

Systematics of characteristics of pygmy dipole resonances in medium-heavy and heavy atomic nuclei with neutron excess

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The systematics of energies and the contribution of the pygmy dipole resonance (PDR) to the energy-weighted sum rule of dipole gamma transitions in medium-heavy and heavy nuclei with an excess of neutrons are considered. The modified macroscopic model of Isacker-Nagarajan-Warner was used for calculating PDR energies with the number of surface neutrons proportional to the thickness of the neutron skin according to the Pethick-Ravenhall expression (PR INW approach). Such modification of the macroscopic approach by Isacker-Nagarajan-Warner enables taking into account microscopic evidence of a direct relationship between skin thickness and low-energy dipole response. The results are compared with the microscopic calculations for the chains of Ni, Sn, and Pb isotopes. It was demonstrated that the dependence of the magnitudes of the energies within the PR INW approach on neutron excess is in rather good agreement with experimental data and microscopic calculations if the absolute value of the strength of the neutron-proton interaction is nearly three times as large as that obtained by Isacker-Nagarajan-Warner by the volume integral of the nucleon-nucleon interaction. While the macroscopic INW PR model can describe the main features of the PDR, the above-mentioned discrepancy of the strength values does not provide reason enough for the conclusion that PDR is a pure collective state. The analytical expressions for the PDR fraction of the energy-weighted sum rule for electric dipole transitions (E1 EWSR) are used. They are based on the “molecular” energy-weighted E1 sum rule, considering the number of surface neutrons as a function of the neutron thickness (PR MSR approach). Systematics for the PDR fraction of E1 EWSR are proposed with parameters obtained by fitting the experimental data and microscopic calculations.

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I. INTRODUCTION

In recent years, much attention has been paid to experimental and theoretical studies of collective states of atomic nuclei, in particular, the pygmy dipole resonance, at energies near the neutron separation energy [1–7].

Although the pygmy dipole resonance (PDR) fraction in the energy-weighted sum rule usually does not exceed a few percent, it can have a significant influence on various nuclear processes [8–14], in particular on the rate of neutron absorption in the r process which is important for analyzing the distribution of elements in the Universe. The account of the PDR significantly improves the agreement of theoretical calculations and experimental values of the distribution of elements in the Universe [10–12].

There are various microscopic and macroscopic methods for the calculation of the properties of the strength functions with PDR excitation and PDR characteristics [5,12,15–32].

Specifically, it was demonstrated that the PDR in the neutron excess nuclei is a mode directly related to the size of the neutron skin [18,21].

The analytical expression for the PDR energy was obtained by Suzuki-Ikeda-Sato [26] in line with the hydrodynamic model by Steinwedel and Jensen for the description of the giant dipole resonances (GDRs). The PDR was considered as an out-of-phase density vibration in the nuclear volume of two liquids that corresponds to nucleons in the core of the nucleus and the excess neutrons in the skin (SIS model). The expression for the PDR energy based on the Goldhaber-Teller model [33] was obtained by Isacker, Nagarajan, and Warner in Ref. [27] (INW model). In this approach, PDR mode corresponds to out-of-phase displacement of the surface density against the neutron-proton core. Nucleon densities were taken in the form of the Fermi function with two parameters (2pF distribution).

The dynamics of the dipole oscillations are governed by the neutron-proton interaction. Here, we use the expression by Pethick and Ravenhall (PR approach) [34] for the calculation of the number of surface neutrons as a linear function of the thickness of the neutron skin. This approach is called the PR INW model. This modification enables us to take into account microscopic observation of the direct relationship between skin thickness and low-energy dipole response, an approach called the PR INW model.

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The thickness of the neutron skin is approximated by the linear function of the neutron-proton asymmetry parameter $I = (N - Z)/A$ with coefficients that depend on the mass numbers of the atomic nucleus. The energies of PDR for isotopes Ni, Sn, and Pb are calculated within the PR INW model. It is shown that, in general, the results of the calculations are consistent with the microscopic calculations after fitting the constants of the neutron-proton interaction.

The PDR energies were also calculated within the SIS model using the expression by Pethick- and Ravenhall (PR SIS approach) [34] for the number N_s of surface neutrons. The general behavior of PDR energies by the PR SIS model as a function of neutron-proton asymmetry contradicts the results of the PR INW model as well as the microscopic calculations. Such behavior of E_p for the SIS-model was already demonstrated in Refs. [15,16] at $N_s = N - Z$.

In the present contribution, the analytical expressions for the PDR fraction of the E1 energy-weighted sum rule (EWSR) are used. They are based on the “molecular” energy-weighted E1 sum rule (MSR) with the number of surface neutrons considered as a function of the neutron thickness (PR MSR approach). Systematics for the PDR fraction of E1 EWSR are proposed with the parameters obtained by fitting experimental data and microscopic calculations.

The paper is organized as follows: In Sec. II, the analytical expression for PDR energy as a function of neutron skin is derived within the framework of the PR INW approach. The simple analytical expressions for neutron skin are presented. The results of the PDR energy calculations within the framework of the macroscopic PR INW and PR SIS models are compared with both the microscopic calculations and experimental data. Systematics of the values of the PDR energies are given by fitting the microscopic and experimental results.

The PDR fraction of the dipole energy-weighted sum rule is considered within the framework of the macroscopic expression in Sec. III. Systematics of this fraction is presented and compared with the experimental data. Section IV contains the summary and the conclusions. In the appendix the procedures for the calculations of the PDR energy within the PR INW and PR SIS models as well as nuclear geometric quantities are outlined.

II. COMPARISON OF PDR ENERGIES AND SYSTEMATICS

Calculated values of the PDR energy are somewhat different in the framework of different microscopic methods. For example, the values of the energy E_p according to microscopic calculations presented in Refs. [16,18,20–23] for isotopes of tin are shown in Fig. 1. The following microscopic methods were used: the relativistic random-phase approximation (RRPA) [16], the relativistic quasiparticle RPA (RQRPA) [18], mean-field (MF) plus the relativistic RPA (MF RRPA) [20], the quasiparticle-phonon model (QPM) [21], the relativistic quasiparticle time blocking approximation (RQTBA) [22], and a self-consistent Hartree-Fock plus continuum RPA approach (HF CRPA) [23].

