

Reflection positivity in Euclidean formulations of relativistic quantum mechanics of particles

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Background: Phenomenological relativistic models of strongly interacting systems have proved to be useful for modeling few-nucleon and few-quark systems at the few-GeV scale. They have the advantage that relativistic invariance is exact; however, satisfying cluster properties requires a complicated construction which has never been used in applications. In addition, there is no straightforward connection to local field theories.

Purpose: To explore an alternative approach for constructing relativistic quantum mechanical models that addresses these two deficiencies. This work examines the general structure of the dynamical input to these models.

Method: Relativistic models that address these two deficiencies can be formulated using reflection positive Euclidean kernels. Relaxing the locality requirement of the axioms of Euclidean field theories leads to a weaker form of reflection positivity. Constraints imposed by the spectral condition and cluster properties are used to determine the structure of a class of kernels satisfying this weaker form of reflection positivity.

Results: The resulting kernels lead to a representation of the quantum mechanical Hilbert space in terms of Euclidean variables that has a positive norm, a set of self-adjoint Poincaré generators with a Hamiltonian satisfying cluster properties and a spectral condition. Relativistic dynamical calculations can be performed directly in the Euclidean representation without analytic continuation. Schwinger functions of a local field theory also satisfy the weakened form of reflection positivity, which provides the desired connection with local field theories.

Conclusion: This work provides the structure of a large class of reflection positive kernels that can be used to test this Euclidean formulation of relativistic quantum mechanics. What is still missing is a dynamical principle that generates model kernels with the desired properties.

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I. INTRODUCTION

The purpose of this work is to continue the exploration of a Euclidean formulation of relativistic quantum mechanics discussed in Refs. [1–5]. The focus of this work is on the structure of the kernel that defines the Hilbert space scalar product.

Since the Euclidean formulation of relativistic quantum mechanics explored in this manuscript is not extensively covered in the literature, relevant background material, including definitions and motivation is discussed below.

The main elements of a relativistic quantum theory are a model Hilbert space and a unitary representation of the

Poincaré group acting on the model Hilbert space. The quantum probability amplitude for a transition from state A to state B is given by the Hilbert space inner product, $\langle A|B\rangle$. The Poincaré group is the group of transformations that relate different inertial coordinate systems in special relativity. A Poincaré transformation, (Λ, a) , is a Lorentz transformation, Λ , followed by a space-time translation by a . In this work relativistic invariance means that experiments that measure quantum observables (probabilities, expectation values, and ensemble averages) cannot be used to distinguish different inertial coordinate systems. These quantum observables will be independent of inertial coordinate system if the representation of the states and operators in different inertial coordinate systems are related by a unitary representation, $U(\Lambda, a)$, of the Poincaré group [6]. Since experimental tests of special relativity are performed on isolated subsystems, this suggests that if two subsystems are separated beyond the range of their mutual interactions, they should behave like independent relativistic systems. This requirement can be satisfied if the unitary representation of the Poincaré group can be approximated by a tensor product of two or more subsystem representations when it acts on states of asymptotically separated subsystems.

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A mathematical expression of this condition is

$$\lim_{(x_i - x_j)^2 \rightarrow \infty} \|(U(\Lambda, a) - (\otimes_i U_i(\Lambda, a)) \otimes_j U_j(I, x_j)|\psi)\| = 0, \\ \forall i \neq j.$$

This condition will be called cluster separability or macroscopic locality. The infinitesimal generator of time-translation in the Poincaré group is the Hamiltonian, H , of the model. For a model to be stable with respect to small uncertainties, the eigenvalues of the Hamiltonian should be bounded from below. This is called the spectral condition. Having a unitary representation of the Poincaré group satisfying macroscopic locality, acting on a model Hilbert space with a Hamiltonian satisfying a spectral condition are minimal requirements of any sensible relativistic quantum theory.

Local quantum field theory provides a natural way to satisfy these requirements. In local quantum field theory the underlying observables are fields integrated over test functions rather than particles. Field operators integrated over test functions that have support in different spacelike separated regions commute (or anticommute for Fermions) and define independent commuting quantum observables. This condition is called microscopic locality. This is a stronger condition than macroscopic locality. It ensures causality on arbitrarily small distance scales. Since there are independent degrees of freedom in any open set of spacetime, local field theories have an infinite number of independent commuting observables. For systems of a finite number of degrees of freedom canonical transformations can be realized by unitary transformations. This is no longer true for local field theories; Haag [7] has demonstrated that even free field theories with different masses, which are related by a canonical transformation, cannot be related by a unitary transformation. This follows because the domain of the generator of the unitary transformation becomes empty in the limit of an infinite number of degrees of freedom. Microscopic locality cannot be formulated for relativistic theories of particles because particles that transform under irreducible representations of the Poincaré group cannot be localized in a region smaller than their Compton wavelength.

Local fields are operator valued distributions. Dynamical Poincaré generators that preserve locality formally include interactions expressed in terms of products of fields at the same point. Products of distributions at the same point, like products of δ functions at the same point, do not make mathematical sense. This makes the construction of interacting local relativistic field theories in $3 + 1$ dimensions a millennium problem [8].

The absence of examples of interacting quantum field theories motivated the formulation of axioms that provide minimal conditions on these distributions that are necessary to satisfy the expected properties of a local relativistic quantum theory. The axioms provide a mathematically well-defined framework where general properties of a local relativistic quantum field theory, like the PCT theorem, can be derived, without the need for specific examples.

While there are several sets of axioms that differ in their mathematical implementation [9–12], they are logically

consistent since they are structured to be satisfied by free field theories.

The success of perturbation theory in quantum electrodynamics makes a compelling case that quantum field theory provides the correct description of the interaction of electrons with photons. Unfortunately the successful perturbative methods used in quantum electrodynamics are not justified for treating strongly interacting local field theories.

The Wightman axioms [10] express the conditions for constructing a local quantum field theory in terms of properties of vacuum expectation values of products of field operators (called Wightman functions). These vacuum expectation values are the kernel of a Lorentz-covariant Hilbert space inner product. By inserting complete sets of intermediate states between the products of fields, Schwinger [13] used the spectral condition on the Hamiltonian to show that time-ordered products of fields can be analytically continued to imaginary relative times.

The resulting Euclidean covariant distributions are called Schwinger functions. The spectral condition also allows the Wightman functions to be analytically continued to a strip in the complex plane [10] called the “tube.” Because the complex orthogonal and complex Lorentz groups are isomorphic, the domain of analyticity can be extended using complex Lorentz covariance. The extended domain of analyticity is called the “extended tube.” The extended tube includes sets of real spacelike separated points called “Jost points.” If the fields are local, then the fields at the Jost points can be permuted, further extending the domain of analyticity to a region called the “extended permuted tube.” This property of Wightman functions implies that the Schwinger functions of a local relativistic field theory are symmetric (antisymmetric) functions of their arguments. The vacuum expectation values of product of fields in Minkowski space can be constructed using different limits of the Schwinger functions. This motivated attempts [14,15] to express the axioms of quantum field theory directly in terms of properties of the Schwinger functions. Osterwalder and Schrader [11,12] discovered a set of axioms on the Schwinger functions that are equivalent to the Wightman axioms.

In the Osterwalder-Schrader axioms Poincaré covariance is replaced by four-dimensional Euclidean covariance. Because the complex orthogonal group and the complex Lorentz group are isomorphic, the real Lorentz group is a complex subgroup of the complex orthogonal group. This means that there is a representation of the Poincaré Lie algebra on the Euclidean Hilbert space. The problem is that the generators of real Lorentz transformations are not self-adjoint operators on this space. Osterwalder and Schrader introduced a new inner product on the space of Euclidean test functions where the generators of the Poincaré Lie algebra become self-adjoint operators. This requires that the Schwinger functions satisfy a condition called reflection positivity. Mathematically reflection positivity has the abstract structure,

$$\sum_{mn} (\Pi f_m, \theta S_{mn} \Pi f_n) \geq 0,$$

where the f_n represent sequences of Euclidean test functions, S_{mn} is a matrix of Schwinger functions, Π projects the test

functions on the subspace of Euclidean test functions with support for positive Euclidean time, and θ reverses the sign of the Euclidean times in the final states. This defines a new Hilbert space inner product (after eliminating 0-norm vectors and including Cauchy sequences) on the space of test functions with Euclidean variables with positive Euclidean time support. This will be referred to as the “physical Hilbert space,” even though vectors in this space are represented by functions of Euclidean variables.

In addition to providing the physical Hilbert space inner product, iterating the Cauchy Schwartz inequality on the physical Hilbert space (called multiple reflections in the literature)

$$\langle e^{-H\tau} \psi | e^{-H\tau} \psi \rangle = \langle \psi | e^{-2H\tau} \psi \rangle \leq \langle \psi | \psi \rangle^{\frac{1}{2}} \langle \psi | e^{-4H\tau} \psi \rangle^{\frac{1}{2}}$$

can be used to show that $e^{-H\tau}$ is a contractive Hermitian semigroup, which implies that H is a positive self-adjoint operator on the physical Hilbert space [16]. This recovers the analyticity that comes from the spectral condition and complex Lorentz invariance.

This structure provides the justification for the Euclidean methods used in lattice calculations and the Schwinger-Dyson equations. These applications utilize the Lagrangian formulation of the theory rather than the Hamiltonian formulation. Quantities of interest are extracted directly from the Euclidean correlation functions without utilizing reflection positivity. While novel methods have been devised for extracting scattering observables and performing Lorentz transformations from the Euclidean correlation functions, these methods are indirect.

The reflection positivity requirement is a nontrivial constraint, particularly in local quantum field theories, where the locality puts nontrivial constraints on the analytic properties of the Schwinger functions due to the symmetry of the arguments. While reflection positivity should be a property of the exact Schwinger functions, it is not necessarily preserved under truncation or approximation.

One possible approach to model strongly interacting relativistic quantum systems is to relax the microscopic locality condition, while preserving the remaining axioms. Microscopic locality is difficult to test since it involves probing the system at arbitrarily small distances. Theories that only satisfy macrolocality can lead to microscopic violations of causality. Experimentally measured quantities are cross sections, and these can be expressed in terms of the scattering operator. The scattering operator in a given energy range does not know about locality; inverse scattering methods imply that measured cross sections can also be described by macrolocal theories with a finite number of degrees of freedom.

Nontrivial relativistic models of directly interacting particles can be formulated by constructing a unitary representation of the Poincaré group satisfying macroscopic locality acting on a Hilbert space of particles [17–49]. Applications are for two- [17,21] and three-particle systems [18]. Satisfying macroscopic locality for more than three particles involves constructing chains of S -matrix preserving unitary transformations that relate tensor product of subsystem representations to representations where interactions can be added

in a manner that preserves the Poincaré symmetry [20,21]. The construction is recursive on particle number and is sufficiently complicated that it has never been used in applications. In addition the construction is representation-dependent so there is no simple connection to an underlying field theory that could be used to construct systematic improvements of the models.

Another possible approach is to relax the locality condition in the Euclidean axioms. The advantage of this approach is that there is a clear connection with the Schwinger functions of an underlying local field theory and macroscopic locality can still be satisfied without a complex construction. To understand the consequence of relaxing microscopic locality note that in a local theory there is one N -point Schwinger function for M final and K initial variables with $M + K = N$. This is because the Osterwalder-Schrader locality axiom implies that the Schwinger functions are symmetric (antisymmetric) functions of their arguments. This is due to the permutation symmetry in the “extended permuted tube” in the Wightman theory. This is strong constraint. If locality is relaxed, then it is possible to have separate “Schwinger” functions with M final and K initial variables. These distributions will be referred to as “quasi-Schwinger functions.” By relaxing the locality constraint it is still possible to use the quasi-Schwinger functions to construct a model Hilbert space, a unitary representation of the Poincaré group satisfying macroscopic locality with a Hamiltonian satisfying a spectral condition. These are the minimal requirements for a relativistic quantum mechanical model mentioned at the beginning of this introduction. As in the local case, models can be formulated in a Euclidean representation, without analytic continuation. These models have the additional advantage that they are systematically improvable since the exact Schwinger functions satisfy these properties.

