

Almost half-quantized planar Hall effects in X -wave magnets with $X = p, d, f, g, i$

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The planar Hall effect is a phenomenon that the Hall conductivity emerges perpendicular to the electric field in the presence of an in-plane magnetic field. We investigate the planar Hall effect in two-dimensional metal coupled with higher symmetric X -wave magnets with $X = p, d, f, g, i$, where those with $X = d, g, i$ are altermagnets. The X -wave magnet is characterized by the number N_X of the nodes in the band structure, where $N_X = 1, 2, 3, 4, 6$ corresponds to $X = p, d, f, g, i$. Although the system is metallic, provided the Dirac gap is tiny, we demonstrate that the Hall conductivities are almost half quantized and well approximated by the formula $\sigma_{xy} = \pm(e^2/2h) \text{sgn}(J \sin N_X \Phi)$, where J is the coefficient of the coupling between the X -wave magnet and the electrons and Φ is the direction of the applied magnetic field. Hence, the Hall conductivity is periodic in Φ , and the periodicity is equal to the number N_X of the nodes. This property may be used to confirm that the target material is indeed an X -wave magnet. Furthermore, the sign of J may be used as a bit for antiferromagnetic spintronics.

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I. INTRODUCTION

The Hall effect is a prominent phenomenon in two-dimensional materials. A current flows into a direction perpendicular to the applied electric field when the magnetic field is applied perpendicular to the plane, as illustrated in Fig. 1(a). Similarly, a current flows into a direction perpendicular to the applied electric field when the magnetic field is applied parallel to the plane, as illustrated in Fig. 1(b). This is the planar Hall effect [1–7]. We are interested in the planar Hall effect in altermagnets and related materials.

An altermagnet is expected to be a promising candidate for antiferromagnetic spintronics [8,9], where the direction of the Néel vector is used as a bit. Antiferromagnets are thought to be ideal for future spintronics memory with ultrafast and ultradense application due to the lack of the stray field in contrast to ferromagnets [10–16]. However, it is hard to detect the direction of the Néel vector because net magnetization is zero in antiferromagnets. Nevertheless, this is possible by measuring anomalous Hall conductivity in altermagnets because the altermagnets break time-reversal symmetry [17–20]. Indeed, it is theoretically predicted [17,19,21,22] and, furthermore, experimentally observed [18,20,23–25]. Very recently, the planar Hall effect in d -wave altermagnets has been discussed in Refs. [26,27], where it was found to be almost half quantized [26].

In general, it is interesting to study the higher symmetric X -wave magnets [8,9,28,29] with $X = p, d, f, g, i$. They are characterized by the specific band structure, where the

number of the nodes is $N_X = 1, 2, 3, 4, 6$, respectively. The X -wave magnets with $X = d, g, i$ are altermagnets. The effective Hamiltonian [29] for the X -wave magnet is concisely described by a two-band model containing the term $J\sigma_z f_X(\mathbf{k})$, where J is the coefficient of the coupling between the X -wave magnet and the electrons, σ_z is the Pauli matrix representing the spin of electrons, and $f_X(\mathbf{k})$ is a symmetric function characterizing the X -wave magnet.

In this paper, we show that Hall conductivities emerge in two-dimensional metallic systems coupled with various X -wave magnets provided the magnetic field is applied along the in-plane direction. It is induced by the momentum shift of the Dirac cone accompanied by the gap opening. The main result is that the Hall conductivity is well approximated by the half-quantization formula

$$\sigma_{xy} = \frac{e^2}{h} \frac{(-1)^{s_X}}{2} \text{sgn}\left(J \frac{B^{N_X}}{\lambda^{N_X}} \sin N_X \Phi\right), \quad (1)$$

where $s_X = 1, 1, -1, -1, 1$ for $X = p, d, f, g, i$, respectively. Here $B > 0$ is the applied magnetic field, Φ is the direction of the applied magnetic field, and λ is the coefficient of the spin-orbit interaction. It is curious that the Hall conductivity is almost half quantized although the system is metallic. This occurs when the Berry curvature is almost localized in the vicinity of the Dirac point with a tiny Dirac gap. The Hall conductivity is proportional to the sign of J . Hence, it is possible to use the sign of J as a bit, while it is possible to detect the sign of J by observing the planar Hall effect. In addition, the number of the nodes, N_X , is detectable by measuring the periodicity of the Hall conductivity as a function of Φ .

II. HIGHER SYMMETRIC X -WAVE MAGNETS

We analyze a metallic system where an X -wave magnet is placed on a substrate as in Fig. 1. The minimal two-band

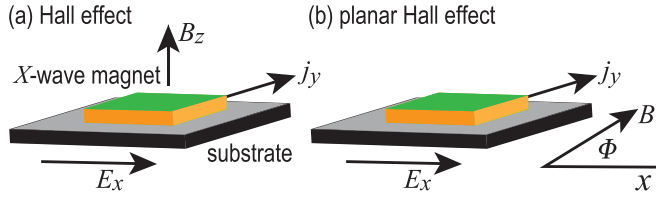


FIG. 1. Illustration for (a) the Hall effect and (b) the planar Hall effect. In the planar Hall effect, when the electric field is applied along the x axis and the magnetic field $(B \cos \Phi, B \sin \Phi, 0)$ is applied parallel to the system, the Hall current flows along the y axis. The Hall conductivity is predicted to be given by the formula (1).

