

Fourth- and sixth-order nonlinear spin current rectifier in three-dimensional h -wave and j -wave odd-parity magnets

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Higher-order symmetric X -wave magnets consist of two groups. One includes d -wave, g -wave, and i -wave altermagnets, while the other includes p -wave and f -wave odd-parity magnets. Recently, the possibility of h -wave magnets has been discussed. Motivated by this development, we systematically construct an X -wave magnet with $(N_X + 1)$ nodes in three dimensions from an X -wave magnet with N_X nodes in two dimensions by means of a dimensional extension, where $N_X = 1, 2, 3, 4, 6$ for $X = p, d, f, g, i$, respectively. Based on this method, we predict j -wave magnets in three dimensions. Then, we argue how to identify each of these X -wave magnets experimentally. We show that the X -wave magnet is completely identified by measuring the nonlinear spin currents. In particular, we predict that there are no spin currents other than the fourth-order ones such as $\sigma_{\text{spin}}^{x^3yz}$ in h -wave odd-parity magnets in three dimensions and the sixth-order ones such as $\sigma_{\text{spin}}^{x^5yz}$ in j -wave odd-parity magnets in three dimensions. They function as spin-current rectifiers because the spin current exhibits unidirectional flow independent of the applied electric field.

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I. INTRODUCTION

Altermagnets [1–7] are one of the most exciting fields in condensed matter physics. They will enable us to realize an ultrafast and ultradense spintronic memory due to a characteristic spin-split band structure and zero net magnetization. Altermagnets are characterized by breaking time-reversal symmetry and preserving inversion symmetry. In addition, p -wave magnets [8–17] and f -wave magnets [9,10,18–20] also attract attention although they are not altermagnets. They are characterized by preserving time-reversal symmetry and breaking inversion symmetry. In this sense, they are odd-parity magnets. These altermagnets and odd-parity magnets are summarized in terms of X -wave magnets [21–23] with $X = p, d, f, g, i$. They are described by the band-splitting functions $f_X(k)$, whose number of nodes are $N_X = 1, 2, 3, 4, 6$, respectively. See Table I. The X -wave magnet with $N_X = 5$ is absent in two dimensions (2D) because the fivefold rotational symmetry is prohibited in crystal. Recently, h -wave magnets with $N_X = 5$ were proposed [24] in three dimensions (3D) but their physical properties are yet to be explored. Indeed, FeSe has been argued to be an h -wave magnet based on the density-functional theory [24].

Nonlinear spin current generation is an important topic in spintronics [25–28]. It is prominent that only a certain order of nonlinear spin currents are generated in an X -wave magnet [19,20]. The order of nonlinearity of spin currents is determined by the number of nodes N_X of the X -wave magnet. For instance, only the second-order nonlinear spin currents are

generated in f -wave magnets [19]. In a similar way, only the third-order (fifth-order) nonlinear spin currents are generated in g -wave (i -wave) altermagnets. See Table I.

In this paper, we propose a method to systematically construct an X -wave magnet with $(N_X + 1)$ nodes in 3D from an X -wave magnet with N_X nodes in 2D by means of a dimensional extension. In this framework, it is naturally understood that h -wave magnets are 3D extensions of g -wave altermagnets. Similarly, we predict j -wave magnets in 3D by a dimensional extension of i -wave altermagnets. A candidate crystal structure is a triangular prism lattice. We represent group-theoretical analysis for X -wave magnets. There are no other X -wave magnets in 3D which can be constructed by this method because there are no X -wave magnets with $N_X > 6$ in 2D. Then, we show that the X -wave magnet is completely identified by measuring the nonlinear spin conductivities. In particular, we predict that only the fourth-order nonlinear spin currents such as $\sigma_{\text{spin}}^{x^3yz}$ emerge in h -wave odd-parity magnets in 3D and only the sixth-order ones such as $\sigma_{\text{spin}}^{x^5yz}$ emerge in j -wave odd-parity magnets in 3D. See Table I. They function as spin-current rectifiers because the spin current exhibits unidirectional flow independent of the applied electric field.

This paper is organized as follows. Section II summarizes known X -wave magnets and proposes three-dimensional h -wave and j -wave magnets based on the layer-construction method. Section III discusses the point groups, magnetic point groups, and irreducible representations relevant to h -wave and j -wave magnets. In Sec. IV, we determine the Fermi surfaces of h -wave and j -wave magnets and present the corresponding tight-binding models. Section V evaluates the nonlinear spin current in these systems. Section VI is devoted to discussion. The Appendix provides point groups, magnetic point groups, and irreducible representations for all X -wave magnets.

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TABLE I. X -wave magnets and nonlinear spin conductivity. The X -wave magnet is characterized by the number of nodes N_X , where only the ℓ th order nonlinear spin currents are generated. Typical spin conductivities are listed in 2D and 3D.

		X					
		p	d	f	g	h	i
N_X	ℓ	1	2	3	4	5	6
			Linear	2nd NL	3rd NL	4th NL	5th NL
2D	0	0	$\sigma_{\text{spin}}^{y;x}$	$\sigma_{\text{spin}}^{x^2;y}$	$\sigma_{\text{spin}}^{y^3;x}$		$\sigma_{\text{spin}}^{y^5;x}$
3D	0	0	$\sigma_{\text{spin}}^{y;x}$	$\sigma_{\text{spin}}^{z;y;x}$	$\sigma_{\text{spin}}^{x^3;z}$	$\sigma_{\text{spin}}^{y^4;x}$	$\sigma_{\text{spin}}^{x^5;y;z}$

II. X -WAVE MAGNETS

The electromagnetic property of the X -wave magnet is characterized by the band splitting depending on the spin. The simplest model is the two-band Hamiltonian [19–23] given by

$$H(\mathbf{k}) = \frac{\hbar^2 k^2}{2m} + J f_X(\mathbf{k}) \sigma_z, \quad (1)$$

where the first term represents the kinetic term of electrons, while the second term represents the band splitting described by the function $f_X(\mathbf{k})$ with the coupling constant J and the Pauli matrix σ_z . The energy is given by

$$\varepsilon_s(\mathbf{k}) = \frac{\hbar^2 k^2}{2m} + sJ f_X(\mathbf{k}), \quad (2)$$

with $s = \pm 1$, where $s = 1$ for up spin and $s = -1$ for down spin.