Figure 1 shows that microscopic values of the PDR energies decrease with growing parameter of the neutron-proton asymmetry. Calculated values of the PDR energies agree

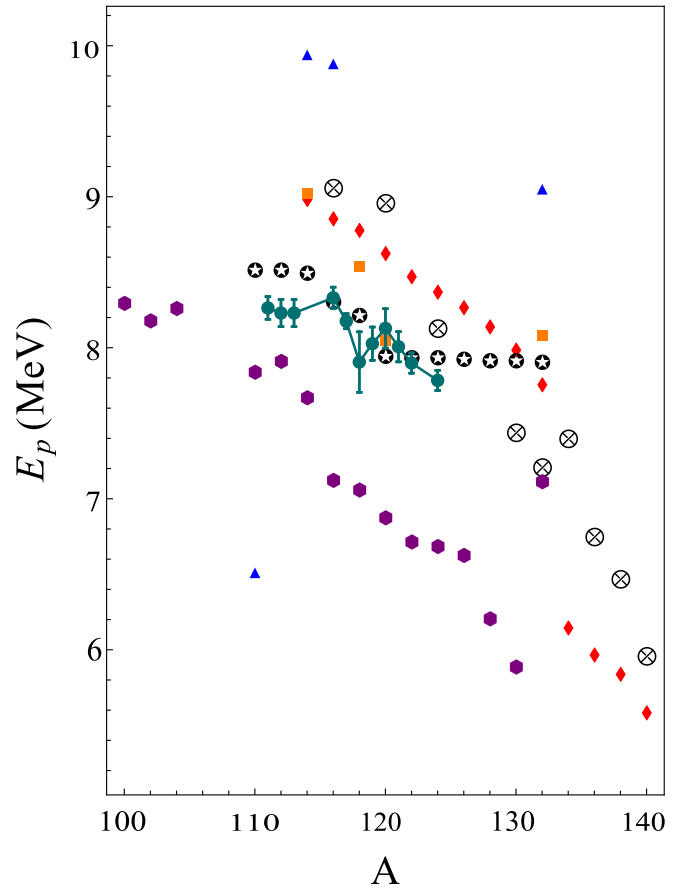


FIG. 1. The energies of the PDR for isotopes of tin $^{A}_{50}\text{Sn}$ with $N \geq Z$ in dependence on the atomic mass number. The symbols correspond to the microscopic calculations by the following methods: squares $\blacksquare$ are for RRPA [16], rhombuses $\blacklozenge$ are for RQRPA [18], circles with an asterisk $\star$ are for MF RRPA [20], hexagons $\blackhexagon$ are for QPM [21], circles with a cross $\otimes$ are for RQTBA [22], triangles $\blacktriangle$ are for HF CRPA [23]. The experimental data are indicated by the shaded circle $\bullet$. They are connected by the lines for the better visualization of the data. These data are taken from Fig. 2 of Ref. [35] and correspond to the average values of the PDR energies.

within $\approx 30\%$. It should be noted that the microscopic value of the PDR energies is, in fact, the energy of the peak of the strength function in the range of the low-energy peak. The magnitude of this energy depends on many calculation parameters, for example, on the number and character of the single-particle states that are involved in the calculations as well as on coupling forces between PDR and other states. Uncertainties in microscopic values of the energy in Fig. 1 may be conditioned by these peculiarities of the microscopic calculations. We compare these results of the microscopic approaches with the results of macroscopic calculations within the framework of PR INW and PR SIS approaches. In macroscopic approaches, the quantity E_p is the energy of the PDR state, and fragmentation of the resonance state is usually considered as a broadening due to the width resulting from one-body dissipation.

The expression for the PDR energy in the INW approach [27] is given by Eq. (A2) in the appendix. As

mentioned before, this method includes two parameters: (1) the number of surface neutrons, N_s , and (2) the thickness y of the neutron skin for 2pF Fermi nuclear distributions (A4), i.e., a difference between neutron and proton radii of the distributions. We use expression by Pethick and Ravenhall (A19): $N_s = 3yN/R_0$, for dependence thickness of surface neutron number on the neutron skin, and Eq. (A10) for the calculation of the quantity y as a linear function of the root-mean-square thickness Δr_{np} : $y = \sqrt{5/3}\Delta r_{np}$. Therefore, we have the following relationship for PDR energy within the PR INW approach:

$$E_p = K_p \left[\frac{1}{3} \frac{Z}{A - N_s} \right]^{1/2} \left[\frac{6a}{yR_m} \Delta F(R_n, R_p) \right]^{1/2} A^{-1/6} \equiv E_{p,1}, \quad (1)$$

where $\Delta F = F(R_n, R_p) - F(R_p, R_p)$ with functions $F(R_p, R_p)$, $F(R_n, R_p)$ that describe the displacement between 2pF distributions of neutrons and protons. The quantities $R_p = R_m - y/2$ and $R_n = R_m + y/2$ are the radii of the 2pF distributions for protons and neutrons, $R_m = (R_p + R_n)/2$ is the average radius, $R_0 = r_0 A^{1/3}$ is the radius of nuclear distributions with a sharp edge. The factor K_p is given by the equation

$$K_p = \left[\frac{\hbar^2}{m} \frac{|\kappa_{np}|}{8\pi a r_0^4} \frac{r_m}{r_0} \right]^{1/2} = 1.34 \sqrt{|\kappa_{np}|} \text{ (MeV)}, \quad (2)$$

where $|\kappa_{np}|$ is the absolute value of the strength of effective neutron-proton interaction, m is the nucleon mass, and a is the diffuseness parameter of 2pF distributions, which is taken identical for protons and neutrons, $r_m = R_m/A^{1/3}$.

In accordance with Eq. (A9), the expression for PDR energy has the following form at $R_m \gg \pi a$:

$$E_p = K_p \left[\frac{1}{3} \frac{Z}{A - N_s} \right]^{1/2} \left(1 - \frac{yR_m}{10a^2} \right)^{1/2} A^{-1/6} \equiv E_{p,2}. \quad (3)$$

and in a linear approximation in y of the function $\Delta F(R_n, R_p)/y$. Therefore, we have for small values of y/a the following equation:

$$E_p \cong K_p \left[\frac{1}{3} \frac{Z}{A - N_s} \right]^{1/2} \left(1 - \frac{yR_m}{20a^2} \right) A^{-1/6} \equiv E_{p,3}. \quad (4)$$

The dependence of PDR energy on the quantities in the brackets given by Eq. (4) is the same as for the INW model [Eq. (10) of Ref. [27] in the leptodermic approximation $R_0 \gg \pi a$].

The skin thickness y for 2pF distributions is determined by Eq. (A13) as a linear function of the parameter neutron-proton asymmetry $I = (N - Z)/A$: $\Delta r_{np} = \alpha + \beta I$. The parameters α , β are given by the Eqs. (A14)–(A18). The results of the calculations of the rms neutron skin with these parameters along the beta-stability line are shown in Fig. 2. The beta-stability line is defined by the Green formula [39,40] $I = 0.4A/(200 + A)$.

It can be seen from the Fig. 2 that the values of the rms thicknesses with different parameters are in rather close agreement and they do not contradict experimental data. In what follows the expression for Δr_{np} with parameters (A15) and (A18) is adopted because it fits the data with the minimal

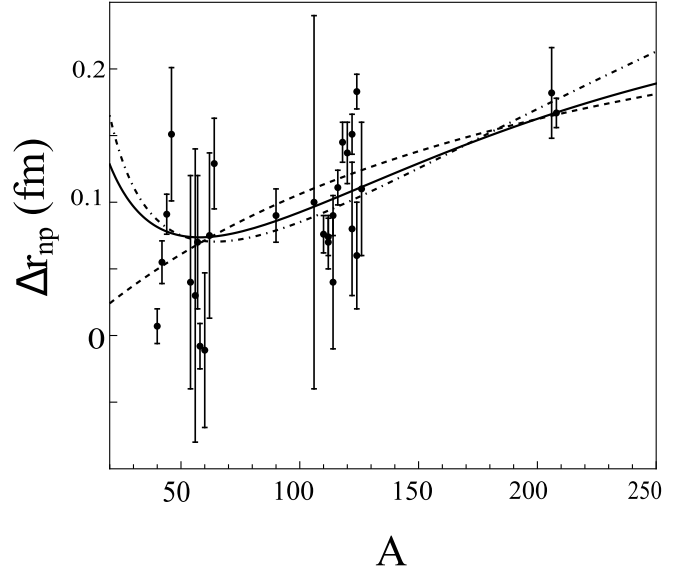


FIG. 2. The dependence of the rms thickness of the neutron skin Δr_{np} on the atomic mass number along the beta-stability line. Denotations of the curves: dashed line is for parameters (A16); dot-dashed line is for parameters (A17); and solid line is for parameters (A18). The experimental values are taken from Refs. [36–38] for isotopes near the beta-stability line.