Hamiltonian is given by

$$H(\mathbf{k}) = \frac{\hbar^2 \mathbf{k}^2}{2m} + \lambda(\mathbf{k} \times \boldsymbol{\sigma})_z + J f_X(\mathbf{k}) \sigma_z + \mathbf{B} \cdot \boldsymbol{\sigma} \quad (2)$$

in two dimensions with the use of the Pauli matrix $\boldsymbol{\sigma}$. The first term represents the kinetic energy, making the system metallic, the second term the Rashba interaction introduced by making an interface between the X-wave magnet and the substrate, the third term the X-wave term with the X-wave function $f_X(\mathbf{k})$, and the last term the applied magnetic field $\mathbf{B} = (B_x, B_y, 0)$. The Rashba interaction forms a Dirac cone at the Γ point. The Rashba interaction is introduced by placing an altermagnet on the substrate [8,9,30–35].

The X-wave function $f_X(\mathbf{k})$ reads as follows [8,9,28,29,36]:

$$f_s = 1, \quad (3)$$

$$f_p = k_x = k \cos \phi, \quad (4)$$

$$f_d = 2k_x k_y = k^2 \sin 2\phi, \quad (5)$$

$$f_f = k_x(k_x^2 - 3k_y^2) = k^3 \cos 3\phi, \quad (6)$$

$$f_g = 4k_x k_y (k_x^2 - k_y^2) = k^4 \sin 4\phi, \quad (7)$$

$$f_i = 2k_x k_y (3k_x^2 - k_y^2)(k_x^2 - 3k_y^2) = k^6 \sin 6\phi, \quad (8)$$

where $k_x = k \cos \phi$ and $k_y = k \sin \phi$.

III. HALL CONDUCTIVITY

We review the Hall conductivity in the two-band Hamiltonian in the form of

$$H = h_0 + \sum_{j=x,y,z} h_j \sigma_j. \quad (9)$$

We define a normalized vector:

$$\mathbf{n}(\mathbf{k}) \equiv \mathbf{h}/|\mathbf{h}| = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta). \quad (10)$$

The Berry curvature is expressed in terms of the solid angle of the vector $\mathbf{n}$ as [37–39]

$$\Omega_{\pm}(\mathbf{k}) = \pm \frac{1}{2} \mathbf{n} \cdot (\partial_{k_x} \mathbf{n} \times \partial_{k_y} \mathbf{n}), \quad (11)$$

for the upper band (+) and the lower band (−). The integration of the Berry curvature over the whole Brillouin zone gives the

Chern number $C_{\pm}$ as

$$\frac{1}{2\pi} \int_{\text{BZ}} \Omega_{\pm}(\mathbf{k}) d\mathbf{k} = C_{\pm}. \quad (12)$$

It is interpreted as the wrapping number of the vector $\mathbf{n}$ known as the Pontryagin number.

According to the Thouless–Kohmoto–Nightingale–den Nijs formula, the Hall conductivity is related to the Berry curvature as

$$\sigma_{xy} = \frac{e^2}{2\pi h} \sum_{n=\pm} \int d^2 k f(E_n(\mathbf{k})) \Omega_n(\mathbf{k}), \quad (13)$$

where $f(E_{\pm}(\mathbf{k})) = 1/[\exp \frac{E_{\pm}(\mathbf{k}) - \mu}{k_B T} + 1]$ is the Fermi distribution function and μ is the chemical potential. In an insulator at zero temperature, the Hall conductivity is proportional to the Chern number C_- of the bottom band,

$$\sigma_{xy} = \frac{e^2}{h} C_-. \quad (14)$$

In the Dirac system, the Chern number is half quantized if the chemical potential is within the Dirac gap. Hence, the Hall conductivity per Dirac cone is given by

$$\sigma_{xy} = \pm \frac{e^2}{2h}. \quad (15)$$

This quantization formula is exact only for insulators.

IV. PLANAR HALL EFFECTS

The above argument is not applicable to a metallic system in general. However, it is possible to define the Berry curvature in the two-band Hamiltonian (2) provided two bands are gapped and nondegenerate.