In 2D, with the use of the lattice constant a , $f_X(k)$ are given by [1,2,19,21,22],

$$f_s^{2D}(\mathbf{k}) = 1, \quad (3)$$

$$f_p^{2D}(\mathbf{k}) = ak_x = ak \cos \phi, \quad (4)$$

$$f_d^{2D}(\mathbf{k}) = 2a^2 k_x k_y = a^2 k^2 \sin 2\phi, \quad (5)$$

$$f_f^{2D}(\mathbf{k}) = a^3 k_x (k_x^2 - 3k_y^2) = a^3 k^3 \cos 3\phi, \quad (6)$$

$$f_g^{2D}(\mathbf{k}) = 4a^4 k_x k_y (k_x^2 - k_y^2) = a^4 k^4 \sin 4\phi, \quad (7)$$

$$f_i^{2D}(\mathbf{k}) = 2a^6 k_x k_y (3k_x^2 - k_y^2)(k_x^2 - 3k_y^2) = a^6 k^6 \sin 6\phi, \quad (8)$$

and

$$f_{p'}^{2D}(\mathbf{k}) = ak_y = ak \sin \phi, \quad (9)$$

$$f_{d'}^{2D}(\mathbf{k}) = a^2 (k_x^2 - k_y^2) = a^2 k^2 \cos 2\phi, \quad (10)$$

$$f_{f'}^{2D}(\mathbf{k}) = a^3 k_y (3k_x^2 - k_y^2) = a^3 k^3 \sin 3\phi, \quad (11)$$

$$\begin{aligned} f_{g'}^{2D}(\mathbf{k}) &= a^4 (k_x^4 - 6k_x^2 k_y^2 + k_y^4) \\ &= a^4 k^4 \cos 4\phi, \end{aligned} \quad (12)$$

$$\begin{aligned} f_{i'}^{2D}(\mathbf{k}) &= 2a^6 (k_x^2 - k_y^2)(k_x^4 + k_y^4 - 14k_x^2 k_y^2) \\ &= a^6 k^6 \cos 6\phi, \end{aligned} \quad (13)$$

each of which has N_X nodes. In 3D, $f_X(k)$ are given by

$$f_d^{3D}(\mathbf{k}) = a^2 k_z (k_x + k_y), \quad (14)$$

$$f_f^{3D}(\mathbf{k}) = 2a^3 k_x k_y k_z = a^2 k^2 \cos \theta \sin 2\phi, \quad (15)$$

$$f_g^{3D}(\mathbf{k}) = a^4 k_z k_x (k_x^2 - 3k_y^2) = a^3 k^3 \cos \theta \cos 3\phi, \quad (16)$$

$$f_i^{3D}(\mathbf{k}) = a^6 (k_x^2 - k_y^2)(k_y^2 - k_z^2)(k_z^2 - k_x^2), \quad (17)$$

and

$$f_{d'}^{3D}(\mathbf{k}) = a^2 k_y (k_x + k_z), \quad (18)$$

$$f_{f'}^{3D}(\mathbf{k}) = a^3 k_z (k_x^2 - k_y^2) = a^2 k^2 \cos \theta \cos 2\phi, \quad (19)$$

$$f_{g'}^{3D}(\mathbf{k}) = a^4 k_z k_y (3k_x^2 - k_y^2) = a^3 k^3 \cos \theta \sin 3\phi. \quad (20)$$

each of which has N_X nodes. There are relations between an X -wave magnet with $N_X + 1$ in 3D and an X -wave magnet with N_X in 2D, such as [23]

$$f_f^{3D}(\mathbf{k}) = ak_z f_d^{2D}(\mathbf{k}), \quad (21)$$

$$f_g^{3D}(\mathbf{k}) = ak_z f_f^{2D}(\mathbf{k}). \quad (22)$$

We propose to construct h -wave magnets and j -wave magnets in 3D from the g -wave and i -wave altermagnets in 2D by a dimensional extension such as

$$f_h^{3D}(\mathbf{k}) = ak_z f_g^{2D}(\mathbf{k}), \quad (23)$$

$$f_j^{3D}(\mathbf{k}) = ak_z f_i^{2D}(\mathbf{k}), \quad (24)$$

provided that there are crystal structures compatible with $f_h^{3D}(\mathbf{k})$ and $f_j^{3D}(\mathbf{k})$. We note that the i -wave altermagnet in 3D is not constructed in this way because there is no h -wave odd-parity magnet in 2D.

III. GROUP THEORETICAL ANALYSIS

We classify X -wave magnets based on spherical-harmonic functions and crystallographic point groups as in the case of classification of multipoles [29,30]. Multipoles are expressed by the spherical-harmonic function as

$$Q_{\ell m} \equiv -e \sqrt{\frac{4\pi}{2\ell+1}} k^\ell Y_{\ell m}(\theta, \phi). \quad (25)$$

The f_X terms correspond to

$$Q_{\ell m}^+ \equiv \frac{(-1)^m}{\sqrt{2}} (Q_{\ell m} + Q_{\ell m}^*), \quad (26)$$

$$Q_{\ell m}^- \equiv \frac{(-1)^m}{i\sqrt{2}} (Q_{\ell m} - Q_{\ell m}^*). \quad (27)$$

The explicit forms are given by

$$Q_{\ell m}^+ \propto k^{\ell-m} P_\ell^m(\cos \theta) \text{Re}[k_+^m], \quad (28)$$

$$Q_{\ell m}^- \propto k^{\ell-m} P_\ell^m(\cos \theta) \text{Im}[k_+^m], \quad (29)$$

TABLE II. Classification table for h -wave and j -wave magnets in terms of the point group (PG), the magnetic point group (MPG), and the irreducible representation (IR).

