

Higher-order bulk photovoltaic effects, quantum geometry, and application to p -wave magnets

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The injection and shift currents are generalized to the ℓ th-order injection and shift currents for the longitudinal conductivities in the two-band model, where ℓ is the power of the applied electric field. In addition, the formulas for the higher-order injection current are expressed in terms of the quantum metric, and those for the higher-order shift current are expressed in terms of the higher-order quantum connection. Then, they are applied to p -wave magnets, which are well described by the two-band Hamiltonian as far as low-energy physics is concerned. It is shown that the injection and shift currents are zero. On the other hand, the ℓ th-order injection and shift currents with odd ℓ are nonzero when the direction of the spin-splitting vector of the p -wave magnet points to an in-plane direction.

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I. INTRODUCTION

Photovoltaic currents are generated under photoirradiation [1–6], which will be useful for solar cell technologies. A p - n junction presents a conventional way to generate photocurrent. On the other hand, the bulk photovoltaic photocurrent generation presents another way without using a junction [1–6]. The injection current [4,6–14] and the shift current [2–4,6–9,12–20] are prominent, which are second-order bulk photovoltaic currents. Similarly, third-order bulk photovoltaic currents are studied [4,6,8,17,21–26]. The jerk current which is the third-order generalization of the injection current is known to be dominant in the clean limit [8,22–24,26].

The quantum metric is a quantum geometric quantity [27,28], which is the real part of the quantum geometric tensor. On the other hand, the Berry curvature is the imaginary part of the quantum geometric tensor. The quantum metric is observable by means of optical absorption [7,8,29–40] and the injection current [8].

The p -wave magnet is characterized by the p -wave spin splitting in the energy spectrum [11,41–46]. Its essence is described by a two-band model [14,47–50]. This simple model is realistic because it is analytically derived [47] from the realistic four-band model obtained based on the density functional theory [41] for CeNiAsO. They were recently realized experimentally in Gd₃Ru₄Al₁₂ [50] and in NiI₂ [51]. However, mainly the combined effect with the Rashba interaction [14,47–49,52] or superconductivity [47,53,54] has been studied theoretically. It is interesting to search for phenomena occurring without the Rashba interaction or superconductivity in the p -wave magnet.

In this paper, we generalize the injection current and the shift current to the ℓ th-order photocurrents with an arbitrary ℓ for the longitudinal conductivities based on the two-band model. The ℓ th-order injection (shift) current dominates the ℓ th-order shift (injection) current in the clean (dirty) sample limit. Then, we express the higher-order injection current in terms of the quantum metric and the higher-order shift current in terms of the higher-order quantum connection. Then, we apply the results to p -wave magnets in the presence of magnetic field. We first show that the injection current and the shift current are exactly zero. Next, we show that the odd-order injection and shift currents takes nonzero value when the direction of the spin-splitting vector of the p -wave magnet points to an in-plane direction. Furthermore, we obtain their analytic form.

II. NONLINEAR CONDUCTIVITY

The current density J^c along the c direction induced by the applied electric field E_x along the x direction is expanded in a power series of E_x as

$$J^c = \sum_{\ell=1}^{\infty} \sigma^{c;x^\ell} E_x^\ell \equiv \sum_{\ell=1}^{\infty} J^{c;x^\ell}, \quad (1)$$

where $c = x, y, z$. We refer to $J^{c;x^\ell} \equiv \sigma^{c;x^\ell} E_x^\ell$ as the ℓ th-order current. The first term is the linear response, the second term is the second-order nonlinear response, and so on. If there is inversion symmetry in the system, the even-order conductivities with even ℓ are prohibited because

$$J^c \mapsto -J^c, \quad E_x \mapsto -E_x \quad (2)$$

under inversion symmetry.

The setup of the system is the same as that in previous works [22–24,26]. In order to study higher-order photocurrent, we apply alternating and monochromatic electric fields in addition to a background static electric field $E_x(0)$:

$$E_x(t) = E_x(\omega)e^{-i\omega t} + E_x(-\omega)e^{i\omega t} + E_x(0) \quad (3)$$

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with $\omega > 0$. Note that we do not apply $E_x(0)$ in the case of the second-order photocurrent. We assume that the electric field is real:

$$E_x(t) = E_x^*(t) \quad (4)$$

or

$$E_x(\omega) = E_x^*(-\omega). \quad (5)$$

Excited electrons from the valence band and the conduction band contribute to the bulk photovoltaic current.

A. Second-order photovoltaic currents

The second-order response has the form

$$J^{c;x^2}(\omega_1 + \omega_2) = \sigma^{c;x^2}(\omega_1 + \omega_2; \omega_1, \omega_2) E_x(\omega_1) E_x(\omega_2). \quad (6)$$

We investigate the direct current generation:

$$J^{c;x^2}(0) = \sigma^{c;x^2}(0; \omega, -\omega) E_x(\omega) E_x(-\omega). \quad (7)$$

In the following, we use the abbreviations $J^{c;x^2} \equiv J^{c;x^2}(0)$ and $\sigma^{c;x^2}(\omega) \equiv \sigma^{c;x^2}(0; \omega, -\omega)$.

The conductivity of the injection current is in general given by the formula [4,6–14]

$$\sigma_{\text{inject}}^{c;x^2} = -\tau \frac{2\pi e^3}{\hbar^2} \sum_{n,m,\mathbf{k}} f_{nm} \Delta_{mn}^c r_{nm}^x r_{mn}^x \delta(\omega_{mn} - \omega), \quad (8)$$

where τ is the relaxation time, $f_{nm} = f_n - f_m$ with $f_n = 1/(\exp \frac{\varepsilon_n - \mu}{k_B T} + 1)$ the Fermi distribution function for the band n , μ is the chemical potential, ε_n is the energy of the band n ,

$$r_{mn}^x = \langle m | i \partial_{k_x} | n \rangle \quad (9)$$

is the Berry connection,

$$\Delta_{mn}^c = v_m^c - v_n^c \quad (10)$$

is the difference of the velocities defined by

$$v_m^c = \frac{1}{\hbar} \langle m | \partial_{k_c} H | m \rangle \quad (11)$$

with the Hamiltonian H , and $\omega_{nm} \equiv (\varepsilon_n - \varepsilon_m)/\hbar$. The injection current is induced when the velocities are different ($\Delta_{mn}^c \neq 0$) between the conduction band n and the valence band m along the c direction.