The Dirac point shifts in the presence of the in-plane magnetic field. The shifted Dirac point is given by $(k'_x, k'_y) = 0$, where

$$k'_x = k_x + \frac{B_y}{\lambda}, \quad k'_y = k_y - \frac{B_x}{\lambda}, \quad (16)$$

by solving the equation

$$\lambda(k_x \sigma_y - k_y \sigma_x) + B_x \sigma_x + B_y \sigma_y = 0. \quad (17)$$

In terms of the shifted momentum, the Hamiltonian (2) is given by

$$H = H'_{\text{kine}} + H'_R + H'_{\text{mass}}, \quad (18)$$

up to the first order of k'_x and k'_y , where H'_{kine} is the kinetic energy at the shifted Dirac point,

$$H'_{\text{kine}} = \frac{\hbar^2}{2m} \left[\left(\frac{B}{\lambda} \right)^2 - \frac{2B_y}{\lambda} k'_x + \frac{2B_x}{\lambda} k'_y \right], \quad (19)$$

H'_R is the Rashba interaction term,

$$H'_R = \lambda(k'_x \sigma_y - k'_y \sigma_x), \quad (20)$$

and H'_{mass} is the Dirac mass term,

$$H'_{\text{mass}} = J(\Delta + g_x k'_x + g_y k'_y) \sigma_z, \quad (21)$$

with the Dirac gap

$$\Delta \equiv f_X(k'_x = 0, k'_y = 0) \quad (22)$$

and

$$g_x \equiv \frac{\partial f_X(k'_x, k'_y)}{\partial k'_x}, \quad g_y \equiv \frac{\partial f_X(k'_x, k'_y)}{\partial k'_y}. \quad (23)$$

The Dirac gap is explicitly determined as

$$\Delta = (-1)^{s_x} \frac{B^{N_x}}{\lambda^{N_x}} \sin N_x \Phi. \quad (24)$$

Then, the Hamiltonian is rewritten as

$$H = \frac{\hbar^2}{2m} \left[\left(\frac{B}{\lambda} \right)^2 - \frac{2B_y}{\lambda} k'_x + \frac{2B_x}{\lambda} k'_y \right] + \lambda(k'_x \sigma_y - k'_y \sigma_x) + J(\Delta + g_x k'_x + g_y k'_y) \sigma_z. \quad (25)$$

The Berry curvature (11) is calculated in the (k'_x, k'_y) space as

$$\Omega_{\pm} = \frac{\mp J \Delta \lambda^2}{2[\lambda^2 k'^2 + (J \Delta + J g_x k'_x + J g_y k'_y)^2]^{3/2}}. \quad (26)$$

We consider a simple case in which the Fermi surface is a circle with the radius k_c . This is realized for small J , where k_c is calculated as

$$k_c = \sqrt{2m} \sqrt{\mu + m\lambda^2 + \sqrt{\Delta^2 + 2\mu m\lambda^2 + m^2\lambda^4}}. \quad (27)$$

We assume that the chemical potential is within the gap. We integrate the Berry curvature (26) with a cutoff k_c for the bottom band as

$$\int_0^{k_c} k dk \int d\phi \Omega_- = -\frac{1}{2} \text{sgn} J \Delta + \int d\phi W(k_c, \phi), \quad (28)$$

where

$$W(k_c, \phi) \equiv -\frac{J \Delta + k_c g_{\parallel}(\phi)}{2\sqrt{\lambda^2 k_c^2 + [J \Delta + k_c g_{\parallel}(\phi)]^2}}, \quad (29)$$

with

$$g_{\parallel}(\phi) \equiv g_x \cos \phi + g_y \sin \phi. \quad (30)$$

The Hall conductivity is given by

$$\sigma_{xy} = -\frac{e^2}{h} \frac{1}{2} \left(1 - \frac{|J \Delta|}{\sqrt{(J \Delta)^2 + \lambda^2 k_c^2}} \right) \text{sgn} J \Delta. \quad (31)$$

It follows from Eq. (31) that

$$|\sigma_{xy}| < \frac{e^2}{h} \frac{1}{2}. \quad (32)$$

The Hall conductivity is not half quantized for a metallic system. However, for $J \Delta / \lambda k_c \ll 1$, we obtain the half quantization

$$\sigma_{xy} = -\frac{e^2}{h} \frac{1}{2} \text{sgn} J \Delta, \quad (33)$$

which is equivalent to the formula (1) with the use of Eq. (24).

This phenomenon is understood as follows. The integral of the Berry curvature over the whole Brillouin zone gives an exactly half-quantized value if the chemical potential is within the gap for the insulating system. However, the present

system is metallic due to the presence of the kinetic term. Then, the integration is done only within the Fermi surface. It gives a non-half-quantized Hall conductivity, which is smaller than the half-quantized value as in Eq. (31). The accuracy of the half quantization depends on how the Berry curvature localizes. It follows from Eq. (26) that

$$\lim_{\Delta \rightarrow 0} \Omega \propto \delta(k'). \quad (34)$$

Namely, if the Dirac gap Δ is tiny, the Berry curvature is sharply localized to the shifted Dirac point $(k'_x, k'_y) = 0$, and the Hall conductivity is almost half quantized as in Eq. (33). On the other hand, if the Dirac gap is large, the Berry curvature widely spreads over the Brillouin zone, and the Hall conductivity is not half quantized as in Eq. (31).

We have derived the formula (31) perturbatively. In the following section, we check numerically the half-quantization formula, Eq. (33), with Eq. (24) without using perturbation based on the tight-binding model.