		Spherical harmonic functions	PG (MPG)	IR
f_h^{3D}	Q_{54}^-	$k_z k_x k_y (k_x^2 - k_y^2)$	$D_{4h} (4/m' m' m')$	A_{1u}
$f_{h'}^{3D}$	Q_{54}^+	$k_z (k_x^4 - 6k_x^2 k_y^2 + k_y^4)$	$D_{4h} (4/mmm')$	A_{2u}
f_j^{3D}	Q_{76}^-	$2k_z k_x k_y (3k_x^2 - k_y^2)(k_x^2 - 3k_y^2)$	$D_{6h} (6/m' m' m')$	A_{2u}
$f_{j'}^{3D}$	Q_{76}^+	$k_z (k_x^6 - 15k_x^4 k_y^2 + 15k_x^2 k_y^4 - k_y^6)$	$D_{6h} (6/mmm')$	A_{1u}

where P_ℓ^m is the associated Legendre polynomial and we have used the relation

$$Y_{\ell m}(\theta, \phi) = (-1)^{\frac{\ell+|m|}{2}} \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-|m|)!}{(\ell+|m|)!}} P_\ell^{|m|}(\cos \theta) e^{im\phi}. \quad (30)$$

There are relations

$$Q_{N_X, N_X-1}^+ \propto k_z Q_{N_X-1, N_X-1}^+, \quad (31)$$

$$Q_{N_X, N_X-1}^- \propto k_z Q_{N_X-1, N_X-1}^-, \quad (32)$$

and

$$f_X^{2D} = Q_{N_X, N_X}^-, \quad f_{X'}^{2D} = Q_{N_X, N_X}^+, \quad (33)$$

$$f_X^{3D} = Q_{N_X, N_X-1}^+, \quad f_{X'}^{3D} = Q_{N_X, N_X-1}^-. \quad (34)$$

We show the classification table for h -wave and j -wave magnets in terms of the point group (PG), the magnetic point group (MPG), and the irreducible representation (IR) (see Table II). First, h -wave magnets are realized in the orthorhombic crystal system, which belongs to the D_{4h} point group. The explicit crystal structures are the perovskite structure, the rutile structure, the olivine structure, and the illite structure. On the other hand, j -wave magnets are realized in the hexagonal crystal, which belongs to the D_{6h} group. The explicit crystal structures are the delafossite-type structure (ABO_2), 1T/2H systems made of transition-metal dichalcogenides, CdI_2 -type, the $NaCoO_2/Na_xCoO_2$ structure, and the $YbMgGaO_4$ structure. Magnetic point groups relevant to X -wave magnets belong to type-III (black-white) magnetic point groups, in which the magnetization is reversed by the N_X rotation. These groups take forms such as N/mmm and $N'/m'm'm'$, where N or N' indicates whether the magnetization is preserved or reversed under a $2\pi/N$ rotation. Likewise, m or m' specifies whether the magnetization is preserved or reversed by the corresponding mirror operation. For the sake of completeness, we classify all X -wave magnets in the Appendix.

IV. h -WAVE AND j -WAVE MAGNETS

A. h -wave magnets in 2D

The continuum form of the h -wave magnet in 2D is described by

$$f_h^{2D}(\mathbf{k}) = k^5 \cos 5\phi. \quad (35)$$

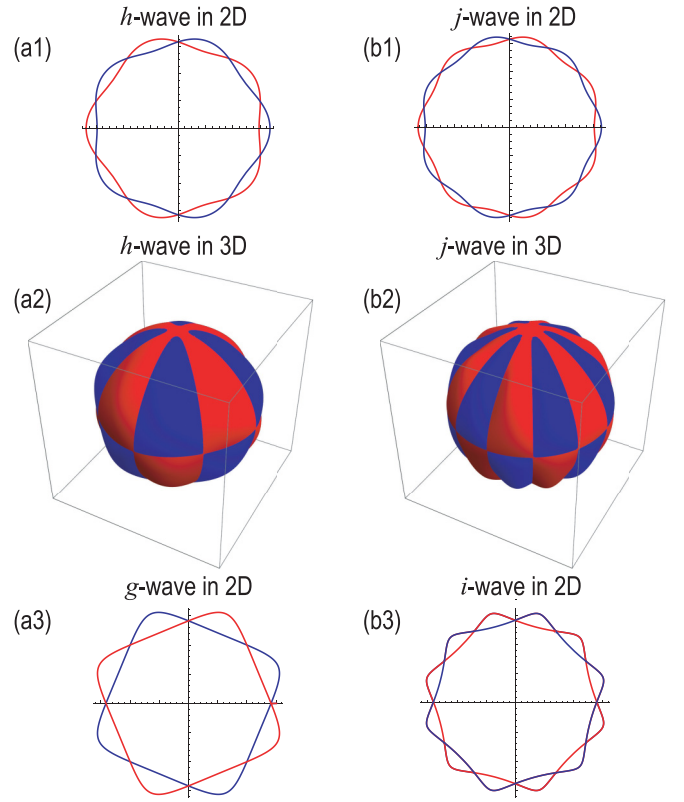


FIG. 1. Illustration of the Fermi surface. (a1) h -wave magnet in 2D. (b1) j -wave magnet in 2D. (a2) h -wave magnet in 3D. (b2) j -wave magnet in 3D. (a3) g -wave alternomagnet in 3D. (b3) i -wave alternomagnet in 2D. There is no lattice realization of h -wave (j -wave) magnets in 2D due to the fivefold (sevenfold) rotational symmetry. The h -wave (j -wave) magnet in 3D is constructed as a dimensional extension of the g -wave (i -wave) alternomagnet in 2D.

The Fermi surface is given by

$$k_F = \frac{\sqrt{2m\mu}}{\hbar} - sJ \frac{4\mu^2 m^3}{\hbar^6} \cos 5\phi \quad (36)$$

at $\varepsilon = \mu$ up to the first order in J and is shown in Fig. 1(a1). Equation (35) is rewritten in the form of

$$f_h^{2D}(\mathbf{k}) = k_x^5 - 7k_x^3 k_y^2 + 2k_x k_y^4. \quad (37)$$

However, it is impossible to implement it in the tight-binding model because the fivefold rotational symmetry is prohibited in crystals.