chi-squared deviation, i.e., the following expression is taken:

$$\Delta r_{np} = 0.62 - 0.22A^{1/3} + 0.003A + (0.47A^{1/3} - 0.009A)I.$$

The factor K_p , Eq. (2), is calculated using the following values of the parameters: $a = 0.57$ fm, $r_0 = 1.15$ fm [41], $r_m = 1.25$ fm (A23), $\hbar^2/m \simeq \hbar^2/m_{\text{proton}} \simeq 41.5$ MeV fm². For the value of the effective proton-neutron interaction, we take the value of the constant component t_0 of the central interaction of the Skyrme force MSk7 [42], $\kappa_{np} = t_0 = -1828.23$ MeV fm³ with the result

$$K_p = 57.3 \text{ MeV}. \quad (5)$$

Figure 3 demonstrates the comparison between the PDR energies calculated for isotopes of Ni, Sn, Pb within the microscopic relativistic quasiparticle RPA (RQRPA) [18] and the results of the macroscopic PR INW and PR SIS models. For the PR INW model, calculations are performed by Eqs. (1)–(3) and (5). Energies of the SIS model are calculated using Eq. (A28) with the PR expression for the number of surface neutrons N_s (PR SIS model). Experimental data from Ref. [35] for isotopes Sn are also shown as well as the results of microscopic calculations within the mean-field plus the relativistic RPA (MF RRPA) [20].

Figure 3 shows that the values of the PDR energies calculated within the macroscopic approach PR INW are rather close to the calculations of the energies by microscopic relativistic RRPA and MF RPA in atomic nuclei with mass numbers $A \gtrsim 100$. Deviation of the macroscopic energies within PR INW from microscopic values does not exceed uncertainties of calculations within different microscopic methods ($\approx 30\%$). The results within the framework of these approaches for isotopes of Sn as a whole are not

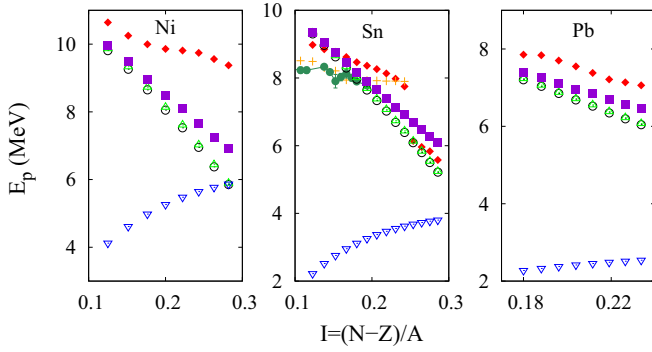


FIG. 3. The comparison of PDR energy values for different theoretical methods and experimental data as a function of the parameter of neutron-proton asymmetry $I = (N - Z)/A$ for isotopes Ni, Sn, Pb. Microscopic models: rhombuses $\blacklozenge$ are for RQRPA [18], crosses $+$ are for MF RRP [20]; experimental data are shown by filled circles $\bullet$ that correspond to the average PDR energies from Fig. 2 of [35]. Macroscopic approaches: circles $\circ$ are for PR INW [$E_{p,1}$, Eq. (1)], triangles $\triangle$ are for approximation (3) for PR INW ($E_{p,2}$); squares $\blacksquare$ are for approximation (4) for PR INW ($E_{p,3}$), inverted triangles ∇ are for PR SIS Eq. (A28) with the PR approach for N_s .

too different from the experimental data. Dependence of the PDR energies on neutron-proton asymmetry for the energies within the PR SIS model contradicts the results of microscopic calculations, as demonstrated for $N_c = Z$ in Refs. [15,16].

For the estimation of the microscopic values of PDR energies for atomic nuclei in the absence of calculations, systematics can be used. For this systematics, we use the following expressions:

$$E_p = e_1 \sqrt{ZA}^{-2/3} / \sqrt{1 + e_2 \Delta r_{np} A^{1/3}} = E_{p,4}, \quad (6)$$

$$E_p = d_1 \sqrt{ZA}^{-2/3} / \sqrt{1 + d_2 \Delta r_{np} A^{1/3}} + d_3 [\Delta r_{np}]^{1/2} A^{-5/6} = E_{p,5}. \quad (7)$$

The first expression is based on the PR INW model, the second one is based on the sum of PR INW and PR SIS approaches. The parameters e_j , d_j are obtained to fit values of microscopic data for the PDR energies to Eqs. (6) and (7). The least squares method is used for the fitting with minimization of the chi-squared deviation of the form $\chi_\alpha^2 = \sum_{j=1}^{N_\Sigma} w_j [E_{p,\text{theor}}(j) - E_{p,\alpha}(j)]^2$, $\alpha = 1-5$. N_Σ is the total number of the data points. The weight factors w_j are the contributions of the square deviations of the fitted PDR energy j from its microscopic value. Microscopic data were taken from calculations within the following methods: RRP [16], RQRPA [18], HB RQRPA [19], MF RRP [20], QPM [21], RQTBA [22], and HF CRPA [23]. We use the constant weight factors $w_j = 1 \text{ MeV}^{-2}$, because systematic uncertainties of the microscopic calculations are not easily specified.

The values of the fitted parameters are given in Table I together with the ratio of the chi-squared value for the given method χ_α^2 to the χ_1^2 for PR INW model with $\kappa_{np} = -1828.23 \text{ MeV fm}^3$: $\bar{\chi}_1^2 \equiv \chi_1^2/N_\Sigma = 2.38$.

It is seen that Eqs. (6) and (7) can be considered as the best systematics of microscopic calculations of the PDR energy

TABLE I. The parameters of systematics of microscopic calculations of the PDR energies $E_{p,\alpha}$, $\alpha = 1-5$, and ratio of the value of the chi-squared deviation for given approach to that one for PR INW model at $\kappa_{np} = -1828.23 \text{ MeV fm}^3$; $\bar{\chi}_1^2 \equiv \chi_1^2/N_\Sigma = 2.38$.

Expression	Parameters	$\bar{\chi}_{E_{p,\alpha}}^2 / \bar{\chi}_{E_{p,\alpha=1}}^2$
$E_{p,1}$, (1)	$\kappa_{np} = -1828.23 \text{ MeV fm}^3$ [42]	1
	$\kappa_{np} = -1930 \pm 78 \text{ MeV fm}^3$	0.98
$E_{p,2}$, (3)	$\kappa_{np} = -1828.23 \text{ MeV fm}^3$ [42]	0.98
$E_{p,3}$, (4)	$\kappa_{np} = -1828.23 \text{ MeV fm}^3$ [42]	0.66
$E_{p,4}$, (6)	$e_1 = 27.3, e_2 = -0.03$	0.38
$E_{p,5}$, (7)	$d_1 = 27.1, d_2 = 0.01, d_3 = 31.1$	0.37

in medium-heavy and heavy nuclei with neutron excess. One can recognize that the value of the proton-neutron interaction $\kappa_{np} = -1930 \pm 78 \text{ MeV fm}^3$ from fitting the PR INW model to microscopic data agrees rather closely with the value of the constant component $t_0 = -1828.23 \text{ MeV fm}^3$ of the central interaction of the Skyrme force MSK7 [42].