V. TIGHT-BINDING MODEL

The tight-binding representation is constructed by replacing $k_j \mapsto \sin ak_j$ and $k_j^2 \mapsto 2(1 - \cos ak_j)$ in the Hamiltonian (2), where $j = x, y$, and a is the lattice constant. The Hamiltonian is defined on the square lattice for the p -wave magnet, the d -wave magnet, and the g -wave magnet, while it is defined on the triangular lattice for the f -wave magnet and the i -wave magnet.

The tight-binding Hamiltonian corresponding to Eq. (2) is given by

$$H = H_{\text{Kine}} + H_R + H_X + \mathbf{B} \cdot \boldsymbol{\sigma}. \quad (35)$$

The kinetic term and the Rashba term are

$$H_{\text{Kine}, \text{Sq}} = \frac{\hbar^2}{ma^2} (2 - \cos ak_x - \cos ak_y), \quad (36)$$

$$H_{R, \text{Sq}} = \lambda (\sigma_y \sin k_x - \sigma_x \sin k_y) \quad (37)$$

on the square lattice. We use them for the p -wave magnet, the d -wave altermagnet, and the g -wave altermagnet.

On the other hand, they are

$$H_{\text{Kine}, \text{Tri}} = \frac{\hbar^2}{ma^2} \left(3 - \cos ak_x - \sum_{\pm} \cos a \frac{k_x \pm \sqrt{3} k_y}{2} \right), \quad (38)$$

$$H_{R, \text{Tri}} = \lambda \sigma_y \frac{2}{3} \left(\sin ak_x + \sin \frac{ak_x}{2} \cos \frac{\sqrt{3} ak_y}{2} \right) - \lambda \frac{2 \cos \frac{ak_x}{2} \sin \frac{\sqrt{3} ak_y}{2}}{\sqrt{3}} \sigma_x \quad (39)$$

on the triangular lattice. We use them for the f -wave magnet and the i -wave altermagnet.

The X -wave terms H_X are given by

$$H_p = J \sigma_z \sin ak_x \quad (40)$$

for the p -wave magnet [28,40–42],

$$H_d = J \sigma_z \sin ak_x \sin ak_y \quad (41)$$

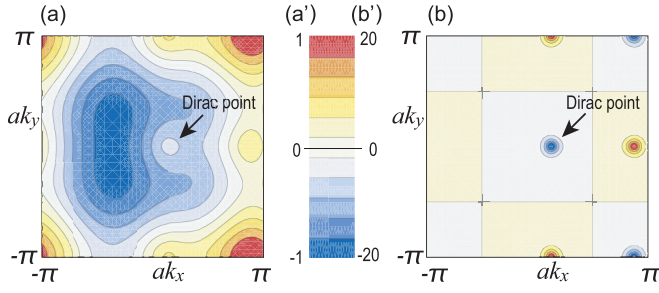


FIG. 2. *p*-wave magnet. (a) Contour plot of the energy of the bottom band. (b) Berry curvature for the bottom band. (a'), (b') Color palette for panels (a) and (b). We have chosen $\Phi = \pi/2$. We have set $Jak_0 = \varepsilon_0/2$, $\lambda k_0 = \varepsilon_0$, and $\hbar^2 k_0^2/(2m) = \varepsilon_0/2$.

for the *d*-wave magnet [8,9,30–35,43],

$$H_f = 4J\sigma_z \sin ak_x \sin \frac{ak_x + \sqrt{3}ak_y}{2} \sin \frac{-ak_x + \sqrt{3}ak_y}{2} \quad (42)$$

for the *f*-wave magnet [29],

$$H_g = 2J\sigma_z \sin ak_x \sin ak_y (\cos ak_y - \cos ak_z), \quad (43)$$

for the *g*-wave magnet [29], and

$$H_i = -\frac{16}{3\sqrt{3}}J\sigma_z \sin ak_x \sin \frac{ak_x + \sqrt{3}ak_y}{2} \sin \frac{-ak_x + \sqrt{3}ak_y}{2} \\ \times \sin \sqrt{3}ak_y \sin \frac{3ak_x + \sqrt{3}ak_y}{2} \sin \frac{-3ak_x + \sqrt{3}ak_y}{2} \quad (44)$$

for the *i*-wave magnet [29].

A. *p*-wave magnets

We show the contour plot of the energy of the bottom band in the Brillouin zone in Fig. 2(a), where the positive-energy and negative-energy regions are made clear with the color palette given in Fig. 2(a'). The negative energy part contains one shifted Dirac point as indicated by an arrow.

The Berry curvature of each band is analytically calculated based on the tight-binding model as $\Omega_{\pm} = \mp\Omega$, with

$$\Omega = \frac{B_y J \lambda \cos k_x \cos k_y}{2[J^2 \sin^2 k_x + (B_y + \lambda \sin k_x)^2 + (B_x - \lambda \sin k_y)^2]^{3/2}}. \quad (45)$$

We show the Berry curvature Ω_- of the bottom band in Fig. 2(b), where it is found to be sharply localized in the vicinity of the shifted Dirac point.