B. h -wave magnets in 3D

The h -wave magnet in 3D is described by the band-splitting function (23) with the use of $f_g^{2D}(\mathbf{k})$, or

$$f_h^{3D}(\mathbf{k}) = 4a^5 k_x k_y k_z (k_x^2 - k_y^2). \quad (38)$$

The Fermi surface is given by

$$k_F = \frac{\sqrt{2m\mu}}{\hbar} - sJ \frac{4\mu^2 m^3}{\hbar^6} \cos \theta \sin 4\phi \quad (39)$$

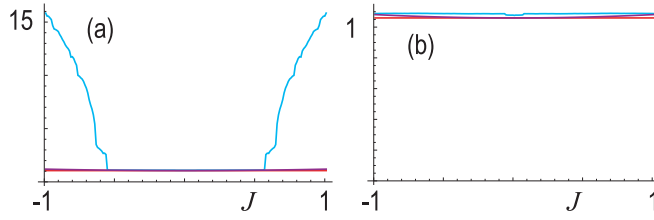


FIG. 2. Fermi volume as a function of J . (a) h -wave magnets in 2D. (b) j -wave magnets in 3D. The vertical axis is the Fermi volume in units of $8\sqrt{2}\pi(m\mu)^{3/2}/(3\hbar^3)$. The horizontal axis is J in units of ε_0 . Cyan curves represent numerical results based on the tight-binding model. Red lines represent the analytic result V_0 based on the continuum model. Purple curves represent analytic results based on the continuum model up to the second order in J . We have set $\hbar^2/(ma^2) = \varepsilon_0/4$ and $\mu = 0.1\varepsilon_0$.

up to the first order in J . Using it, the Fermi volume is given by

$$V_h^{3D} = V_0^{3D} + \frac{176\sqrt{2}\pi\mu^{9/2}m^{13/2}}{3\hbar^{13}}J^2, \quad (40)$$

with the Fermi volume in the absence of magnetic order,

$$V_0^{3D} = \frac{8\sqrt{2}\pi}{3} \left(\frac{\mu m}{\hbar^2} \right)^{3/2}. \quad (41)$$

In contrast to the h -wave magnet in 2D, it is possible to construct a tight-binding model corresponding to the continuum model (1) with Eq. (38) on the cubic lattice. It is given by

$$H = \frac{\hbar^2}{ma^2} (3 - \cos ak_x - \cos ak_y - \cos ak_z) \sigma_0 + 2J\sigma_z \sin ak_x \sin ak_y \sin ak_z (\cos ak_y - \cos ak_x). \quad (42)$$

The Fermi surface is shown in Fig. 1(a2) based on the tight-binding model. The Fermi volume is shown as a function of J based on the tight-binding model without using the perturbative expansion of J in Fig. 2(a). It has almost no dependence on J even for large J , where it is a good approximation to use $V_h^{3D} = V_0^{3D}$ with Eq. (41).

C. i -wave magnets in 2D

The continuum form of the h -wave magnets in 2D is given by

$$f_j^{2D}(\mathbf{k}) = k^7 \cos 7\phi. \quad (43)$$

The Fermi surface is given by

$$k_F = \frac{\sqrt{2m\mu}}{\hbar} - sJ \frac{8\mu^3 m^4}{\hbar^8} \cos 7\phi \quad (44)$$

up to the first order in J and is shown in Fig. 1(b1). Equation (43) is rewritten in the form of

$$f_j^{2D}(\mathbf{k}) = k_x^7 - 18k_x^5 k_y^2 + 25k_x^3 k_y^4 - 4k_x k_y^6. \quad (45)$$

However, it is impossible to implement it in the tight-binding model because the sevenfold rotational symmetry is prohibited in crystals.

D. j -wave magnets in 3D

We predict j -wave magnets in 3D based on the band-splitting function (24) with the use of $f_i^{2D}(\mathbf{k})$, or

$$f_j^{3D}(\mathbf{k}) = 2a^7 k_x k_y k_z (3k_x^2 - k_y^2)(k_x^2 - 3k_y^2). \quad (46)$$

The Fermi surface is given by

$$k_F = \frac{\sqrt{2m\mu}}{\hbar} - sJ \frac{8\mu^3 m^4}{\hbar^8} \cos \theta \sin 6\phi \quad (47)$$

up to the first order in J . Using it, the Fermi volume is given by

$$V_j^{3D} = V_0^{3D} + \frac{128\sqrt{2}\pi\mu^{13/2}m^{17/2}}{3\hbar^{17}}J^2. \quad (48)$$

In contrast to the i -wave magnet in 2D, it is possible to construct a tight-binding model corresponding to the continuum model (1) with Eq. (46) on the triangular prism lattice. It is given by

$$H = \left\{ \frac{-2\hbar^2}{3ma^2} \left[\sum_{j=0}^2 \cos(\mathbf{a}\mathbf{n}_j^A \cdot \mathbf{k}) - 3 \right] + \frac{\hbar^2}{m_z a^2} (1 - \cos ak_z) \right\} \sigma_0 + 16J\sigma_z \sin ak_z \prod_{j=0}^2 \sin(\mathbf{a}\mathbf{n}_j^A \cdot \mathbf{k}) \prod_{j=0}^2 \sin(\sqrt{3}\mathbf{a}\mathbf{n}_j^B \cdot \mathbf{k}), \quad (49)$$

where we have defined $\mathbf{n}_j^A = (\cos \frac{2\pi j}{3}, \sin \frac{2\pi j}{3})$ and $\mathbf{n}_j^B = (\sin \frac{2\pi j}{3}, \cos \frac{2\pi j}{3})$ with $j = 0, 1, 2$.

The Fermi surface is shown in Fig. 1(b2) based on the tight-binding model. The Fermi volume is shown as a function of J based on the tight-binding model without using the perturbative expansion of J in Fig. 2(b). It has almost no dependence on J even for large J , where it is a good approximation to use $V_j^{3D} = V_0^{3D}$.

V. NONLINEAR SPIN CURRENT GENERATION

X -wave magnets have a characteristic nonlinear spin conductivity, where only a certain order of nonlinear spin conductivity is nonzero [19]. For example, only the second-order nonlinear spin conductivity is nonzero for f -wave magnets [19].