The shift current is in general given by the formula [2–4,6–9,12–20]

$$\sigma_{\text{shift}}^{c;x^2} = -\frac{\pi e^3}{\hbar^2} \sum_{n,m,\mathbf{k}} f_{nm} R_{mn}^{c,x} r_{nm}^x r_{mn}^x \delta(\omega_{mn} - \omega), \quad (12)$$

where

$$R_{mn}^{c,x} = r_{mm}^x - r_{nn}^x + i \partial_{k_c} \ln r_{mn}^x \quad (13)$$

is the shift vector [4]. The shift vector is gauge invariant although the Berry connection is not gauge invariant. The shift vector describes the difference of the mean position of the Wannier function between two bands m and n . The integrand is rewritten as

$$R_{mn}^{c,x} r_{nm}^x r_{mn}^x = i r_{mn}^x r_{nm,c}^x, \quad (14)$$

where we have defined the covariant derivative:

$$\nabla_{k_c} r_{nm}^x \equiv r_{nm,c}^x \equiv \frac{\partial r_{nm}^x}{\partial k_c} - i r_{nm}^x (r_{nn}^x - r_{mm}^x). \quad (15)$$

The shift current is induced when the mean positions are different ($R_{mn}^{c,x} \neq 0$) between the conduction band n and the valence band m .

In the following, we concentrate on the longitudinal conductivities by setting $c = x$.

B. Jerk current and snap current

The jerk current is a third-order bulk photocurrent [22–24]. In the presence of the applied electric field Eq. (3), it is calculated from

$$\frac{d^2 J_{\text{jerk}}^{x;x^3}}{dt^2} = \frac{2e}{V} \sum_n \frac{df_n}{dt} \frac{dv_n^x}{dt} E_x(\omega) E_x(-\omega) E_x(0). \quad (16)$$

By assuming that the right-hand terms are constant, this equation is solved as

$$J_{\text{jerk}}^{x;x^3} \equiv \sigma_{\text{jerk}}^{x;x^3} E_x(\omega) E_x(-\omega) E_x(0), \quad (17)$$

where

$$\frac{\sigma_{\text{jerk}}^{x;x^3}}{\sigma_{\text{jerk}}^0} = \tau^2 \sum_{n,m,\mathbf{k}} f_{nm} \frac{\partial^2 \omega_{mn}}{\partial k_x^2} r_{mn}^x r_{nm}^x \delta(\omega_{mn} - \omega), \quad (18)$$

with $\sigma_{\text{jerk}}^0 \equiv 2\pi e^4 / \hbar^3 V$. It is proportional to τ^2 [8,22].

The snap current is a fourth-order bulk photocurrent [26]. It is calculated from

$$\frac{d^3 J_{\text{snap}}^{x;x^4}}{dt^3} = \frac{3e}{V} \sum_n \frac{df_n}{dt} \frac{d^2 v_n^x}{dt^2} E_x(\omega) E_x(-\omega) [E_x(0)]^2. \quad (19)$$

By assuming that the right-hand terms are constant, this equation is solved as

$$J_{\text{snap}}^{x;x^4} \equiv \sigma_{\text{snap}}^{x;x^4} E_x(\omega) E_x(-\omega) [E_x(0)]^2, \quad (20)$$

where

$$\frac{\sigma_{\text{snap}}^{x;x^4}}{\sigma_{\text{snap}}^0} = \tau^3 \sum_{n,m,\mathbf{k}} f_{nm} \frac{\partial^3 \omega_{mn}}{\partial k_x^3} r_{mn}^x r_{nm}^x \delta(\omega_{mn} - \omega), \quad (21)$$

with $\sigma_{\text{snap}}^0 \equiv 3\pi e^5 / \hbar^4 V$. It is proportional to τ^3 .

C. Higher-order photovoltaic currents

We generalize the injection and shift currents to the ℓ -th-order photovoltaic currents. We analyze the ℓ -th-order photocurrent in the form of

$$J^{x;x^\ell} = \sigma^{x;x^\ell} E_x(\omega) E_x(-\omega) [E_x(0)]^{\ell-2}, \quad (22)$$

with $\ell - 2$ components being static fields $E_x(0)$. We focus on the two-band model with $n = \pm$ in the following.

1. Higher-order injection currents

We generalize the injection current to the ℓ -th-order photovoltaic currents. We consider an acceleration due to the static electric field and an optical excitation from the valence band

to the conduction band due to the oscillatory electric field, separately [22,26]. The equation of motion of electrons is

$$\hbar \frac{d\mathbf{k}}{dt} = -e \frac{\partial \mathbf{A}_0}{\partial t}, \quad (23)$$

where $\mathbf{A}_0 = (A_0, 0, 0)$ is the vector potential. The static electric field is obtained from the vector potential as

$$E_x(0) \equiv -\frac{\partial A_0}{\partial t}, \quad (24)$$

or $A_0 = -E_x(0)t + \text{const.}$ The Bloch velocity under electric field is given by the minimal substitution:

$$v_n(k_x) \rightarrow v_n(k_x - eA_0/\hbar). \quad (25)$$

A comment is in order. We have only considered the contribution from the static electric field in the vector potential. Even if we take into account the contribution from the oscillatory electric field in the vector potential, the time evolution of the mean momentum does not change. It is understood as follows. The equation of motion under the oscillatory electric field reads

$$\hbar \frac{dk}{dt} = eE_x(0) + eE_x(\omega) \cos \omega t, \quad (26)$$

the solution of which is given by

$$k = \frac{e}{\hbar} \left(tE_x(0) + \frac{E_x(\omega)}{\omega} \sin \omega t \right) + k_0. \quad (27)$$

Its mean for one period with respect to the time t is zero. Hence, it is enough only to consider the static electric field in the vector potential.