We calculate numerically the Hall conductivity (13) by integrating the Berry curvatures $\Omega_{\pm}$ below the chemical potential. The result is shown as a function of the chemical potential μ in Fig. 3(a), where we have set $\Phi = \pi/2$. There is a plateau where the Hall conductivity is almost half quantized, when the chemical potential is within the Dirac gap, as is found in the energy spectrum given in Fig. 3(b).

We show the Hall conductivity as a function of the angle Φ in Fig. 3(c), where we have set the chemical potential to the

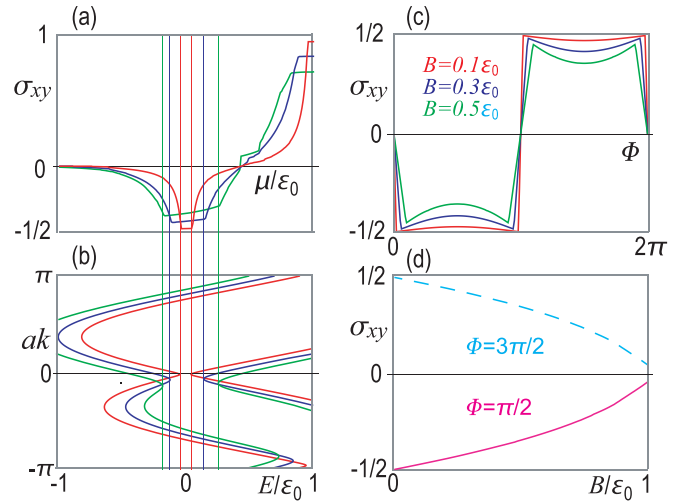


FIG. 3. *p*-wave magnet. (a) Hall conductivity in units of e^2/h as a function of chemical potential μ . (b) Energy spectra for $-\pi < ak_x < \pi$ and $k_y = 0$. (c) Hall conductivity as a function of angle Φ . In these figures, red curves indicate $B = 0.1\varepsilon_0$, blue curves indicate $B = 0.3\varepsilon_0$, and green curves indicate $B = 0.5\varepsilon_0$. (d) Hall conductivity as a function of magnetic field B . A magenta curve indicates $\Phi = \pi/2$ and a cyan curve indicates $\Phi = 3\pi/2$. We have chosen $\Phi = \pi/2$ in panels (a), (b), and (d). See also the caption of Fig. 2.

Dirac energy, $\mu = E_{\text{Dirac}}$, given by

$$E_{\text{Dirac}} \equiv \frac{\hbar^2 B^2}{2m\lambda^2}. \quad (46)$$

The Hall conductivity is almost half quantized, $\sigma_{xy} = -\frac{e^2}{h} \frac{1}{2}$ for $0 < \Phi < \pi$ and $\sigma_{xy} = \frac{e^2}{h} \frac{1}{2}$ for $\pi < \Phi < 2\pi$. It well reproduces the angle dependence of the Hall conductivity (1).

We also show the Hall conductivity as a function of the magnetic field B in Fig. 3(d), where a magenta curve describes it for $\Phi = \pi/2$, and a dashed cyan curve describes it for $\Phi = 3\pi/2$. It is around $\pm 1/2$ for small B .

B. *d*-wave altermagnets

The band structure is shown in Fig. 4(a). The Berry curvature is sharply localized as in Fig. 4(b). The Hall conductivity is numerically calculated as a function of the chemical potential μ in Fig. 5(a), while the energy spectrum is shown in

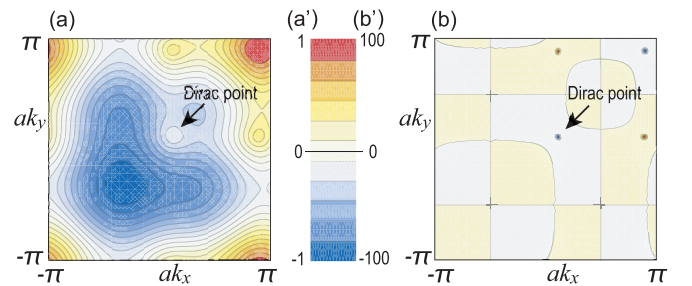


FIG. 4. *d*-wave altermagnet. (a) Contour plot of the energy of the bottom band. (b) Berry curvature for the bottom band. (a'), (b') Color palette for panels (a) and (b). We have chosen $\Phi = \pi/4$. We have set $2Ja^2k_0^2 = \varepsilon_0/2$, $\lambda k_0 = \varepsilon_0$, and $\hbar^2 k_0^2/(2m) = \varepsilon_0/2$.