The formula describing the nonlinear spin current by the application of E_x and E_y has been studied [19]. It is generalized to

$$j_b = \sum_{\ell_x, \ell_y, \ell_z} \sigma^{x^{\ell_x} y^{\ell_y} z^{\ell_z}; b} (E_x)^{\ell_x} (E_y)^{\ell_y} (E_z)^{\ell_z} \quad (50)$$

by the application of E_x , E_y , and E_z . Then, the $(\ell_x + \ell_y + \ell_z)$ th order conductivity is defined by

$$\sigma^{x^{\ell_x} y^{\ell_y} z^{\ell_z}; b} = \frac{1}{\ell_x! \ell_y! \ell_z!} \frac{\partial^{\ell_x + \ell_y + \ell_z} j_b}{\partial E_x^{\ell_x} \partial E_y^{\ell_y} \partial E_z^{\ell_z}}. \quad (51)$$

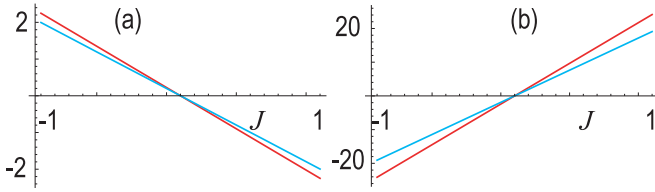


FIG. 3. Spin conductivity as a function of J . (a) h -wave magnets in 2D. We have set $\mu = 0.05\varepsilon_0$. (b) j -wave magnets in 3D. The vertical axis is σ_{spin} in units of $e^{N_X} \tau^{N_X-1} \varepsilon_0 / (\hbar^{N_X} k_0)$. The horizontal axis is J in units of ε_0 . We have set $\hbar^2/(ma^2) = \varepsilon_0/4$ and $\mu = 0.01\varepsilon_0$.

The spin-dependent conductivity is explicitly given by

$$\sigma_s^{x^{\ell_x} y^{\ell_y} z^{\ell_z}; b} = \frac{(-e/\hbar)^{\ell_x + \ell_y + \ell_z + 1}}{(i\omega + \frac{1}{\tau})^{\ell_x + \ell_y + \ell_z}} \int d\mathbf{k} f_s^{(0)} \frac{\partial^{\ell_x + \ell_y + \ell_z + 1} \varepsilon_s}{\partial k_x^{\ell_x} \partial k_y^{\ell_y} \partial k_z^{\ell_z} \partial k_b}. \quad (52)$$

Based on this formula, we find that the spin conductivity defined as

$$\sigma_{\text{spin}}^{x^{\ell_x} y^{\ell_y} z^{\ell_z+1}; b} = \frac{\sigma_{\uparrow}^{x^{\ell_x} y^{\ell_y} z^{\ell_z+1}; b} - \sigma_{\downarrow}^{x^{\ell_x} y^{\ell_y} z^{\ell_z+1}; b}}{2} \quad (53)$$

for $(X+1)$ -wave odd-parity magnets in 3D is identical to $\sigma_{\text{spin}}^{x^{\ell_x} y^{\ell_y} z^{\ell_z}; b}$ for X -wave altermagnets in 2D.

The energy of the X -wave magnets is expanded in the form of

$$\varepsilon_s = \frac{\hbar^2 k^2}{2m} + sJ \sum_{n_x + n_y + n_z = N_X} c^{n_x, n_y, n_z} k_x^{n_x} k_y^{n_y} k_z^{n_z}. \quad (54)$$

Then, the spin conductivity $\sigma_{\text{spin}}^{x^{\ell_x} y^{\ell_y} z^{\ell_z}; b}$ is nonzero only for nonzero c^{n_x, n_y, n_z} , where the spin conductivity is given by

$$\begin{aligned} \sigma_{\text{spin}}^{x^{n_x-1} y^{n_y} z^{n_z}; x} &= \sigma_{\text{spin}}^{x^{n_x} y^{n_y-1} z^{n_z}; y} = \sigma_{\text{spin}}^{x^{n_x} y^{n_y} z^{n_z-1}; z} \\ &= \frac{(-e/\hbar)^{n_x + n_y + n_z + 1}}{(i\omega + \frac{1}{\tau})^{n_x + n_y + n_z}} V \frac{n_x! n_y! n_z!}{(n_x - \ell_x)!(n_y - \ell_y)!(n_z - \ell_z)!}, \end{aligned} \quad (55)$$

with the Fermi volume $V \equiv \int d\mathbf{k} f_s^{(0)}$. Hence, the function $f_X(\mathbf{k})$ of the X -wave magnet is experimentally determined by measuring $\sigma_{\text{spin}}^{x^{n_x} y^{n_y} z^{n_z}; b}$ for various n_x, n_y, n_z satisfying $n_x + n_y + n_z = N_X$. Namely, the X -wave magnet is completely identified by measuring the nonlinear spin conductivities.

A. h -wave magnets in 3D

Using Eq. (52) and the expansion Eq. (54), all of the nonzero contributions are listed as $\sigma_{\text{spin}}^{x^3 y; z}$, $\sigma_{\text{spin}}^{x y^3; z}$, $\sigma_{\text{spin}}^{x^2 y^2; z}$, $\sigma_{\text{spin}}^{y^3 z; x}$, $\sigma_{\text{spin}}^{x^3 z; y}$ and $\sigma_{\text{spin}}^{x y^2; y}$, where

$$\begin{aligned} \sigma_{\text{spin}}^{x^3 y; z} &= \sigma_{\text{spin}}^{x^2 y^2; x} = \sigma_{\text{spin}}^{x^3 z; y} = -\sigma_{\text{spin}}^{x y^3; z} \\ &= -\sigma_{\text{spin}}^{y^3 z; x} = -\sigma_{\text{spin}}^{x y^2; y} = 6 \frac{(-e/\hbar)^5}{(i\omega + 1/\tau)^4} V. \end{aligned} \quad (56)$$

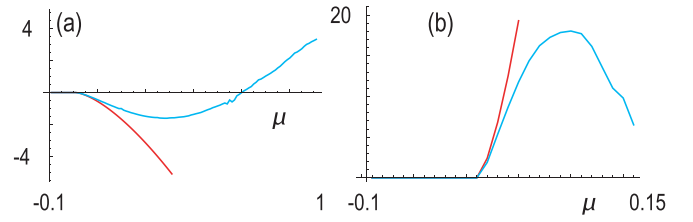


FIG. 4. Spin conductivity as a function of μ . (a) h -wave magnets in 2D. (b) j -wave magnets in 3D. The vertical axis is σ_{spin} in units of $e^{N_X} \tau^{N_X-1} \varepsilon_0 / (\hbar^{N_X} k_0)$. The horizontal axis is μ in units of ε_0 . We have set $\hbar^2/(ma^2) = \varepsilon_0/4$ and $J = 0.1\varepsilon_0$.