Previous work on this approach has led to some interesting surprises that motivate further explorations. In Ref. [1] it was demonstrated, using a solvable model, that differential cross sections could be directly computed over a wide range of energies by combining time-dependent methods with the invariance principle [50–52], replacing H by $-e^{-\delta\tau H}$ in the wave operators. The important observation is that because $H \geq 0$, the spectrum of $-e^{-\delta\tau H} \in [0, 1]$. This means that $e^{ine^{-\delta\tau H}}$ can be uniformly approximated by a polynomial in $e^{-\delta\tau H}$, which is a translation to the right in Euclidean time by $\delta\tau$ in the physical Hilbert space. Accurate polynomial approximations to e^{inx} for $x \in [0, 1]$ and large n can be computed using Chebyshev polynomials.

To formulate scattering asymptotic conditions for composite particles using strong limits [53–55], it is necessary to have solutions of the one-body problem. These are point-spectrum eigenstates of a subsystem mass operator. In a Wightman theory these are isolated using functions of the mass that are 1 on point mass eigenvalue and 0 on the continuous spectrum [56]. In the Euclidean representation on the physical Hilbert space the mass squared operator is a Euclidean 4-Laplacian. In Ref. [4] it was shown, using a condition due to Carleman [57,58], that polynomials in the Euclidean Laplacian are complete on the physical Hilbert space, which can in principle be used to isolate one-particle states that are needed to formulate

the scattering asymptotic conditions for nontrivial reaction products.

These observations suggest this approach merits further investigation. To exploit these methods it is necessary to have a large class of candidates for models of reflection positive quasi-Schwinger functions. The purpose of this paper is to investigate the general structure of these distributions. While the results are short of the ideal goal of a structure theorem, they provide a robust class of examples. An important element that is still missing is a dynamical principle that can be used to derive quasi-Schwinger functions for realistic systems. This is beyond the scope of this paper, and will be investigated in future work.

Section II introduces the notation and conventions that will be used in the rest of this manuscript, including a discussion of the complex Poincaré and Euclidean groups. More detailed aspects of this relationship are covered in the Appendix. Section III discusses a theorem due to Widder that provides the mathematical motivation for the structure of reflection positive kernels. Section IV discusses the structure of positive-mass positive-energy unitary irreducible representations of the Poincaré group. These are the building blocks of general unitary irreducible representations. This section is divided into three subsections. The first one constructs these representations starting with complete sets of commuting observables constructed from the generators of the Poincaré group. The second subsection uses the Poincaré covariant representations of the first subsection to construct Lorentz covariant representations of these irreducible representations. In the Lorentz covariant representations the Hilbert space inner product has Lorentz covariant kernels. These kernels are representations of the Wightman distributions of a free quantum field theory. The third subsection uses reflection positivity to construct the Lorentz covariant kernels directly from the Euclidean representations. This demonstrates the equivalence of all three representations. Section V generalizes the result of the third subsection of Sec. IV to illustrate the Euclidean covariance requirements on many-body kernels. Section VI discusses the structure of connected Euclidean invariant distributions. These are the elementary building blocks of dynamical representations and this is the most important section of the paper. Section VII discusses how use the connected distributions of section six to construct Euclidean covariant distributions that satisfy macrolocality. Section VIII summarizes the results and discussed the outlook for this approach.

II. NOTATION

The following notation will be used. Euclidean four vectors are denoted with a subscript e :

$$x_e = (x_e^0, x_e^1, x_e^2, x_e^3). \quad (1)$$

Minkowski 4 vectors are denoted with a subscript m :

$$x_m = (x_m^0, x_m^1, x_m^2, x_m^3). \quad (2)$$

The signature of the Minkowski metric is $(-+++)$. The matrices (Pauli matrices)

$$\sigma_{m\mu} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right), \quad (3)$$

$$\sigma_{e\mu} = \left(\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \quad (4)$$

are used to construct the 2×2 matrix representations of Euclidean or Minkowski four vectors:

$$X_e = \sum_{\mu=0}^3 x_e^\mu \sigma_{e\mu} = \begin{pmatrix} ix_e^0 + x_e^3 & x_e^1 - ix_e^2 \\ x_e^1 + ix_e^2 & ix_e^0 - x_e^3 \end{pmatrix},$$

$$x_e^\mu = \frac{1}{2} \text{Tr}(\sigma_{e\mu}^\dagger X_e), \quad (5)$$

$$X_m = \sum_{\mu=0}^3 x_m^\mu \sigma_{m\mu} = \begin{pmatrix} x_m^0 + x_m^3 & x_m^1 - ix_m^2 \\ x_m^1 + ix_m^2 & x_m^0 - x_m^3 \end{pmatrix},$$

$$x_m^\mu = \frac{1}{2} \text{Tr}(\sigma_{m\mu} X_m). \quad (6)$$

The Euclidean time-reflection operator, θ , which changes the sign of the Euclidean time, is defined by

$$\theta x_e = (-x_e^0, x_e^1, x_e^2, x_e^3). \quad (7)$$

The Euclidean or Minkowski parity operator Π is defined by

$$\Pi x_e = (x_e^0, -x_e^1, -x_e^2, -x_e^3),$$

$$\Pi x_m = (x_m^0, -x_m^1, -x_m^2, -x_m^3). \quad (8)$$

The relation between the complex Lorentz and orthogonal groups in four dimensions plays a central role in the connection between the Euclidean and Minkowski representation of relativistic quantum mechanics. This relationship can be seen by noting that the determinants of the matrices (5) and (6) are related to the Minkowski and Euclidean invariant line elements, respectively:

$$\det(X_m) = (x_m^0)^2 - \mathbf{x}_m \cdot \mathbf{x}_m, \quad \det(X_e) = -((x_e^0)^2 + \mathbf{x}_e \cdot \mathbf{x}_e). \quad (9)$$

In both cases the determinants are preserved under linear transformations of the form

$$X \rightarrow X' = AXB', \quad (10)$$

where A and B are both $SL(2, \mathbb{C})$ matrices and X can be X_m or X_e . In both cases the relevant symmetry group is $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$. For arbitrary elements $(A, B) \in SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ the transformation (10) defines a complex Lorentz or complex orthogonal transformation that preserves the real invariant line elements. For real Lorentz transformations $B = A^*$ and for real orthogonal transformations A and B are independent $SU(2)$ matrices.

III. WIDDER'S THEOREM

The goal of this work is to understand the general structure of reflection positive distributions. The simplest prototype of

a reflection positive kernel is a $k(t)$ satisfying

$$\int_0^\infty dt dt' f(t) k(t+t') f(t') \geq 0 \quad (11)$$

for continuous functions $f(t)$ with compact support on $(0, \infty)$.

The general form of continuous $k(t)$ satisfying Eq. (11) follows from a theorem due to Widder [59]:

$$k(t) = \int \rho(\lambda) d\lambda e^{-t\lambda}, \quad (12)$$

where $\rho(\lambda)$ is nondecreasing and the integral converges for $0 \leq t \leq 2T$ where $\text{support}(f(t)) \in [0, T]$.

For $t > 0$ this can be written as

$$k(t) = \int \tilde{\rho}(\lambda) \frac{e^{-itp}}{\lambda^2 + p^2} dp d\lambda, \quad (13)$$

where $\tilde{\rho}(\lambda) = \rho(\lambda) \frac{\lambda}{\pi}$ and the restriction on the support of $f(t)$ means that the p integral can be computed using the residue theorem. The relevant observation is that Eq. (13) has a structure similar to a Källén-Lehmann representation. In what follows it is shown that all positive mass, positive energy unitary irreducible representations of the Poincaré group can be constructed using a generalization of this construction. Since any unitary representation of the Poincaré group can be decomposed into direct integrals of irreducible representations, this provides the general structure of reflection positive distributions.

There are several related theorems due to Widder with slightly different assumptions. Since smooth functions of compact support are dense in the space of Schwartz functions, the compact support requirement is not critical for motivating the general structure of the quasi-Schwinger functions.

IV. IRREDUCIBLE REPRESENTATIONS

A. Poincaré covariant representations

The elementary building blocks of a relativistic theory of particles are unitary irreducible representations of the Poincaré group [6]. In this section transformation properties of single-particle states of mass m and spin s are used to construct equivalent states in the Euclidean representation. This work is limited to positive-mass positive-energy irreducible representations. These representations are also the building blocks of general unitary representations of the Poincaré group, which can be decomposed into direct integrals of irreducible representations.

In this section two-point quasi-Schwinger functions that describe particles of mass m and spin s are constructed. This construction motivates the structure of multipoint reflection positive quasi-Schwinger functions, which are discussed in the following section.

The starting point is the standard quantum mechanical description of a particle of mass m and spin s . The state of the particle is determined by a complete set of compatible measurements. Compatible quantities that can be measured are a particle's linear momentum, $\mathbf{p}$, and the projection, μ , of its spin on a fixed axis in a fixed reference frame. Lorentz boosts can change the momentum to any value and, because the spin

satisfies $SU(2)$ commutation relations, the spin projection can take on $2s + 1$ values in integer steps between $-s$ and s . These considerations determine the spectrum of the momentum and spin projection operators. A Hilbert space representation for such a particle is the space of square integrable functions of the eigenvalues of the linear momentum and a spin projection operator,

$$\langle (m, s) \mathbf{p}, \mu | \psi \rangle, \quad (14)$$

satisfying

$$\sum_{\mu=-s}^s \int_{\mathbb{R}^3} d\mathbf{p} |\langle (m, s) \mathbf{p}, \mu | \psi \rangle|^2 < \infty. \quad (15)$$

Poincaré transformation properties of these wave functions follow from the transformation properties of the basis of simultaneous eigenstates of mass, spin, linear momentum and spin projection. For particles, m and s are fixed. In the general case the mass spectrum is no longer discrete and there are additional invariant degeneracy quantum numbers. The group representation property implies that a general Poincaré transformation can be expressed as a Lorentz transformation followed by a spacetime translation. It follows that the unitary representation of the Poincaré group, $U(\Lambda, a)$, can be expressed as

$$U(\Lambda, a) = U(I, a) U(\Lambda, 0), \quad (16)$$

where $U(I, a)$ is a four-parameter unitary representation of the group of spacetime translations and $U(\Lambda, 0)$ is a unitary representation of $SL(2, \mathbb{C})$. The notation Λ is used to represent the $SL(2, \mathbb{C})$ matrix or the corresponding 4×4 Lorentz transformation (A1). The interpretation should be clear from the context.

The spacetime translation operator $U(I, a)$ is a multiplication operator acting on the basis states

$$U(I, a) |(m, s) \mathbf{p}, \mu\rangle = |(m, s) \mathbf{p}, \mu\rangle e^{ip \cdot a}. \quad (17)$$

For rotations, $\Lambda = R$, on a $\mathbf{0}$ linear momentum eigenstate, the rotation does not change the momentum; it only transforms the magnetic quantum number. This means that the transformed state is a linear combination of zero momentum spin states:

$$\begin{aligned} U(R, 0) |(m, s) \mathbf{0}, \mu\rangle &= \sum_{v=-s}^s |(m, s) \mathbf{0}, v\rangle \langle s, v | U(R, 0) | s, \mu \rangle \\ &= \sum_{v=-s}^s |(m, s) \mathbf{0}, v\rangle D_{v\mu}^s[R], \end{aligned} \quad (18)$$

where $D_{v\mu}^s[R]$ is the $2s + 1$ -dimensional unitary representation of $SU(2)$ in the basis of eigenstates of the spin projection $\hat{\mathbf{z}} \cdot \mathbf{s}$, $D_{\mu\nu}^s[R] = \langle s, \mu | U(R, 0) | s, \nu \rangle$.

The matrix, $D_{\mu\nu}^s[R]$, can be computed explicitly [13]:

$$\begin{aligned} D_{\mu, \nu}^s[R] &= \langle s, \mu | U(R, 0) | s, \nu \rangle \\ &= \sum_{k=0}^{s+\mu} \frac{\sqrt{(s+\mu)!(s+\nu)!(s-\mu)!(s-\nu)!}}{k!(s+\nu-k)!(s+\mu-k)!(k-\mu-\nu)!} \\ &\quad \times R_{++}^k R_{+-}^{s+\nu-k} R_{-+}^{s+\mu-k} R_{--}^{k-\mu-\nu}, \end{aligned} \quad (19)$$

where

$$R = \begin{pmatrix} R_{++} & R_{+-} \\ R_{-+} & R_{--} \end{pmatrix} = e^{-\frac{i}{2}\theta \cdot \sigma} = \sigma_0 \cos\left(\frac{\theta}{2}\right) - i\hat{\theta} \cdot \sigma \sin\left(\frac{\theta}{2}\right) \quad (20)$$

is a $SU(2)$ matrix. $D_{\mu\nu}^s[R]$ in Eq. (19) is a homogeneous polynomial of degree $2s$ in the $SU(2)$ matrix elements, R_{ij} , with real coefficients, while the matrix elements R_{ij} are entire functions of angles, θ . This means that $D_{\mu\nu}^s[R]$ is an entire function of the rotation angles.