Figure 4 compares the microscopic data systematics of the PDR energy given by Eqs. (6) and (7) with the results of the PR INW model for isotopes Ni, Sn, Pb in dependence on the parameter of the neutron-proton asymmetry $I = (N - Z)/A$.

One can see that the values of the PDR energies within microscopic data systematics (6) and (7) are in very close agreement. The PDR energies by systematics fall more slowly with the growth of the neutron-proton asymmetry than that within the PR INW model.

III. PDR FRACTION OF THE E1 EWSR

Another important characteristic of PDR excitation is the fraction of PDR [$\sigma_{\text{int}}(\text{PDR})$] of the integrated photoabsorption cross section σ_{int} for electric dipole gamma quanta. This component of cross-section $\sigma_{\text{int}}(\text{PDR})$ is proportional to the PDR fraction [$m_{L=1}^1(\text{PDR})$] of the E1 EWSR $m_{L=1}^1$ [43]. The ratio of the PDR to GDR sum rule is proportional to the number of

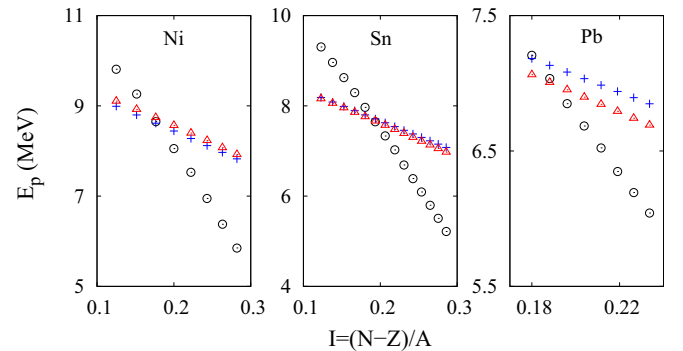


FIG. 4. Comparison of the systematics of the PDR energy with the results of the PR INW model for isotopes Ni, Sn, Pb in dependence on the parameter of the neutron-proton asymmetry. Denotations: circles $\circ$ are for PR INW model ($E_{p,1}$), crosses $+$ are for microscopic data systematics $E_{p,4}$, Eq. (6), triangles $\triangle$ are for microscopic data systematics $E_{p,5}$, Eq. (7).

neutrons, N_s , in the neutron skin [26,27,43–45]:

$$\bar{s}_p = \frac{m_{L=1}^1(\text{PDR})}{m_{L=1}^1(\text{GDR})} = \frac{\sigma_{\text{int}}(\text{PDR})}{\sigma_{\text{int}}(\text{GDR})} = \frac{Z}{N} \frac{N_s}{A - N_s}. \quad (8)$$

This relationship corresponds to the ratio with molecular energy-weighted E1 sum rule for the PDR fraction [43–45]. It is denoted below as the MSR expression.

With allowance for Eq. (A19) for a number near surface neutrons, this equation equals (expression PR MSR)

$$\begin{aligned} \bar{s}_p &= \frac{\sigma_{\text{int}}(\text{PDR})}{\sigma_{\text{int}}(\text{GDR})} = \frac{m_{L=1}^1(\text{PDR})}{m_{L=1}^1(\text{GDR})} \\ &= \frac{Z}{A - N\sqrt{15}\Delta r_{np}/R_0} \frac{\sqrt{15}\Delta r_{np}}{R_0} = \bar{s}_{p,1}. \end{aligned} \quad (9)$$

In the leptodermic approximation and in the case $N \cong Z \cong A/2$, formula (9) takes the form

$$\begin{aligned} \bar{s}_{p,1} &\cong \frac{1}{2 - \sqrt{15}\Delta r_{np}/R_0} \frac{\sqrt{15}\Delta r_{np}}{R_0} \\ &\approx 1.68 \frac{\Delta r_{np}}{A^{1/3}} = \bar{s}_{p,2}. \end{aligned} \quad (10)$$

We suppose, that E1 EWSR (integral cross section σ_{int}) can be taken to be the sum of two components from excitations of the GDR and PDR. Therefore, the expression for the PDR fraction of the E1 EWSR within PR MSR is considered in the form

$$\begin{aligned} s_p &= \frac{m_{L=1}^1(\text{PDR})}{m_{L=1}^1} = \frac{\sigma_{\text{int}}(\text{PDR})}{\sigma_{\text{int}}} \\ &= \frac{\bar{s}_{p,1}}{1 + \bar{s}_{p,1}} = s_{p,1}. \end{aligned} \quad (11)$$

For the estimation of the fractions of PDR to GDR in E1 EWSR (integral cross section σ_{int}) for atomic nuclei in the absence of theoretical calculations or experimental data, systematics can be used. For systematics, we use the following expressions:

$$\begin{aligned} \bar{s}_p &= \frac{\sigma_{\text{int}}(\text{PDR})}{\sigma_{\text{int}}(\text{GDR})} = \frac{m_{L=1}^1(\text{PDR})}{m_{L=1}^1(\text{GDR})} \\ &= \delta \left(\frac{\Delta r_{np}}{A^{1/3}} \right)^\gamma \equiv \bar{s}_{p,\text{sys}}. \end{aligned} \quad (12)$$

Equation (12) is similar to Eq. (10) regarding the dependence on $\Delta r_{np}/A^{1/3}$. By analogy with Eq. (11), we take the following relationship for the systematics of the PDR fraction of E1 EWSR (integral cross section σ_{int}):

$$\begin{aligned} s_{p,\text{sys}} &= \frac{m_{L=1}^1(\text{PDR})}{m_{L=1}^1} = \frac{\sigma_{\text{int}}(\text{PDR})}{\sigma_{\text{int}}} \\ &= \frac{\bar{s}_{p,\text{sys}}}{1 + \bar{s}_{p,\text{sys}}}. \end{aligned} \quad (13)$$

The parameters δ , γ in Eq. (12) are obtained from fitting the values of microscopic or experimental data for ratio $\bar{s}_p$ by systematics $\bar{s}_{p,\text{sys}}$. The least squares method is used for the fitting with minimization of the chi-squared deviation $\chi_{\bar{s}_{p,\text{sys}}}^2$

TABLE II. Parameters of systematics (12) for the ratio of fractions of PDR to GDR in E1 EWSR from fitting the results of the microscopic calculations within RRPA [16], HB RQRPA [19], MF RRPA [20] as well as experimental data from Refs. [1,35] (in brackets). The last column is the relative chi-squared values for $\bar{s}_{p,\text{sys}}$, $\bar{s}_{p,2}$ to the value $\chi_{\bar{s}_{p,1}}^2$ for the PR MSR expression given by Eq. (9).