We expand Eq. (25) in powers of $E_x(0)$, and obtain

$$\frac{d^\ell v_n^x}{dt^\ell} = \left(\frac{e}{\hbar} \right)^\ell \frac{\partial^{\ell+1} \omega_n}{\partial k_x^{\ell+1}} [E_x(0)]^\ell, \quad (28)$$

where we have used

$$\frac{dv_n^x}{dt} = \frac{dk_x}{dt} \frac{dv_n^x}{dk_x} = -\left(\frac{e}{\hbar} \right) \frac{\partial A_0}{\partial t} \frac{d^2 \omega_n}{dk_x^2} = \frac{e}{\hbar} \frac{d^2 \omega_n}{dk_x^2} E_x(0). \quad (29)$$

The current is given by

$$J = \frac{e}{V} \sum_{n,\mathbf{k}} f_n v_n^x, \quad (30)$$

with the velocity along the x direction:

$$v_n^x \equiv \frac{\partial \omega_n}{\partial k_x}. \quad (31)$$

The injection current originates in $dJ^{x;x^2}/dt$, the jerk current originates in $d^2 J^{x;x^3}/dt^2$, and the snap current originates in $d^3 J^{x;x^4}/dt^3$. Hence, it is natural to derive the ℓ th-order injection current from Eq. (30) as

$$\frac{d^{\ell-1} J_{\text{inject}}^{x;x^\ell}}{dt^{\ell-1}} = \frac{e}{V} \sum_n \sum_{s=0}^{\ell-1} \binom{\ell-1}{s} \frac{d^s f_n}{dt^s} \frac{d^{\ell-1-s} v_n^x}{dt^{\ell-1-s}}. \quad (32)$$

The term proportional to $E_x(\omega)E_x(-\omega)[E_x(0)]^{\ell-2}$ is given by taking the terms with $s = 1$ from Eq. (32) as

$$\frac{d^{\ell-1} J_{\text{inject}}^{x;x^\ell}}{dt^{\ell-1}} = (\ell-1) \frac{e}{V} \sum_n \frac{df_n}{dt} \frac{d^{\ell-2} v_n^x}{dt^{\ell-2}}. \quad (33)$$

The Fermi golden rule reads

$$\frac{df_\pm}{dt} = \pm \frac{2\pi e^2}{\hbar^2} |\mathbf{E}(\omega) \cdot \mathbf{r}_{+-}|^2 \delta(\omega_{+-} - \omega), \quad (34)$$

where

$$\begin{aligned} |\mathbf{E}(\omega) \cdot \mathbf{r}_{+-}|^2 &= |E_x(\omega) r_{+-}^x|^2 = E_x(\omega) r_{+-}^x E_x^*(\omega) r_{+-}^{x*} \\ &= -E_x(\omega) E_x(-\omega) r_{+-}^x r_{-+}^x, \end{aligned} \quad (35)$$

with the use of the relation (5).

Inserting it into Eq. (33), we have

$$\begin{aligned} \frac{d^{\ell-1} J_{\text{inject}}^{x;x^\ell}}{dt^{\ell-1}} &= \frac{(\ell-1)2\pi e^2}{\hbar^2} \frac{e}{V} \left(\frac{e}{\hbar} \right)^{\ell-2} r_{-+}^x r_{+-}^x \delta(\omega_{+-} - \omega) \\ &\quad \times \frac{\partial^{\ell+1} \omega_{+-}}{\partial k_x^{\ell+1}} E_x(\omega) E_x(-\omega) [E_x(0)]^{\ell-2}. \end{aligned} \quad (36)$$

By assuming a monochromatic oscillation $J^{x;x^\ell} \propto e^{-i\omega_0 t}$, we obtain the ℓ th-order current $J^{x;x^\ell}$:

$$\begin{aligned} J_{\text{inject}}^{x;x^\ell} &= \frac{\ell-1}{(i\omega_0 + 1/\tau)^{\ell-1}} \frac{2\pi e^{\ell+1}}{\hbar^\ell V} r_{-+}^x r_{+-}^x \delta(\omega_{+-} - \omega) \\ &\quad \times \frac{\partial^{\ell-1} \omega_{+-}}{\partial k_x^{\ell-1}} E_x(\omega) E_x(-\omega) [E_x(0)]^{\ell-2}, \end{aligned} \quad (37)$$

where we have introduced a cutoff by the relaxation time τ . When we concentrate on the direct current component $\omega_0 = 0$, it is proportional to $\tau^{\ell-1}$. The longitudinal component is

$$\frac{\sigma_{\text{inject}}^{x;x^\ell}}{\sigma_{\text{inject}}^{(\ell)}} = \sum_{\mathbf{k}} f_{-+} \frac{\partial^{\ell-1} \omega_{+-}}{\partial k_x^{\ell-1}} r_{-+}^x r_{+-}^x \delta(\omega_{+-} - \omega), \quad (38)$$

with

$$\sigma_{\text{inject}}^{(\ell)} \equiv 2\pi(\ell-1)V \frac{e^{\ell+1}}{\hbar^\ell} \tau^{\ell-1}. \quad (39)$$

This formula is reduced to the injection current ($\ell = 2$), the jerk current ($\ell = 3$), and the snap current ($\ell = 4$). They are named the crackle current ($\ell = 5$), the pop current ($\ell = 6$), the lock current ($\ell = 7$), the drop current ($\ell = 8$), the shot current ($\ell = 9$), and the put current ($\ell = 10$) [26,55].

2. Higher-order shift currents

The shift current is induced by the difference of the position between the valence band and the conduction band when electrons are excited from the valence band to the conduction band. It induces the electric dipole $E_x(t)r_{+-}^x$. Hence, the shift current originates in a quantum interference [26] between the oscillation of ρ_{mn} and the oscillation of the dipole velocity $E_x r_{nm;x}^x$:

$$J_{\text{shift}}^{x;x^2} = -\frac{e^2}{\hbar V} \sum_{\mathbf{k}} E_x r_{+-;x}^x \rho_{+-}^{(1)}, \quad (40)$$

which is defined by the interband contributions $r_{+-;x}^x$ and $\rho_{+-}^{(1)}(t)$.

It is natural to define a higher-order shift current by

$$J_{\text{shift}}^{x;x^\ell} = -\frac{e^2}{\hbar V} \sum_{\mathbf{k}} E_x r_{+-;x}^x \rho_{+-}^{(\ell-1)}, \quad (41)$$

where $\rho_{mn}^{(\ell-1)}$ is the ℓ th-order solution of the density matrix with the von Neumann equation [4,6,26]:

$$\begin{aligned} \frac{\partial \rho_{mn}}{\partial t} + i\omega_{mn}\rho_{mn} \\ = \frac{e}{i\hbar} \sum_s E_x (\rho_{ms} r_{sn}^x - r_{ms}^x \rho_{sn}) - \frac{e}{\hbar} E_x \rho_{mn;x}. \end{aligned} \quad (42)$$

For the two-band system, it is rewritten as

$$i(\omega_{+-} - \omega_0)\rho_{+-} = \frac{e}{i\hbar} E_x r_{+-}^x (\rho_{++} - \rho_{--}) - \frac{e}{\hbar} E_x \rho_{+-;x}, \quad (43)$$

where we have used $r_{nn}^x = 0$ and assumed a monochromatic solution $\rho_{mn} \propto e^{-i\omega_0 t}$. This definition recovers properly the definition of the shift current for $\ell = 2$.

