

Demagnetization in micromagnetics: Magnetostatic self-interactions of bulk chiral magnetic skyrmions

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We develop a theoretical and numerical framework for three-dimensional bulk chiral magnets that includes the full magnetostatic dipole-dipole interaction and its back-reaction on the magnetization. Assuming translational invariance along one spatial direction, we analyze the effect of dipolar interactions on three Dzyaloshinskii-Moriya interaction (DMI) terms—Dresselhaus, Rashba, and Heusler—corresponding to Bloch, Néel, and antiskyrmion textures. In the absence of the dipolar interaction, these three DMI terms are gauge equivalent and yield degenerate skyrmion energies. Incorporating the nonlocal dipole-dipole interaction breaks this degeneracy: Bloch skyrmions remain unaffected, Néel skyrmions shrink slightly, and Heusler antiskyrmions lose axial symmetry and stabilize into square lattice crystals. The system is solved using a nonlocal numerical relaxation method that self-consistently computes the magnetostatic potential from Poisson's equation. Our results show that long-range dipolar interactions can stabilize bulk antiskyrmion crystals in translationally invariant three-dimensional chiral magnets.

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I. INTRODUCTION

Topological spin textures arise in magnetic systems due to the competition between the Heisenberg exchange interaction and the Dzyaloshinskii-Moriya interaction (DMI). The Heisenberg exchange energy promotes parallel alignment of spins, whereas the DMI energy favors noncollinear alignment of spins [1]. Magnetic skyrmions are one such type of topological spin texture. They are topological defects in the magnetization configuration and behave as magnetic quasiparticles. Such vortexlike configurations were first observed in [2,3] in the context of condensed matter systems, and in particular, the authors found skyrmions in easy-axis chiral magnetic materials with the Dzyaloshinskii-Moriya interaction. The easy-axis, or Zeeman, interaction is essential for stability because it provides a scale for the magnetic skyrmion.

Three different skyrmion types have been investigated in the literature, each associated with a particular DMI term. The first two are the Bloch and Néel skyrmions, corresponding to the Dresselhaus and Rashba spin-orbit couplings (SOCs), respectively. The final type is the antiskyrmion, which appears in Heusler compounds [4–6], so we label the associated DMI term the Heusler DMI. The chiral magnet can also be formulated as a $U(1) \times SU(2)$ gauge theory with a constant background gauge field [7]. Within this formulation, the three

DMI terms are related by gauge transformations, which are controlled by a single rotation parameter [8]. The energy functional is invariant under the relevant gauge transformations, so the three skyrmion types are degenerate in energy. An in-depth review of the current state of the theoretical and experimental realizations of magnetic skyrmions in condensed matter systems can be found in [9].

The inclusion of the magnetic dipole-dipole interaction (DDI) introduces frustration into a ferromagnetic state, turning it into a frustrated spin system. This frustration between the DDI and Heisenberg exchange interaction can lead to interesting phenomena. The nucleation of skyrmions using an external magnetic dipole, which modifies the Zeeman interaction term, was demonstrated numerically by Garanin *et al.* [10]. Depending on how the magnetic dipole approaches the film, a skyrmion-antiskyrmion pair can be created in the bulk, or a skyrmion can be created at the edge of the film. It was also shown that magnetic biskyrmions can be stabilized by dipolar interactions in centrosymmetric materials where the DMI term is forbidden [11].

Kashuba and Pokrovsky [12] presented a theory on the effect of long-range dipole interactions in thin ferromagnetic films. In particular, they studied stripe domain structures in ordinary ferromagnets with no interfacial DMI. An analytical investigation into the stabilization of magnetic skyrmions by dipolar interactions in thin films was conducted by Ezawa [13]. He considered both ordinary and chiral ferromagnets and employed the Dresselhaus DMI term in the latter, which favors Bloch skyrmions. In both ferromagnet types, domain wall stripes and skyrmions were constructed analytically. However, in his analysis, Ezawa considered the spin texture of a Néel skyrmion, which is incompatible with the Dresselhaus DMI.

A numerical study on the effect of the dipolar interaction on chiral ferromagnets was performed by Kwon *et al.*

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[14]. They also considered the Dresselhaus DMI term, for which they investigated stripe domains and Bloch skyrmion lattices in the presence of an applied external magnetic field and perpendicular uniaxial anisotropy. As the external magnetic field is increased, the microscopic spin configurations transform from a spin-spiral stripe to a skyrmion crystalline structure. Sui and Hu [15] conducted a similar investigation (but with no anisotropies), wherein they considered the DMI term corresponding to Rashba SOC, which gives rise to Néel-type skyrmions. In both studies [14,15], magnetic skyrmion crystals were investigated using simulated annealing with a Monte Carlo method on small grids. Furthermore, the authors of both studies included a tunable constant that allowed them to alter the strength of the dipolar interaction, and Kwon *et al.* chose to make the dipolar interaction strength comparable to the anisotropy. Sui and Hu [15] found that, as the DDI strength increases, the skyrmion crystal structure changes from a hexagonal arrangement to one with a square form.

Our approach differs from that of [14,15]. Instead of considering a tunable dipole interaction parameter, we focus on the effect the dipolar interaction has on the different DMI-related skyrmion types in the bulk of a chiral ferromagnet. That is, we do not model a thin film; instead, we model a three-dimensional bulk system and impose translation invariance along the z direction. We consider three DMI terms and their associated skyrmion types: Dresselhaus (Bloch), Rashba (Néel), and Heusler (antiskyrmion). Without the dipolar interaction, the magnetic skyrmion types are equivalent up to redefinitions of the field. However, the dipole-dipole interaction breaks this equivalence, and there is a split in the energies. Furthermore, the axial symmetry is broken for the antiskyrmion (Heusler DMI), resulting in a change in shape of both isolated antiskymions and their crystal structure.

II. THE MODEL

The system we wish to model is a large three-dimensional chiral ferromagnet whose magnetization field is assumed to vary smoothly and have constant magnitude, so it can be written as $\mathbf{m}(\mathbf{x}) = M_s \mathbf{n}(\mathbf{x})$, where $|\mathbf{n}| = 1$ and M_s is a constant parameter called the magnetic saturation density. The total energy of such a field is assumed to take the form

$$E = E_{\text{exch}} + E_{\text{DMI}} + E_{\text{pot}} + E_{\text{DDI}}, \quad (1)$$

where the exchange, DMI, and potential energies are

$$E_{\text{exch}} = \frac{J}{2} \int_{\mathbb{R}^3} d^3x |\mathbf{d}\mathbf{n}|^2, \quad (2)$$

$$E_{\text{DMI}} = \mathcal{D} \int_{\mathbb{R}^3} d^3x \sum_{i=1}^3 \mathbf{d}_i \cdot (\mathbf{n} \times \partial_i \mathbf{n}), \quad (3)$$

$$E_{\text{pot}} = \int_{\mathbb{R}^3} d^3x [K_m(1 - n_3^2) + M_s B_{\text{ext}}(1 - n_3)] \quad (4)$$

and E_{DDI} is the dipole-dipole interaction energy, which we will consider in detail in Sec. III. The potential has two terms: an intrinsic anisotropy term, with coupling constant K_m (which can be any real constant), and a Zeeman term representing the interaction of $\mathbf{m}$ with a constant external applied magnetic field $\mathbf{B}_{\text{ext}} = (0, 0, B_{\text{ext}})$.

The ground state configuration is found by minimizing the potential energy E_{pot} , which yields the vacuum configuration $\mathbf{n}_{\text{vac}} = \mathbf{n}_{\uparrow} = (0, 0, 1)$. In general, for the energy to remain finite, we must impose the vacuum boundary condition $\lim_{|\mathbf{x}| \rightarrow \infty} \mathbf{n} = \mathbf{n}_{\uparrow}$ on configurations. Throughout we will consider configurations invariant under translation in the x_3 direction (the direction of the applied magnetic field), so that the domain effectively reduces to the plane $\mathbb{R}^2$. For these translationally invariant cases, with the vacuum boundary condition, there is a compactification of space $\mathbb{R}^2 \cup \{\infty\} \cong S^2$ such that the magnetization can be interpreted as a map $\mathbf{n} : S^2 \rightarrow S^2$. Associated with this map is an integer-valued topological degree $N_{\text{Sk}} \in \pi_2(S^2) = \mathbb{Z}$, which is given by

$$N_{\text{Sk}} = \int_{\mathbb{R}^2} d^2x \rho(\mathbf{x}), \quad \rho(\mathbf{x}) = \frac{1}{4\pi} \mathbf{n} \cdot \left(\frac{\partial \mathbf{n}}{\partial x_1} \times \frac{\partial \mathbf{n}}{\partial x_2} \right). \quad (5)$$

Following the usual convention in the literature, magnetization fields with $N_{\text{Sk}} < 0$ are referred to as skyrmions, while those with $N_{\text{Sk}} > 0$ are antiskyrmions. The skyrmion type is determined by the choice of the DMI term. Magnetic skyrmions with $N_{\text{Sk}} = 0$, such as a chiral magnetic bag enclosing a domain wall, are referred to as skyrmionium [16] or a 2π -vortex [17]. We will, however, not consider skyrmion bags in this article.

In addition to magnetic skyrmions having a topological degree, they also have an associated size [18]. There is some choice in how it is defined [19]. For a field theory permitting topological solitons, one can introduce the root-mean-square radius of the soliton, defined by

$$R_{\text{Sk}} = \left(\frac{1}{N_{\text{Sk}}} \int_{\mathbb{R}^2} d^2x r^2 |\rho(\mathbf{x})| \right)^{\frac{1}{2}}. \quad (6)$$

We will use this definition of skyrmion size throughout to compare what happens to the size of the skyrmion as the DDI is included for various DMI terms.