The positive matrix, $P(p/m) = (AA^\dagger)^{1/2} = e^{\frac{1}{2}\rho \cdot \sigma}$, in the polar decomposition, Eqs. (A19) and (A20), of the $SL(2, \mathbb{C})$ matrix, A , represents a rotationless Lorentz boost with rapidity ρ . It can be used to define states $|(m, s)\mathbf{p}, \mu\rangle$ with nonzero linear momentum in terms of zero momentum states,

$$|(m, s)\mathbf{p}, \mu\rangle := U(P(p/m), 0)|(m, s)\mathbf{0}, \mu\rangle N(\mathbf{p}), \quad (21)$$

where $N(\mathbf{p})$ is a normalization factor that is chosen to ensure that this transformation is unitary. The definition (21)

means that the eigenvalue of the magnetic quantum number in the state (21) is the value that would be measured in the particle's rest frame if it was boosted to its rest frame by the inverse of a rotationless boost. The spin defined by Eq. (21) is called the canonical spin. (Different spins, like helicity or light-front spin, can be defined by replacing $P(p/m)$ in Eq. (21) with a different boost, $A(p/m) := P(p/m)R(p/m)$, where $R(p/m)$ is a momentum-dependent rotation.)

For basis states with a Dirac δ -function normalization

$$\langle (m, s)\mathbf{p}', \mu' | (m, s)\mathbf{p}, \mu \rangle = \delta(\mathbf{p}' - \mathbf{p})\delta_{\mu'\mu}, \quad (22)$$

the transformation (21) is unitary for

$$N(\mathbf{p}) = \sqrt{\frac{m}{\omega_m(\mathbf{p})}}, \quad \omega_m(\mathbf{p}) := \sqrt{m^2 + \mathbf{p}^2}. \quad (23)$$

The unitary transformations, Eqs. (17), (18), and (21), can be combined to construct a unitary representation of the Poincaré group on the single-particle Hilbert space

$$\begin{aligned} U(\Lambda, a)|(m, s)\mathbf{p}, \mu\rangle &= U(I, a)U(\Lambda, 0)|(m, s)\mathbf{p}, \mu\rangle \\ &= U(I, a)U(P(\Lambda p/m), 0)U(P^{-1}(\Lambda p/m), 0)U(\Lambda, 0)U(P(p/m), 0)|(m, s)\mathbf{0}, \mu\rangle \sqrt{\frac{m}{\omega_m(\mathbf{p})}} \\ &= \sum_{v=-s}^s U(I, a)U(P(\Lambda p/m), 0)|(m, s)\mathbf{0}, v\rangle \sqrt{\frac{m}{\omega_m(\mathbf{p})}} D_{v\mu}^s[(P^{-1}(\Lambda p/m))\Lambda(P(p/m))] \\ &= \sum_{v=-s}^s U(I, a)|(m, s)\Lambda p, v\rangle \sqrt{\frac{\omega_m(\Lambda p)}{\omega_m(\mathbf{p})}} D_{v\mu}^s[(P^{-1}(\Lambda p/m))\Lambda(P(p/m))] \\ &= \sum_{v=-s}^s e^{i\Lambda p \cdot a} |(m, s)\Lambda p, v\rangle \sqrt{\frac{\omega_m(\Lambda p)}{\omega_m(\mathbf{p})}} D_{v\mu}^s[(P^{-1}(\Lambda p/m))\Lambda(P(p/m))], \end{aligned} \quad (24)$$

where $R_w(\Lambda, p) := P^{-1}(\Lambda p/m)\Lambda P(p/m)$ is a canonical-spin Wigner rotation. The representation (24) is referred to as a Poincaré covariant unitary representation of the Poincaré group. This representation is irreducible since $U(\Lambda, a)$ generates all of the basis vectors, Eq. (21), from $|(m, s)\mathbf{0}, s\rangle$.

B. Lorentz covariant representations

The next step is to transform (24) into a Lorentz covariant representation of the Poincaré group. Because the Wigner functions (19) are entire functions of angles, both the group representation property and equations for adding angular momenta with real Clebsch-Gordan coefficients also hold for complex angles—or equivalently for all $\Lambda \in SL(2, \mathbb{C})$ [10].

Specifically, since

$$\sum_{\alpha=-s}^s D_{\mu\alpha}^s[R_2]D_{\alpha\nu}^s[R_1] - D_{\mu\nu}^s[R_2R_1] = 0 \quad (25)$$

and

$$\begin{aligned} D_{\mu_1\nu_1}^{s_1}[R]D_{\mu_2\nu_2}^{s_2}[R] - \sum_{s, \mu, \nu} < s_1, \mu_1, s_2, \mu_2 | s, \mu > D_{\mu\nu}^s[R] \\ < s_1, \nu_1, s_2, \nu_2 | s, \mu > = 0 \end{aligned} \quad (26)$$

are entire and vanish for all real angles, they vanish for complex angles by analyticity (i.e., for $R = e^{-i\frac{1}{2}\theta \cdot \sigma} \rightarrow e^{\frac{z}{2}\sigma}$). The coefficients $< s_1, \mu_1, s_2, \mu_2 | s, \mu >$ in Eq. (26) are real $SU(2)$ Clebsch-Gordan coefficients. This means that the $2s + 1$ -dimensional representation of the Wigner rotations can be factored into a matrix product of three $2s + 1$ -dimensional Wigner functions of $SL(2, \mathbb{C})$ matrices [see Eq. (19)]:

$$\begin{aligned} D_{\nu\mu}^s[R_w(\Lambda, p)] &= \sum_{\nu'\nu''} D_{\nu\nu'}^s[P^{-1}(\Lambda p/m)]D_{\nu'\nu''}^s[\Lambda]D_{\nu''\mu}^s \\ &\times [P(p/m)]. \end{aligned} \quad (27)$$

Using this decomposition equation (24) can be rewritten as

$$\begin{aligned} U(\Lambda, 0) \sum_{\mu'=-s}^s |(m, s)\mathbf{p}, \mu'\rangle D_{\mu'\mu}^s [P^{-1}(p/m)] \sqrt{\omega_m(\mathbf{p})} \\ = \sum_{\mu', \mu''=-s}^s |(m, s)\Lambda p, \mu'\rangle D_{\mu'\mu''}^s [P^{-1}(\Lambda p/m)] \sqrt{\omega_m(\Lambda p)} \\ \times D_{\mu''\mu}^s [\Lambda]. \end{aligned} \quad (28)$$

This leads to the definition of “Lorentz covariant” basis states

$$\begin{aligned} |(m, s)\mathbf{p}, \mu\rangle_{\text{cov}} := \sum_{\mu'=-s}^s |(m, s)\mathbf{p}, \mu'\rangle D_{\mu'\mu}^s [P^{-1}(p/m)] \\ \times \sqrt{\omega_m(\mathbf{p})}, \end{aligned} \quad (29)$$

where the spins transform under a $2s + 1$ -dimensional representation of $SL(2, \mathbb{C})$:

$$U(\Lambda) |(m, s)\mathbf{p}, \mu\rangle_{\text{cov}} = \sum_{\mu'=-s}^s |(m, s)\Lambda p, \mu'\rangle_{\text{cov}} D_{\mu'\mu}^s [\Lambda]. \quad (30)$$

Since for $SU(2)$ matrices, $R = (R^\dagger)^{-1}$, it also follows that

$$\begin{aligned} |(m, s)\mathbf{p}, \mu\rangle_{\overline{\text{cov}}} := \sum_{\mu'=-s}^s |(m, s)\mathbf{p}, \mu'\rangle D_{\mu'\mu}^s [P(p/m)] \\ \times \sqrt{\omega_m(\mathbf{p})} \end{aligned} \quad (31)$$

transforms like

$$U(\Lambda) |(m, s)\mathbf{p}, \mu\rangle_{\overline{\text{cov}}} = \sum_{\mu'=-s}^s |(m, s)\Lambda p, \mu'\rangle_{\overline{\text{cov}}} D_{\mu'\mu}^s [(\Lambda^\dagger)^{-1}]. \quad (32)$$

Equations (30) and (32) define inequivalent [see Eq. (A7)] unitary representations of the Lorentz group.

In terms of the notation for the transformation properties of spinors, the spins in Eq. (30) transform like right-handed spinors (A1), ξ^μ , while the spins in Eq. (32) transform like dual (A6) left-handed spinors (A2), ξ_μ .

Since $D_{\mu'\mu}^s [P(p/m)]$ is invertible, both Lorentz covariant representations are isomorphic to the original Poincaré covariant representation. While either Lorentz covariant representation can be used, these inequivalent representations get transformed into each other under the discrete transformation of space reflection. This is because complex conjugation changes the sign of σ_2 which results in X_m being reflected about the $x - z$ plane. This means that to treat space reflections and Lorentz transformations consistently in the Lorentz covariant representations, both representations (29) and (31) must appear either as tensor products, as they do for four vectors, or direct sums, as they do for Dirac spinors.

These covariant basis vectors can be used to construct Lorentz covariant wave functions:

$${}_{\text{cov}} \langle (m, s)\mathbf{p}, \mu | \psi \rangle \quad (33)$$

or

$${}_{\overline{\text{cov}}} \langle (m, s)\mathbf{p}, \mu | \psi \rangle. \quad (34)$$

The Poincaré covariant wave functions are related to the Lorentz covariant wave functions by

$$\begin{aligned} \langle (m, s)\mathbf{p}, \mu | \psi \rangle &= \sum_{\mu'=-s}^s D_{\mu\mu'}^s [P(p/m)] {}_{\text{cov}} \langle (m, s)\mathbf{p}, \mu' | \psi \rangle \\ &\times \sqrt{\frac{1}{\omega_m(\mathbf{p})}} \end{aligned} \quad (35)$$

or

$$\begin{aligned} \langle (m, s)\mathbf{p}, \mu | \psi \rangle &= \sum_{\mu'=-s}^s D_{\mu\mu'}^s [(P^{-1})(p/m)] {}_{\overline{\text{cov}}} \langle (m, s)\mathbf{p}, \mu' | \psi \rangle \\ &\times \sqrt{\frac{1}{\omega_m(\mathbf{p})}}. \end{aligned} \quad (36)$$