The zeroth-order solution is given by

$$\rho_{\pm}^{(0)} = f_{\pm}, \quad (44)$$

because electric field is not applied. The first-order solution is given by

$$\begin{aligned} \rho_{+-}^{(1)} &= \frac{ie}{\hbar(\omega_{+-} - \omega_0)} E_x r_{+-}^x (\rho_{++}^{(0)} - \rho_{--}^{(0)}) e^{-i\omega_0 t} \\ &= \frac{ie}{\hbar(\omega_{+-} - \omega_0)} E_x r_{+-}^x f_{-+} e^{-i\omega_0 t}. \end{aligned} \quad (45)$$

Especially, we have $\rho_{++}^{(1)} = 0$ because $f_{++} \equiv f_{+} - f_{+} = 0$.

The recursive equation reads

$$\begin{aligned} \rho_{+-}^{(\ell)} &= \frac{e}{i\hbar} E_x r_{+-}^x (\rho_{++}^{(\ell-1)} - \rho_{--}^{(\ell-1)}) + \frac{ieE_x}{\hbar(\omega_{+-} - \omega_0)} \rho_{+-;x}^{(\ell-1)} \\ &= \frac{ieE_x}{\hbar(\omega_{+-} - \omega_0)} \nabla_{k_x} \rho_{+-}^{(\ell-1)} \\ &= \left(\frac{ieE_x}{\hbar(\omega_{+-} - \omega_0)} \right)^{\ell-1} \nabla_{k_x}^{\ell-1} \rho_{+-}^{(1)} \\ &= \left(\frac{ieE_x}{\hbar(\omega_{+-} - \omega_0)} \right)^{\ell-1} E_x f_{-+} \nabla_{k_x}^{\ell-1} r_{+-}^x, \end{aligned} \quad (46)$$

where we have used $\rho_{++}^{(\ell-1)} = \rho_{--}^{(\ell-1)}$. Hence, the ℓ th-order shift current is obtained as

$$\begin{aligned} J_{\text{shift}}^{x;x^\ell} &= -\frac{e^2}{\hbar V} \sum_{\mathbf{k}} E_x r_{+-;x}^x \rho_{+-}^{(\ell-1)} \\ &= -\frac{e^2}{\hbar V} \left(\frac{ie}{\hbar(\omega_{+-} - \omega_0)} \right)^{\ell-2} e^{-i\omega_0 t} \\ &\quad \times \sum_{\mathbf{k}} r_{+-;x}^x f_{-+} \nabla_{k_x}^{\ell-2} r_{+-}^x E_x^l. \end{aligned} \quad (47)$$

We use the monochromatic condition (3) and study the direct current component:

$$\begin{aligned} J_{\text{shift}}^{x;x^\ell} &= \sigma_{\text{shift}}^{x;x^\ell}(0; \omega, -\omega, 0, 0, \dots, 0) \\ &\quad \times E_x(\omega) E_x(-\omega) [E_x(0)]^{\ell-2}. \end{aligned} \quad (48)$$

Then, the ℓ th-order shift current is obtained as

$$\begin{aligned} J_{\text{shift}}^{x;x^\ell} &= -\frac{e^2}{\hbar V} \left(\frac{ie}{\hbar\omega} \right)^{\ell-2} \sum_{\mathbf{k}} r_{+-;x}^x \nabla_{k_x}^{\ell-2} r_{+-}^x \\ &\quad \times f_{-+} E_x(\omega) E_x(-\omega) [E_x(0)]^{\ell-2} \delta(\omega_{+-} - \omega). \end{aligned} \quad (49)$$

It is independent of τ . The ℓ th-order shift conductivity is given by

$$\frac{\sigma_{\text{shift}}^{x;x^\ell}}{\sigma_{\text{shift}}^{(\ell)}} = \frac{1}{V} \sum_{\mathbf{k}} f_{-+} r_{+-;x}^x \nabla_{k_x}^{\ell-2} r_{+-}^x \delta(\omega_{+-} - \omega), \quad (50)$$

with

$$\sigma_{\text{shift}}^{(\ell)} \equiv -\frac{e^3}{\hbar^2} \left(\frac{ie}{\hbar\omega} \right)^{\ell-2}. \quad (51)$$

It recovers an ordinary shift current for $\ell = 2$.

III. QUANTUM GEOMETRY

Quantum geometry is differential geometry based on the normalized complex vectors in quantum mechanics. The quantum geometric tensor is defined by [27,28]

$$\mathcal{F}_{mn}^{\mu\nu}(\mathbf{k}) = \langle \partial_{k_\mu} \psi_m(\mathbf{k}) | 1 - \mathcal{P}(\mathbf{k}) | \partial_{k_\nu} \psi_n(\mathbf{k}) \rangle, \quad (52)$$

where $\mu = x, y, z$, and

$$\mathcal{P}(\mathbf{k}) \equiv \sum_{n \in \text{Occupied}} |u_n(\mathbf{k})\rangle \langle u_n(\mathbf{k})| \quad (53)$$

is the projection operator to the occupied band.

In the two-band model, its diagonal component is simply given by the quantum metric:

$$\begin{aligned} \mathcal{F}_{--}^{\mu\nu}(\mathbf{k}) &= \langle \partial_{k_\mu} \psi_-(\mathbf{k}) | \psi_+(\mathbf{k}) \rangle \langle \psi_+(\mathbf{k}) | \partial_{k_\nu} \psi_-(\mathbf{k}) \rangle \\ &= r_{+-}^{\mu*}(\mathbf{k}) r_{+-}^\nu(\mathbf{k}) = g_{\mu\nu}(\mathbf{k}). \end{aligned} \quad (54)$$