The DMI energy is determined by three constant vectors $\mathbf{d}_i$, $i = 1, 2, 3$. As we restricted our attention to translationally invariant fields, only $\mathbf{d}_1$ and $\mathbf{d}_2$ are relevant. The overall DMI strength $\mathcal{D}$ is set by choosing, without loss of generality, the longer of these two vectors, $\mathbf{d}_1$, to have length 1. We will consider three different choices of the DMI vectors $\mathbf{d}_1$ and $\mathbf{d}_2$. The most standard choice, deriving from Dresselhaus spin-orbit coupling, yields the DMI vectors $\{\mathbf{d}_1 = -\mathbf{e}_1, \mathbf{d}_2 = -\mathbf{e}_2\}$, and the corresponding DMI term is

$$\sum_i \mathbf{d}_i \cdot (\mathbf{n} \times \partial_i \mathbf{n}) = \mathbf{n} \cdot (\nabla \times \mathbf{n}) = \Lambda_{xz}^{(y)} - \Lambda_{yz}^{(x)}, \quad (7)$$

where we have introduced the Lifshitz invariants [20]

$$\Lambda_{ij}^{(k)} = n_i \frac{\partial n_j}{\partial x_k} - n_j \frac{\partial n_i}{\partial x_k}. \quad (8)$$

Rashba spin-orbit coupling corresponds to the DMI vectors $\{\mathbf{d}_1 = \mathbf{e}_2, \mathbf{d}_2 = -\mathbf{e}_1\}$, which gives

$$\sum_i \mathbf{d}_i \cdot (\mathbf{n} \times \partial_i \mathbf{n}) = n_3 (\nabla \cdot \mathbf{n}) - \mathbf{n} \cdot \nabla n_3 = \Lambda_{zx}^{(x)} - \Lambda_{yz}^{(y)}. \quad (9)$$

The final DMI term we consider arises from the vectors $\{\mathbf{d}_1 = \mathbf{e}_1, \mathbf{d}_2 = -\mathbf{e}_2\}$ and corresponds to tetragonal compounds with

TABLE I. Summary of the three different DMI terms we consider and their corresponding skyrmion types. Here, we consider only axially symmetric configurations without the dipole-dipole interaction (DDI) present. Also shown is the divergence of each skyrmion $\nabla \cdot \mathbf{n}$, which is relevant because it is the source $\rho = -(\nabla \cdot \mathbf{n})$ in the Poisson equation $\Delta\psi = \mu\rho$ for the magnetic potential and it provides information about the nature of DDI. For example, we expect the Bloch skyrmion to be unaffected by the DDI, the magnetic potential for the Néel skyrmion to be axially symmetric, and the Heusler antiskyrmion to lose axial symmetry.

	Dresselhaus	Rashba	Heusler
$\{\mathbf{d}_1, \mathbf{d}_2\}$	$\{-\mathbf{e}_1, \mathbf{e}_2\}$	$\{\mathbf{e}_2, -\mathbf{e}_1\}$	$\{\mathbf{e}_1, -\mathbf{e}_2\}$
DMI	$\Lambda_{xz}^{(y)} - \Lambda_{yz}^{(x)}$	$\Lambda_{zx}^{(x)} - \Lambda_{yz}^{(y)}$	$\Lambda_{xz}^{(y)} + \Lambda_{yz}^{(x)}$
Skyrmion	Bloch	Néel	Antiskyrmion
N_{Sk}	-1	-1	+1
$\mathbf{n}$	$(-\sin f(r) \sin \theta, \sin f(r) \cos \theta, \cos f(r))$	$(\sin f(r) \cos \theta, \sin f(r) \sin \theta, \cos f(r))$	$(-\sin f(r) \sin \theta, -\sin f(r) \cos \theta, \cos f(r))$
$\nabla \cdot \mathbf{n}$	0	$f'(r) \cos f(r) + r^{-1} \sin f(r)$	$[-f'(r) \cos f(r) + r^{-1} \sin f(r)] \sin 2\theta$

D_{2d} symmetry (such as Heusler compounds) [21],

$$\sum_i \mathbf{d}_i \cdot (\mathbf{n} \times \partial_i \mathbf{n}) = \Lambda_{xz}^{(y)} + \Lambda_{yz}^{(x)}. \quad (10)$$

In the absence of E_{DDI} all these choices are equivalent up to field redefinitions. That is,

$$E_{\text{Dresselhaus}}(n_1, n_2, n_3) = E_{\text{Rashba}}(n_2, -n_1, n_3), \\ E_{\text{Heusler}}(n_1, n_2, n_3) = E_{\text{Rashba}}(-n_2, -n_1, n_3). \quad (11)$$

The Rashba DMI term favors skyrmions of the Néel type, that is,

$$\mathbf{n}_{\text{Néel}}(r, \theta) = (\sin f(r) \cos \theta, \sin f(r) \sin \theta, \cos f(r)), \quad (12)$$

where $f(r)$ decreases monotonically from π to 0. Note that this field has degree $N_{\text{Sk}} = -1$, so we call it a skyrmion, following the standard convention in the magnetism literature. It follows from (11) that the Dresselhaus DMI term favors Bloch skyrmions,

$$\mathbf{n}_{\text{Bloch}}(r, \theta) = (-\sin f(r) \sin \theta, \sin f(r) \cos \theta, \cos f(r)), \quad (13)$$

while the Heusler DMI term favors *antiskyrmions* of the form

$$\mathbf{n}_{\text{Heusler}}(r, \theta) = (-\sin f(r) \sin \theta, -\sin f(r) \cos \theta, \cos f(r)), \quad (14)$$

and that these three solutions are exactly degenerate in energy. As we will see, dipole-dipole interactions break this degeneracy, producing particularly striking changes to the Heusler system. A summary of the different DMI terms and their corresponding skyrmion types is provided in Table I, and the three different skyrmion types are displayed in Fig. 1 with the color scheme detailed below.

The magnetization vector $\mathbf{n} \in S^2$ can be represented graphically with a Hue-Saturation-Value (HSV) color scheme associated with the Runge color sphere and is presented as laid out in [22]. The phase $\arg(n_1 + in_2)$ determines the hue, where $\arg(n_1 + in_2) = 0$ is identified with red, $\arg(n_1 + in_2) = 2\pi/3$ is identified with green, and $\arg(n_1 + in_2) = 4\pi/3$ is identified with blue [23]. Finally, the value of n_3 determines the brightness [24], such that spin-up states $\mathbf{n}_\uparrow = (0, 0, 1)$ are white and the spin-down states $\mathbf{n}_\downarrow = (0, 0, -1)$ are black.

III. THE INTERACTION ENERGY OF A DISTRIBUTION OF MAGNETIC DIPOLES

In this section, we define the magnetostatic self-energy, or dipole-dipole interaction energy, E_{DDI} and compute its first variation. The magnetic field induced by an isolated dipole of moment $\mathbf{m}$ at $\mathbf{0}$ is [25]

$$\mathbf{B} = -\frac{\mu_0}{4\pi r^3} \left(\mathbf{m} - 3 \frac{\mathbf{m} \cdot \mathbf{x}}{r^2} \mathbf{x} \right), \quad (15)$$

where μ_0 is the permeability of free space. This is often referred to as the *demagnetizing* field and can be expressed as the gradient of a magnetic potential $\psi : \mathbb{R}^3 \rightarrow \mathbb{R}$, that is,

$$\mathbf{B} = -\nabla \psi, \quad (16)$$

where

$$\psi = -\mu_0 \mathbf{m} \cdot \nabla \left(\frac{1}{4\pi r} \right). \quad (17)$$

Consider now a continuous distribution of magnetic dipole density $\mathbf{m} : \Omega \rightarrow \mathbb{R}^3$, where $\Omega \subseteq \mathbb{R}^3$ is some domain. The magnetic field it induces, at a point $\mathbf{x} \in \mathbb{R}^3$, is given by integrating the field induced at $\mathbf{x}$ by $\mathbf{m}(\mathbf{y})$ at $\mathbf{y} \in \Omega$ over $\mathbf{y} \in \Omega$:

$$\mathbf{B}(\mathbf{x}) = -\frac{\mu_0}{4\pi} \int_{\Omega} d^3 \mathbf{y} \frac{1}{|\mathbf{x} - \mathbf{y}|^3} \left\{ \mathbf{m}(\mathbf{y}) - 3 \frac{\mathbf{m}(\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|^2} (\mathbf{x} - \mathbf{y}) \right\}. \quad (18)$$

Again, this is the gradient of the magnetic potential $\psi : \mathbb{R}^3 \rightarrow \mathbb{R}$,

$$\psi(\mathbf{x}) = -\mu_0 \int_{\Omega} d^3 \mathbf{y} \left\{ \mathbf{m}(\mathbf{y}) \cdot \nabla_{\mathbf{x}} \left(\frac{1}{4\pi |\mathbf{x} - \mathbf{y}|} \right) \right\}. \quad (19)$$

We now note that

$$G(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi |\mathbf{x} - \mathbf{y}|} \quad (20)$$

is the Green's function for the Laplacian $\Delta = -\nabla^2$ on $\mathbb{R}^3$, that is,

$$\Delta_{\mathbf{x}} G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}). \quad (21)$$

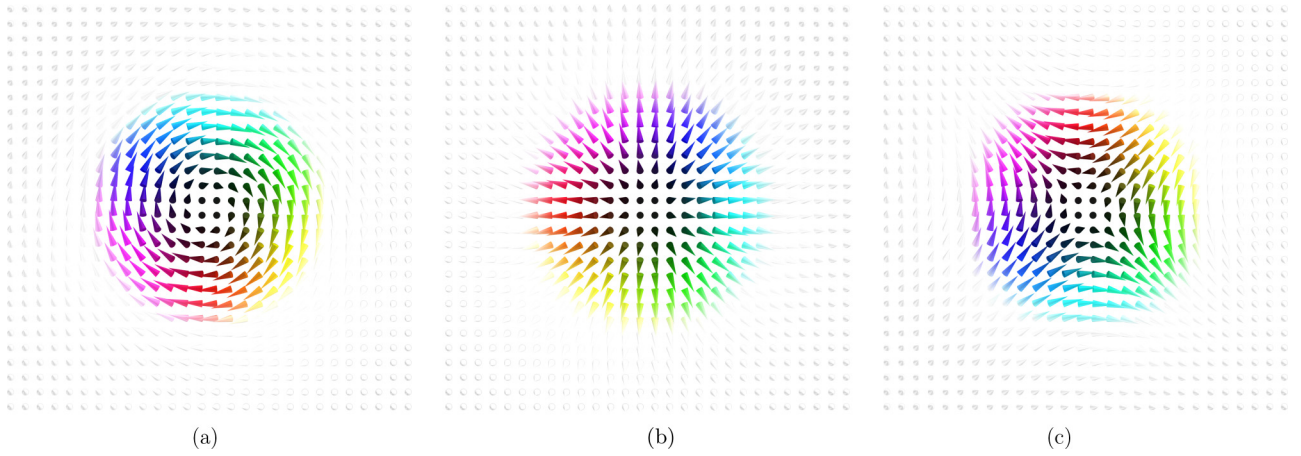


FIG. 1. Three different skyrmion types consider in the text. These are the spin textures of the (a) Bloch skyrmion, (b) Néel skyrmion, and (c) antiskyrmion. They correspond to skyrmions with Dresselhaus, Rashba, and Heusler DMI terms, respectively, and are obtained by minimizing the energy without including the magnetostatic self-interaction. The magnetization $\mathbf{n} = (n_1, n_2, n_3) \in S^2$ is colored using the Runge color sphere. Spin-up states $\mathbf{n}_\uparrow = (0, 0, 1)$ are white, whereas spin-down states $\mathbf{n}_\downarrow = (0, 0, -1)$ are black. The hue is determined by the phase $\arg(n_1 + in_2)$. Plotted on top of the skyrmion is the projection of the magnetization onto the plane $\mathbb{R}^2$, showing the in-plane vector $(n_1(x_1, x_2), n_2(x_1, x_2)) \in \mathbb{R}^2$.