Expression	Parameters	$\bar{\chi}_{\bar{s}_{p,\alpha}}^2 / \bar{\chi}_{\bar{s}_{p,1}}^2$
$\bar{s}_{p,\text{sys}}$, (12)	$\delta = 0.37$, $\gamma = 0.55$ ($\delta = 0.37$, $\gamma = 0.78$)	0.45(0.37)
$\bar{s}_{p,2}$, (12)	$\delta = 0.435$, $\gamma = 1.0$ ($\delta = 0.435$, $\gamma = 1.0$)	0.85(1.45)

$= \sum_{j=1}^{\bar{N}_\Sigma} w_j [\bar{s}_{p,\text{data}}(j) - \bar{s}_{p,\text{sys}}(j)]^2$. Constant weight factors $w_j = 1/10^{-4}$ are taken. The values of the parameters δ , γ are found from the fitting results of microscopic calculations and experimental data separately. The theoretical data of the following microscopic methods are used: RRPA [16], HB RQRPA [19] and MF RRPA [20]. Experimental data are taken from Fig. 26 of Ref. [1] and Fig. 2 of Ref. [35]. Data in Ref. [1] were collected from Refs. [46–61]. All these experimental data correspond to the fraction $s_{p,\text{data}}$ of PDR to E1 EWSR. Before fitting, we recalculate $s_{p,\text{data}}$ to $\bar{s}_{p,\text{data}}$ by the relationship $\bar{s}_{p,\text{data}} = s_{p,\text{data}}/(1 - s_{p,\text{data}})$. Fitted parameters are presented in Table II. The last column in Table II shows the ratio of the chi-squared values $\chi_{\bar{s}_{p,k}}^2$ for ratios $\bar{s}_{p,\text{sys}}$, $\bar{s}_{p,2}$ to the value $\chi_{\bar{s}_{p,1}}^2$ [$\bar{\chi}_{\bar{s}_{p,1}}^2 = \chi_{\bar{s}_{p,1}}^2 / \bar{N}_\Sigma = 2.8$ (4.0)] for an analytical expression of PR MSR (9).

As is seen from Table II, experimental data and microscopic calculations for the ratio $\bar{s}_p$ of fractions of PDR to GDR in E1 EWSR are statistically better described by systematics (12) than by analytical expressions of PR MSR.

Figure 5 demonstrates the dependence of the ratio $\bar{s}_p$ of the PDR to GDR sum rules for the isotopes of Ni, Sn, Pb on the parameter of neutron-proton asymmetry $I = (N - Z)/A$. The results of the calculations by PR MSR expres-

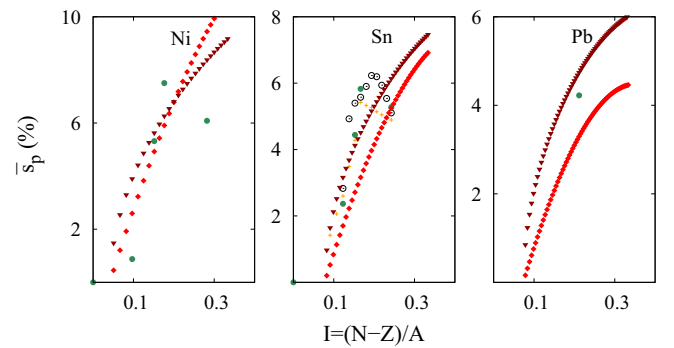


FIG. 5. The values $\bar{s}_{p,1}$ (9) and $\bar{s}_{p,\text{sys}}$ (12) of the ratio of the PDR to GDR fractions in E1 EWSR for isotopes of Ni, Sn, Pb as a function of the neutron-proton asymmetry parameter in comparison with microscopic data. Denotations: rhombuses $\blacklozenge$ are for calculation by expression (9) of PR MSR, inverted triangles $\blacktriangledown$ are for systematics (12) for microscopic calculations ($\delta = 0.37$, $\gamma = 0.55$), filled circles $\bullet$ are for RRPA [16], circles $\odot$ are for HB RQRPA [19], crosses $+$ are for MF RRPA [20].

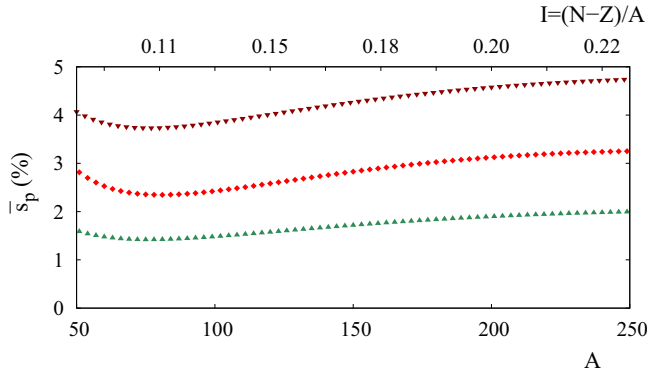


FIG. 6. Ratio $\bar{s}_p$ of the PDR to GDR sum rules of E1 EWSR along the beta-stability line. Denotation: rhombuses $\blacklozenge$ are for calculation by expression (9) of PR MSR, triangles $\blacktriangle$ are for systematics (12) of the experimental data ($\delta = 0.37$, $\gamma = 0.78$), inverted triangles $\blacktriangledown$ are for systematics (12) of microscopic calculations ($\delta = 0.37$, $\gamma = 0.55$).

sion Eq. (9) and systematics (12) of microscopic calculations ($\delta = 0.37$, $\gamma = 0.55$) are presented in comparison with the microscopic results within the RRPA [16], HB RQRPA [19], and MF RRPA [20].

Figure 5 shows that the ratio of the PDR to GDR sum rules in accordance with the PR MSR expression (9) does not exceed $\approx 6\%$ – 8% in neutron-rich nuclei with $A \gtrsim 100$. These values are less in $\approx 20\%$ – 30% than the values of microscopic calculations and systematics (12) of the microscopic calculations. The values of the fractions of the PDR to GDR sum rules calculated by the PR MSR expression (9) and systematics (12) of microscopic data are monotonically increasing with an excess of neutrons. In microscopic calculations such behavior of $\bar{s}_p$ is demonstrated at a small neutron excess.

Figure 6 shows the ratio $\bar{s}_{p,\text{sys}}$ of systematics of the PDR to GDR sum rules of E1 EWSR along the beta-stability line as a function of the atomic mass number and parameter of neutron-proton asymmetry $I = (N - Z)/A$. The ratio of the fractions is calculated according to PR MSF expression (9) and systematics (12) with the parameters from Table II of fitting the results of microscopic calculations and experimental data. The beta-stability line is determined by the Green formula [39,40]: $I = 0.4A/(200 + A)$.

It can be seen that, along the beta-stability line, the values of the ratio $\bar{s}_p$ of the PDR to GDR sum rules calculated within different methods do not exceed $\approx 5\%$. The magnitude of $\bar{s}_p$ is placed between values for systematics for microscopic and experimental data. The values of the systematics of microscopic calculations are placed higher than for the systematics of experimental data.

Figure 7 compares the experimental data of the PDR fraction s_p of E1 EWSR with calculations within both PR MSR, Eqs. (11) and (9), and systematics (13) with (12) of experimental and microscopic data. Experimental data are taken from Fig. 26 of Ref. [1] and Fig. 2 of Ref. [35]. Data in Ref. [1] were collected from Refs. [46–61].