The spin conductivity is shown as a function of J in Fig. 3(a). It is linear even for the numerical calculation without using the perturbation theory with respect to J . The spin conductivity is shown as a function of μ in Fig. 4(a). The analytic formula based on the continuum model well reproduces the numerical result based on the tight-binding model at the bottom of the band structure.

B. j -wave magnets in 3D

Using Eq. (52) and the expansion, Eq. (54), all of the nonzero contributions are $\sigma_{\text{spin}}^{x^5 y; z}$, $\sigma_{\text{spin}}^{x^3 y^3; z}$, $\sigma_{\text{spin}}^{x y^5; z}$, $\sigma_{\text{spin}}^{x^4 y^2; x}$, $\sigma_{\text{spin}}^{x^2 y^3; z}$, $\sigma_{\text{spin}}^{y^5 z; x}$, $\sigma_{\text{spin}}^{x^5 z; y}$, $\sigma_{\text{spin}}^{x^3 y^2; y}$, and $\sigma_{\text{spin}}^{x y^4; y}$, where

$$\begin{aligned} \sigma_{\text{spin}}^{x^5 y; z} &= \sigma_{\text{spin}}^{x^4 y^2; x} = \sigma_{\text{spin}}^{x^5 z; y} = \sigma_{\text{spin}}^{x y^5; z} = \sigma_{\text{spin}}^{y^5 z; x} \\ &= \sigma_{\text{spin}}^{x y^4; y} = -\sigma_{\text{spin}}^{x^3 y^3; z} = -\sigma_{\text{spin}}^{x^2 y^3; x} = -\sigma_{\text{spin}}^{x^3 y^2; y} \\ &= 6! \frac{(-e/\hbar)^7}{(i\omega + 1/\tau)^6} V. \end{aligned} \quad (57)$$

The spin conductivity is shown as a function of J in Fig. 3(b). It is linear even for the numerical calculation without using the perturbation theory with respect to J . The spin conductivity is shown as a function of μ in Fig. 4(b). The analytic formula based on the continuum model well reproduces the numerical result based on the tight-binding model at the bottom of the band structure.

VI. DISCUSSION

We have proposed a scheme to construct X -wave magnets in 3D by a dimensional extension of X -wave magnets in 2D. Then, h -wave magnets are the 3D extension of g -wave altermagnets, while j -wave magnets are the 3D extension of i -wave altermagnets. They can be identified experimentally by measuring nonlinear spin currents by the application of an electric field. Only the fourth-order (sixth-order) spin currents flow in h -wave (j -wave) magnets, which function as spin-current rectifiers because the spin current exhibits unidirectional flow independent of the applied electric field.

A group-theoretical framework for identifying odd-parity magnets has recently been proposed [24]. Using this framework, h -wave spin splitting was theoretically predicted in FeSe through density functional theory [24]. In a similar way, j -wave magnets are expected to be discovered in the

near future. Both h -wave and j -wave odd-parity magnets in three dimensions have been theoretically proposed using the layer-construction method, where candidate materials include layered g -wave and i -wave altermagnets. It is known that p -wave magnets arise from a helical spin structure, in which a linear k_z term appears when the helix rotates along the z direction. By analogy, we expect that h -wave and j -wave odd-parity magnets can emerge from layered structures composed of g -wave and i -wave altermagnets, where the spins rotate along the z direction. Two-dimensional g -wave and i -wave altermagnets have been theoretically proposed in twisted magnetic van der Waals bilayers [31], and an i -wave altermagnet has also been proposed in $\text{MnP}(\text{S}, \text{Se})_3$ [32]. Therefore, stacking such systems may provide promising candidates for realizing h -wave and j -wave odd-parity magnets.

We estimate the magnitude of the electric field required to generate nonlinear spin currents [19]. The spin conductivity with the dimensionless electric field is E/E_{cr} , with the critical electric field $E_{\text{cr}} \equiv \frac{\hbar k}{e\tau} = 0.66 \text{ V}/\mu\text{m}$. Here we have used the relaxation time [33,34] $\tau = 3.4 \times 10^{-12} \text{ s}$ and the typical momentum [35] $k = 0.35 \text{ \AA}^{-1}$. A typical experimental electric

field [36,37] is $E = 1 \text{ V}/\mu\text{m}$, which satisfies the condition $E/E_{\text{cr}} > 1$. Therefore, higher-order nonlinear spin currents can be generated under realistic experimental conditions.

A strong electric field is required to induce magnetization, which may lead to Joule-heating issues. This problem can be mitigated by applying pulsed electric fields. In addition, cooling with a refrigerator is effective for suppressing Joule heating.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX: SPHERICAL HARMONIC FUNCTIONS

For the sake of the completeness, we classify all X -wave magnets in the following tables. They are classification tables for X -wave magnets in terms of the point group (PG), the magnetic point group (MPG) and the irreducible representation (IR). “NO” in the tables given by Eqs. (A8) and (A10) represents that the spherical harmonic function is not compatible with the lattice structure due to the fivefold and sevenfold rotational symmetries.