Comparing (21) with (19), we see that the magnetic potential ψ satisfies the Partial Differential Equation (PDE) [26]

$$\Delta\psi = -\nabla^2\psi = \mu_0\rho, \quad \rho = -(\nabla \cdot \mathbf{m}), \quad (22)$$

that is, Poisson's equation with source $\rho = -(\nabla \cdot \mathbf{m})$. Up to a multiplicative constant, the magnetic field induced by the dipole distribution $\mathbf{m}$ coincides, therefore, with the magnetic field induced by the charge distribution $-(\nabla \cdot \mathbf{m})$ [27]. Roughly speaking, we may think of $-(\nabla \cdot \mathbf{m})$ as the “magnetic charge density.”

The interaction energy of a pair of magnetic dipoles $\mathbf{m}^{(1)}$ and $\mathbf{m}^{(2)}$ is $-\mathbf{m}^{(1)} \cdot \mathbf{B}^{(2)}$, where $\mathbf{B}^{(2)}$ is the magnetic field induced by $\mathbf{m}^{(2)}$ at the position of $\mathbf{m}^{(1)}$. Hence, the total dipole-dipole interaction energy of a continuous dipole density distribution is the magnetostatic energy

$$E_{\text{DDI}} = -\frac{1}{2} \int_{\Omega} d^3\mathbf{x} (\mathbf{B}(\mathbf{x}) \cdot \mathbf{m}(\mathbf{x})), \quad (23)$$

where $\mathbf{B}$ is the demagnetizing field (18) induced by the magnetization $\mathbf{m}$. (The factor of 1/2 arises since we sum over dipole pairs.) It is useful to rewrite this as a functional of the magnetic potential ψ :

$$\begin{aligned} E_{\text{DDI}} &= \frac{1}{2} \int_{\Omega} d^3\mathbf{x} \mathbf{m} \cdot \nabla\psi \\ &= -\frac{1}{2} \int_{\Omega} d^3\mathbf{x} (\nabla \cdot \mathbf{m})\psi + \frac{1}{2} \oint_{\partial\Omega} d\mathbf{s} \cdot (\psi\mathbf{m}) \\ &= \frac{1}{2\mu_0} \int_{\Omega} d^3\mathbf{x} \psi \Delta\psi + \frac{1}{2} \oint_{\partial\Omega} d\mathbf{s} \cdot (\psi\mathbf{m}) \end{aligned} \quad (24)$$

by the divergence theorem. We will use this formula only in situations where the boundary conditions ensure that the boundary term vanishes. In this case, E_{DDI} coincides with the “electrostatic” self-energy of the “charge” distribution $-(\nabla \cdot \mathbf{m})$.

We now specialize to the case of immediate interest: the dipole-dipole interaction energy of a chiral ferromagnet in a

translationally invariant configuration. The dipole field has constant length M_s , so $\mathbf{m} = M_s\mathbf{n}$, where $\mathbf{n}$ is valued on the unit sphere. Further, we impose translation invariance in the direction $\mathbf{e}_3 = (0, 0, 1)$, so $\mathbf{n}$ is independent of x_3 . The boundary conditions require some care. We assume that some anisotropy in the system (either intrinsic or generated by an external applied magnetic field) imposes an energetic preference for the dipole orientation $\mathbf{n} = \mathbf{e}_3$ and consider fields $\mathbf{n} : \mathbb{R}^2 \rightarrow S^2$ which have compact support in the sense that an $R_0 > 0$ exists such that, for all $r := |(x_1, x_2)| \geq R_0$, $\mathbf{n}(x_1, x_2) = \mathbf{e}_3$. Since the field $\mathbf{n}$ is translationally invariant, the total dipole interaction energy either vanishes (for example, if $\mathbf{n}$ is constant) or diverges. The energy per unit length (in the $\mathbf{e}_3$ direction) may be finite, however, and this coincides with the total energy of the slab $\Omega = \mathbb{R}^2 \times [0, 1]$, which we compute as the limit of the energy of the thick disk $\Omega_R = \{\mathbf{x} : x_1^2 + x_2^2 \leq R^2, 0 \leq x_3 \leq 1\}$ as $R \rightarrow \infty$. Note that, in this case, the boundary term in (24) vanishes identically for all $R > R_0$: the flux of $\psi\mathbf{m}$ through the cylindrical wall vanishes since $\mathbf{m}$ has no normal component on this part of the boundary, and the flux through the top disk (at $x_3 = 1$) is exactly canceled by the flux through the bottom disk (at $x_3 = 0$) since $\psi\mathbf{m}$ is translationally invariant. Hence,

$$E_{\text{DDI}} = \frac{1}{2\mu_0} \int_{\mathbb{R}^2} d^2x \psi \Delta\psi, \quad (25)$$

where $\Delta = -\partial_1^2 - \partial_2^2$ is the Laplacian on $\mathbb{R}^2$ and ψ satisfies

$$\Delta\psi = -\mu_0 M_s \left(\frac{\partial n_1}{\partial x_1} + \frac{\partial n_2}{\partial x_2} \right). \quad (26)$$

Consider the large r behavior of the magnetic potential $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}$. Any solution of the Poisson equation $\Delta\psi = \mu_0\rho$ on the plane has a multipole expansion

$$\psi = -\frac{q\mu_0}{2\pi} \ln(r) + O(r^{-1}), \quad (27)$$

where $q = \int_{\mathbb{R}^2} d^2x \rho$ is the total “charge.” In general, such functions are logarithmically unbounded. In our case

$$q = \int_{\mathbb{R}^2} d^2x \rho = - \int_{\mathbb{R}^2} d^2x (\nabla \cdot \mathbf{m}) = 0, \quad (28)$$

and we conclude that ψ is (at least) $1/r$ localized.

IV. EVADING THE HOBART-DERRICK THEOREM

So far we have shown how to include the dipolar interaction and computed its variation with respect to the magnetization. Now, we want to understand how the dipolar interaction transforms under a coordinate rescaling to see the role it plays in evading the Hobart-Derrick theorem [28].

Let us consider an energy and length rescaling with $E = \frac{J^2}{D} \hat{E}$ and $x = \frac{J}{D} \hat{x}$. Further, let us also consider the rescaling of the magnetic potential $\psi = \frac{D}{M_s} \hat{\psi}$. Then the dimensionless energy $\hat{E}$ is (dropping all hats)

$$E = \int_{\mathbb{R}^3} \left\{ \frac{1}{2} |\mathbf{d}\mathbf{n}|^2 + \mathbf{d}_i \cdot (\mathbf{n} \times \partial_i \mathbf{n}) + K(1 - n_3^2) + h(1 - n_3) + \frac{1}{2} \mathbf{n} \cdot \nabla \psi \right\} d^3x, \quad (29)$$

where the magnetic potential satisfies the dimensionless Poisson equation

$$\Delta \psi = -\mu \nabla \cdot \mathbf{n}. \quad (30)$$

The dimensionless coupling constants are

$$K = \frac{K_m J}{D^2}, \quad h = \frac{M_s B_{\text{ext}} J}{D^2}, \quad \mu = \frac{\mu_0 J M_s^2}{D^2}. \quad (31)$$

We now return to the case of interest in which we have translation invariance in the $\mathbf{e}_3$ direction. Before, we were considering the energy per unit length in the translationally invariant direction. So our domain was $\Omega = \mathbb{R}^2 \times [0, 1]$, which now becomes $\Omega' = \mathbb{R}^2 \times [0, t]$, where $t = L_0^{-1}$ is the thickness of the slab under the rescaling. To maintain adimensionality, let us consider the energy per unit thickness E/t . Therefore, the adimensional energy functional we are really considering is

$$E = \int_{\mathbb{R}^2} \left\{ \frac{1}{2} |\mathbf{d}\mathbf{n}|^2 + \mathbf{d}_i \cdot (\mathbf{n} \times \partial_i \mathbf{n}) + K(1 - n_3^2) + h(1 - n_3) + \frac{1}{2} \mathbf{n} \cdot \nabla \psi \right\} d^2x. \quad (32)$$

With the dimensionless total energy (32) in hand, we now turn to evading the Hobart-Derrick theorem [28]. Let us consider, for a given field $\mathbf{n} : \mathbb{R}^2 \rightarrow S^2$, the one parameter variation $\mathbf{n}_\lambda(\mathbf{x}) := \mathbf{n}(\lambda \mathbf{x})$, $\lambda > 0$, obtained by isotropic dilation. The “magnetic charge density” of $\mathbf{n}_\lambda$ is $\rho_\lambda(x) = -\nabla \cdot \mathbf{n}_\lambda(\mathbf{x}) = -\lambda \rho(\lambda \mathbf{x})$, so its magnetic potential ψ_λ satisfies

$$(\Delta \psi_\lambda)(\mathbf{x}) = \mu \lambda \rho(\lambda \mathbf{x}). \quad (33)$$

Comparing (33) with (30), we see that $\psi_\lambda(\mathbf{x}) = \psi(\lambda \mathbf{x})/\lambda$. Combining this scaling behavior of the magnetic potential $\psi_\lambda(\mathbf{x})$ with the well-known scaling behavior of the exchange,

DMI, and potential terms, we see that

$$E(\mathbf{n}_\lambda) = E_{\text{exch}}(\mathbf{n}) + \frac{1}{\lambda} E_{\text{DMI}}(\mathbf{n}) + \frac{1}{\lambda^2} [E_{\text{pot}}(\mathbf{n}) + E_{\text{DDI}}(\mathbf{n})]. \quad (34)$$

So the dipole-dipole interaction energy behaves like a potential term under the spatial scaling.