In general, the Hamiltonian of the two-band model is given by

$$H(\mathbf{k}) = \sigma_0 h_0(\mathbf{k}) + \sigma \cdot \mathbf{h}(\mathbf{k}), \quad (55)$$

where $\mathbf{h}(\mathbf{k})$ is parametrized as

$$\mathbf{n}(\mathbf{k}) \equiv \mathbf{h}/|\mathbf{h}| = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \quad (56)$$

and σ_j is the Pauli matrix with $j = x, y, z$. The eigenfunction of the Hamiltonian (55) is given by

$$\psi_+(\mathbf{k}) = e^{i\alpha(\mathbf{k})} \begin{pmatrix} e^{-i\phi(\mathbf{k})} \cos \frac{\theta(\mathbf{k})}{2} \\ \sin \frac{\theta(\mathbf{k})}{2} \end{pmatrix}, \quad (57)$$

$$\psi_-(\mathbf{k}) = e^{i\alpha(\mathbf{k})} \begin{pmatrix} -e^{-i\phi(\mathbf{k})} \sin \frac{\theta(\mathbf{k})}{2} \\ \cos \frac{\theta(\mathbf{k})}{2} \end{pmatrix}, \quad (58)$$

where $\alpha(\mathbf{k})$ is a real function representing gauge degrees of freedom. Then, the Berry connections are calculated as

$$\begin{aligned} r_{+-}^{\mu}(\mathbf{k}) &= -i \langle \psi_+(\mathbf{k}) | \partial_{k_\mu} \psi_-(\mathbf{k}) \rangle \\ &= -ie^{-i\alpha} \left(e^{i\phi} \cos \frac{\theta}{2}, \sin \frac{\theta}{2} \right) \\ &\quad \times \left[e^{i\alpha} \begin{pmatrix} -ie^{i\phi} \partial_{k_\mu} \phi \sin \frac{\theta}{2} - \frac{\partial_{k_\mu} \theta}{2} e^{-i\phi} \cos \frac{\theta}{2} \\ \frac{\partial_{k_\mu} \theta}{2} \sin \frac{\theta}{2} \end{pmatrix} \right. \\ &\quad \left. + e^{i\alpha} (\partial_{k_\mu} \alpha) \begin{pmatrix} -e^{-i\phi} \sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{pmatrix} \right] \\ &= \partial_{k_\mu} \phi \sin \frac{\theta}{2} \cos \frac{\theta}{2} + i \frac{\partial_{k_\mu} \theta}{2} \cos^2 \frac{\theta}{2} + i \frac{\partial_{k_\mu} \theta}{2} \sin^2 \frac{\theta}{2} \\ &= i \frac{\partial_{k_\mu} \theta}{2} + \frac{\partial_{k_\mu} \phi}{2} \sin \theta, \end{aligned} \quad (59)$$

and

$$\begin{aligned}
r_{-+}^{\mu}(\mathbf{k}) &= -i\langle\psi_{-}(\mathbf{k})|\partial_{\mu}\psi_{+}(\mathbf{k})\rangle \\
&= -ie^{-i\alpha}\left(-e^{i\phi}\sin\frac{\theta}{2}, \cos\frac{\theta}{2}\right) \\
&\quad \times \left[e^{i\alpha}\left(i e^{i\phi}\partial_{\mu}\phi\cos\frac{\theta}{2} - \frac{\partial_{\mu}\theta}{2}e^{-i\phi}\sin\frac{\theta}{2}\right) \right. \\
&\quad \left. + e^{i\alpha}(\partial_{\mu}\alpha)\left(\frac{e^{-i\phi(\mathbf{k})}\cos\frac{\theta(\mathbf{k})}{2}}{\sin\frac{\theta(\mathbf{k})}{2}}\right)\right] \\
&= \partial_{\mu}\phi\sin\frac{\theta}{2}\cos\frac{\theta}{2} - i\frac{\partial_{\mu}\theta}{2}\sin^2\frac{\theta}{2} - i\frac{\partial_{\mu}\theta}{2}\cos^2\frac{\theta}{2} \\
&= -i\frac{\partial_{\mu}\theta}{2} + \frac{\partial_{\mu}\phi}{2}\sin\theta.
\end{aligned} \tag{60}$$

We have

$$r_{+-}^{\mu*}(\mathbf{k}) = r_{-+}^{\mu}(\mathbf{k}). \tag{61}$$

Then, we have

$$r_{-+}^{\mu}r_{+-}^{\mu} = r_{+-}^{\mu*}(\mathbf{k})r_{-+}^{\mu}(\mathbf{k}) = g_{\mu\mu}, \tag{62}$$

irrespective of α . We use this relation to rewrite the injection current, the shift current, and the jerk current in terms of the quantum metric.

The injection current is rewritten in terms of the quantum metric [7,8]:

$$\sigma_{\text{inject}}^{x;x^2} = -\tau \frac{2\pi e^3}{\hbar^2} \sum_{\mathbf{k}} f_{-+} \Delta_{+-}^x g_{xx} \delta(\omega_{+-} - \omega). \tag{63}$$

The shift current is rewritten in terms of the quantum metric [12]:

$$\sigma_{\text{shift}}^{x;x^2} = -\frac{\pi e^3}{\hbar^2 V} \sum_{\mathbf{k}} f_{-+} R_{+-}^{x,x} g_{xx} \delta(\omega_{+-} - \omega). \tag{64}$$

In a similar way, the higher-order injection current is rewritten in terms of the quantum metric as

$$\frac{\sigma_{\text{inject}}^{x;x^{\ell}}}{\sigma_{\ell}^0} = \sum_{\mathbf{k}} f_{-+} \frac{\partial^{\ell-1}\omega_{+-}}{\partial k_x^{\ell-1}} g_{xx} \delta(\omega_{+-} - \omega). \tag{65}$$

There is another representation [8] equivalent to Eq. (64):

$$\sigma_{\text{shift}}^{x;x^2} = -\frac{\pi e^3}{\hbar^2 V} \sum_{\mathbf{k}} f_{-+} i C_{xxx}^{-+} \delta(\omega_{+-} - \omega), \tag{66}$$

where the quantum connection is defined by

$$\begin{aligned}
C_{xxx}^{mn} &\equiv (\hat{e}_x^{mn}, \nabla_x \hat{e}_x^{mn}) \equiv r_{nm}^x r_{mn,x}^x \\
&= r_{nm}^x \left[\frac{\partial r_{nm}^x}{\partial k_x} - i r_{nm}^x (r_{nn}^x - r_{mm}^x) \right]
\end{aligned} \tag{67}$$

with [8]

$$\hat{e}_x^{mn}(\mathbf{k}) \equiv r_{nm}^x |u_m(\mathbf{k})\rangle \langle u_n(\mathbf{k})|. \tag{68}$$

We generalize this relation by introducing the quantum connection to the higher-order quantum connection

defined by

$$C_{xxx}^{mn(\ell)} \equiv (\nabla_x^{\ell} \hat{e}_x^{mn}, \nabla_x \hat{e}_x^{mn}) \equiv \frac{\nabla_x^{\ell} r_{nm}^x}{\nabla k_x^{\ell}} r_{nm,x}^x, \tag{69}$$

where (A, B) represents the Hilbert-Schmidt inner product of the tangent basis vectors [8] defined by

$$(A, B) \equiv \text{Tr}[A^{\dagger} B] \equiv \sum_{\alpha, \beta} A_{\alpha\beta}^* B_{\alpha\beta}. \tag{70}$$