One can see that for the experimental data and the calculations within different approaches, the contributions of the

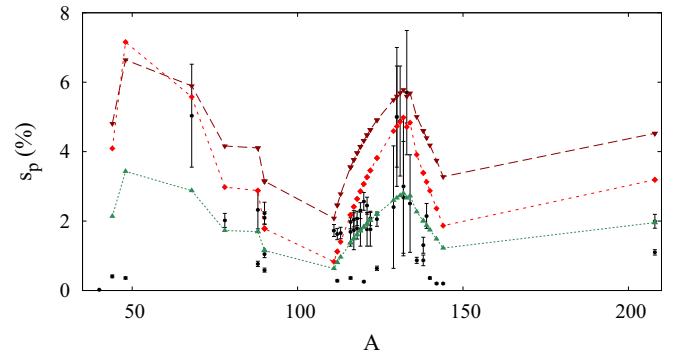


FIG. 7. The comparison of the experimental data for the PDR fractions of E1 EWSR with calculations within different methods. Denotations: black points (with or without uncertainties) are for experimental data, rhombuses $\blacklozenge$ are for $s_{p,1}$ calculation by PR MSR, Eqs. (11) and (9), triangles $\blacktriangle$ are for systematics (13) and (12) with the parameters $\delta = 0.37$, $\gamma = 0.78$ by fitting experimental data, inverted triangles $\blacktriangledown$ are for systematics (13) and (12) with the parameters $\delta = 0.37$, $\gamma = 0.55$ by fitting microscopic calculations. The lines are drawn for better visualization of the corresponding data.

PDR sum rule of E1 EWSR do not exceed $\approx 5\%$ in neutron-rich nuclei with $A \gtrsim 100$. The magnitudes of s_p are placed between values for the systematics for microscopic and experimental data. The values of the systematics of microscopic calculations are higher than for the systematics of experimental data. It should be noted that there are ambiguities in the values of PDR sum rules for the experimental data and microscopic calculations in neutron-rich nuclei. The low-energy part of the excitation of the isoscalar giant dipole resonance is placed near isovector E1 PDR and the experimental data can be a mixture of E1 isovector and isoscalar excitations especially in (α, α') experiment. The theoretical values of the PDR sum rule that were used for their systematics are also depend on the energy interval, over which they are determined. Here we consider that these effects are small from a statistical point of view.

Note that, in the PR INW approach, the value of the PDR energy (3) and (4) is finite in the absence of the neutron skin, like in the INW method for $N_s = 0$ that is in contrast with the model SIS Eq. (A28). The fraction of the PDR to the E1 EWSR equals zero in these situations, and there is no excitation of the PDR.

IV. SUMMARY AND CONCLUSIONS

The relationship for the PDR energy based on the macroscopic model of Isacker-Nagarajan-Warner [27] with the number of near-surface neutrons related to the thickness of the neutron skin is presented. The expression by Pethick and Ravenhall [34] is used for the calculation of the neutron skin (PR INW model). Such modification of the macroscopic approach by Isacker, Nagarajan, and Warner enables us to take into account microscopic observations of the direct relationship between skin thickness and low-energy dipole response.

The neutron skin thickness is approximated by the linear function of the neutron excess parameter $I = (N - Z)/A$ with

coefficients depending on the atomic mass number A . Within the framework of this macroscopic PR INW model, it is demonstrated that the values of the PDR energies for isotopes of Ni, Sn, and Pb are consistent with microscopic calculations. In these situations, the strength of effective proton-neutron interaction was taken to be equal to the value of the constant component $t_0 = \kappa_{np} = -1828.23 \text{ MeV fm}^3$ of the central interaction of the Skyrme force MSk7 [42]. Simple expressions of systematics for the PDR energies in neutron-rich atomic nuclei are proposed and tested. The results agree rather well with those for PR INW, experimental data and microscopic calculations.

The PDR fraction of the E1 energy-weighted sum rule (integrated photoabsorption cross-section for E1 gamma quanta) is considered for the medium-heavy and heavy atomic nuclei. The systematics of the PDR fractions of E1 EWSR is proposed on the basis of the molecular energy-weighted E1 sum rule with a number of near-surface neutrons considered as a function of the thickness of the neutron skin (PR MSR model). The comparison of the experimental data with the calculations within the framework of PR MSR, systematics and microscopic results are given. They demonstrate that contributions of the PDR sum rule to E1 EWSR do not exceed $\approx 5\%$ in medium-heavy and heavy neutron-rich nuclei. The presented systematics of the PDR characteristics can be used for the estimation of the magnitudes of the corresponding quantities in the absence of experimental data and microscopic calculations.

It should be noted that that microscopic results and experimental data for the PDR energies in the medium-heavy and heavy nuclei with neutron excess can be described by the INW PR approach as a whole if the absolute value of the strength of the neutron-proton interaction ($|\kappa_{np}| = 1828.23 \text{ MeV fm}^3$) is three times as large as that ($|\kappa_{np}| = 555.0 \text{ MeV fm}^3$) obtained by Isacker, Nagarajan, and Warner by the volume integral of the nucleon-nucleon interaction. Therefore, while the macroscopic INW PR model can describe the main features of the PDR, the above-mentioned discrepancy of the strength values does not provide reason enough for the conclusion that PDR is pure collective state. This question can be resolved by the microscopic methods in our opinion.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [1,35,36–38,46–61].

APPENDIX: MACROSCOPIC MODELS FOR THE PDR ENERGY AND THEIR PARAMETERS

The relationship for the square of the PDR energy by Isacker-Nagarajan-Warner approach [27] is based on the Goldhaber-Teller mechanism of the GDR description [33]. It can be obtained from Eq. (9) in Ref. [27] and the equation of the frequency of the neutron surface oscillations relative to the nuclear core. It can be presented in the form

$$E_p^2 = \frac{4\pi|\kappa_{np}|}{3} \frac{\hbar^2 \rho_{n0} \rho_{p0}}{\mu_p} \Delta F(R_n, R_p),$$

$$\Delta F = F(R_n, R_p) - F(R_p, R_p). \quad (\text{A1})$$

Here, $|\kappa_{np}|$ is the absolute value of the strength of the effective neutron-proton interaction, $\mu_p = mN_s A_c / A$ is the reduced mass of the oscillating neutron-proton subsystems in the PDR, N_s is the number of neutrons in the surface layer, $A_c = A - N_s$ the mass number of the nuclear core, $\rho_{n0} = N/V$, $\rho_{p0} = Z/V$ are the central values of the density of protons and neutrons in the nucleus with volume $V = (4\pi/3)R_0^3$ with the sharp nuclear radius $R_0 = r_0 A^{1/3}$ for the density of distribution of nucleons with a sharp edge. One can get the following general expression for the PDR energy:

$$E_p = \left[\frac{\hbar^2 |\kappa_{np}|}{m} \frac{3}{A} \frac{1}{4\pi r_0^6} \right]^{1/2} \left[\frac{Z}{A_c} \frac{N}{N_s} \right]^{1/2} [\Delta F(R_n, R_p)]^{1/2}. \quad (\text{A2})$$