If $\mathbf{n}$ is a static solution of the model, it extremizes E locally with respect to all variations, including $\mathbf{n}_\lambda$, so

$$\left. \frac{dE}{d\lambda} \right|_{\lambda=1} = E_{\text{DMI}} + 2(E_{\text{pot}} + E_{\text{DDI}}) = 0, \quad (35)$$

which is the Derrick scaling constraint. Therefore, it is possible to evade the Hobart-Derrick theorem without the inclusion of the anisotropy and Zeeman interactions, provided the dipolar interaction is present.

We note that the energy contribution from the DMI term must be negative semidefinite because both the potential and DDI energies are positive semidefinite. To see this, we use the general identity $\psi \Delta \phi = \nabla \psi \cdot \nabla \phi - \nabla \cdot (\psi \nabla \phi)$ and the divergence theorem to express the DDI energy as the electrostatic self-energy of the charge distribution $\nabla \cdot \mathbf{n}$, that is,

$$E_{\text{DDI}} = \frac{1}{2\mu} \int_{\mathbb{R}^2} d^2x |\nabla \psi|^2 \geq 0. \quad (36)$$

Hence, the dipolar interaction can stabilize magnetic skyrmions in chiral ferromagnets where the DMI is present but does not provide stability in ordinary ferromagnets.

V. THE EULER-LAGRANGE EQUATIONS

We seek fields $\mathbf{n} : \mathbb{R}^2 \rightarrow S^2$ which (at least locally) minimize E , so for all smooth variations $\mathbf{n}_t$ of $\mathbf{n} = \mathbf{n}_0$ through fields of compact support, we require that

$$\left. \frac{d}{dt} \right|_{t=0} E(\mathbf{n}_t) = 0. \quad (37)$$

This condition is equivalent to the Euler-Lagrange equations for E , which we now derive.

Let $\boldsymbol{\varepsilon} = \partial_t \mathbf{n}_t|_{t=0}$ be the generator of the variation $\mathbf{n}_t$, and note that $\boldsymbol{\varepsilon} \cdot \mathbf{n} = 0$ everywhere since $|\mathbf{n}| = 1$. The induced variations of E_{exch} , E_{DMI} , and E_{pot} are easily computed:

$$\left. \frac{d}{dt} \right|_{t=0} E_{\text{exch}}(\mathbf{n}_t) = \int_{\mathbb{R}^2} d^2x \boldsymbol{\varepsilon} \cdot \Delta \mathbf{n}, \quad (38)$$

$$\left. \frac{d}{dt} \right|_{t=0} E_{\text{DMI}}(\mathbf{n}_t) = \int_{\mathbb{R}^2} d^2x \boldsymbol{\varepsilon} \cdot (-2\mathbf{d}_i \times \partial_i \mathbf{n}), \quad (39)$$

$$\left. \frac{d}{dt} \right|_{t=0} E_{\text{pot}}(\mathbf{n}_t) = \int_{\mathbb{R}^2} d^2x \boldsymbol{\varepsilon} \cdot (0, 0, -h - 2Kn_3). \quad (40)$$

We will also need the variation of E_{DDI} , which is rather more subtle.

Denote by ψ_t the unique solution of (30) with source $-\text{div } \mathbf{n}_t$ decaying to zero at infinity and $\dot{\psi} = \partial_t \psi_t|_{t=0}$. It is important to note that, while $\boldsymbol{\varepsilon}$ has compact support, neither $\psi = \psi_0$ nor $\dot{\psi}$ does; as argued above, they are $1/r$ localized. Care must therefore be taken with boundary terms when computing the variation of E_{DDI} . The variation of E_{DDI} induced by

$\mathbf{n}_t$ is

$$\begin{aligned}
\left. \frac{d}{dt} \right|_{t=0} E_{\text{DDI}}(\mathbf{n}_t) &= \frac{1}{2\mu} \int_{\mathbb{R}^2} d^2x (\dot{\psi} \Delta \psi + \psi \Delta \dot{\psi}) \\
&= \frac{1}{\mu} \int_{\mathbb{R}^2} d^2x \psi \Delta \dot{\psi} \\
&\quad + \lim_{R \rightarrow \infty} \frac{1}{2\mu} \int_{\partial B_R(0)} (\dot{\psi} * d\psi - \psi * d\dot{\psi}) \\
&= \frac{1}{\mu} \int_{\mathbb{R}^2} d^2x \psi \Delta \dot{\psi} \\
&\quad + \lim_{R \rightarrow \infty} \frac{R}{2\mu_0} \int_0^{2\pi} (\dot{\psi} \psi_r - \psi \dot{\psi}_r) d\theta \\
&= \frac{1}{\mu} \int_{\mathbb{R}^2} d^2x \psi \Delta \dot{\psi} \tag{41}
\end{aligned}$$

since $\psi, \dot{\psi} = O(r^{-1})$ and $\psi_r, \dot{\psi}_r = O(r^{-2})$. Differentiating (30) with respect to t , we deduce that

$$\Delta \dot{\psi} = -\mu \nabla \cdot \boldsymbol{\varepsilon}, \tag{42}$$

and hence,

$$\begin{aligned}
\left. \frac{d}{dt} \right|_{t=0} E_{\text{DDI}}(\mathbf{n}_t) &= - \int_{\mathbb{R}^2} d^2x \psi (\nabla \cdot \boldsymbol{\varepsilon}) = \int_{\mathbb{R}^2} d^2x \boldsymbol{\varepsilon} \cdot \nabla \psi \\
&\quad - \lim_{R \rightarrow \infty} \int_{\partial B_R(0)} \psi * \boldsymbol{\varepsilon} = \int_{\mathbb{R}^2} d^2x \boldsymbol{\varepsilon} \cdot \nabla \psi \tag{43}
\end{aligned}$$

by Stokes's theorem since $\boldsymbol{\varepsilon} = \varepsilon_1 dx_1 + \varepsilon_2 dx_2$ has compact support.

Combining (38), (39), (40), and (43), we see that

$$\begin{aligned}
\left. \frac{d}{dt} \right|_{t=0} E(\mathbf{n}_t) &= \int_{\mathbb{R}^2} d^2x \boldsymbol{\varepsilon} \cdot [\Delta \mathbf{n} - 2\mathbf{d}_i \times \partial_i \mathbf{n} \\
&\quad - (h + 2K\mathbf{n} \cdot \mathbf{e}_3)\mathbf{e}_3 + \nabla \psi], \tag{44}
\end{aligned}$$

and this must vanish for arbitrary $\boldsymbol{\varepsilon}$ pointwise orthogonal to $\mathbf{n}$. Hence, the Euler-Lagrange equation for E is

$$\begin{aligned}
\text{grad}E(\mathbf{n}) &:= P_{\mathbf{n}}[\Delta \mathbf{n} - 2\mathbf{d}_i \times \partial_i \mathbf{n} \\
&\quad - (h + 2K\mathbf{n}_3)\mathbf{e}_3 + \nabla \psi] = 0, \tag{45}
\end{aligned}$$

where the projector $P_{\mathbf{n}}(\mathbf{u}) := \mathbf{u} - (\mathbf{u} \cdot \mathbf{n})\mathbf{n}$ accounts for the constraint that $|\mathbf{n}| \equiv 1$ and ψ is determined by the Poisson equation (30). Our notation calls attention to the fact that $\text{grad}E$ is formally the L^2 gradient of E , that is, the direction in field space in which $E(\mathbf{n})$ grows fastest.

VI. NUMERICAL METHOD

We now detail our method for finding magnetization fields $\mathbf{n}$ that simultaneously minimize the energy (32) and solve Poisson's equation (30). This method is implemented for NVIDIA CUDA architecture and is adapted from a similar nonlocal method developed to determine (nuclear) skyrmion crystals stabilized by ω -mesons [29,30]. Similar methods were also recently developed to study knot solitons in an extended version of the standard model of particle physics [31] and the back-reaction of the Coulomb force in an extension

of the Skyrme model [32]. We follow the same methodology, separating our algorithm into two stages, detailed as follows.

Magnetic skyrmions are local minimizers of the energy functional (32). We choose to numerically relax the energy (32) using an accelerated gradient descent based method with flow-arresting criteria, known as arrested Newton flow. It consists of a fourth-order central finite difference method, yielding a discrete approximation $E_{\text{dis}}(\mathbf{n})$ of the energy $E(\mathbf{n})$, and is performed on a mesh grid of N^2 lattice sites with lattice spacing h_{latt} , so the field may be thought of as a point in the discretized configuration space $\mathcal{C} = (S^2)^{N^2} \subset \mathbb{R}^{3N^2}$. Hence, our discrete approximant of the energy functional is the function $E_{\text{dis}} : \mathcal{C} \rightarrow \mathbb{R}$. To find a minimum of E_{dis} , we start at some initial field $\mathbf{n}(0) \in \mathcal{C}$ and solve Newton's equation for the potential E_{dis} ,

$$\ddot{\mathbf{n}}(t) = -\text{grad}E_{\text{dis}}(\mathbf{n}(t)), \tag{46}$$

with initial velocity $\dot{\mathbf{n}}(0) = 0$ using an appropriate time-stepping scheme (we use a fourth-order Runge-Kutta scheme). This curve will accelerate “downhill” initially but will not converge to a minimum of E_{dis} (since the flow conserves total energy, including kinetic energy). To remedy this, after each time step, we check whether $E_{\text{dis}}(t + \delta t) > E_{\text{dis}}(t)$: If it is, we set $\dot{\mathbf{n}}(t) = 0$ and restart the flow at $\mathbf{n}(t)$. The flow is deemed to have converged when $\|\text{grad}E_{\text{dis}}(\mathbf{n})\|_{\infty} < \varepsilon_{\text{tol}}$, where ε_{tol} is some threshold tolerance (we use $\varepsilon_{\text{tol}} = 10^{-6}$). This scheme is simple and robust and converges much faster than conventional gradient flow.