The Hilbert space inner product in the Lorentz covariant representation follows from relations (35) and (36) and the inner product in the Poincaré covariant irreducible representations (24):

$$\begin{aligned} \langle \phi | \psi \rangle &= \sum_{\mu=-s}^s \int_{\mathbb{R}^3} d\mathbf{p} \langle \phi | (m, s)\mathbf{p}, \mu \rangle \langle (m, s)\mathbf{p}, \mu | \psi \rangle \\ &= \sum_{\mu, v=-s}^s \int_{\mathbb{R}^3} d\mathbf{p} \langle \phi | (m, s)\mathbf{p}, \mu \rangle_{\text{cov}} \\ &\times \frac{D_{\mu v}^s [P(p/m) P^\dagger(p/m)]}{\omega_m(\mathbf{p})} {}_{\text{cov}} \langle (m, s)\mathbf{p}, v | \psi \rangle \\ &= \sum_{\mu, v=-s}^s \int_{\mathbb{R}^4} d^4 p \langle \phi | (m, s)\mathbf{p}, \mu \rangle_{\text{cov}} 2\delta(m^2 + p^2) H(p^0) \\ &\times D_{\mu v}^s [P(p/m) P(p/m)] {}_{\text{cov}} \langle (m, s)\mathbf{p}, v | \psi \rangle, \end{aligned} \quad (37)$$

where $H(P^0)$ is the Heaviside function, with a similar expression for the inner product in the representation (32):

$$\begin{aligned} \langle \phi | \psi \rangle &= \sum_{\mu, v=-s}^s \int_{\mathbb{R}^3} d\mathbf{p} \langle \phi | (m, s)\mathbf{p}, \mu \rangle_{\overline{\text{cov}}} \\ &\times \frac{D_{\mu v}^s [P^{-1}(p/m) (P^\dagger(p/m))^{-1}]}{\omega_m(\mathbf{p})} {}_{\overline{\text{cov}}} \langle (m, s)\mathbf{p}, v | \psi \rangle \\ &= \sum_{\mu, v=-s}^s \int_{\mathbb{R}^4} d^4 p \langle \phi | (m, s)\mathbf{p}, \mu \rangle_{\overline{\text{cov}}} 2\delta(m^2 + p^2) H(p^0) \\ &\times D_{\mu v}^s [P^{-1}(p/m) P^{-1}(p/m)] {}_{\overline{\text{cov}}} \langle (m, s)\mathbf{p}, v | \psi \rangle, \end{aligned} \quad (38)$$

where Eq. (A20), $P = P^\dagger$, was used in Eqs. (37) and (38). Because $P(p/m) = e^{\frac{1}{2}\rho\sigma}$, where ρ is the rapidity of the boost, it follows that

$$P(p/m)P(p/m) = e^{\rho\sigma} = \frac{\sigma_m \cdot p}{m} = \cosh(\rho) + \hat{\rho} \cdot \sigma \sinh(\rho) \quad (39)$$

and

$$P^{-1}(p/m)P^{-1}(p/m) = \frac{(\sigma_2 \sigma_m^t \sigma_2) \cdot p}{m} = \frac{\sigma_m \cdot \Pi p}{m} = \cosh(\rho) - \hat{\rho} \cdot \sigma \sinh(\rho), \quad (40)$$

where Π is the parity operator. These are called “right” and “left-handed” representations because they differ by a space

reflection. The resulting kernel is independent of the type of boost used to define the type of spin (21) since for a general $SL(2, \mathbb{C})$ boost, $A(p/m)$, the polar decomposition (A19) gives

$$\begin{aligned} A(p/m)A^\dagger(p/m) &= P(p/m)R(p/m)R^\dagger(p/m)P^\dagger(p/m) \\ &= P(p/m)P^\dagger(p/m) = P(p/m)P(p/m) \\ &= \frac{\sigma_m \cdot p}{m}. \end{aligned} \quad (41)$$

Using Eqs. (39) and (40) in Eqs. (37) and (38) gives the following expression for the inner product of right- or left-handed Lorentz covariant wave functions:

$$\begin{aligned} \langle \phi | \psi \rangle &= \sum_{\mu, \nu=-s}^s \int_{\mathbb{R}^3} d^3 p \langle \phi | (m, s) \mathbf{p}, \mu \rangle_{\text{cov}} \frac{D_{\mu\nu}^s[\sigma_m \cdot p/m]}{\omega_m(\mathbf{p})} \langle (m, s) \mathbf{p}, \nu | \psi \rangle \\ &= \sum_{\mu, \nu=-s}^s \int_{\mathbb{R}^4} d^4 p \langle \phi | (m, s) \mathbf{p}, \mu \rangle_{\text{cov}} 2\delta(m^2 + p^2) H(p^0) D_{\mu\nu}^s[\sigma_m \cdot p/m]_{\text{cov}} \langle (m, s) \mathbf{p}, \nu | \psi \rangle \end{aligned} \quad (42)$$

and

$$\begin{aligned} \sum_{\mu, \nu=-s}^s \int_{\mathbb{R}^3} d^3 p \langle \phi | (m, s) \mathbf{p}, \mu \rangle_{\text{cov}} \frac{D_{\mu\nu}^s[\sigma_m \cdot \Pi p/m]}{\omega_m(\mathbf{p})} \langle (m, s) \mathbf{p}, \nu | \psi \rangle \\ = \sum_{\mu, \nu=-s}^s \int_{\mathbb{R}^4} d^4 p \langle \phi | (m, s) \mathbf{p}, \mu \rangle_{\text{cov}} 2\delta(m^2 + p^2) H(p^0) D_{\mu\nu}^s[\sigma_m \cdot \Pi p/m]_{\text{cov}} \langle (m, s) \mathbf{p}, \nu | \psi \rangle. \end{aligned} \quad (43)$$

Note that both $D_{\mu\nu}^s[\sigma_m \cdot p/m]$ and $D_{\mu\nu}^s[\sigma_m \cdot \Pi p/m]$ are positive since each one can be expressed as the square of a Hermitian matrix.

The Fourier transforms of the kernels of these inner products are

$$W_{R\mu\nu}^s(x, y) = \int_{\mathbb{R}^3} \frac{d^3 p}{(2\pi)^3} e^{ip \cdot (x-y)} \frac{D_{\mu\nu}^s[\sigma_m \cdot p/m]}{\omega_m(\mathbf{p})} \quad (44)$$

and

$$W_{L\mu\nu}^s(x, y) = \int_{\mathbb{R}^3} \frac{d^3 p}{(2\pi)^3} e^{ip \cdot (x-y)} \frac{D_{\mu\nu}^s[\sigma_2 \sigma_m^t \sigma_2 \cdot p/m]}{\omega_m(\mathbf{p})}, \quad (45)$$

where $p^0 = \omega_m(\mathbf{p})$ and R and L stand for “right” and “left.” The Lorentz covariance properties of these distributions are

$$W_{R\mu\nu}^s(\Lambda x, \Lambda y) = \sum_{\mu', \nu'=-s}^s D_{\mu\mu'}^s[\Lambda] W_{R\mu'\nu'}^s(x, y) D_{\nu'\nu}^s[\Lambda^\dagger] \quad (46)$$

and

$$W_{L\mu\nu}^s(\Lambda x, \Lambda y) = \sum_{\mu', \nu'=-s}^s D_{\mu\mu'}^s[\Lambda^*] W_{L\mu'\nu'}^s(x, y) D_{\nu'\nu}^s[(\Lambda)^{-1}]. \quad (47)$$

In terms of the spinor transformation properties of the matrices in the kernel, $D_{\mu\nu}^s[\sigma \cdot p/m]$ transforms a left-handed spinor to a right-handed dual spinor while $D_{\mu\nu}^s[\sigma \cdot \Pi p/m]$

transforms a right-handed spinor to a left-handed dual spinor. This is because complex conjugation transforms right-handed spinors to left-handed spinors and left-handed spinors to right-handed spinors. Because of this property of the Wigner functions, the dotted index notation, Eqs. (A1) and (A2), will not be used, instead $D_{\mu\nu}^s[\sigma \cdot p/m]$ will be referred to as the right-handed representation while $D_{\mu\nu}^s[\sigma \cdot \Pi p/m]$ will be referred to as the left-handed representation. The invariance of the inner product is due to the momentum dependence in the kernel.

The kernels of these Lorentz covariant inner products are representations of two-point Wightman distributions for right- or left-handed particles of spin s [10]. The price paid to have wave functions with spins that transform under finite-dimensional representations of $SL(2, \mathbb{C})$ is that the inner product has a momentum and spin-dependent kernel. In the covariant representations (29) and (31) the mass dependence has been moved from the Hamiltonian to the kernel of the inner product. This is necessary since time translations and rotationless boosts, which are dynamical, transform trivially in the Lorentz covariant representation.

A general covariant wave function can transform as a product or direct sum of right- and left-handed spinor representations. For product representations the basis states are replaced by

$$\langle \phi | (m, s) \mathbf{p}, \mu \rangle_{\text{cov}} \rightarrow \langle \phi | (m, s_r, s_l) \mathbf{p}, \mu_r, \mu_l \rangle_{\text{cov}}, \quad (48)$$

where the inner product of Lorentz covariant wave functions has the form

$$\begin{aligned}
 \langle \phi | \psi \rangle &= \int_{\mathbb{R}^3} d\mathbf{p} \sum_{\mu_r, \nu_r = -s_r}^{s_r} \sum_{\mu_l, \nu_l = -s_l}^{s_l} \langle \phi | (m, s_r, s_l) \mathbf{p}, \mu_r, \mu_l \rangle_{\text{cov}} \\
 &\times \frac{D_{\mu_r \nu_r}^{s_r} [\sigma_m \cdot \mathbf{p}/m] D_{\mu_l \nu_l}^{s_l} [\sigma_m \cdot \mathbf{p}/m]}{\omega_m(\mathbf{p})} \\
 &\times \langle (m, s_r, s_l) \mathbf{p}, \nu_r, \nu_l | \psi \rangle_{\text{cov}} \\
 &= \int_{\mathbb{R}^4} d^4 p \sum_{\mu_r, \nu_r = -s_r}^{s_r} \sum_{\mu_l, \nu_l = -s_l}^{s_l} \langle \phi | (m, s_r, s_l) \mathbf{p}, \mu_r, \mu_l \rangle_{\text{cov}} \\
 &\times 2\delta(m^2 + p^2) H(p^0) D_{\mu_r \nu_r}^{s_r} [\sigma \cdot \mathbf{p}/m] \\
 &\times D_{\mu_l \nu_l}^{s_l} [\sigma \cdot \mathbf{p}/m] \langle (m, s_r, s_l) \mathbf{p}, \nu_r, \nu_l | \psi \rangle_{\text{cov}}. \quad (49)
 \end{aligned}$$

The kernel of this inner product

$$\begin{aligned}
 W_{\mu_r \mu_l \nu_r \nu_l}^{s_r s_l}(x, y) &= \int \frac{d^4 p}{(2\pi)^4} e^{ip \cdot (x-y)} 2\delta(m^2 + p^2) H(p^0) \\
 &\times D_{\mu_r \nu_r}^{s_r} [\sigma \cdot \mathbf{p}/m] D_{\mu_l \nu_l}^{s_l} [\sigma \cdot \mathbf{p}/m] \quad (50)
 \end{aligned}$$

is a two-point Wightman distribution. The notation (s_r, s_l) will be used to denote the spin representations in Eq. (50).

C. Euclidean covariant representations

The next step is to relate the Lorentz covariant representation of the Hilbert space inner product given in terms of the Wightman distributions to a Hilbert space inner product given in terms of Euclidean covariant distributions. The representation (50) of the Wightman distributions can be related to a reflection positive Euclidean two-point distribution.

To show this consider the following Euclidean covariant distribution:

$$\begin{aligned}
 S_{\mu_r \mu_l \nu_r \nu_l}^{s_r s_l}(x_e, y_e) &:= \int \frac{d^4 p_e}{(2\pi)^4} \frac{D_{\mu_r \nu_r}^{s_r} [p_e \cdot \sigma_e] D_{\mu_l \nu_l}^{s_l} [\Pi p_e \cdot \sigma_e]}{p_e^2 + m^2} \\
 &\times e^{ip_e \cdot (x_e - y_e)}. \quad (51)
 \end{aligned}$$

Because the Wigner functions are polynomials in the components of p_e , the p_e integral in Eq. (51) will not generally converge, however this expression represents a distribution, where it is necessary to perform the x_e and y_e integrals before computing the p_e integrals. If the test functions are Schwartz functions, then their Fourier transforms are Schwartz functions. This means that the p_e integrals converge as distributions. For test functions satisfying the Euclidean time-support condition, $f(\tau, \mathbf{x}) = 0$ for $\tau \leq 0$, it follows that

$$\begin{aligned}
 h_{\mu_r \mu_l \nu_r \nu_l}^{s_r s_l}(\mathbf{x}, \mathbf{y}, p_e^0) &:= \int f^*(\theta x_e) e^{ip_e^0 x_e^0} dx_e^0 \int g(y_e) e^{-ip_e^0 y_e^0} dy_e^0 \\
 &\times D_{\mu_r \nu_r}^{s_r} [p_e \cdot \sigma_e] D_{\mu_l \nu_l}^{s_l} [\Pi p_e \cdot \sigma_e] \quad (52)
 \end{aligned}$$

is analytic in the lower-half p_e^0 plane, and the p_e^0 integral in Eq. (51) can be computed using the residue theorem, closing the contour in the lower half plane. The poles in the p_e^0 integration are at $p_e^0 = \pm i\sqrt{\mathbf{p}^2 + m^2}$. The integral over the contour in the lower half plane gets a contribution from the pole at $-i\sqrt{\mathbf{p}^2 + m^2}$.