The functions $F(R_n, R_p)$, $F(R_p, R_p)$ describe displacement between the distributions of neutrons and protons. They are defined by Eqs. (6) in Ref. [27] which can be presented in the form

$$F(R_n, R_p) = \frac{1}{4a} \text{csch}^2\left(\frac{y}{2a}\right) \times \left[\coth\left(\frac{y}{2a}\right) \frac{c_n - c_p}{a} - (c'_n + c'_p) \right]. \quad (\text{A3})$$

Here, $c_i = (R_i^3 + \pi^2 a^2 R_i)/3$, $c'_i \equiv c'(R_i, a) = \partial c_i / \partial R_i = R_i^2 + \pi^2 a^2/3$, and R_n , R_p are the radii of the neutron and proton density distributions in the form of two-parameter Fermi functions (2pF distribution),

$$\rho_j(r) = \rho_{0,j} \left[1 + \exp\left(\frac{r - R_j}{a_j}\right) \right]^{-1}, \quad (\text{A4})$$

on half of the magnitude; the diffusenesses of the distribution of protons and neutrons are considered the same $a_n = a_p = a$. The parameter $y = R_n - R_p$ is the neutron skin thickness for 2pF nuclear distributions (2pF neutron skin thickness). For $y = 0$,

$$F(R_p, R_p) = \frac{3R_p^2 + (\pi^2 - 6)a^2}{18a}. \quad (\text{A5})$$

The functions $\text{csch}(x)$, $\text{coth}(x)$ in (A3) are the hyperbolic cosecant and hyperbolic cotangent. These functions have a singularity as the argument goes to zero. We transform Eq. (A3) to the following expression without the functions

with such singularities:

$$F(R_n, R_p) = \bar{F} + \frac{y^2}{72a} + \frac{1}{4a} H_1\left(\frac{y}{2a}\right) \times \left[H_2\left(\frac{y}{2a}\right) \frac{y}{a} \left(\bar{c}' + \frac{y^2}{12}\right) + \frac{y^2}{a} \left(\bar{F} + \frac{y^2}{72a}\right) \right] + H_2\left(\frac{y}{2a}\right) \frac{1}{y} \left(\bar{c}' + \frac{y^2}{12}\right), \quad (\text{A6})$$

where $\bar{F} \equiv F(R_m, R_m)$, $\bar{c}' = c'(R_m, a)$ with the mean radius $R_m = (R_n + R_p)/2$. The functions $H_1(x)$ and $H_2(x)$ with $x = y/(2a)$ are equal to $H_1(x) = \text{csch}^2(x) - \frac{1}{x^2} = -\frac{1}{3} + \frac{x^2}{15} + O(x^4)$, $H_2(x) = \coth^2(x) - \frac{1}{x^2} - \frac{x}{3} = -\frac{x^3}{45} + O(x^5)$, and they have no singularity at $x = 0$.

One gets formulas in the second order of y : for functions F ,

$$F(R_p, R_p) = \bar{F} - \frac{yR_m}{6a} + \frac{y^2}{24a}, \quad \bar{F} = \frac{3R_m^2 + (\pi^2 - 6)a^2}{18a} \cong \frac{R_m^2}{6a}, \quad (\text{A7})$$

$$F(R_n, R_p) = \bar{F} \left[1 - \frac{y^2}{10a^2} + \frac{y^2}{120a\bar{F}} \right] \cong \frac{R_m^2}{6a} \left[1 - \frac{y^2}{10a^2} \right]. \quad (\text{A8})$$

As a result, we find the following expression for the PDR energy (A2) in the second order of y

$$E_p = \left[\frac{\hbar^2}{m} \frac{|\kappa_{np}|}{A} \frac{3}{4\pi r_0^6} \right]^{1/2} \left[\frac{Z}{A_c} \right]^{1/2} \times \left[\frac{R_m}{6a} \frac{yN}{N_s} \left(1 - \frac{y}{30a^2} \frac{3R_m^2 + \pi^2 a^2}{R_m} \right) \right]^{1/2}. \quad (\text{A9})$$

We calculate a value of 2pF neutron skin thickness y using the values of a root mean-square (rms) thickness [62–65], i.e., the difference $\Delta r_{np} = R_{\text{rms},n} - R_{\text{rms},p}$ of roots of the mean-square radii $R_{\text{rms},j} = \sqrt{\langle r^2 \rangle_j}$ (rms radii) for neutron ($j = n$) and proton ($j = p$) distributions.

For the medium-heavy and heavy atomic nuclei, the relationship $a/R_j \ll 1$ is fulfilled, therefore we use below leptodermic expansion. One can get the following expressions for the 2pF neutron skin thickness y and the 2pF mean radius R_m as functions of the corresponding mean-square nuclear characteristics:

$$y = R_n - R_p \cong \sqrt{\frac{5}{3}} \left[1 + \frac{7\pi^2}{10} \left(\frac{a}{R_{\text{rms},p}} \right)^2 \right] \Delta r_{np} \cong \sqrt{\frac{5}{3}} \Delta r_{np}, \quad (\text{A10})$$

$$R_m = \frac{R_n + R_p}{2} \cong \sqrt{\frac{5}{3}} \left[R_{\text{rms},p} + \frac{\Delta r_{np}}{2} \right]. \quad (\text{A11})$$

The rms radius of the proton distribution $R_{\text{rms},p}$ is calculated according to Refs. [66–68]

$$R_{\text{rms},p} = \left[r_1 + \frac{r_2}{A^{2/3}} + \frac{r_3}{A^{4/3}} \right] A^{1/3}, \quad (\text{A12})$$

where the values of the parameters were obtained from the fit of experimental data: (in fm) $r_1 = 0.9071$, $r_2 = 1.105$, $r_3 = -0.548$. These values are rather close to those used in modified drop model [41] (in fm): $r_1 = 0.891$, $r_2 = 1.394$, $r_3 = -0.930$. We have used the value of $a = a_p \cong 0.57$ fm [63] for the parameter of diffuseness.

In accordance with theoretical studies and fitting experimental data [36–38, 69–72], the rms thickness Δr_{np} of the neutron skin for medium-heavy and heavy nuclei can be approximated by a linear function of the parameter of the neutron-proton asymmetry $I = (N - Z)/A$

$$\Delta r_{np} = \alpha + \beta I. \quad (\text{A13})$$

In various studies, the values of the parameters α , β are slightly different, but the average values are in rather close agreement. It was shown from fitting experimental data in Ref. [38], that chi-squared deviation χ^2 for the linear function (A13) has a smaller value if the parameters α , β are different for various chains of isotopes. We have found the dependence of the parameters α , β on the atomic mass number using the relationships for neutron and proton rms radii determined by the microscopic approaches from Refs. [73, 74]. In accordance with these studies, we have obtained overall dependencies of the parameters of α , β on the atomic mass number for the first order of I in the following form:

$$\alpha \equiv \alpha_W = A^{1/3} (a_1 + a_2/A), \quad \beta \equiv \beta_W = b_1 A^{1/3}, \quad (\text{A14})$$

$$\alpha \equiv \alpha_S = A^{1/3} (\bar{a}_1 + \bar{a}_2 A^{2/3} + \bar{a}_3 A^{-1/3}), \quad \beta \equiv \beta_S = \bar{b}_1 A^{1/3} + \bar{b}_2 A, \quad (\text{A15})$$