In practice, we compute $\text{grad}E_{\text{dis}}(\mathbf{n})$ by discretizing our formula for $\text{grad}E$, (45). Of course, constructing $\text{grad}E_{\text{dis}}(\mathbf{n}(t))$ requires us to compute the magnetic potential $\psi(t)$ induced by $\mathbf{n}(t)$; that is, we must solve the PDE (30) on our numerical mesh. We achieve this by thinking of (30) for fixed $\mathbf{n}$ as the Euler-Lagrange equation for the functional

$$F(\psi) = \frac{1}{2} \|\text{d}\psi\|_{L^2}^2 + \mu \int_{\mathbb{R}^2} d^2x \psi (\nabla \cdot \mathbf{n}), \tag{47}$$

which we minimize using a conjugate gradient method. (Such methods are particularly efficient for minimizing quadratic functionals such as this.)

To summarize, the full numerical algorithm is implemented as follows.

(1) Perform a step of the arrested Newton flow method for the magnetization $\mathbf{n}$ using a fourth-order Runge-Kutta method.

(2) Solve Poisson's equation for the magnetic potential ψ using nonlinear conjugate gradient descent with the Polak-Ribiere-Polyak method.

(3) Compute the total energy of the configuration $(\mathbf{n}_n, \psi_n)$ and compare the result to the energy of the previous configuration $(\mathbf{n}_{n-1}, \psi_{n-1})$. If the energy has increased, arrest the flow.

(4) Check the convergence criterion: $\|E_{\text{dis}}(\mathbf{n})\|_{\infty} < \varepsilon$. If the convergence criterion has been satisfied, then stop the algorithm.

(5) Repeat the process (return to step 1).

Landau-Lifshitz-Gilbert dynamics

As a consistency check, we verify our numerics by comparing a selection of our results with those obtained using Landau-Lifshitz-Gilbert (LLG) dynamics. The dimensionless LLG equation is

$$\frac{d\mathbf{n}}{d\tau} = -\frac{\gamma}{1+\alpha^2} \mathbf{n} \times (\mathbf{h}_{\text{eff}} + \alpha \mathbf{n} \times \mathbf{h}_{\text{eff}}), \quad (48)$$

where γ is the gyromagnetic ratio, α is the Gilbert damping parameter, the effective magnetic field is

$$\begin{aligned} \mathbf{h}_{\text{eff}} &= -\frac{\delta E}{\delta \mathbf{n}} \\ &= -\Delta \mathbf{n} + 2\mathbf{d}_i \times \partial_i \mathbf{n} + (h + 2K\mathbf{n} \cdot \mathbf{e}_3)\mathbf{e}_3 + \mathbf{h}_{\text{demag}}, \end{aligned} \quad (49)$$

and $\mathbf{h}_{\text{demag}} = -\nabla\psi$ is the demagnetization field. The first term in the LLG equation (48) represents precession, and the second term represents damping. To find static solutions we set $\alpha = 1$ and $\gamma = 1$. We evolve the system using a fourth-order Runge-Kutta method. During the evolution we determine the optimal/appropriate magnetostatic potential ψ at each intermediate Runge-Kutta step, i.e., when we compute the slopes $\mathbf{k}_i$. We find that the results are consistent, with the LLG method being significantly slower than the arrested Newton flow method (roughly 1–2 orders of magnitude slower).

VII. RESULTS

In the previous section, we detailed our numerical algorithm that constructs magnetic skyrmions in the presence of the long-range dipolar interaction from some starting choice of the initial configuration. The method is optimized for the dimensionless energy (32) and Poisson equation (30) with the choice of free coupling constants $\{\mu, K, h\}$. Although these coupling constants are dimensionless, they are defined in terms of the micromagnetic parameters $\{\mathcal{D}, J, M_s, K_m, B_{\text{ext}}\}$.

The micromagnetic parameters employed in this study are chosen to be representative of experimentally realizable chiral ferromagnets that host nanoscale skyrmions. Specifically, we take the exchange stiffness $J = 40 \text{ pJ m}^{-1}$, Dzyaloshinskii-Moriya interaction strength $\mathcal{D} = 4 \text{ mJ m}^{-2}$, saturation magnetization $M_s = 580 \text{ kA m}^{-1}$, and anisotropy constant $K_m = 0.8 \text{ MJ m}^{-3}$, with no external magnetic field ($B_{\text{ext}} = 0 \text{ T}$). This gives us the dimensionless coupling constants $K = 2$, $\mu = 1.057$, and $h = 0$. The chosen regime ensures a realistic competition between exchange, Dzyaloshinskii-Moriya, and dipole-dipole energies, allowing all three DMI terms to be compared on equal footing. The inclusion of finite magnetostatic and anisotropy energies further reflects conditions relevant to both thin-film and bulk chiral magnets where dipolar effects significantly influence skyrmion stability.

In particular, these parameters correspond to the energy and length scales typical of soft ferromagnetic multilayers and Heusler-type alloys, such as CoFeB [18,33] and Mn-based compounds [4,21], which are widely used as benchmark materials in micromagnetic simulations of chiral spin textures. The length scale $L_0 = J/\mathcal{D} = 10 \text{ nm}$ defines the intrinsic magnetic length, yielding skyrmion diameters in the experimentally accessible range of 10–100 nm. The chosen DMI

TABLE II. The effect of the DDI on the skyrmion size R_{Sk} . The Bloch skyrmion is unchanged, whereas the Néel and antiskyrmion shrink relative to their size without the DDI ($R_{\text{Sk}} = 8.422 \text{ nm}$).

	Dresselhaus	Rashba	Heusler
Skyrmion	Bloch	Néel	Antiskyrmion
R_{Sk} (nm)	8.422	8.415	8.418

magnitude ensures that chiral twisting competes effectively with exchange and dipolar energies, allowing all three DMI terms—Dresselhaus, Rashba, and Heusler—to produce stable topological textures under comparable conditions.

The saturation magnetization and anisotropy values are consistent with thin-film and bulk ferromagnetic materials in which demagnetizing fields are non-negligible, ensuring that the magnetostatic self-interaction plays a physically relevant role [20]. These values thus provide a realistic and well-balanced parameter regime for exploring how dipole-dipole interactions influence skyrmion and antiskyrmion stability in three-dimensional translationally invariant chiral magnets.

For simulations involving isolated magnetic skyrmions, all numerics carried out were simulated on a dimensionless mesh grid of size 10×10 with 1024^2 lattice sites and vacuum boundary conditions. Physically, this corresponds to a $100 \times 100 \text{ nm}^2$ grid with $0.098 \times 0.098 \text{ nm}^2$ lattice spacing. The energy scale for the simulations is $E_0 = J^2/\mathcal{D} = 4 \times 10^{-19} \text{ J}$.

A. Isolated magnetic skyrmions

First of all, let us consider the effect of the dipolar interaction on a single isolated magnetic skyrmion. Recall that we are considering three DMI terms—Dresselhaus, Rashba and Heusler—each of which has an associated skyrmion type (Bloch, Néel, and antiskyrmion, respectively). Before implementing the numerical algorithm, we can gain some intuition by computing the divergence of the magnetization *Ansätze* (12), (13), and (14), given that it is the source for the Poisson equation (30). That is,

$$\nabla \cdot \mathbf{n}_{\text{Dresselhaus}} = 0, \quad (50a)$$

$$\nabla \cdot \mathbf{n}_{\text{Rashba}} = \frac{df}{dr} \cos f(r) + \frac{1}{r} \sin f(r), \quad (50b)$$

$$\nabla \cdot \mathbf{n}_{\text{Heusler}} = \left(\frac{1}{r} \sin f(r) - \frac{df}{dr} \cos f(r) \right) \sin 2\theta. \quad (50c)$$

So we see that the dipolar interaction has no effect on Bloch skyrmions because the Bloch *Ansatz* (13) is solenoidal. Since $E_{\text{DDI}} \geq 0$ and the Bloch skyrmion minimizes E without E_{DDI} , it follows that the Bloch skyrmion also minimizes E with E_{DDI} in this case: The dipole-dipole interaction has no effect on isolated skyrmions in the Dresselhaus case.

However, it does have an effect on Néel skyrmions and antiskyrmions. Within the Néel *Ansatz* (12) the magnetic potential is found to be radially symmetric, whereas it has a D_2 symmetry within the Heusler *Ansatz* (14), so the DDI breaks the radial symmetry of antiskyrmions in the Heusler case. These observations are confirmed numerically; the relaxed solutions are plotted in Fig. 2 and the resulting skyrmion sizes are displayed in Table II.