By using it, the ℓ th-order shift conductivity is expressed in terms of it as

$$\sigma_{\text{shift}}^{x;x^{\ell}} = -\frac{e^3}{\hbar^2 V} \left(\frac{ie}{\hbar\omega} \right)^{\ell-2} \sum_{\mathbf{k}} f_{-+} C_{xxx}^{+-(\ell)} \delta(\omega_{+-} - \omega). \tag{71}$$

IV. *p*-WAVE MAGNET

We apply these results on photovoltaic currents to the *p*-wave magnet as a typical example in the following, where various formulas are analytically obtained without using approximations. A prominent feature of the *p*-wave magnet is that there are no second-order photovoltaic currents but there are nonzero higher-order photovoltaic currents for $\ell \geq 3$.

The *p*-wave magnet is well described by the two-band Hamiltonian given by

$$H = H_0 + H_p + H_B. \tag{72}$$

The first term represents the kinetic energy of electrons:

$$H_0 = \frac{\hbar^2 k_x^2}{2m} \sigma_0, \tag{73}$$

where m is the electron mass, and σ_0 is the 2×2 identity matrix. The second term

$$H_p(\mathbf{k}) = J(\mathbf{s} \cdot \boldsymbol{\sigma}) k_x \tag{74}$$

describes the effect of the *p*-wave magnet on the electrons [14,41,47,48,52,53,56], where $J\mathbf{s}$ is the spin-splitting vector with J being its magnitude and $\mathbf{s} = (\sin\Theta \cos\Phi, \sin\Theta \sin\Phi, \cos\Theta)$ being its direction. We set $J_x = J \sin\Theta \cos\Phi$, $J_y = J \sin\Theta \sin\Phi$, and $J_z = J \cos\Theta$. The third term is the magnetic field term:

$$H_B = B\sigma_z. \tag{75}$$

Then, the Hamiltonian (72) is of the form of Eq. (55) with

$$\begin{aligned}
h_0 &= \frac{\hbar^2 k_x^2}{2m}, \quad h_x = J k_x \sin\Theta \cos\Phi, \\
h_y &= J k_x \sin\Theta \sin\Phi, \quad h_z = J k_x \cos\Theta + B.
\end{aligned} \tag{76}$$

It is to be remarked that the two-band Hamiltonian (72) is realistic although it looks too simple. There is a realistic four-band model [41] derived for the *p*-wave magnet CeNiAsO based on the density functional theory. On the other hand, it is possible to prove [47] that this specific two-band model is the effective model describing the lowest two bands of the four-band model around the Γ point.

There is inversion symmetry for $\Theta = \pi/2$:

$$\sigma_z H_p(k_x) \sigma_z = H_p(-k_x), \tag{77}$$

where the spin-splitting vector points to an in-plane direction.

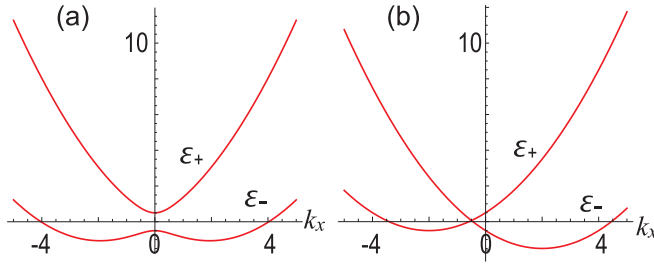


FIG. 1. Energy spectrum in units of ε_0 . (a) The Néel vector is along the x direction ($\Theta = \pi/2$; $\Phi = 0$). (b) The Néel vector is along the z direction ($\Theta = 0$). The horizontal axis is k_x . We have set $J = \varepsilon_0/k_0$, $\hbar^2 k_0^2/(2m) = \varepsilon_0/2$, and $B = 0.5\varepsilon_0$.

The energy spectrum of the Hamiltonian (72) is given by

$$\varepsilon_{\pm} = \frac{\hbar^2 k_x^2}{2m} \pm \sqrt{J^2 k_x^2 \sin^2 \Theta + (B + J k_x \cos \Theta)^2}, \quad (78)$$

which demonstrates the p -wave spin splitting. By solving $\omega_{+-}(k_{\pm}) - \omega = 0$, we obtain

$$k_{\pm} = -\frac{BJ_z}{J} \pm \frac{\sqrt{J^2 \omega^2 - 4B^2 J_{\parallel}^2}}{2J^2}, \quad (79)$$

with

$$J_{\parallel}^2 = J_x^2 + J_y^2. \quad (80)$$

The band structure is shown in Figs. 1(a) and 1(b) when the spin-splitting vector is taken along the x axis and along the z axis, respectively.

The quantum metric for the two-band system is explicitly given by [35,36,38,39,57,58]

$$g_{\mu\nu}(\mathbf{k}) = \frac{1}{2} (\partial_{k_{\mu}} \mathbf{n}) \cdot (\partial_{k_{\nu}} \mathbf{n}), \quad (81)$$

with the normalized vector (56). Explicitly, the quantum metric is obtained as

$$g_{xx} = \frac{B^2 J_{\parallel}^2}{2(B^2 + J^2 k_x^2 + 2BJ_z)^2} \quad (82)$$

with $J_{\parallel}^2 \equiv J_x^2 + J_y^2$.

The quantum connection (67) is calculated as

$$C_{xxx}^{--} = \frac{B^2 J_{\parallel}^2 (J^2 k_x + BJ_z)}{(B^2 + J^2 k_x^2 + 2BJ_z)^3}, \quad (83)$$

and the higher-order Hermitian connection (69) is calculated as

$$C_{xxx}^{--(1)} = \frac{B^2 J_{\parallel}^2 (J^2 k_x + BJ_z)^2}{(B^2 + J^2 k_x^2 + 2BJ_z)^4}, \quad (84)$$

$$C_{xxx}^{--(2)} = \frac{B^2 J_{\parallel}^2 (J^2 k_x + BJ_z)}{(B^2 + J^2 k_x^2 + 2BJ_z)^5} \times (3J^4 k_x^2 + 3B^2 J_z^2 - B^2 J_{\parallel}^2 + 6BJ^2 J_z k_x), \quad (85)$$

and so on. We find it is proportional to

$$C_{xxx}^{--(\ell)} \propto \frac{B^2 J_{\parallel}^2 (J^2 k_x + BJ_z)}{(B^2 + J^2 k_x^2 + 2BJ_z)^{\ell+3}}. \quad (86)$$

At zero temperature, the Fermi distribution difference is reduced to be

$$f_{-+} = \theta\left(\frac{\omega}{2} - \frac{\hbar^2 k_x^2}{2m} + \mu\right), \quad (87)$$

where $\theta(x)$ is the step function $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$. In the following, we set $f_{-+} = 1$ at zero temperature. The validity of this assumption is discussed in Sec. IV E.