where the factors a_j , b_j and $\bar{a}_j$, $\bar{b}_j$ are constant. These factors are obtained from fitting of the least squares method of the experimental data to Eqs. (A10)–(A13). The experimental data are taken from mixed database, which includes the evaluated data for even-even nuclei from Ref. [38] and data for other nuclei from Refs. [36, 37]. The following results are found:

$$\alpha = -0.007 \pm 0.012, \quad \beta = 0.85 \pm 0.08, \quad \chi^2 = 1.85, \quad (\text{A16})$$

$$a_1 = -0.039 \pm 0.006, \quad a_2 = 1.8 \pm 0.2, \quad b_1 = 0.29 \pm 0.03, \quad \chi^2 = 1.17, \quad (\text{A17})$$

$$\bar{a}_1 = -0.22 \pm 0.07, \quad \bar{a}_2 = 0.003 \pm 0.002, \quad \bar{a}_3 = 0.62 \pm 0.18, \quad \bar{b}_1 = 0.47 \pm 0.09, \quad \bar{b}_2 = -0.009 \pm 0.005, \quad \chi^2 = 0.96. \quad (\text{A18})$$

Here, χ^2 is the mean-squared deviation of the theoretical values (A13)–(A15) of the rms thickness from the

experimental data,

$$\chi^2 = \frac{1}{N_{\text{point}} - N_{\text{par}}} \sum_{k=1}^{N_{\text{point}}} \left(\frac{(\Delta r_{np,k})_{\text{theor}} - (\Delta r_{np,k})_{\text{expt}}}{\sigma(\Delta r_{np,k})_{\text{expt}}} \right)^2,$$

where N_{point} is the number of experimental points for the thickness of the neutron skin, N_{par} is the number of parameters, and $\sigma(\Delta r_{np,k})_{\text{expt}}$ is the standard deviation of data point k .

We calculate the number of surface neutrons, N_s , as a function of neutron thickness by using the relationship by Pethick and Ravenhall [34] which for the medium-heavy and heavy atomic nuclei can be presented in the following form:

$$\frac{N_s}{N} = \frac{3y}{R_0} = \frac{\sqrt{15}\Delta r_{np}}{R_0}, \quad (\text{A19})$$

which was used at a later in Refs. [25,75,76]. The mean radius $R_m = (R_n + R_p)/2$ in the Eqs. (A6)–(A9) is approximated by the expression $R_m = r_m A^{1/3}$ with a constant value of the parameter r_m . The exact value of the mean radius can be calculated by the formula $R_m = [R_{\text{rms},p} + \Delta r_{np}/2]\sqrt{5/3}$. We obtain the magnitude of the r_m to fit $r_m A^{1/3}$ to exact value of the mean radius with Eqs. (A13) and (A18) for rms thickness Δr_{np} . The least squares method is used for fitting with the minimization of the mean-squared deviation χ_m^2 :

$$\chi_m^2 = \sum_j w_j (r_m g_j - R_{m,j})^2, \quad g_j = A_j^{1/3}, \quad (\text{A20})$$

$$R_{m,j} = R_m(A_j).$$

The weight factor w_j of the deviation of data point j is taken as $1/\sigma_j^2$ with $\sigma_j = 0.1R_{m,j}$. The minimum χ_m^2 is realized at the following value of the parameter r_m with standard deviation σ_{r_m} :

$$r_m = \frac{\sum_j w_j g_j R_{m,j}}{\sum_j w_j g_j^2}, \quad \sigma_{r_m}^2 = \frac{\sum_j [w_j g_j]^2 \sigma_j^2}{[\sum_j w_j g_j^2]^2}. \quad (\text{A21})$$

The calculations were performed for atomic mass numbers A_j along the beta-stability line with the proton numbers $20 \leq Z_j \leq 93$. The atomic mass numbers are taken in accordance with the Green formula $N - Z = 0.4A^2/(A + 200)$ [39,40,77], namely,

$$A_j = 5[Z_j + \sqrt{Z_j^2 + 40Z_j + 10000} - 100]/3. \quad (\text{A22})$$

As a result, the following value for the parameter r_m is obtained:

$$r_m \cong R_m/A^{1/3} = 1.25 \pm 0.02 \text{ fm}. \quad (\text{A23})$$

Note that the dependence of the GDR energy on atomic mass number for the method by Isacker, Nagarajan, and Warner [27] at constant r_m is proportional to $A^{-1/6}$, i.e., it corresponds the one for the Goldhaber-Teller model [33,78], if the parameter $r_m = R_m/A^{1/3}$ is a constant. Indeed, the relationship for the GDR energy in the INW method has the following form:

$$E_g = \left[\frac{1}{A} \frac{\hbar^2}{m} |\kappa_{np}| \frac{3}{4\pi r_0^6} \right]^{1/2} [F(R_n, R_p)]^{1/2}. \quad (\text{A24})$$

We obtain the GDR energy in the second order of thickness y

$$E_g = \left[\frac{\hbar^2}{m} |\kappa_{np}| \frac{3}{4\pi r_0^6} \right]^{1/2} \left[\frac{F(R_n, R_p)}{A} \right]^{1/2} \\ \cong E_{\text{GT}} \left[1 - \frac{y^2}{20a^2} \right], \quad (\text{A25})$$

with

$$E_{\text{GT}} = \sqrt{\frac{\hbar^2}{m} \frac{|\kappa_{np}| R_m^2}{8\pi a r_0^6 A^{2/3}}} A^{-1/6} \\ \cong \sqrt{\frac{\hbar^2}{m} \frac{|\kappa_{np}| r_m^2}{8\pi a r_0^6}} A^{-1/6} = K_g A^{-1/6}. \quad (\text{A26})$$

For values $a = 0.57 \text{ fm}$, $r_0 = 1.15 \text{ fm}$ [41], $r_m = 1.25 \text{ fm}$ (A23), $\hbar^2/m \cong \hbar^2/m_{\text{proton}} \cong 41.5 \text{ MeV fm}^2$, and $|\kappa_{np}| = 1828.23 \text{ MeV fm}^3$, the factor K_g in (A26) equals

$$K_g = 1.40\sqrt{|\kappa_{np}|} = 59.9 \text{ MeV}. \quad (\text{A27})$$

The expression for the PDR energy was first proposed by Suzuki-Ikeda-Sato in Ref. [26]. It can be presented in the following form (in MeV):

$$E_{p,\text{sis}} = \frac{2.082}{A^{1/3}} \left[\frac{\hbar^2 8a_{\text{sym}}}{m r_0^2} \frac{ZN}{A^2} \right]^{1/2} \left[\frac{N_s}{A - N_s} \frac{Z}{N} \right]^{1/2} \\ = 79 \left[\frac{4ZN}{A^2} \right]^{1/2} \left[\frac{N_s}{A - N_s} \frac{Z}{N} \right]^{1/2} A^{-1/3}, \quad (\text{A28})$$

where the values used are $\hbar^2/m \cong \hbar^2/m_{\text{proton}} \cong 41.5 \text{ MeV fm}^2$, $r_0 = 1.15 \text{ fm}$, $a_{\text{sym}} = 23 \text{ MeV}$.

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