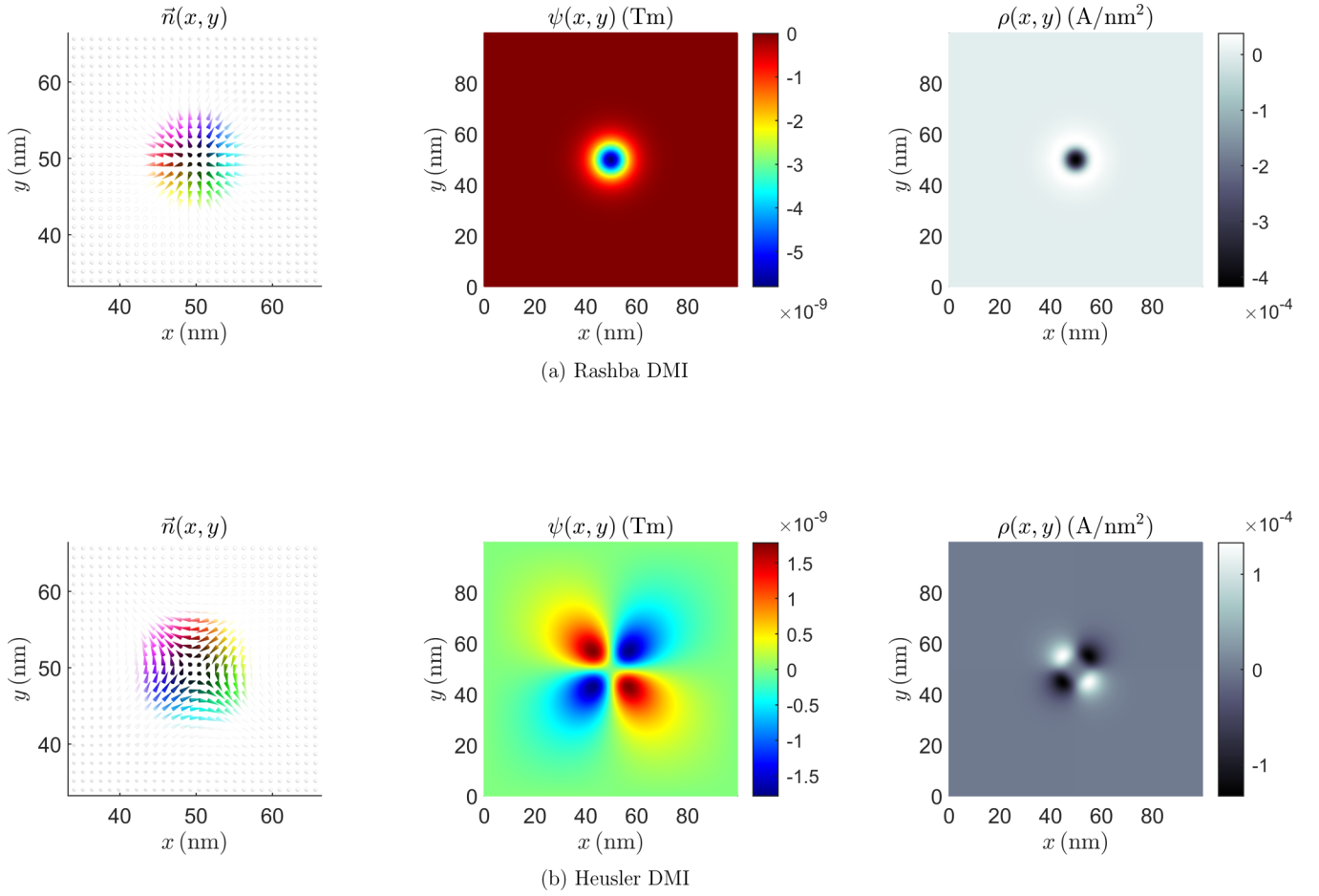


FIG. 2. Effect of the dipolar interaction on the (a) Néel skyrmion and (b) Heusler antiskyrmion. The magnetic potential $\psi(x, y)$ is shown alongside the magnetic charge density $\rho(x, y) = -M_s(\nabla \cdot \mathbf{n})$. The Dresselhaus DMI solution is omitted because the DDI has no effect on the Bloch skyrmion. Numerical relaxation is carried out using the arrested Newton flow and nonlinear conjugate gradient descent methods for the magnetization $\mathbf{n} : \mathbb{R}^2 \rightarrow S^2$ and magnetic potential $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}$, respectively. We impose vacuum boundary conditions, $\mathbf{n} \rightarrow \mathbf{n}_1$ and $\psi \rightarrow 0$, as $r \rightarrow \infty$.

B. Magnetic skyrmion crystals

Next, we investigate the DDI effect on magnetically ordered crystals. In the absence of DDI, there is no optimal lattice geometry for the parameter set we have chosen—the skyrmions repel one another and want to be infinitely separated. Our method is simpler than that of [34]: We do not vary over the lattice geometry. Instead, we restrict our geometry to that of two lattice types, an equianharmonic (triangular) lattice and a lemniscatic (square) lattice. In the triangular case, for which the unit cell is a rhombus of side length L and apex angle $\frac{\pi}{3}$, it is convenient to combine unit cells into pairs and work on a rectangular lattice of side length $L \times \sqrt{3}L$, with topological charge $|N_{\text{sk}}| = 2$ per unit cell (see Fig. 3). For both lattice geometries, we vary the lattice parameter L . In all cases, as $L \rightarrow \infty$, the energy per unit topological charge approaches that of the isolated single magnetic skyrmion. This time, we include the results for the Dresselhaus DMI term. For large lattice values L , the DDI will have no effect on the Bloch skyrmions. However, for small L , where the individual skyrmions are packed close together, the DDI could have a small effect. Asymptotically, in the limit $L \rightarrow \infty$, the

energy per unit topological charge will approach that of two skyrmions/antiskyrmions in the absence of the DDI.

In the absence of the DDI, an optimal lattice size L_* which minimizes the energy $E(L)$ does not exist. The energy is minimized for two infinitely separated skyrmions/antiskyrmions.

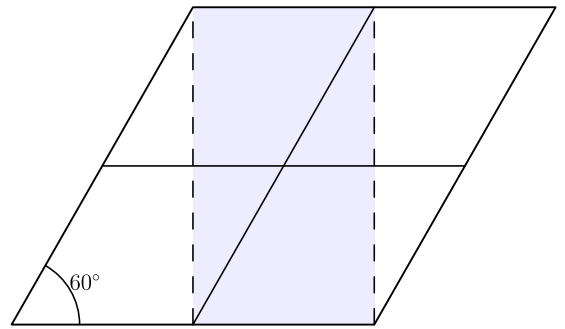


FIG. 3. Fields periodic with respect to an equianharmonic lattice of side length L are also periodic with respect to a rectangular lattice of size $L \times \sqrt{3}L$.

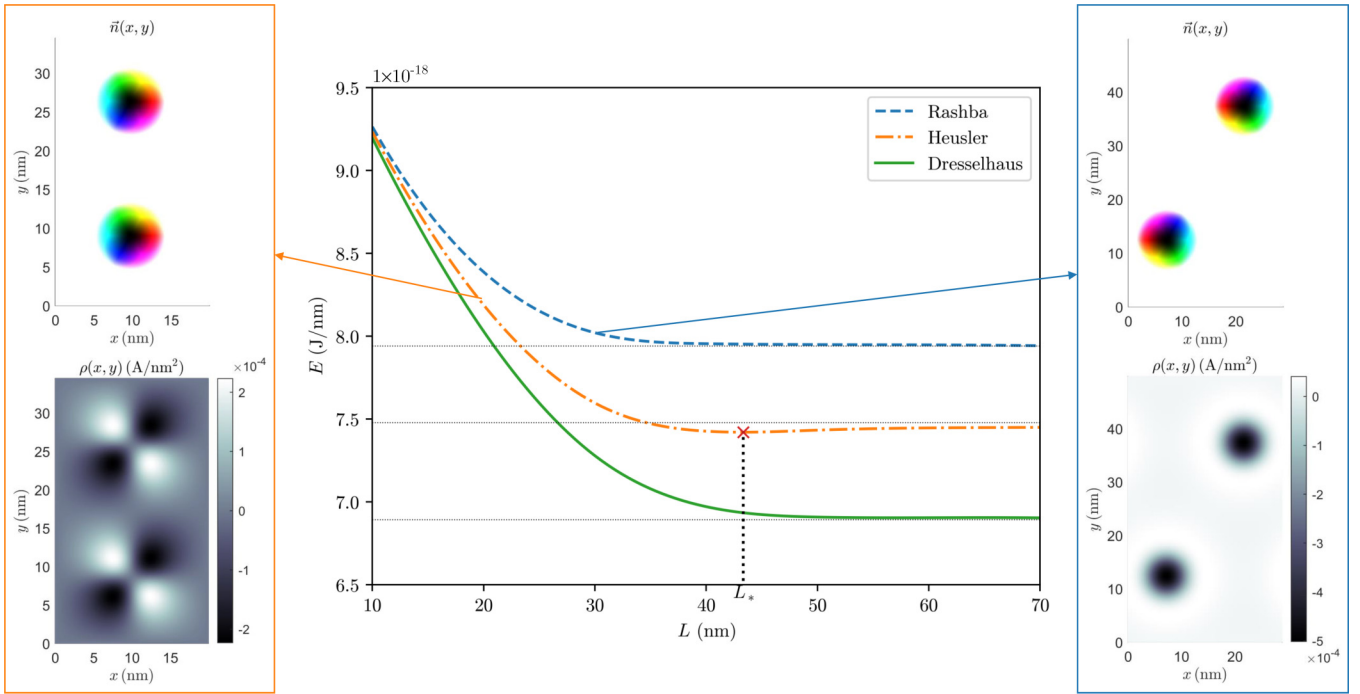


FIG. 4. The effect of the dipolar interaction on hexagonal magnetic skyrmion crystals as a function of the lattice size L . The unit cell used is of size $L \times \sqrt{3}L$, which favors an equianharmonic lattice. Initially, two magnetic skyrmions are placed in a hexagonal arrangement, and then they are relaxed using the numerical algorithm detailed in the text. It can be seen that including the DDI and its back-reaction lifts the energy degeneracy, resulting in distinct $E(L)$ curves. The Rashba (Néel) and Dresselhaus (Bloch) skyrmion lattices maintain a hexagonal arrangement, whereas the Heusler antiskyrmions relax into one with a square form. An example Heusler antiskyrmion lattice is shown at small $L = 20$ nm, and a Rashba Néel skyrmion lattice is shown at a higher $L = 30$ nm. Within these subplots are the magnetic charge density $\rho(x, y)$ and the magnetization $\mathbf{n}(x, y)$. Also plotted is the asymptotic energy of two well-separated skyrmions for each DMI.

Consideration of the DDI breaks the energy degeneracy of the crystal configurations, resulting in separate E against L curves for the three DMI terms considered. Qualitatively, the results are similar to those with no dipolar interaction for the Rashba and Dresselhaus DMIs. This is not surprising since isolated skyrmions induce no magnetizing field in the Dresselhaus case and no long-range field in the Rashba case: The “charge” distribution $\nabla \cdot \mathbf{n}$ is rotationally symmetric and has a total charge of zero, so all terms in its multipole expansion vanish, and the demagnetizing field, like $\mathbf{n}$, is exponentially spatially localized.