A. Injection current

The injection current at zero temperature is calculated in the Hamiltonian (72) as

$$\sigma_{\text{inject}}^{xx} = -\tau \frac{2\pi e^3}{\hbar^2} \sum_{k_x} \Delta_{+-}^x g_{xx} \frac{\delta(k_x - k_{\pm})}{2|\partial_{k_x} \omega_{+-}|}. \quad (88)$$

By using the Hamiltonian (72), we have

$$\begin{aligned} & \sum_{k_x} \Delta_{+-}^x g_{xx} \frac{\delta(k_x - k_0)}{2|\partial_{k_x} \omega_{+-}|} \\ &= \sum_{k_x=k_{\pm}} \frac{2(J^2 k_x + BJ_z)}{\sqrt{B^2 + J^2 k_x^2 + 2BJ_z}} \frac{B^2 J_{\parallel}^2}{4(B^2 + J^2 k_x^2 + 2BJ_z)^2} \\ & \quad \times \frac{\sqrt{B^2 + J^2 k_x^2 + 2BJ_z}}{4|J^2 k_x + BJ_z|} \\ &= \sum_{k_x=k_{\pm}} \frac{(J^2 k_x + BJ_z)}{2|J^2 k_x + BJ_z|} \frac{B^2 J_{\parallel}^2}{4(B^2 + J^2 k_x^2 + 2BJ_z)^2}, \end{aligned} \quad (89)$$

where we have used

$$\Delta_{+-}^x = \frac{2(J^2 k_x + BJ_z)}{\sqrt{B^2 + J^2 k_x^2 + 2BJ_z}}. \quad (90)$$

By using

$$B^2 + J^2 k_0^2 + 2BJ_z = \frac{\hbar^2 \omega^2}{4}, \quad (91)$$

and

$$2(J^2 k_0 + BJ_z) = \eta \sqrt{J^2 \hbar^2 \omega^2 - 4B^2 J_{\parallel}^2}, \quad (92)$$

we have

$$\begin{aligned} & \sum_{k_x} \Delta_{mn}^x g_{xx} \delta(\omega_{+-} - \omega) \\ &= \sum_{\eta=\pm} \frac{\eta \sqrt{J^2 \hbar^2 \omega^2 - 4B^2 J_{\parallel}^2}}{2|\eta \sqrt{J^2 \hbar^2 \omega^2 - 4B^2 J_{\parallel}^2}|} \frac{B^2 J_{\parallel}^2}{4\left(\frac{\hbar^2 \omega^2}{4}\right)^2} \\ &= \sum_{\eta=\pm} \eta \frac{2B^2 J_{\parallel}^2}{(\hbar \omega)^4} = 0. \end{aligned} \quad (93)$$

Then, the injection current is exactly zero.

B. Shift current

The shift current at zero temperature is zero in the Hamiltonian (72) because

$$\begin{aligned}\sigma_{\text{shift}}^{x;x^2} &= -\frac{\pi e^3}{\hbar^2} \sum_{k_x} iC_{xxx} \frac{\delta(k_x - k_{\pm})}{2|\partial_{k_x}\omega_{+-}|} \\ &= -\frac{\pi e^3}{\hbar^2} \sum_{n,m,k_x=k_{\pm}} i \frac{B^2 J_{\parallel}^2 (J^2 k_x + BJ_z)}{(B^2 + J^2 k_x^2 + 2BJ_z)^3} \\ &\quad \times \frac{\sqrt{B^2 + J^2 k_x^2 + 2BJ_z}}{4|J^2 k_x + BJ_z|} \\ &= -\frac{\pi e^3}{\hbar^2} \sum_{n,m,\eta=\pm} \eta \frac{2iB^2 J_{\parallel}^2}{\omega^5} = 0,\end{aligned}\quad (94)$$

where we have used Eq. (83).

C. Higher-order injection current

1. Jerk current

Since the injection current and the shift current are zero, we study the jerk current. The jerk current is calculated as

$$\frac{\sigma_{\text{jerk}}^{x;x^3}}{\sigma_{\text{jerk}}^{(3)}} = \sum_{k_x} f_{-+} \frac{\partial^2 \omega_{+-}}{\partial k_x^2} g_{xx} \frac{\delta(k_x - k_0)}{2|\partial_{k_x}\omega_{+-}|}. \quad (95)$$

By using the Hamiltonian (72), we have

$$\begin{aligned}\frac{\partial^2 \omega_{+-}}{\partial k_x^2} \frac{g_{xx}}{2|\partial_{k_x}\omega_{+-}|} &= \frac{2B^2 J_{\parallel}^2}{(B^2 + J^2 k_x^2 + 2BJ_z)^{3/2}} \frac{B^2 J_{\parallel}^2}{4(B^2 + J^2 k_x^2 + 2BJ_z)^2} \\ &\quad \times \frac{\sqrt{B^2 + J^2 k_x^2 + 2BJ_z}}{4|J^2 k_x + BJ_z|} \\ &= \frac{B^4 J_{\parallel}^4}{8(B^2 + J^2 k_x^2 + 2BJ_z)^3 |J^2 k_x + BJ_z|},\end{aligned}\quad (96)$$

where we have used

$$\frac{\partial^2 \omega_{+-}}{\partial k_x^2} = \frac{2B^2 J_{\parallel}^2}{(B^2 + J^2 k_x^2 + 2BJ_z)^{3/2}}, \quad (97)$$

and

$$\partial_{k_x} \omega_{+-} = \frac{2(J^2 k_x + BJ_z)}{\sqrt{B^2 + J^2 k_x^2 + 2BJ_z}}, \quad (98)$$

for

$$\omega > 2 \left| \frac{BJ_{\parallel}}{\hbar J} \right| \equiv \omega_{\text{cr}}. \quad (99)$$