The distribution, Eq. (51), transforms covariantly under the complex orthogonal group

$$\begin{aligned}
 S_{\mu_r \mu_l \nu_r \nu_l}^{s_r s_l}(O(A, B)x_e, O(A, B)y_e) \\
 &= \sum_{\mu'_r, \nu'_r = -s_r}^{s_r} \sum_{\mu'_l, \nu'_l = -s_l}^{s_l} D_{\mu_r \mu'_r}^{s_r} [A] D_{\mu_l \mu'_l}^{s_l} [B] (B^T)^{-1} S_{\mu'_r \mu'_l \nu'_r \nu'_l}^{s_r s_l}(x_e, y_e) \\
 &\times D_{\nu'_r \nu_r}^{s_r} [B^T] D_{\nu'_l \nu_l}^{s_l} [A^{-1}]. \quad (53)
 \end{aligned}$$

Note that as in the $SL(2, \mathbb{C})$ case, $D_{\mu_r \nu_r}^{s_r} [p_e \cdot \sigma_e]$ maps a right-handed $SU(2) \times SU(2)$ spinor to a left-handed dual spinor while $D_{\mu_l \nu_l}^{s_l} [\Pi p_e \cdot \sigma_e]$ maps a left-handed $SU(2) \times SU(2)$ spinor to a right-handed dual spinor.

When the test functions have support for positive Euclidean time and the Euclidean time on the final test function is reflected, the Euclidean time difference in the exponent $\theta x_e^0 - y_e^0$ is strictly negative. In this case the result of this integration (for $\theta x_e^0 - y_e^0 < 0$) is

$$\begin{aligned}
 \sum_{\mu_r, \nu_r = -s_r}^{s_r} \sum_{\mu_l, \nu_l = -s_l}^{s_l} \int d^4 x_e d^4 y_e f_{\mu_r \mu_l}^{s_r s_l*}(\theta x_e) S_{\mu_r \mu_l \nu_r \nu_l}^{s_r s_l}(x_e, y_e) g_{\nu_r \nu_l}^{s_r s_l}(y_e) &= \sum_{\mu_r, \nu_r = -s_r}^{s_r} \sum_{\mu_l, \nu_l = -s_l}^{s_l} \int d\mathbf{p} f_{\mu_r \mu_l}^{s_r s_l*}(x_e) d^4 x_e \frac{e^{-\omega_m(\mathbf{p})x_e^0 + i\mathbf{p} \cdot \mathbf{x}}}{(2\pi)^{3/2}} \\
 &\times \frac{D_{\mu_r \nu_r}^{s_r} [p_m \cdot \sigma_m] D_{\mu_l \nu_l}^{s_l} [\Pi p_m \cdot \sigma_m]}{2\omega_m(\mathbf{p})} d^4 y_e g_{\nu_r \nu_l}^{s_r s_l}(y_e) \\
 &\times \frac{e^{-\omega_m(\mathbf{p})y_e^0 - i\mathbf{p} \cdot \mathbf{y}}}{(2\pi)^{3/2}}, \quad (54)
 \end{aligned}$$

where

$$\begin{aligned}
 (-i\omega_m(\mathbf{p}), \mathbf{p}) \cdot (\sigma_e, \boldsymbol{\sigma}) &= (-i\omega_m(\mathbf{p}), \mathbf{p}) \cdot (i\sigma_0, \boldsymbol{\sigma}) \\
 &= (\omega_m(\mathbf{p}), \mathbf{p}) \cdot (\sigma_0, \boldsymbol{\sigma}) \\
 &= p_m \cdot \sigma_m \quad (55)
 \end{aligned}$$

was used.

By defining the wave functions

$$\psi_{\nu_r \nu_l}(\mathbf{p}) := \int_{\mathbb{R}^4} d^4 y_e g_{\nu_r \nu_l}(y_e) \frac{e^{-\omega_m(\mathbf{p})y_e^0 - i\mathbf{p} \cdot \mathbf{y}}}{(2\pi)^{3/2}} \quad (56)$$

and

$$\phi_{\mu_r \mu_l}^*(\mathbf{p}) = \int_{\mathbb{R}^4} d^4 x_e f_{\mu_r \mu_l}^*(x_e) \frac{e^{-\omega_m(\mathbf{p})x_e^0 + i\mathbf{p} \cdot \mathbf{x}}}{(2\pi)^{3/2}}, \quad (57)$$

Eq. (54) becomes

$$\begin{aligned} & \sum_{\mu_r, \nu_r = -s_r}^{s_r} \sum_{\mu_l, \nu_l = -s_l}^{s_l} \int_{\mathbb{R}^3} (\phi_{\mu_r \mu_l}^{s_r s_l}(\mathbf{p}))^* \\ & \times \frac{d\mathbf{p} D_{\mu_r \nu_r}^{s_r} [p_m \cdot \sigma_m] D_{\mu_l \nu_l}^{s_l} [\Pi p_m \cdot \sigma_m]}{2\omega_m(\mathbf{p})} \psi_{\nu_r \nu_l}^{s_r s_l}(\mathbf{p}), \end{aligned} \quad (58)$$

which is exactly the expression for the inner product in the Lorentz covariant representation with wave functions (49)

$$(\phi_{\mu_r \mu_l}^{s_r s_l}(\mathbf{p}))^* = \langle \phi | (m, s_r, s_l) \mathbf{p}, \mu_r, \mu_l \rangle_{\text{cov}} \quad (59)$$

and

$$\psi_{\nu_r \nu_l}^{s_r s_l}(\mathbf{p}) = \langle (m, s_r, s_l) \mathbf{p}, \nu_r, \nu_l | \psi \rangle_{\text{cov}}. \quad (60)$$

Since the Wigner functions of $p_m \cdot \sigma$ and $\Pi p_m \cdot \sigma$ that appear in these expressions are squares of Hermitian matrices, it follows that Eq. (58) is nonnegative. When $B \rightarrow A^*$ the complex Euclidean covariance condition (53) becomes the Lorentz covariance conditions (46) and (47):

$$\begin{aligned} & S_{\mu_r \mu_l; \nu_r \nu_l}^{s_r s_l}(O(A, A^*)x_e, O(A, A^*)y_e) \\ & \times \sum_{\mu'_r, \nu'_r = -s_r}^{s_r} \sum_{\mu'_l, \nu'_l = -s_l}^{s_l} D_{\mu_r \mu'_r}^{s_r} [A] D_{\mu_l \mu'_l}^{s_l} [(A^\dagger)^{-1}] S_{\mu'_r \mu'_l; \nu'_r \nu'_l}^{s_r s_l}(x_e, y_e) \\ & \times D_{\nu'_r \nu_r}^{s_r} [A^\dagger] D_{\nu'_l \nu_l}^{s_l} [A^{-1}]. \end{aligned} \quad (61)$$

This means that this complex subgroup of the complex orthogonal transformations defines a unitary representation of the Poincaré group on the Hilbert space defined by the inner product (55).

It follows from Eq. (61) that

$$\begin{aligned} & \sum_{\mu'_r, \nu'_r = -s_r}^{s_r} \sum_{\mu'_l, \nu'_l = -s_l}^{s_l} \int f^*(\theta x_e, \mu'_r, \mu'_l) S_{\mu'_r \mu'_l; \nu'_r \nu'_l}^{s_r s_l}(x_e, y_e) f(y_e, \nu_r, \nu_l) d^4 x_e d^4 y_e \\ & = \int \sum_{\mu'_r, \nu'_r = -s_r}^{s_r} \sum_{\mu'_l, \nu'_l = -s_l}^{s_l} f^*(O(A, A^*)\theta x_e, \mu'_r, \mu'_l) D_{\mu_r \mu'_r}^{s_r} [A] D_{\mu_l \mu'_l}^{s_l} [(A^\dagger)^{-1}] S_{\mu'_r \mu'_l; \nu'_r \nu'_l}^{s_r s_l}(x_e, y_e) \\ & \times D_{\nu'_r \nu_r}^{s_r} [A^\dagger] D_{\nu'_l \nu_l}^{s_l} [A^{-1}] f(O(A, A^*)y_e, \nu'_r, \nu'_l) d^4 x_e d^4 y_e, \end{aligned} \quad (62)$$

or that

$$\begin{aligned} f(y_e, \nu_r, \nu_l) & \rightarrow \sum_{\nu'_r, \nu'_l = -s_l}^{s_l} f(O(A, A^*)y_e, \nu'_r, \nu'_l) D_{\nu'_r \nu_r}^{s_r} [A^\dagger] D_{\nu'_l \nu_l}^{s_l} \\ & \times [A^{-1}] \end{aligned} \quad (63)$$

is a unitary representation of $SL(2, \mathbb{C})$.

To understand how the support conditions work for finite Lorentz transformations it is useful to express the Euclidean coordinates in terms of the 2×2 matrices X_e . X_e can be decomposed into the sum of Hermitian and anti-Hermitian parts

$$X_e = \frac{1}{2}(X_e + X_e^\dagger) + \frac{1}{2}(X_e - X_e^\dagger) = X_h + X_a, \quad (64)$$

where

$$X_h = X_h^\dagger \quad X_a = -X_a^\dagger. \quad (65)$$

Under real Lorentz transformations

$$AX_e A^\dagger = AX_h A^\dagger + AX_a A^\dagger, \quad (66)$$

the Hermitian and anti-Hermitian parts transform independently

$$X'_h = AX_h A^\dagger, \quad X'_a = AX_a A^\dagger. \quad (67)$$

For real Euclidean X_e , $X_a = ix_e^0 I$, so

$$AX_a A^\dagger = AA^\dagger X_a = ix_e^0 AA^\dagger, \quad (68)$$

which means $\text{Tr}(AX_e A^\dagger) = ix_e^0 \text{Tr}(AA^\dagger)$, or that the sign of the imaginary part of X_e is unchanged, while the Hermitian part, associated with real Lorentz transformations transforms independently. This means that the sign of the real part of the Euclidean time is preserved under real Lorentz transformations.

Since the integrals are over four-dimensional Euclidean variables

$$|\det AX_e A^\dagger| = |\det X_e| = |\det X_e^\dagger| = |\det \theta X_e|, \quad (69)$$

which implies $d^4 x_e = d^4 \theta x_e = d^4 x'_e$, where x'_e is the real Lorentz transformed x_e .

The Fourier transforms of these distributions can be computed explicitly (see Ref. [60])

$$\begin{aligned} & S_{\mu_r \mu_l; \nu_r \nu_l}^{s_r s_l}(x_e, y_e) \\ & = \int \frac{d^4 p_e}{(2\pi)^4} \int \frac{e^{ip_e \cdot (x_e - y_e)} D_{\mu_r \nu_r}^{s_r} [p_e \cdot \sigma_e] D_{\mu_l \nu_l}^{s_l} [p_e \cdot \sigma_e^t \sigma_2]}{p_e^2 + m^2} \\ & = \frac{m^2}{(2\pi)^2} D_{\mu_r \nu_r}^{s_r} [-i\sigma_e \cdot \nabla_{x_e}] D_{\mu_l \nu_l}^{s_l} [-i\sigma_2 \sigma_e^t \sigma_2 \cdot \nabla_{x_e}] \\ & \times \frac{K_1(m|x_e - y_e|)}{m|x_e - y_e|}. \end{aligned} \quad (70)$$

The Wigner functions are polynomial in the derivatives. While this kernel is singular as $x_e \rightarrow y_e$, the Euclidean time support condition along with the Euclidean time reflection ensures that $|x_e - y_e|$ never vanishes.

V. EUCLIDEAN COVARIANCE

The spinor transformation properties of the Schwinger functions in the previous section, Eq. (53), were determined by requiring that the Lorentz invariant inner product is recovered when the Euclidean wave functions have support for positive Euclidean time and the final Euclidean time is reflected.