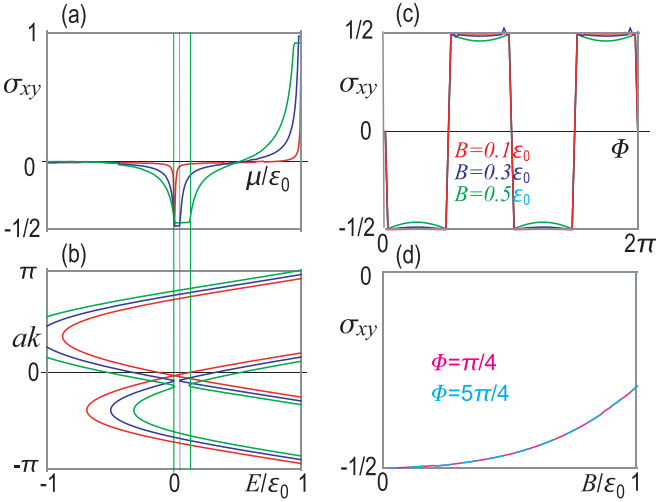


FIG. 5. *d*-wave altermagnet. (a) Hall conductivity in units of e^2/h as a function of chemical potential μ . (b) Energy spectrum for $(k_x, k_y) = k(\cos \frac{\pi}{4}, \sin \frac{\pi}{4})$, with $-\pi < ak < \pi$. (c) Hall conductivity as a function of angle Φ . (d) Hall conductivity as a function of magnetic field B . A magenta curve indicates $\Phi = \pi/4$ and a cyan curve indicates $\Phi = 5\pi/4$. We have chosen $\phi = \pi/4$ in panels (a), (b), and (d). We have set $2Ja^2k_0^2 = \varepsilon_0/2$. See also the caption of Fig. 4.

Fig. 5(b). There is a plateau where the Hall conductivity is almost half quantized.

We show the Hall conductivity as a function of the angle Φ in Fig. 5(c). It is almost half quantized, $\sigma_{xy} = -\frac{e^2}{h} \frac{1}{2}$ for $0 < \Phi < \pi/2$ and $\pi < \Phi < 3\pi/2$. On the other hand, $\sigma_{xy} = \frac{e^2}{h} \frac{1}{2}$ for $\pi/2 < \Phi < \pi$ and $3\pi/2 < \Phi < 2\pi$. It well reproduces the angle dependence in the formula (1).

We also show the Hall conductivity as a function of the magnetic field B in Fig. 5(d), where a magenta curve describes it for $\Phi = \pi/4$, and a dashed cyan curve describes it for $\Phi = 5\pi/4$. It is around $\pm 1/2$ for small B .

C. *f*-wave magnets, *g*-wave altermagnets, and *i*-wave magnets

The Hall conductivity is numerically calculated and shown in Fig. 6 for the *f*-wave magnet, in Fig. 7 for the *g*-wave altermagnet, and in Fig. 8 for the *i*-altermagnet, respectively. They well reproduce the formula (1).

For the *g*-wave altermagnets, the Hall conductivity is zero for $B = 0$. It is around $\pm 1/2$ for small B .

For the *i*-wave altermagnets, the Hall conductivity is around zero for small B . It contradicts to the half quantization in Eq. (1). It would be an artifact in numerical calculations because the Dirac gap is proportional to B^6 . It is too small for small B , where the Dirac energy (46) is away from the gap in the tight-binding model.

VI. FINITE-TEMPERATURE EFFECT

We show the Hall conductivity as a function of temperature in Fig. 9 for the *p*-wave magnet in panel (a) and for the *d*-wave altermagnet in panel (b). Its magnitude decreases with the increase of temperature and leaves the half-quantized value.

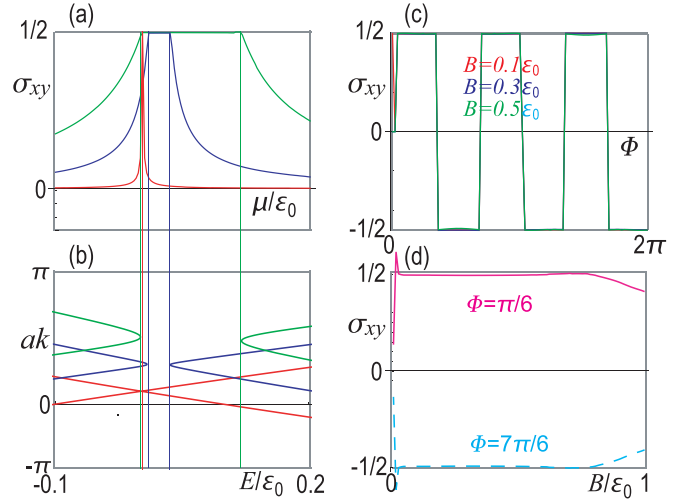


FIG. 6. *f*-wave magnet. (a) Hall conductivity in units of e^2/h as a function of chemical potential μ . (b) Energy spectrum for $(k_x, k_y) = k(\cos \frac{\pi}{6}, \sin \frac{\pi}{6})$, with $-\pi < ak < \pi$. (c) Hall conductivity as a function of angle Φ . (d) Hall conductivity as a function of magnetic field B . A magenta curve indicates $\Phi = \pi/6$ and a cyan curve indicates $\Phi = 7\pi/6$. We have chosen $\Phi = \pi/6$. We have set $Ja^3k_0^3 = \varepsilon_0/2$, $\lambda k_0 = \varepsilon_0$, and $\hbar^2 k_0^2/(2m) = \varepsilon_0/2$.