1. p wave

p waves	Spherical harmonic functions		PG(MPG)	IR
	Q_{10}^-	0		
	Q_{10}^+	k_z	$D_{2h}(mmm')$	B_{1u}
f_p^{2D}	Q_{11}^-	k_y	$D_{2h}(mm'm)$	B_{2u}
f_p^{2D}	Q_{11}^+	k_x	$D_{2h}(m'mm)$	B_{3u}

(A1)

2. d wave

d waves	Spherical harmonic functions		PG(MPG)	IR
	Q_{20}^-	0		
	Q_{20}^+	$3k_z^2 - k^2$	(mmm)	A_{1g}
	Q_{21}^-	$k_y k_z$	$D_{2h}(mm'm')$	E_g
	Q_{21}^+	$k_x k_z$	$D_{2h}(m'mm')$	E_g
f_d^{2D}	Q_{22}^-	$k_x k_y$	$D_{4h}(4'/m'm'm)$	B_{2g}
f_d^{2D}	Q_{22}^+	$k_x^2 - k_y^2$	$D_{4h}(4'mmm)$	B_{1g}

(A2)

There are relations

$$Q_{21}^+ \propto k_z Q_{11}^+, \quad Q_{21}^- \propto k_z Q_{11}^-.$$
(A3)

3. *f* wave

<i>f</i> waves	Spherical harmonic functions	PG(MPG)	IR
	0		
	Q_{30}^-		
	Q_{30}^+	$k_z(5k_z^2 - 3k^2)$	(mmm') A_{1g}
	Q_{31}^-	$k_y(k^2 - 5k_z^2)$	$D_{2h}(mm'm)$ B_{2u}
	Q_{31}^+	$k_x(k^2 - 5k_z^2)$	$D_{2h}(m'mm)$ B_{3u}
f_f^{3D}	Q_{32}^-	$2k_x k_y k_z$	$O_h(4'/m'm'm')$ A_{2u}
$f_{f'}^{3D}$	Q_{32}^+	$(k_x^2 - k_y^2)k_z$	$D_{4h}(4'/mmm')$ B_{1g}
$f_{f'}^{2D}$	Q_{33}^-	$k_y(3k_x^2 - k_y^2)$	$D_{6h}(6'/mm'm)$ E_{2u}
f_f^{2D}	Q_{33}^+	$k_x(k_x^2 - 3k_y^2)$	$D_{6h}(6'/m'mm)$ E_{2u}

(A4)

There are relations

$$Q_{32}^+ \propto k_z Q_{22}^+, \quad Q_{32}^- \propto k_z Q_{22}^-.$$

(A5)

4. *g* wave

<i>g</i> waves	Spherical harmonic functions	PG(MPG)	IR
	0		
	Q_{40}^-		
	Q_{40}^+	$35k_z^4 - 30k_z^2 k^2 + 3k^4$	(mmm) A_{1g}
	Q_{41}^-	$k_z k_y (7k_z^2 - 3k^2)$	$D_{2h}(mm'm)$ B_{2u}
	Q_{41}^+	$k_z k_x (7k_z^2 - 3k^2)$	$D_{2h}(m'mm)$ B_{3u}
	Q_{42}^-	$2k_x k_y (7k_z^2 - k^2)$	$D_{4h}(4'/m'm'm')$ B_{2g}
	Q_{42}^+	$(k_x^2 - k_y^2)(7k_z^2 - k^2)$	$D_{4h}(4'/mmm)$ B_{1g}
$f_{g'}^{3D}$	Q_{43}^-	$k_z k_y (3k_x^2 - k_y^2)$	$D_{6h}(6'/mm'm')$ E_{2g}
f_g^{3D}	Q_{43}^+	$k_z k_x (k_x^2 - 3k_y^2)$	$D_{6h}(6'/m'mm')$ E_{2g}
f_g^{2D}	Q_{44}^-	$k_x k_y (k_x^2 - k_y^2)$	$D_{4h}(4'/m'm'm)$ A_{2g}
$f_{g'}^{2D}$	Q_{44}^+	$k_x^4 - 6k_x^2 k_y^2 + k_y^4$	$D_{4h}(4'/mmm)$ A_{1g}

(A6)

There are relations

$$Q_{43}^+ \propto k_z Q_{33}^+, \quad Q_{43}^- \propto k_z Q_{33}^-.$$

(A7)

5. *h* wave

<i>h</i> waves	Spherical harmonic functions	PG(MPG)	IR
	0		
	Q_{50}^-		
	Q_{50}^+	$k_z(63k_z^4 - 70k_z^2 k^2 + 15k^4)$	(mmm') A_{1g}
	Q_{51}^-	$k_y(21k_z^4 - 14k_z^2 k^2 + k^4)$	$D_{2h}(mm'm)$ B_{2u}
	Q_{51}^+	$k_x(21k_z^4 - 14k_z^2 k^2 + k^4)$	$D_{2h}(m'mm)$ B_{3u}
	Q_{52}^-	$2k_z k_x k_y (k_z^2 - k^2)$	$D_{4h}(4'/m'm'm')$ B_{2g}
	Q_{52}^+	$k_z(k_x^2 - k_y^2)(k_z^2 - k^2)$	$D_{4h}(4'/mmm')$ B_{1g}
	Q_{53}^-	$k_y(3k_x^2 - k_y^2)(9k_z^2 - k^2)$	$D_{6h}(6'/mm'm)$ E_{2u}
	Q_{53}^+	$k_x(k_x^2 - 3k_y^2)(9k_z^2 - k^2)$	$D_{6h}(6'/m'mm)$ E_{2u}
f_h^{3D}	Q_{54}^-	$k_z k_x k_y (k_x^2 - k_y^2)$	$D_{4h}(4'/m'm'm')$ A_{1u}
$f_{h'}^{3D}$	Q_{54}^+	$k_z(k_x^4 - 6k_x^2 k_y^2 + k_y^4)$	$D_{4h}(4'/mmm')$ A_{2u}
NO	Q_{55}^-	$5k_x^4 k_y - 10k_x^2 k_y^3 + k_y^5$	NO
NO	Q_{55}^+	$k_x^5 - 10k_x^3 k_y^2 + 5k_x k_y^4$	NO

(A8)

There are relations

$$Q_{54}^+ \propto k_z Q_{44}^+, \quad Q_{54}^- \propto k_z Q_{44}^-.$$

(A9)