The covariance condition for the Schwinger functions follow from Eq. (53), which can be expressed in the form

$$\begin{aligned} & S_{\mu_r \mu_l; \nu_r \nu_l}^{s_r s_l} ((O(A, B)x_e, (O(A, B)y_e) \\ &= \sum_{\mu'_r, \nu'_r = -s_r}^{s_r} \sum_{\mu'_l, \nu'_l = -s_l}^{s_l} S_{\mu'_r \mu'_l; \nu'_r \nu'_l}^{s_r s_l} (x_e, y_e) D_{\mu'_r \mu_r}^{s_r} [A^t] D_{\nu'_r \nu_r}^{s_r} [B^t] \\ & \times D_{\mu'_l \mu_l}^{s_l} [(B)^{-1}] D_{\nu'_l \nu_l}^{s_l} [A^{-1}], \end{aligned} \quad (71)$$

where A and B are independent $SU(2)$ matrices. It is straightforward to extend this condition to Euclidean covariant kernels with more initial and final Euclidean coordinates and spins. The distributions that replace the Schwinger functions of axiomatic field theory will be referred to as quasi-Schwinger functions, since they are not required to satisfy the locality (symmetry) requirement.

In the Euclidean case, when spinors are involved, this kernel is no longer positive. This is most easily seen in Eq. (51), where $D_{\mu_r \nu_r}^{s_r} [p_e \cdot \sigma_e]$ and $D_{\mu_l \nu_l}^{s_l} [p_e \cdot \Pi \sigma_e]$ are not Hermitian for real p_e .

VI. CONNECTED DISTRIBUTIONS—DYNAMICS

The construction of positive-mass positive-energy irreducible representations of the Poincaré group using Euclidean covariant distributions was discussed in Sec. IV. Tensor products of these distributions describe noninteracting many-particle systems.

The quasi-Schwinger functions for interacting particles have cluster expansions, which are sums of tensor products of connected quasi-Schwinger functions. Since reflection positivity is preserved under addition and tensor products, reflection positivity follows if the connected quasi-Schwinger

functions are reflection positive. Connected quasi-Schwinger functions are the building blocks of general quasi-Schwinger distributions. The purpose of this section is to show that, in the absence of the symmetry requirement it is straightforward to construct connected quasi-Schwinger functions satisfying the conditions needed to for a relativistic quantum theory.

The structure of connected quasi-Schwinger functions is motivated by the construction of the quasi-Schwinger functions for positive-mass positive-energy irreducible representations derived in Sec. IV. The general structure of a connected quasi-Schwinger function can be understood by considering the example of a connected four-point quasi-Schwinger function with two initial and two final coordinates and right handed spinors. It is assumed that it has a contribution from an “intermediate state” with mass λ and spin $(s, 0)$. This exhibits the structural elements of a general connected quasi-Schwinger functions.

The following structure of the connected quasi-Schwinger function is assumed:

$$\begin{aligned} & S_4^c(x_{e1}, s_1, \mu_1, x_{e2}, s_2, \mu_2; y_{e1}, s_1, \nu_1, y_{e2}, s_2, \nu_2) \\ &= \sum_{\mu, \nu = -s}^s \int S_2^{*s_1, s_2; s} \left(\frac{1}{2}(x_{e1} - x_{e2}), p_e \right) \\ & \times \langle s_1, \mu_1, s_2, \mu_2 | s, \mu \rangle \frac{D_{\mu\nu}^s(p_e \cdot \sigma_e)}{p_e^2 + m^2} \\ & \times e^{ip_e \cdot (x_{e1} + x_{e2} - y_{e1} - y_{e2})} d^4 p_e \langle s_1, \nu_1, s_2, \nu_2 | s, \nu \rangle S_2^{s_1, s_2; s} \\ & \times \left(\frac{1}{2}(y_{e1} - y_{e2}), p_e \right), \end{aligned} \quad (72)$$

where

$$S_2^{s_1 s_2; s} \left(\frac{1}{2}(x_{e1} - x_{e2}), p_e \right) \quad (73)$$

is a connected Euclidean invariant function of $\frac{1}{2}(x_{e1} - x_{e2})$ and p_e . It is assumed to vanish as $(x_{e1} - x_{e2})^2 \rightarrow \infty$, but be analytic in p_e . For identical particles the coefficient

$$\langle s_1, \nu_1, s_2, \nu_2 | s, \nu \rangle S_2^{s_1 s_2; s} \left(\frac{1}{2}(y_{e1} - y_{e2}), p_e \right) \quad (74)$$

is assumed to be either symmetric or antisymmetric with respect to interchange of $1 \leftrightarrow 2$.

To show that this expression satisfies the Euclidean covariance condition (71) note

$$S_4^c(O(A, B)x_{e1}, s_1, \mu_1, O(A, B)x_{e2}, s_2, \mu_2; O(A, B)y_{e1}, s_1, \nu_1, O(A, B)y_{e2}, s_2, \nu_2) \quad (75)$$

$$\begin{aligned} &= \sum_{\mu, \nu = -s}^s S_2^{*s_1 s_2; s} \left(\frac{1}{2}O(A, B)(x_{e1} - x_{e2}), p_e \right) \langle s_1, \mu_1, s_2, \mu_2 | s, \mu \rangle \frac{D_{\mu\nu}^s(p_e \cdot \sigma_e)}{p_e^2 + m^2} \\ & \times e^{ip_e \cdot O(A, B)(x_{e1} + x_{e2} - y_{e1} - y_{e2})} d^4 p_e \langle s_1, \nu_1, s_2, \nu_2 | s, \nu \rangle \\ & \times S_2^{s_1 s_2; s}(O(A, B) \left(\frac{1}{2}(y_{e1} - y_{e2}), p_e \right). \end{aligned} \quad (76)$$

Euclidean invariance of the dot product in the exponent can be used to move $O(A, B)$ to p_e ,

$$= \int \sum_{\mu, \nu=-s}^s \mathcal{S}_2^{*s_1 s_2; s} \left(\frac{1}{2} O(A, B)(x_{e1} - x_{e2}), p_e \right) \langle s_1, \mu_1, s_2, \mu_2, |s, \mu \rangle \frac{D_{\mu\nu}^s(p_e \cdot \sigma_e)}{p_e^2 + m^2} e^{i(O(A, B)^{-1} p_e) \cdot (x_{e1} + x_{e2} - y_{e1} - y_{e2})} d^4 p_e \\ \times \langle s_1, \nu_1, s_2, \nu_2 | s, \nu \rangle \mathcal{S}_2^{s_1 s_2; s} \left(\frac{1}{2} O(A, B)(y_{e1} - y_{e2}), p_e \right), \quad (77)$$

while changing variables $p'_e = O^{-1}(A, B)p_e$ gives

$$= \int \sum_{\mu, \nu=-s}^s \mathcal{S}_2^{*s_1 s_2; s} \left(\frac{1}{2} O(A, B)(x_{e1} - x_{e2}), O(A, B)p_e \right) \langle s_1, \mu_1, s_2, \mu_2, |s, \mu \rangle \times \frac{D_{\mu\nu}^s(O(A, B)p_e \cdot \sigma_e)}{p_e^2 + m^2} e^{ip_e \cdot (x_{e1} + x_{e2} - y_{e1} - y_{e2})} d^4 p_e \\ \times \langle s_1, \nu_1, s_2, \nu_2 | s, \nu \rangle \mathcal{S}_2^{s_1 s_2; s} \left(\frac{1}{2} O(A, B)(y_{e1} - y_{e2}), O(A, B)p_e \right). \quad (78)$$

The Euclidean invariance, $\mathcal{S}_2^{s_1 s_2; s}(\frac{1}{2}(y_{e1} - y_{e2}), p_e) = \mathcal{S}_2^{s_1 s_2; s}(O(A, B)\frac{1}{2}(y_{e1} - y_{e2}), O(A, B)p_e)$ means that the factors of $O(A, B)$ can be removed while Eq. (A10), $(O(A, B)(p_e \cdot \sigma_e) = A(p_e \cdot \sigma_e)B^t$, gives

$$= \sum_{\mu, \nu=-s}^s \mathcal{S}_2^{*s_1 s_2; s} \left(\frac{1}{2}(x_{e1} - x_{e2}), p_e \right) \langle s_1, \mu_1, s_2, \mu_2 | s, \mu \rangle \times D_{\mu\mu'}^s(A) \frac{D_{\mu'\nu'}^s(p_e \cdot \sigma_e)}{p_e^2 + m^2} D_{\nu'\nu}^s(B^t) e^{ip_e \cdot (x_{e1} + x_{e2} - y_{e1} - y_{e2})} d^4 p_e \\ \times \langle s_1, \nu_1, s_2, \nu_2 | s, \nu \rangle \mathcal{S}_2^{s_1 s_2; s} \left(\frac{1}{2}(y_{e1} - y_{e2}), p_e \right). \quad (79)$$

Finally, the property (26) of the Clebsch-Gordan coefficients can be used to get

$$= \int \sum_{\mu, \nu=-s}^s \sum_{\mu'_1, \nu'_1=-s_1}^{s_1} \sum_{\mu'_2, \nu'_2=-s_2}^{s_2} D_{\mu_1 \mu'_1}^{s_1}(A) D_{\mu_2 \mu'_2}^{s_2}(A) \mathcal{S}_2^{*s_1 s_2; s} \left(\frac{1}{2}(x_{e1} - x_{e2}), p_e \right) \langle s_1, \mu'_1, s_2, \mu'_2 | s, \mu \rangle \\ \times \frac{D_{\mu\nu}^s(p_e \cdot \sigma_e)}{p_e^2 + m^2} e^{ip_e \cdot (x_{e1} + x_{e2} - y_{e1} - y_{e2})} d^4 p_e \langle s_1, \nu'_1, s_2, \nu'_2 | s, \nu \rangle \mathcal{S}_2^{s_1 s_2; s} \left(\frac{1}{2}(y_{e1} - y_{e2}), p_e \right) D_{\nu'_2 \nu_2}^{s_2}(B^t) D_{\nu'_1 \nu_1}^{s_1}(B^t), \quad (80)$$

which shows that the two-point quasi-Schwinger function (72) satisfies the Euclidean covariance condition (71):

$$= \sum_{\mu'_1, \nu'_1=-s_1}^{s_1} \sum_{\mu'_2, \nu'_2=-s_2}^{s_2} D_{\mu_1 \mu'_1}^{s_1}(A) D_{\mu_2 \mu'_2}^{s_2}(A) S_4^c(x_{e1}, s_1, \mu'_1, x_{e2}, s_2, \mu'_2; y_{e1}, s_1, \nu'_1, y_{e2}, s_2, \nu'_2) D_{\nu'_2 \nu_2}^{s_2}(B^t) D_{\nu'_1 \nu_1}^{s_1}(B^t). \quad (81)$$

The other requirement is reflection positivity. In this case it is sufficient to assume that the test functions have support for positive Euclidean times. It follows that they have support for positive $X_e^0 = \frac{1}{2}(x_{e1}^0 + x_{e2}^0)$.