VII. DISORDER EFFECT

The Chern number is rewritten as [44–48]

$$\mathcal{C} = (2\pi)^{-2} \int d^2k \int_{-\infty}^{\infty} d\omega \Omega, \quad (47)$$

where the Berry curvature is rewritten as

$$\Omega = \frac{1}{6} \varepsilon_{\mu\nu\rho} \text{Tr}[G\partial_\mu G^{-1} G\partial_\nu G^{-1} G\partial_\rho G^{-1}] \quad (48)$$

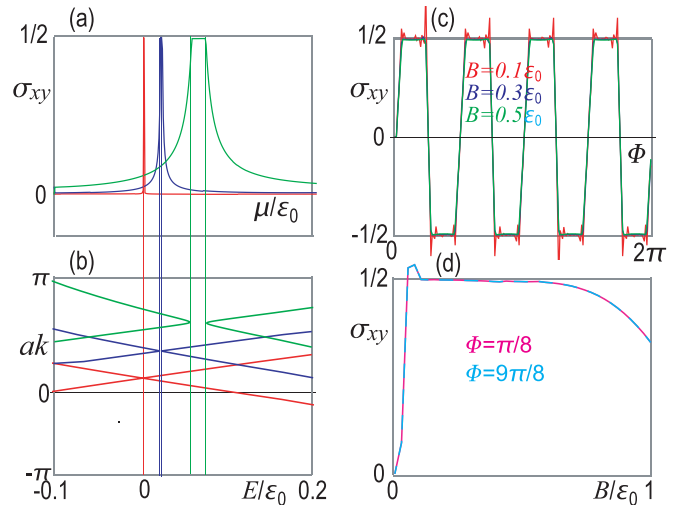


FIG. 7. *g*-wave altermagnet. (a) Hall conductivity in units of e^2/h as a function of chemical potential μ . We have chosen $\Phi = \pi/8$. (b) Energy spectrum for $(k_x, k_y) = k(\cos \frac{\pi}{8}, \sin \frac{\pi}{8})$, with $-\pi < ak < \pi$. (c) Hall conductivity as a function of angle Φ . (d) Magnetic field B dependence of the Hall conductivity. A magenta curve indicates $\Phi = \pi/8$ and a cyan curve indicates $\Phi = 9\pi/8$. We have set $4Ja^4k_0^4 = \varepsilon_0/2$, $\lambda k_0 = \varepsilon_0$, and $\hbar^2 k_0^2/(2m) = \varepsilon_0/2$.

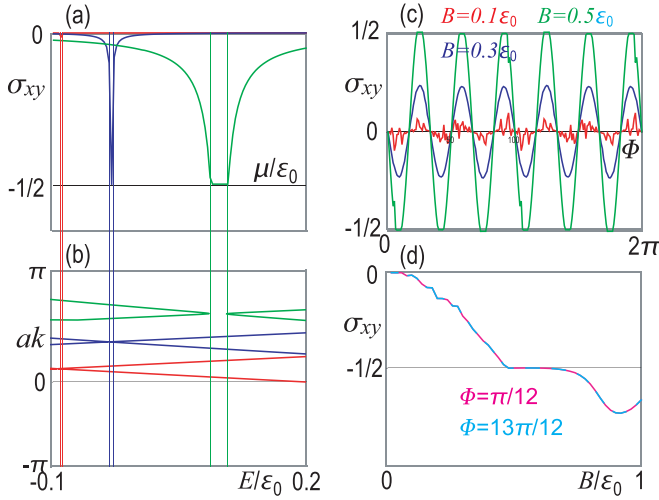


FIG. 8. *i*-wave altermagnet. (a) Hall conductivity in units of e^2/h as a function of chemical potential μ . (b) Energy spectrum for $(k_x, k_y) = k(\cos \frac{\pi}{12}, \sin \frac{\pi}{12})$, with $-\pi < ak < \pi$. (c) Hall conductivity dependence on angle Φ . (d) Hall conductivity as a function of magnetic field B . A magenta curve indicates $\Phi = \pi/12$ and a cyan curve indicates $\Phi = 13\pi/12$. We have chosen $\Phi = \pi/12$ for panels (a), (b), and (d). We have set $2Ja^2k_0^2 = \epsilon_0/2$, $\lambda k_0 = \epsilon_0$, and $\hbar^2 k_0^2/(2m) = \epsilon_0/2$.

in terms of the Green function

$$G(\mathbf{k}) = [i\omega - H(\mathbf{k}) - i\Gamma]^{-1}, \quad (49)$$

where k_μ , k_ν , and k_ρ run through $k_0 \equiv i\omega$, k_x , and k_y , with $i\omega$ referring to the Matsubara frequency (ω : real), and Γ is the self-energy representing the effect of disorder or interactions. Γ is renormalized by shifting the frequency as $\omega' = \omega + \Gamma$. Then, the Berry curvature is irrelevant to Γ because the integral of ω is taken as $-\infty < \omega < \infty$.