In more detail, we find that the associated skyrmions maintain their repulsive behavior and want to be infinitely separated, but the Rashba crystals have higher energy than the Dresselhaus ones. Another key difference is that the energy of the Rashba lattice asymptotically approaches the energy of two well-separated Néel skyrmions at a smaller lattice parameter L than the Dresselhaus lattice does for two Bloch skyrmions. This can be seen in Figs. 4 and 5. This is most likely due to the fact that the DDI slightly shrinks the size of the Néel skyrmion, making it more localized than the Bloch skyrmion. Even for a low lattice parameter L , the DDI has a minimal effect on the crystalline structure of Dresselhaus Bloch skyrmions.

However, a rather striking outcome for the Heusler antiskyrmion lattice is observed. First, the antiskyrmions prefer a square lattice arrangement over a triangular one at all lattice sizes L . Furthermore, the DDI introduces an attractive force so that an optimal lattice size, $L_* = 39$ nm, that minimizes the

energy $E(L)$ does exist. We have also checked that the square lattice with spacing L_* minimizes the energy per unit cell with respect to arbitrary variations of the *geometry* of the period lattice, not just its unit cell size. This calculation is quite sophisticated, and we present the details in the Appendix. We are very confident, therefore, that Heusler antiskyrmions bind together to form a square lattice at the chosen parameter set.

As the lattice size is increased, the energy of the Heusler lattice asymptotically approaches the energy of two infinitely separated antiskyrmions from below, meaning it has positive binding energy $E_{\text{bind}} = 2E_1 - E_{\text{latt}} > 0$. By contrast, the Rashba and Dresselhaus lattices approach their respective asymptotic energies from above; that is, they have negative binding energies $2E_1 < E_{\text{latt}}$. So it is energetically preferable for the Heusler antiskyrmions to form a crystalline structure, whereas it is preferable for Bloch and Néel skyrmions to be infinitely separated. The results of the effect of the dipolar interaction on the magnetic skyrmion crystals are summarized in Figs. 4 and 5.

VIII. CONCLUSION

We developed a theory of magnetic skyrmions that includes the effect of the dipole-dipole interaction in the bulk of chiral ferromagnets. In other words, we included the magnetizing field and its associated back-reaction on the magnetization. The theory is formulated in terms of an arbitrary DMI and can be extended easily to include any choice of potential. We focused on three DMI terms (Dresselhaus, Rashba, and

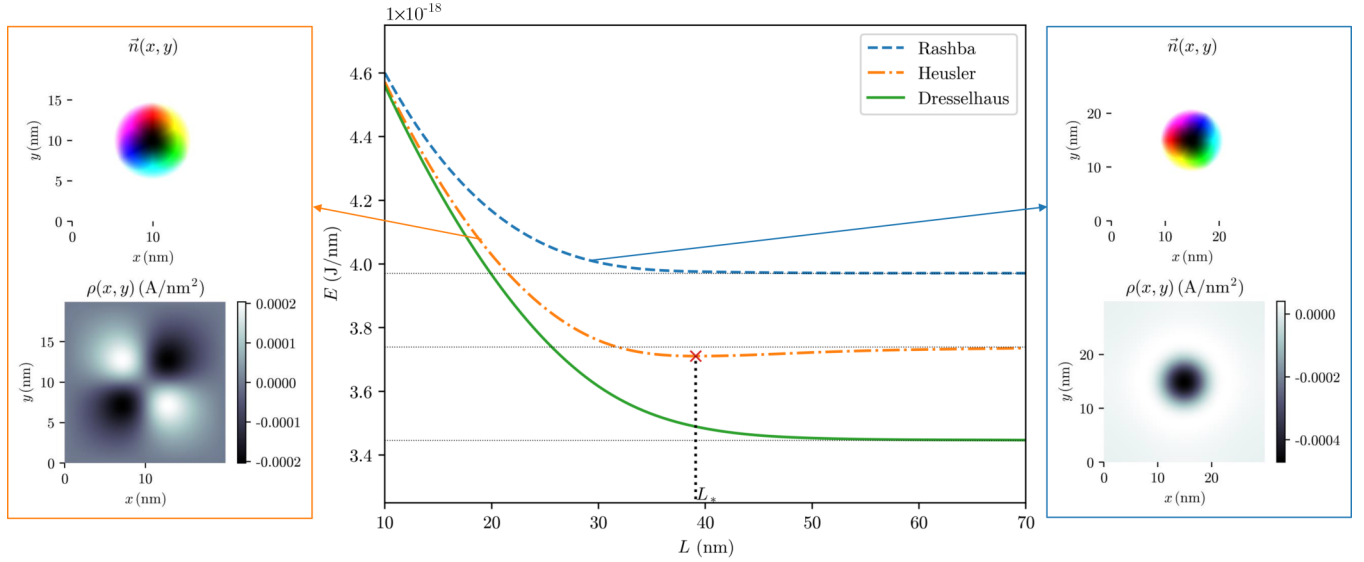


FIG. 5. The effect of the dipolar interaction on square magnetic skyrmion crystals as a function of the lattice size L . The unit cell used is of size $L \times L$, which favors a lemniscatic lattice. Initially, two magnetic skyrmions are placed in a square arrangement, and then they are relaxed using the numerical algorithm detailed in the text. It can be seen that including the DDI and its back-reaction lifts the energy degeneracy, resulting in distinct $E(L)$ curves. The Rashba (Néel) and Dresselhaus (Bloch) skyrmion lattices maintain a hexagonal arrangement, whereas the Heusler antiskyrmions relax into one with a square form. An example Heusler antiskyrmion lattice is shown at small $L = 20$ nm, and a Rashba Néel skyrmion lattice is shown at a higher $L = 30$ nm. Within these subplots are the magnetic charge density $\rho(x, y)$ and the magnetization $\mathbf{n}(x, y)$. Also plotted is the asymptotic energy of two well-separated skyrmions for each DMI.

Heusler) and included the Zeeman interaction and anisotropy. In conventional chiral magnets without the DDI, all three DMI terms (and their associated skyrmion types) are equivalent up to redefinitions of the fields. Therefore, they are energy degenerate and, for the parameter set we chose, do not form a stable crystalline structure. The inclusion of the dipolar interaction and its back-reaction breaks the energy degeneracy and the symmetry group of the individual magnetic skyrmion. Furthermore, it provides stability to Heusler antiskyrmions, allowing them to form into a crystalline arrangement.

The micromagnetic crystal simulation method we employed was very simple. We restricted the geometry of our unit cell to be equianharmonic/lemniscatic, which would normally favor a hexagonal/square crystalline structure. In both cases, the Dresselhaus and Rashba skyrmions preferred to be infinitely separated; however, the Heusler DMI preferred antiskyrmions in a square lattice arrangement. Additionally, this remained the case for a different choice of the parameter set, one which results in larger skyrmions with a size of ~ 100 nm and is typically used to model Heusler compounds.

An applied external magnetic field was also considered, but the results were qualitatively the same for our parameter set. However, an external magnetic field in the absence of anisotropies was not considered in this study. For this case, as the applied magnetic field strength is increased, one would expect to see a transition from the spin-stripe domain to a lattice of skyrmions, as found in [14].

This method can be adapted to the analog problem of skyrmions in chiral liquid crystals [35], that is, to study the flexoelectric polarization effect on skyrmions in liquid crystals. The problem there is slightly more difficult because the Poisson equation depends on the divergence of the

polarization $\mathbf{P}(\mathbf{n})$ instead of the divergence of the order parameter $\mathbf{n} \in S^2$ [36].

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DATA AVAILABILITY

The data that support the findings of this article are openly available [37].

APPENDIX: LATTICE CRITICALITY

We have described a numerical scheme that, for a given fixed period lattice

$$\Lambda = \{\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 : \alpha_1, \alpha_2 \in \mathbb{Z}\} \quad (\text{A1})$$

and topological degree, produces the minimizer among all fields $\mathbf{n} : \mathbb{R}^2 \rightarrow S^2$ periodic with respect to Λ ,

$$\mathbf{n}(\mathbf{x} + \mathbf{v}_1) \equiv \mathbf{n}(\mathbf{x} + \mathbf{v}_2) \equiv \mathbf{n}(\mathbf{x}), \quad (\text{A2})$$

of the energy of a unit cell,

$$E_{\text{cell}} = \int_{\Sigma} d^2x \left\{ \frac{1}{2} |\mathbf{d}\mathbf{n}|^2 + \mathbf{d}_i \cdot (\mathbf{n} \times \partial_i \mathbf{n}) + V(\mathbf{n}) + \frac{1}{2\mu} \psi \Delta \psi \right\}. \quad (\text{A3})$$

Here $\Sigma = \mathbb{R}^2 / \Lambda \equiv T^2$ is the torus with period lattice Λ (or, in more elementary terms, its unit cell, which is the parallel-

ogram spanned by the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$). We have left the potential terms $V(\mathbf{n})$ arbitrary since our argument does not rely on the specific potential (4) used in this paper. One should note that such a minimizer is likely to exist for *any* choice of the period lattice Λ . If this solution is to have any physical significance, it should also be a local minimizer of E_{cell} with respect to variations of the period lattice Λ . Previous studies of soliton crystals in other systems showed that, in general, the energetically optimal period lattice may have unexpectedly low symmetry, being neither square nor triangular [22,38], so this issue is not negligible. In this appendix, we formulate this criticality condition as a set of explicit integral constraints suitable for numerical verification by adapting ideas introduced in [39].

Let $\mathbf{n} : \Sigma \rightarrow S^2$ be a fixed field on the torus $\Sigma = \mathbb{R}^2/\Lambda$, assumed to minimize E_{cell} among all fields in its homotopy class. Denote by $\psi : \Sigma \rightarrow \mathbb{R}$ its associated magnetic potential. Any other period lattice $\tilde{\Lambda}$ is generated by $M\mathbf{v}_1$ and $M\mathbf{v}_2$, where $M \in \text{GL}(2, \mathbb{R})$ is some invertible matrix. Furthermore, there is a canonical magnetization field on the new torus $\tilde{\Sigma} = \mathbb{R}^2/\tilde{\Lambda}$, namely, $\tilde{\mathbf{n}}(\mathbf{x}) := \mathbf{n}(A\mathbf{x})$, where $A = M^{-1}$. This has an associated magnetic potential $\tilde{\psi} : \tilde{\Sigma} \rightarrow \mathbb{R}$, which, by definition, satisfies the equation

$$-(\partial_1^2 + \partial_2^2)\tilde{\psi} = -\mu(\partial_1\tilde{\mathbf{n}} + \partial_2\tilde{\mathbf{n}}). \quad (\text{A4})$$

Note that $\tilde{\psi}$ is *not* simply $\psi(A\mathbf{x})$ in general. With fixed mapping $\mathbf{n}$, E_{cell} is a smooth function of $A \in \text{GL}(2, \mathbb{R})$. We require that this function is locally minimal at $A = \mathbb{I}_2$. In particular, $\mathbb{I}_2$ must be a critical point of $E_{\text{cell}} : \text{GL}(2, \mathbb{R}) \rightarrow \mathbb{R}$.