To show reflection positivity note that $(\theta f, S f)$ is

$$\int \sum_{\mu_1, \nu_1=-s_1}^{s_1} \sum_{\mu_2, \nu_2=-s_2}^{s_2} f^*(\theta x_{e1}, s_1, \mu_1, \theta x_{e2}, s_2, \mu_2) S_4^c(x_{e1}, s_1, \mu_1, x_{e2}, s_2, \mu_2; y_{e1}, s_1, \nu_1, y_{e2}, s_2, \nu_2) f(y_{e1}, s_1, \nu_1, y_{e2}, s_2, \nu_2). \quad (82)$$

Moving the reflection operators from the final test functions to the quasi-Schwinger function gives

$$= \int \sum_{\mu_1, \nu_1=-s_1}^{s_1} \sum_{\mu_2, \nu_2=-s_2}^{s_2} f^*(x_{e1}, s_1, \mu_1, x_{e2}, s_2, \mu_2) \mathcal{S}_2^{*s_1 s_2; s} \left(\frac{1}{2}(\theta(x_{e1} - x_{e2}), p_e) \right) \langle s_1, \mu_1, s_2, \mu_2 | s, \mu \rangle \\ \times \frac{D_{\mu\nu}^s(p_e \cdot \sigma_e)}{p_e^2 + \lambda^2} e^{ip_e \cdot (\theta x_{e1} + \theta x_{e2} - y_{e1} - y_{e2})} \langle s_1, \nu_1, s_2, \nu_2 | s, \nu \rangle d^4 p_e \mathcal{S}_2^{s_1 s_2; s} \left(\frac{1}{2}(y_{e1} - y_{e2}), p_e \right) f(y_{e1}, s_1, \nu_1, y_{e2}, s_2, \nu_2). \quad (83)$$

Using the fact that $\theta(x_{e1} - x_{e2}) \cdot p_e = ((x_{e1} - x_{e2}) \cdot p_e)^*$ gives

$$= \sum_{\mu_1, \nu_1=-s_1}^{s_1} \sum_{\mu_2, \nu_2=-s_2}^{s_2} \sum_{\mu, \nu=-s}^s \int f^*(x_{e1}, s_1, \mu_1, x_{e2}, s_2, \mu_2) \left(\mathcal{S}_2^{s_1 s_2; s} \left(\frac{1}{2}(x_{e1} - x_{e2}), p_e \right) \langle s_1, \mu_1, s_2, \mu_2 | s, \mu \rangle \right)^* \\ \times \frac{D_{\mu\nu}^s(p_e \cdot \sigma_e)}{p_e^2 + \lambda^2} e^{ip_e \cdot (\theta x_{e1} + \theta x_{e2} - y_{e1} - y_{e2})} \times \langle s_1, \nu_1, s_2, \nu_2 | s, \nu \rangle d^4 p_e \mathcal{S}_2^{s_1 s_2; s} \left(\frac{1}{2}(y_{e1} - y_{e2}), p_e \right) f(y_{e1}, s_1, \nu_1, y_{e2}, s_2, \nu_2). \quad (84)$$

Because the $\mathcal{S}_2^{s_1 s_2; s}$ are analytic in p_e and the Wigner functions are polynomials in the components of p_e , as in the single-particle case, after integrating over test functions in X_e and Y_e satisfying the Euclidean time support condition, the integral over p_e^0 can

be computed using the residue theorem. The only contributing pole is at $p^0 = -i\omega_\lambda(\mathbf{p})$:

$$\begin{aligned}
&= \int \sum_{\mu_1, \nu_1 = -s_1}^{s_1} \sum_{\mu_2, \nu_2 = -s_2}^{s_2} \sum_{\mu, \nu = -s}^s \left(\int dx_{e1} dx_{e2} f((x_{e1}, s_1, \mu_1, x_{e2}, s_2, \mu_2) \mathcal{S}_2^{s_1, s_2; s} \left(\frac{1}{2}(x_{e1} - x_{e2}), p_e \right) \langle s_1, \mu_1, s_2, \mu_2 | s, \mu \rangle \right) \\
&\times e^{-\omega_\lambda(\mathbf{p})(x_{e1}^0 + x_{e2}^0)} \left(\frac{\pi d\mathbf{p}}{\omega_\lambda(\mathbf{p})} D_{\mu\nu}^s(p_m \cdot \sigma_m) \left(\int dy_{e1} dy_{e2} f(y_{e1}, s_1, \nu_1, y_{e2}, s_2, \nu_2) \mathcal{S}_2^{s_1, s_2; s} \left(\frac{1}{2}(y_{e1} - y_{e2}), p_e \right) \langle s_1, \nu_1, s_2, \nu_2 | s, \nu \rangle \right) \right) \\
&\times e^{-\omega_\lambda(\mathbf{p})(y_{e1}^0 + y_{e2}^0)} \Bigg), \tag{85}
\end{aligned}$$

which is nonnegative since it has the form $\sum_{mn} a_m^* P_{mn} a_n$ where P_{mn} is positive definite, so it is positive.

Note that unlike the local case, the Euclidean time differences $x_1^0 - x_2^0$ do not have to be different from zero. This is because in the nonlocal case there are different N -point quasi-Schwinger functions with different numbers of initial and final coordinates.

This four-point example is easy to generalize:

(1) The connected four point function can be replaced by a square matrix of $m + n$ -point connected distributions with the same “intermediate states.”

(2) The spinors of rank $(s, 0)$ can be replaced by spinors of rank (s_r, s_l) or direct sums of rank $(s, 0) \oplus (0, s)$.

(3) A single “intermediate state” of mass, $\lambda > 0$, and spin (s_r, s_l) can be replaced by linear superpositions of states with different λ 's and spins with a positive weight.

(4) The initial and final distributions $\mathcal{S}_2^{s_1, s_2; s}(\frac{1}{2}(y_1 - y_2), p_e) \langle s_1, \nu_1, s_2, \nu_2 | s, \nu \rangle$ are replaced sums of products of invariant distributions and constant coupling coefficients of the form

$$\begin{aligned}
&\sum_a \mathcal{S}_{n,a}^{(s_{r1}, s_{l1}), \dots, (s_{rn}, s_{ln}); (s_r, s_l)}(X_e - x_{e1} \cdots X_e - x_{en-1}; p_e^c) \\
&\times C_{(\mu_{r1}, \mu_{l1}), \dots, (\mu_{rn}, \mu_{ln}); (\mu_r, \mu_l)}^{(s_{r1}, s_{l1}), \dots, (s_{rn}, s_{ln}); (s_r, s_l)}(a), \tag{86}
\end{aligned}$$

where $X = \frac{1}{n}(x_{e1} + \cdots + x_{en})$, each $\mathcal{S}_{n,a}$ is a connected Euclidean invariant function of the $X_e - x_{ei}$ and p_e , analytic in p_e , and the coefficients $C_{(\mu_{r1}, \mu_{l1}), \dots, (\mu_{rn}, \mu_{ln}); (\mu_r, \mu_l)}^{(s_{r1}, s_{l1}), \dots, (s_{rn}, s_{ln}); (s_r, s_l)}(a)$ decompose the tensor products of $D(A)$, $D(B)$ to direct sums with spin (s_r, s_l) , similar to Eqs. (81) and (26).

It is straightforward to construct $\mathcal{S}_{n,a}$ with these properties. In the general case a single value of m is replaced by a linear superposition of states with different values of m with a positive weight, $\rho(m)$:

$$m \rightarrow \int \rho(m) dm. \tag{87}$$

The combinations (86) should be symmetric or antisymmetric with respect to exchange of identical particles. This can be realized by projecting the initial and final states on the symmetric or antisymmetric subspace of the Hilbert space.

With these generalizations the proof of Euclidean covariance and reflection positivity follows the proof in the four-point case.

The discussion in this section shows that it is not difficult to construct connected quasi-Schwinger functions that satisfy Euclidean covariance and reflection positivity. These can be

used to construct a Hilbert space inner product where the vectors are function of Euclidean variables with support for positive Euclidean times.

VII. CLUSTER EXPANSIONS

The motivation for exploring the Euclidean approach to relativistic quantum mechanics is the difficulty in satisfying cluster properties in relativistic direct interaction models.

In this section a generalization of the linked cluster theorem is used to construct quasi-Schwinger functions that satisfy cluster properties using the connected quasi-Schwinger functions introduced in the previous section.

In this section reflection positive quasi-Schwinger functions will be expressed as linear combinations of products of connected reflection positive quasi-Schwinger functions. The connected reflection positive quasi-Schwinger functions are the elementary building blocks of reflection positive quasi-Schwinger functions that satisfy cluster properties. For systems of identical particles the sums have to include all combinations of tensor products of connected quasi-Schwinger functions that are generated by permutations.

Let

$$\{S_{mn}(x_{me}, \dots, x_{le} : y_{1e}, \dots, y_{ne})\} \tag{88}$$

be a collection of reflection-positive quasi-Schwinger distribution ($1 \leq m, n \leq N \leq \infty$) for a system of identical particles. In this and the following expressions the spinor indices are suppressed.

Each S_{mn} can be expanded as a sum of tensor products of connected reflection positive kernels. The tensor products contributing to this sum are products of a total of l kernels; k_1 connected kernels of type $S_{m_1 n_1}^c$, k_2 connected kernels of the type $S_{m_2 n_2}^c$, $\dots$, k_l connected kernels of the type $S_{m_l n_l}^c$, where

$$n = \sum_{i=1}^l k_i n_i, \quad m = \sum_{i=1}^l k_i m_i. \tag{89}$$

For systems of identical particles the sums include all distributions generated by $m!$ permutations of the final coordinates and $n!$ permutations of the initial coordinates.

Assume each kernel, $S_{m_i n_i}^c$, is invariant up to sign under $m_i! n_i!$ permutations that separately interchange the initial and final arguments. If a given kernel appears k_i times in the product, then there are $k_i!$ additional permutations that exchange

the k_i identical terms in the product. After accounting for these invariances there remain

$$N = \frac{m!n!}{k_1! \cdots k_l! (n_1!m_1!)^{k_1} (n_2!m_2!)^{k_2} \cdots (n_l!m_l!)^{k_l}} \quad (90)$$

kernels with this structure that differ by permutations, where the integers in Eq. (90) are constrained by Eq. (89). For identical particles for each product that contributes to S_{mn} , the sum must also include all N distributions that are generated by these additional permutations.

It is possible to construct a generating function for the S_{mn} in terms of the individual connected S_{mn}^c . To do this consistently for Bosons and Fermions define formal creation and annihilation operators. The $a^\dagger(y_i)$ operators create initial states and $b^\dagger(x_i)$ create final states. These operators satisfy

$$[a(x_i), a^\dagger(y_i)]_\pm = \delta^4(x_i - y_i) \delta_{\mu_{ir}\mu'_{ir}} \delta_{\mu_{il}\mu'_{il}}, \quad (91)$$

$$[b(x_i), b^\dagger(y_i)]_\pm = \delta^4(x_i - y_i) \delta_{\mu_{ir}\mu'_{ir}} \delta_{\mu_{il}\mu'_{il}}, \quad (92)$$

$$[a(x_i), b(y_i)]_\pm = 0, \quad (93)$$

$$a(x_i)|0\rangle = b(x_i)|0\rangle = 0, \quad \langle 0|0\rangle = 1. \quad (94)$$

In these expressions, $[A, B]_-$ is a commutator and $[A, B]_+$ is an anticommutator. The operators $a(x_i)$ and $b(y_i)$ are just formal operators that are useful for bookkeeping purposes. The same is true for the formal “vacuum,” $|0\rangle$; it has nothing to do with the ground state of the theory. They are just for the purpose of constructing quasi-Schwinger functions from the connected distributions.

With this notation define the generating functional, S , as the formal sum

$$S = \sum_{mn} \frac{1}{m!n!} \int d^{4m}x d^{4n}y \int b^\dagger(x_{me}) \cdots b^\dagger(x_{1e}) \times S_{mn}(x_{me}, \cdots, x_{1e} : y_{1e}, \cdots, y_{ne}) a^\dagger(x_{1e}) \cdots a^\dagger(x_{ne}). \quad (95)$$

The individual symmetrized $S_{mn}(X : Y)$ can be extracted from S using products of annihilation operators and the formal vacuum vector

$$S_{mn}(X : Y) = \langle 0|a(y_{ne}) \cdots a(y_{1e})b(x_{1e}) \cdots b(x_{me})S|0\rangle. \quad (96)$$

Note that in Eq. (96) there are $n!m!$ pairings of creation and annihilation operators of the same type that are equivalent to adding all possible exchanges. If S_{mn} is already symmetric or antisymmetric, then each product of pairings gives the same result. This results in an overcounting by $n!m!$ which is canceled by the denominator in Eq. (95).

Each kernel S_{mn} is a sum of all allowed products of connected kernels that satisfy (89):

$$S_{mn} = \sum \prod_{i=1}^l (S_{m_i n_i}^c)^{k_i}, \quad (97)$$

where the sum is over all products of connected kernels satisfying Eq. (89). It is useful to express each one of these as

operators in terms of the creation and annihilation operators:

$$S_i^c = \sum \frac{1}{m_i!n_i!} \int d^{4m_i}x_e d^{4n_i}y_e \int b^\dagger(x_{me}) \cdots b^\dagger(x_{1e}) \times S_{m_i n_i}^c(x_{me} \cdots x_{1e} : y_{1e} \cdots y_{n_i e}) a^\dagger(y_{1e}) \cdots a^\dagger(y_{n_i e}), \quad (98)$$

where the $S_{m_i n_i}^c$ are connected factors that appear in the product.