VIII. DISCUSSIONS

Studying planar Hall effects in the *X*-wave magnets, we have revealed a universal feature common to all of them. When the Dirac gap is tiny, the Hall conductivity induced by the in-plane magnetic field is almost half quantized as in Eq. (1). This formula implies that the magnitude of the Hall conductivity is proportional to $\text{sgn}(J \sin N_X \Phi)$. It is

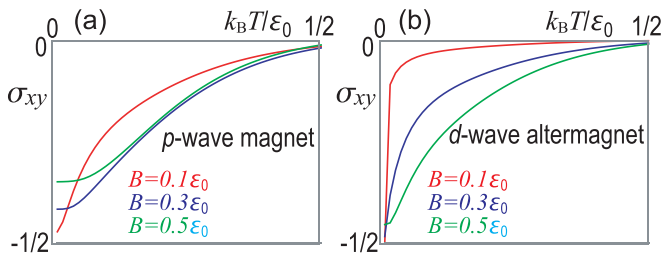


FIG. 9. Finite-temperature effect of the Hall conductivity for various values of B . The Hall conductivity in units of e^2/h as a function of temperature $k_B T/\epsilon_0$. We have set $\lambda k_0 = \epsilon_0$ and $\hbar^2 k_0^2/(2m) = \epsilon_0/2$. (a) *p*-wave magnet. We have set $2Ja^2k_0^2 = \epsilon_0/2$. (b) *d*-wave altermagnet. We have set $2Ja^2k_0^2 = \epsilon_0/2$.

possible to determine N_X by measuring the Φ dependence. On the other hand, the Dirac gap is given by $\Delta \propto B^{N_X}$. Then, the fine tuning of the chemical potential is necessary to adjust the chemical potential within the Dirac gap for large N_X and for small B .

The absolute value of the Hall conductivity takes the maximum value at $\Phi = (\pm\pi/2 + 2n\pi)/N_X$, with $n = 0, \dots, N_X - 1$ in the *X*-wave magnet. In particular, we have studied the Hall conductivity as a function of the magnetic field B by fixing $\Phi = \pi/(2N_X)$ and $\pi/(2N_X) + \pi$ in Figs. 3(c), 5(c), 6(c), 7(c), and 8(c) by the following reason. The external magnetic field is introduced by applying current to a solenoid. By changing the direction of the applied current continuously, the angle $\Phi = \pi/(2N_X)$ is continuously flipped to $\Phi = \pi/(2N_X) + \pi$. The Hall conductivity is odd for $\Phi = \pi/(2N_X)$ and $\pi/(2N_X) + \pi$ for the *p*-wave magnet and the *f*-wave magnet. On the other hand, it is even for the *d*-wave altermagnet, the *g*-wave altermagnet, and the *i*-wave magnet.

The *d*-wave, *g*-wave, and *i*-wave altermagnets have collinear antiferromagnetic spin texture, while *p*-wave magnets have helical spin texture. By applying an external magnetic field, the spin texture is canted to the direction of the magnetic field. It produces an additional term, $\mathbf{M} \cdot \boldsymbol{\sigma}$, where $\mathbf{M}$ is the additional magnetization induced by the magnetic field. However, it is renormalized by setting a new magnetic field, $\mathbf{B}' = \mathbf{B} + \mathbf{M}$. Hence, the effect of the magnetization can be included in the magnetic field term as $\mathbf{B}' \cdot \boldsymbol{\sigma}$ in the Hamiltonian [Eq. (2)].

It was theoretically proposed that CeNiAsO is a *p*-wave magnet [28]. It was also theoretically proposed that a *p*-wave magnet is realized in graphene by introducing spin nematic order [49]. Experiments on *p*-wave magnets in $\text{Gd}_3\text{Ru}_4\text{Al}_{12}$ [50] and in NiI_2 [51] have been reported. A *d*-wave magnet has been realized in RuO_2 [18,52–55], Mn_5Si_3 [20], FeSb_2 [56], organic materials [57], perovskite materials [58,59], and twisted magnetic Van der Waals bilayers [60]. It has been theoretically proposed that an *f*-wave magnet is realized in rhombohedral trilayer graphene by introducing spin nematic order [49]. A *g*-wave altermagnet and an *i*-wave altermagnet have been realized in twisted magnetic Van der Waals bilayers [60].

The $\text{sgn} J = \pm 1$ of the *d*-wave altermagnet has been proposed to be used as an Ising bit, where the bit is expected to flip very fast [8,9,30]. In a similar way, we may use the $\text{sgn} J$ of other *X*-wave magnets as a bit. As in the case of the *d*-wave altermagnet, there is zero-net magnetization in the *X*-wave magnets, which would provide us with ultra-fast and ultradense nonvolatile memories.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not

technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of

this research project. The data are available from the authors upon reasonable request.

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