All tori are diffeomorphic, and it is helpful to identify the new torus $\tilde{\Sigma}$ with the reference torus Σ via the linear diffeomorphism $M : \Sigma \rightarrow \tilde{\Sigma}$. From this viewpoint, the magnetization field $\mathbf{n} : \Sigma \rightarrow S^2$ remains fixed, but the metric on Σ , its area form, the DMI vectors, and the source of the magnetic potential $\tilde{\psi}$ depend on M . The metric is the pullback of the Euclidean metric on $\tilde{\Sigma}$ by M ,

$$\tilde{g} = (M^T M)_{ij} dx_i dx_j, \quad (\text{A5})$$

its area form is $\det M dx_1 \wedge dx_2$, the DMI vectors are

$$\tilde{\mathbf{d}}_i = A_{ij} \mathbf{d}_j, \quad (\text{A6})$$

and $\hat{\psi} := \psi \circ L : \Sigma \rightarrow \mathbb{R}$ satisfies Poisson's equation for $(\Sigma, \tilde{g})$,

$$\tilde{\Delta}\hat{\psi} = \hat{\rho}, \quad (\text{A7})$$

with the source

$$\hat{\rho} = -\mu A_{ij} \partial_i n_j. \quad (\text{A8})$$

In Eq. (A7),

$$\tilde{\Delta} = -A_{ik} A_{jk} \frac{\partial^2}{\partial x_i \partial x_j} \quad (\text{A9})$$

is the Laplacian on Σ with respect to the deformed metric $\tilde{g}$. We find that

$$E_{\text{exch}}^{\text{cell}}(A) = \frac{1}{2 \det A} \text{tr}(A^T H A), \quad (\text{A10})$$

$$E_{\text{DMI}}^{\text{cell}}(A) = \frac{1}{\det A} \text{tr}(A^T \mathcal{D}), \quad (\text{A11})$$

$$E_{\text{pot}}^{\text{cell}}(A) = \frac{1}{\det A} \int_{\Sigma} d^2x V(\mathbf{n}) = \frac{E_{\text{pot}}^{\text{cell}}(\mathbb{I}_2)}{\det A}, \quad (\text{A12})$$

$$E_{\text{DDI}}^{\text{cell}}(A) = \frac{1}{2\mu} \int_{\Sigma} \frac{d^2x}{\det A} \hat{\psi} \tilde{\Delta} \hat{\psi}, \quad (\text{A13})$$

where H and $\mathcal{D}$ are the matrices

$$H_{ij} = \int_{\Sigma} d^2x \partial_i \mathbf{n} \cdot \partial_j \mathbf{n}, \quad (\text{A14})$$

$$\mathcal{D}_{ij} = \int_{\Sigma} d^2x \mathbf{d}_i \cdot (\mathbf{n} \times \partial_j \mathbf{n}). \quad (\text{A15})$$

Although we do not have an explicit formula for $E_{\text{DDI}}^{\text{cell}}(A)$ like we do for each of the other terms, we will still be able to compute its derivative at $\mathbb{I}_2$ explicitly, and this will suffice for our purposes.

As a consistency check, let us derive the condition for E_{cell} to be critical with respect to isotropic dilations of the period lattice. They are generated by the curve $A(t) = e^{-t} \mathbb{I}_2$. The corresponding magnetic potential satisfies

$$e^{-2t} \Delta \hat{\psi} = -\mu e^{-t} \nabla \cdot \mathbf{n}, \quad (\text{A16})$$

so $\hat{\psi} = e^t \psi$. Hence, the condition for criticality is

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} & \left(\frac{1}{2} \text{tr} H + e^t \text{tr} \mathcal{D} + e^{2t} E_{\text{pot}}^{\text{cell}}(\mathbb{I}_2) \right. \\ & \left. + \frac{1}{2\mu} \int_{\Sigma} d^2x e^t \psi e^{-2t} \Delta e^t \psi e^{2t} \right) \\ & = \text{tr} \mathcal{D} + 2E_{\text{pot}}^{\text{cell}}(\mathbb{I}_2) + \frac{1}{2\mu} \int_{\Sigma} d^2x \psi \Delta \psi \\ & = E_{\text{DMI}}^{\text{cell}}(\mathbb{I}_2) + 2[E_{\text{pot}}^{\text{cell}}(\mathbb{I}_2) + E_{\text{DDI}}^{\text{cell}}(\mathbb{I}_2)] = 0, \end{aligned} \quad (\text{A17})$$

which coincides precisely with the Derrick scaling constraint (35) evaluated on a single lattice cell. This constraint is automatically satisfied by our numerical solutions since our scheme minimizes E_{cell} with respect to the area of the lattice cell.

We now analyze the three-dimensional space of lattice variations complementary to isotropic dilations. Let $A(t)$ be a curve in $\text{SL}(2, \mathbb{R})$ with $A(0) = \mathbb{I}_2$ generating a variation of Λ through lattices whose unit cells have fixed area. Then $\varepsilon := \dot{A}(0)$ is traceless. The induced variations of the various terms in $E_{\text{cell}}(\mathbb{I}_2)$ are

$$\left. \frac{d}{dt} \right|_{t=0} E_{\text{exch}}^{\text{cell}}(A(t)) = \text{tr}(\varepsilon^T H), \quad (\text{A18})$$

$$\left. \frac{d}{dt} \right|_{t=0} E_{\text{DMI}}^{\text{cell}}(A(t)) = \text{tr}(\varepsilon^T \mathcal{D}), \quad (\text{A19})$$

$$\left. \frac{d}{dt} \right|_{t=0} E_{\text{pot}}^{\text{cell}}(A(t)) = 0. \quad (\text{A20})$$

The variation of $E_{\text{DDI}}^{\text{cell}}$ is considerably subtler. Let us denote by Δ_t the Laplacian on Σ with respect to the metric $\tilde{g}(t)$, that is,

$$\Delta_t = -[A(t)A(t)^T]_{ij} \partial_i \partial_j, \quad (\text{A21})$$

and by ψ_t the solution of

$$\Delta_t \psi_t = \rho_t, \quad (\text{A22})$$

where

$$\rho_t = -\mu A_{ij}(t) \partial_i n_j. \quad (\text{A23})$$

Note that $\psi_0 = \psi$. Further, let an overdot denote the derivative with respect to t at $t = 0$. Then

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} E_{\text{DDI}}^{\text{cell}}(A(t)) &= \frac{1}{2\mu} \int_{\Sigma} d^2x (\dot{\psi} \Delta \psi + \psi \Delta \dot{\psi} + \psi \dot{\Delta} \psi) \\ &= \frac{1}{\mu} \int_{\Sigma} d^2x \left(\psi \Delta \dot{\psi} + \frac{1}{2} \psi \dot{\Delta} \psi \right). \end{aligned} \quad (\text{A24})$$

Differentiating (A22), we see that

$$\Delta \dot{\psi} = \dot{\rho} - \dot{\Delta} \psi = -\mu \varepsilon_{ij} \partial_i n_j - 2\varepsilon_{ij} \partial_i \partial_j \psi. \quad (\text{A25})$$

Hence,

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} E_{\text{DDI}}^{\text{cell}}(A(t)) &= \frac{1}{\mu} \int_{\Sigma} d^2x \varepsilon_{ij} \psi (-\mu \partial_i n_j + \partial_i \partial_j \psi) \\ &= \text{tr}(\varepsilon^T Q), \end{aligned} \quad (\text{A26})$$

where

$$Q_{ij} = \int_{\Sigma} d^2x \psi \left(\frac{1}{\mu} \partial_i \partial_j \psi - \partial_j n_i \right). \quad (\text{A27})$$

Note that Q is traceless by (30).

The condition that E_{cell} is critical with respect to all area-preserving lattice variations is therefore

$$\text{tr}[\varepsilon(H + \mathcal{D} + Q)] = 0 \quad (\text{A28})$$

for all traceless matrices ε . This is equivalent to the condition that

$$H + \mathcal{D} + Q = \lambda \mathbb{I}_2 \quad (\text{A29})$$

for some $\lambda \in \mathbb{R}$. Taking traces of both sides, we see that

$$\lambda = E_{\text{exch}}^{\text{cell}} + \frac{1}{2} E_{\text{DMI}}^{\text{cell}}. \quad (\text{A30})$$

To summarize, a doubly periodic map $\mathbf{n} : \mathbb{R}^2/\Lambda \rightarrow S^2$, inducing magnetic potential ψ , is critical with respect to all variations of the period lattice Λ if it satisfies the scaling identity (A17) and the angular virial constraint (A29), where the matrices H , $\mathcal{D}$, and Q are defined in (A14), (A15), and (A27) and all integrals are over a single unit cell of the lattice Λ . These conditions are easily checked numerically.

To numerically verify the extended virial constraints [(A17) and (A29)], we compute the energy-normalized constraints $\{V_1 = 0, V_2 = 0\}$, where

$$V_1 = \frac{E_{\text{DMI}}^{\text{cell}} + 2(E_{\text{pot}}^{\text{cell}} + E_{\text{DDI}}^{\text{cell}})}{E_{\text{cell}}}, \quad (\text{A31})$$

$$V_2 = \frac{H + \mathcal{D} + Q - \lambda \mathbb{I}_2}{E_{\text{cell}}}. \quad (\text{A32})$$

We find that both of these normalized constraints are satisfied to within $\text{err} = 10^{-5}$ for the square Heusler antiskyrmion lattice at a lattice parameter of $L_* = 39$ nm.

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