relativistic viscous hydrodynamics to Boltzmann descriptions, *Phys. Rev. C* **82**, 044901 (2010).
- [85] J. E. Bernhard, Bayesian parameter estimation for relativistic heavy-ion collisions, Ph.D. thesis, Duke University, [arXiv:1804.06469](https://arxiv.org/abs/1804.06469).
- [86] M. Martinez and M. Strickland, Dissipative dynamics of highly anisotropic systems, *Nucl. Phys. A* **848**, 183 (2010).
- [87] W. Florkowski and R. Ryblewski, Highly-anisotropic and strongly-dissipative hydrodynamics for early stages of relativistic heavy-ion collisions, *Phys. Rev. C* **83**, 034907 (2011).
- [88] M. Alqahtani, M. Nopoush, R. Ryblewski, and M. Strickland, Anisotropic hydrodynamic modeling of 2.76 TeV Pb-Pb collisions, *Phys. Rev. C* **96**, 044910 (2017).
- [89] A. Kurkela, A. Mazeliauskas, J.-F. Paquet, S. Schlichting, and D. Teaney, Matching the nonequilibrium initial stage of heavy ion collisions to hydrodynamics with QCD kinetic theory, *Phys. Rev. Lett.* **122**, 122302 (2019).
- [90] P. Romatschke, Relativistic (lattice) Boltzmann equation with non-ideal equation of state, *Phys. Rev. D* **85**, 065012 (2012).
- [91] L. Tinti, A. Jaiswal, and R. Ryblewski, Quasiparticle second-order viscous hydrodynamics from kinetic theory, *Phys. Rev. D* **95**, 054007 (2017).
- [92] S. Barrera Cabodevila, A. Kurkela, and F. Lindenbauer, Solving the QCD effective kinetic theory with neural networks, *Eur. Phys. J. C* **85**, 1304 (2025).