6. *i* wave

i waves	Spherical harmonic functions	PG(MPG)	IR
	Q_{60}^-	0	
	Q_{60}^+	$35k_z^6 - 105k_z^4k^2 + 63k_z^2k^4 - 5k^6$	(mmm) A_{1g}
	Q_{61}^-	$k_zk_y(33k_z^4 - 30k_z^2k^2 + 5k^4)$	$D_{2h}(mm'm)$ B_{2u}
	Q_{61}^+	$k_zk_x(33k_z^4 - 30k_z^2k^2 + 5k^4)$	$D_{2h}(m'mm)$ B_{3u}
	Q_{62}^-	$2k_xk_y(33k_z^4 - 18k_z^2k^2 + k^4)$	$D_{4h}(4'/m'm'm')$ B_{2g}
	Q_{62}^+	$(k_x^2 - k_y^2)(33k_z^4 - 18k_z^2k^2 + k^4)$	$D_{4h}(4'/mmm')$ B_{1g}
	Q_{63}^-	$k_zk_y(3k_x^2 - k_y^2)(11k_z^2 - 3k^2)$	$D_{6h}(6'/mm'm')$ E_{2u}
	Q_{63}^+	$k_zk_x(k_x^2 - 3k_y^2)(11k_z^2 - 3k^2)$	$D_{6h}(6'/m'mm')$ E_{2u}
	Q_{64}^-	$4k_xk_y(k_x^2 - k_y^2)(11k_z^2 - k^2)$	$D_{4h}(4/m'm'm)$ A_{2g}
	Q_{64}^+	$(k_x^4 - 6k_x^2k_y^2 + k_y^4)(11k_z^2 - k^2)$	$D_{4h}(4/mmm)$ A_{1g}
NO	Q_{65}^-	$k_z(5k_x^4k_y - 10k_x^2k_y^3 + k_y^5)$	NO E_g
NO	Q_{65}^+	$k_z(k_x^5 - 10k_x^3k_y^2 + 5k_xk_y^4)$	NO E_g
f_i^{2D}	Q_{66}^-	$2k_xk_y(3k_x^2 - k_y^2)(k_x^2 - 3k_y^2)$	$D_{6h}(6/m'm'm)$ A_{2g}
$f_{i'}^{2D}$	Q_{66}^+	$(k_x^2 - k_y^2)(k_x^4 + k_y^4 - 14k_x^2k_y^2)$	$D_{6h}(6/mmm)$ A_{1g}

(A10)

(A10)

There are relations

$$Q_{65}^+ \propto k_z Q_{55}^+, \quad Q_{65}^- \propto k_z Q_{55}^-.$$

(A11)

A comment is in order with respect to $f_i^{3D}(\mathbf{k})$ in Eq. (17), which is not included in the Eq. (A10). It is explicitly given by

$$f_i^{3D}(\mathbf{k}) \propto Q_{66}^+ - Q_{62}^+.$$

(A12)

7. *j* wave

j waves	Spherical harmonic functions	PG(MPG)	IR	
	Q_{70}^-	0		
	Q_{70}^+	$k_z(429k_z^6 - 693k_z^4k^2 + 315k_z^2k^4 - 35k^6)$	(mmm')	A_{1g}
	Q_{71}^-	$k_y(429k_z^6 - 495k_z^4k^2 + 135k_z^2k^4 - 5k^6)$	$D_{2h}(mm'm)$	B_{2u}
	Q_{71}^+	$k_x(429k_z^6 - 495k_z^4k^2 + 135k_z^2k^4 - 5k^6)$	$D_{2h}(m'mm)$	B_{3u}
	Q_{72}^-	$k_zk_xk_y(143k_z^4 - 110k_z^2k^2 + 15k^4)$	$D_{4h}(4'm'm'm')$	B_{2g}
	Q_{72}^+	$k_z(k_x^2 - k_y^2)(143k_z^4 - 110k_z^2k^2 + 15k^4)$	$D_{4h}(4'/mmm')$	B_{1g}
	Q_{73}^-	$k_y(3k_x^2 - k_y^2)(143k_z^4 - 66k_z^2k^2 + 3k^4)$	$D_{6h}(6'mm'm)$	E_{2u}
	Q_{73}^+	$k_x(k_x^2 - 3k_y^2)(143k_z^4 - 66k_z^2k^2 + 3k^4)$	$D_{6h}(6'mm'm)$	E_{2u}
	Q_{74}^-	$k_zk_xk_y(k_x^2 - k_y^2)(13k_z^2 - 3k^2)$	$D_{4h}(4/m'm'm')$	A_{2g}
	Q_{74}^+	$k_z(k_x^4 - 6k_x^2k_y^2 + k_y^4)(13k_z^2 - 3k^2)$	$D_{4h}(4/mmm')$	A_{1g}
NO	Q_{75}^-	$(5k_x^4k_y - 10k_x^2k_y^3 + k_y^5)(13k_z^2 - k^2)$	NO	E_{1u}
NO	Q_{75}^+	$(k_x^5 - 10k_x^3k_y^2 + 5k_xk_y^4)(13k_z^2 - k^2)$	NO	E_{1u}
f_j^{3D}	Q_{76}^-	$2k_zk_xk_y(3k_x^2 - k_y^2)(k_x^2 - 3k_y^2)$	$D_{6h}(6/m'm'm')$	A_{2u}
$f_{j'}^{3D}$	Q_{76}^+	$k_z(k_x^2 - k_y^2)(k_x^4 + k_y^4 - 14k_x^2k_y^2)$	$D_{6h}(6/mmm')$	A_{1u}
NO	Q_{77}^-	$7k_x^6k_y - 35k_x^4k_y^3 + 21k_x^2k_y^5 - 7k_xk_y^6$	NO	E_{1u}
NO	Q_{77}^+	$k_x^7 - 21k_x^5k_y^2 + 35k_x^3k_y^4 - 7k_xk_y^6$	NO	E_{1u}

(A13)

(A13)

There are relations

$$Q_{76}^+ \propto k_z Q_{66}^+, \quad Q_{76}^- \propto k_z Q_{66}^-.$$

(A14)

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