In this notation the operator S is a sum of products of these different types of connected operators. While the factor $\frac{1}{m_i!n_i!}$ divides by the number of permutations give the same kernel when $S_{m_i n_i}^c$ is symmetric, if a given S_i^c appears k_i times in the product, there are $k_i!$ permutations that exchange all of the coordinates in each factor in the product. In terms of the different types of connected operators the generating function has the form

$$S = \sum \frac{1}{k_1!} (S_1^c)^{k_1} \cdots \frac{l}{k_l!} (S_l^c)^{k_l} = e^{\sum_l S_l^c}, \quad (99)$$

and the individual S_{mn} can be extracted from this expression using Eq. (96) which can be expressed as

$$S_{mn}(X : Y) = \langle 0|a(y_{en}) \cdots a(y_{e1})b(x_{e1}) \cdots b(x_{em})e^{\sum_l S_l^c}|0\rangle, \quad (100)$$

which is the form of the linked cluster theorem for the quasi-Schwinger distributions. The convergence of the series is irrelevant since only a finite number of terms contribute to a given $S_{mn}(X : Y)$.

Note that while this generating function involves systems of arbitrary numbers of degrees of freedom, it is possible the individual connected components can involve a finite number of degrees of freedom.

VIII. CONCLUSION

The purpose of this work is to show that by relaxing the requirement of locality it is possible to construct a set of reflection positive Euclidean covariant distributions satisfying cluster properties. The new feature is that a single N -point Schwinger function is replaced by $N - 1$ distributions with m final and k initial degrees of freedom with $m + k = N$. This simplifies the reflection positivity requirement. These Euclidean distributions define the kernel of a Hilbert space inner product. There is a representation of the Poincaré Lie algebra on this Hilbert space [5]. These generators are self-adjoint operators satisfying cluster properties. The spectrum of the Hamiltonian is bounded from below. It follows that there is a unitary representation of the Poincaré group that satisfies spacelike cluster properties and a spectral condition. Given these distributions it is possible to perform any kind of quantum calculation without the need for analytic continuation.

While establishing the existence of a large class of reflection positive quasi-Schwinger functions is an important first step for constructing dynamical models, the problem is that this construction assumed an acceptable spectral density, which is dynamical information that should be calculated rather than assumed.

The formulation of a dynamical principle that could generate these distributions is beyond the scope of this paper, but it is an important question that needs to be addressed in the future.

Some aspects of this program are discussed elsewhere. They all assume the existence of a set of distributions with the properties shown in this work. Reference [1] provides a computational justification that this formalism may be applied to compute scattering cross sections. The scattering computations used a variation of the time-dependent formulation of scattering that utilized the invariance principle [50–52] with narrow wave packets to approximate sharp-momentum transition matrix elements. Reference [4] provided a formulation of scattering with composite particles, using a generalization of Haag-Ruelle scattering. This required a Euclidean construction to isolate composite one-body states using functions of the mass operator, which is represented by the Euclidean Laplacian, with compact support. This was done by establishing the completeness of polynomials in the Euclidean Laplacian using the Carleman condition [57]. Reference [5] provides explicit expressions for the Poincaré generators with any spin and proved their self-adjointness.

One of the interesting observations about this approach is that given reflection positive Euclidean distributions, it is possible to perform quantum mechanical calculations directly in a Euclidean representation, without analytic continuation.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: RELATIVISTIC COVARIANCE: $SL(2, \mathbb{C})$ AND $SU(2) \times SU(2)$

This Appendix provides more details on the relation between the Lorentz and orthogonal groups in four dimensions.

Since the covering group of the complex Lorentz and the four-dimensional complex orthogonal group is $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$, there are formal expressions for the Poincaré generators as linear combinations of the generators of the Euclidean group. These operators satisfy the commutation relations of the Poincaré Lie algebra, but they are not self-adjoint on the Hilbert space where the Euclidean generators are self-adjoint. Reflection positivity leads to a Hilbert space inner product where the Poincaré generators constructed from the Euclidean generators are represented by self-adjoint operators.

The purpose of this section is to review the relation between the complex Lorentz and complex orthogonal groups. The starting point is the matrix representations of the Euclidean and Minkowski 4-vectors (5) and (6).

The complex 4×4 Lorentz and orthogonal matrices corresponding to the transformation (10) are

$$\Lambda(A, B)^\alpha_\beta = \frac{1}{2} \text{Tr}(\sigma_\alpha A \sigma_\beta B^\dagger), \quad (\text{A1})$$

$$O(A, B)^\alpha_\beta = \frac{1}{2} \text{Tr}(\sigma_{\epsilon\alpha}^\dagger A \sigma_{\epsilon\beta} B^\dagger). \quad (\text{A2})$$

For real Lorentz transformations $B = A^*$ and for real orthogonal transformations A and B are independent $SU(2)$ matrices. When the left sides of Eqs. (A1) and (A2) are real, the 4×4 transformations can also be expressed by taking complex conjugates

$$\Lambda(A, B)^\alpha_\beta = \frac{1}{2} \text{Tr}(\sigma_\alpha^* A^* \sigma_\beta^* B^\dagger), \quad (\text{A3})$$

$$O(A, B)^\alpha_\beta = \frac{1}{2} \text{Tr}(\sigma_{\epsilon\alpha}^\dagger A^* \sigma_{\epsilon\beta}^* B^\dagger). \quad (\text{A4})$$

There are two kinds of spinors associated with $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ that are distinguished by their $SL(2, \mathbb{C})$ transformation properties

$$\xi^a \rightarrow \xi^{a'} = \sum_b A_{ab} \xi^b, \quad (\text{A5})$$

$$\xi^{\dot{a}} \rightarrow \xi^{\dot{a}'} = \sum_b B_{\dot{a}\dot{b}} \xi^{\dot{b}}. \quad (\text{A6})$$

For reasons that will be discussed later spinors transforming like Eq. (A5) are called right-handed spinors while spinors transforming like Eq. (A6) are called left-handed spinors. Dot superscripts are used to distinguish left handed spinors from right handed spinors. For real Lorentz transformations $B_{\dot{a}\dot{b}} = A_{ab}^*$ while for real orthogonal transformations A_{ab} and $B_{\dot{a}\dot{b}}$ are independent $SU(2)$ matrices. In both cases the two representations are inequivalent; which means that a general A and B or A and A^* cannot be related by a single constant similarity transformation:

$$\nexists M \text{ satisfying } MAM^{-1} = A^* \text{ or } MAM^{-1} = B \quad \forall A, B. \quad (\text{A7})$$

Because they have determinant 1, a general $SL(2, \mathbb{C})$ matrix can be expressed as the exponential of the dot product of a complex 3-vector $\mathbf{z}$ with the vector of Pauli matrices:

$$A = \pm e^{\mathbf{z} \cdot \boldsymbol{\sigma}}, \quad \det(A) = e^{\mathbf{z} \cdot \text{tr}(\boldsymbol{\sigma})} = 1. \quad (\text{A8})$$

It follows from the representation (A8) and properties of the Pauli matrices that

$$\sigma_2 A^\dagger \sigma_2 = A^{-1}, \quad \sigma_2 B^\dagger \sigma_2 = B^{-1}. \quad (\text{A9})$$

This means that σ_2 behaves like a metric tensor with respect to these spinors. Dual right- and left-handed spinors are identified by lower indices:

$$\xi_a := \sum_b (\sigma_2)_{ab} \xi^b, \quad \xi_{\dot{a}} := \sum_{\dot{b}} (\sigma_2)_{\dot{a}\dot{b}} \xi^{\dot{b}}. \quad (\text{A10})$$

The transformation properties of the dual spinors follow from the transformation properties of the corresponding upper index spinors

$$\begin{aligned}\xi_a &\rightarrow \xi'_a = \sum_{bc} (\sigma_2)_{ab} A_{bc} \xi^c = \sum_{bc} (\sigma_2)_{ab} A_{bc} (\sigma_2)_{cd} \xi_d \\ &= \sum_b (A^t)_{ab}^{-1} \xi_b,\end{aligned}\quad (\text{A11})$$

$$\begin{aligned}\xi_{\dot{a}} &\rightarrow \xi'_{\dot{a}} = \sum_{\dot{b}\dot{c}} (\sigma_2)_{\dot{a}\dot{b}} B_{\dot{b}\dot{c}} \xi^{\dot{c}} = \sum_{\dot{b}\dot{c}} B_{\dot{a}\dot{b}} (\sigma_2)_{\dot{b}\dot{c}} \xi^{\dot{c}} \\ &= \sum_{\dot{b}} (B^t)_{\dot{a}\dot{b}}^{-1} \xi_{\dot{b}}.\end{aligned}\quad (\text{A12})$$

The dual spinors define invariant linear functionals on the corresponding spinors. This follows since the contraction of a spinor and a dual spinor of the same type is invariant under $SL(2, \mathbb{C})$:

$$\begin{aligned}\sum_a \xi^{a'} \chi'_a &= \sum_{abc} A_{ab} \xi^b (A^t)_{ac}^{-1} \chi_c = \sum_{abc} x^a (A^t)_{ab} ((A^t)^{-1})_{bc} \chi_c \\ &= \sum_a \xi^a \chi_a,\end{aligned}\quad (\text{A13})$$

and replacing A by B ,

$$\sum_{\dot{a}} \xi^{\dot{a}'} \chi'_{\dot{a}} = \sum_{\dot{a}} \xi^{\dot{a}} \chi_{\dot{a}}. \quad (\text{A14})$$

For Lorentz transformations when $\mathbf{z} = -i\frac{\phi}{2}$, $A = \pm e^{\mathbf{z} \cdot \boldsymbol{\sigma}}$ represents a rotation through an angle $|\phi|$ about the $\hat{\phi}$ axis, while for $\mathbf{z} = \frac{\rho}{2}$, $A = \pm e^{\mathbf{z} \cdot \boldsymbol{\sigma}}$ represents a rotationless Lorentz boost with rapidity ρ . For rotations A is unitary while for rotationless boosts A is a positive Hermitian matrix. Both (A, B) and $(-A, -B)$ correspond to the same 4×4 complex Lorentz or orthogonal transformation.

The 4×4 matrix representation of the complex orthogonal group can be expressed in a number of different ways in terms of the $SL(2, \mathbb{C})$ matrices A and B using properties of the Pauli matrices

$$\sum_{\mu} (O(A, B)x_e)^{\mu} \cdot \sigma_{e\mu} = \sum_{\mu} x_e^{\mu} \cdot A \sigma_{e\mu} B^t. \quad (\text{A15})$$

Taking transposes gives

$$\sum_{\mu} (O(A, B)x_e)^{\mu} \cdot \sigma_{e\mu}^t = \sum_{\mu} x_e^{\mu} \cdot B \sigma_{e\mu}^t A^t. \quad (\text{A16})$$

Multiplying by σ_2 on the right and left using $\sigma_2 A \sigma_2 = (A^t)^{-1}$ on Eqs. (A15) and (A16) gives

$$\sum_{\mu} (O(A, B)x_e)^{\mu} \cdot \sigma_2 \sigma_{e\mu} \sigma_2 = \sum_{\mu} x_e^{\mu} \cdot (A^t)^{-1} \sigma_2 \sigma_{e\mu} \sigma_2 B^{-1} \quad (\text{A17})$$

and

$$\sum_{\mu} (O(A, B)x_e)^{\mu} \cdot \sigma_2 \sigma_{e\mu}^t \sigma_2 = \sum_{\mu} x_e^{\mu} \cdot (B^t)^{-1} \sigma_2 \sigma_{e\mu}^t \sigma_2 A^{-1}. \quad (\text{A18})$$

In all four cases the complex orthogonal matrix, $O(A, B)^{\mu}_{\nu}$, remains unchanged.

A general $SL(2, \mathbb{C})$ matrix has a polar decomposition of the form

$$A = RP = P'R', \quad (\text{A19})$$

where

$$P := (A^{\dagger}A)^{1/2}, \quad R := A(A^{\dagger}A)^{-1/2},$$

$$P' := (AA^{\dagger})^{1/2}, \quad R' := (A^{\dagger}A)^{-1/2}A, \quad (\text{A20})$$

and P and P' are positive Hermitian matrices and R and R' are $SU(2)$ matrices. This implies that any Lorentz transformation can be expressed as the product of a rotation and a “rotationless” boost in either order. Since rotations leave a rest four-momentum invariant, both A and P' in Eq. (A19) transform the rest four-momentum to the same final four-momentum, p , but the “rotationless” boost, P' , is distinguished by being a positive matrix. The rotationless boost is a function of the 4-velocity, $v := p/m$. The notation $P(p/m)$ is used for the positive $SL(2, \mathbb{C})$ matrix corresponding to a boost that transforms $p_0 := (m, \mathbf{0})$ to p .

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