By using (91) and (92), we have

$$\left. \frac{\partial^2 \omega_{+-}}{\partial k_x^2} \frac{g_{xx}}{2|\partial_{k_x}\omega_{+-}|} \right|_{k_x=k_{\pm}} = \frac{8B^4 J_{\parallel}^4}{(\hbar\omega)^6 \sqrt{J^2 \hbar^2 \omega^2 - 4B^2 J_{\parallel}^2}}. \quad (100)$$

Then, we obtain the jerk current as

$$\frac{\sigma_{\text{jerk}}^{x;x^3}}{\sigma_{\text{jerk}}^{(3)}} \equiv \frac{\sigma_{\text{inject}}^{x;x^3}}{\sigma_{\text{inject}}^{(3)}} = \theta(\omega - \omega_{\text{cr}}) \frac{64B^4 J_{\parallel}^4}{\hbar^6 \omega^6 \sqrt{J^2 \hbar^2 \omega^2 - 4B^2 J_{\parallel}^2}}. \quad (101)$$

It diverges at the optical band edge $\omega = \omega_{\text{cr}}$. This is due to the fact that the band is flat,

$$|\partial_{k_x} \omega_{+-}|_{k_x=k_{\pm}} = \frac{2\sqrt{J^2 \hbar^2 \omega^2 - 4B^2 J_{\parallel}^2}}{\hbar^2 \omega} \Big|_{\omega=\omega_{\text{c}}} = 0, \quad (102)$$

at the optical band edge. The origin of this divergence is the divergence of the density of states

$$\begin{aligned}\rho(\omega) &\equiv \sum_{k_x} f_{-+} \delta(\omega_{+-} - \omega) \\ &= \sum_{k_x} f_{-+} \frac{\delta(k_x - k_0)}{2|\partial_{k_x}\omega_{+-}|} \\ &= \frac{\hbar^2 \omega}{4\sqrt{J^2 \hbar^2 \omega^2 - 4B^2 J_{\parallel}^2}}\end{aligned}\quad (103)$$

at the optical band edge:

$$\lim_{\omega \rightarrow \omega_{\text{c}}} \rho(\omega) = \infty. \quad (104)$$

When $J_z = 0$,

$$\frac{\sigma_{\text{jerk}}^{x;x^3}}{\sigma_{\text{jerk}}^{(3)}} = \theta(\omega - 2|B|) \frac{64B^4 J_{\parallel}^3}{\hbar^6 \omega^6 \sqrt{\hbar^2 \omega^2 - 4B^2}}. \quad (105)$$

On the other hand, when $J_{\parallel} = 0$, we have $\sigma_{\text{jerk}}^{x;x^3} = 0$.

2. Crackle current

The crackle current is similarly calculated as

$$\begin{aligned}\frac{\sigma_{\text{crackle}}^{x;x^5}}{\sigma_{\text{crackle}}^{(5)}} &\equiv \frac{\sigma_{\text{inject}}^{x;x^5}}{\sigma_{\text{inject}}^{(5)}} \\ &= \theta(\omega - \omega_{\text{cr}}) \frac{3072B^4 J_{\parallel}^4 (J^2 \hbar^2 \omega^2 - 5B^2 J_{\parallel}^2)}{\hbar^{10} \omega^{10} \sqrt{J^2 \hbar^2 \omega^2 - 4B^2 J_{\parallel}^2}}.\end{aligned}\quad (106)$$

3. Higher-order injection current

The even-order injection current is zero when $J_z = 0$ because there is inversion symmetry in the system, as discussed in Eq. (2). On the other hand, it is zero when $J_{\parallel} = 0$ because the quantum metric is proportional to $J_{\parallel}$ as in Eq. (86). Explicit calculations show that they are zero even for $J_z \neq 0$ and $J_{\parallel} \neq 0$. Higher-order injection current is calculated in a similar manner. The ω dependence of $\log_{10} |\sigma_{\text{inject}}^{x;x^{\ell}} / \sigma_{\text{inject}}^{(\ell)}|$ is shown in Fig. 2(a). The asymptotic behavior for large ω is given by

$$\frac{\sigma_{\text{inject}}^{x;x^{2\ell'+1}}}{\sigma_{\text{inject}}^{(2\ell'+1)}} \propto \theta(\omega - \omega_{\text{cr}}) \frac{B^4 J_{\parallel}^2 J^{2\ell'-2}}{(\hbar\omega)^{2\ell'+5}} \quad (107)$$

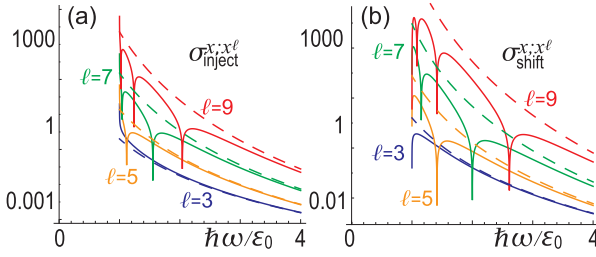


FIG. 2. (a) Higher-order injection currents $\log_{10}|\sigma_{\text{inject}}^{xx^{l'}}/\sigma_{\text{inject}}^{(\ell)}|$ in units of $\sigma_{\text{inject}}^{(\ell)}$, where τ dependence is included in $\sigma_{\text{inject}}^{(\ell)}$. (b) Higher-order shift currents $\log_{10}|\sigma_{\text{shift}}^{xx^{l'}}/\sigma_{\text{shift}}^{(\ell)}|$ in units of $\sigma_{\text{shift}}^{(\ell)}$, which is independent of τ . Blue curves indicate $\ell = 3$, orange curves indicate $\ell = 5$, green curves indicate $\ell = 7$, and red curves indicate $\ell = 9$. Dashed curves indicate asymptotic behaviors for large ω . We have set $J = \varepsilon_0/k_0$, $\hbar^2 k_0^2/(2m) = \varepsilon_0/2$, and $B = 0.5\varepsilon_0$.

for $\ell' \geq 1$ with Eq. (39). There are several dips in Fig. 2(a), where $\sigma_{\text{inject}}^{xx^{l'}} = 0$. In general, the conductivity $\sigma_{\text{inject}}^{xx^{l'}}$ is a polynomial of ω and B , which has several zeros as a function of ω . It results in the negative divergence of $\log_{10}|\sigma_{\text{inject}}^{xx^{l'}}/\sigma_{\text{inject}}^{(\ell)}|$. There are $(\ell - 1)$ dips in the $(2\ell + 1)$ th-order injection current.

D. Higher-order shift current

The even-order shift current is zero when $J_z = 0$ because there is inversion symmetry in the system, as discussed in Eq. (2). On the other hand, it is zero when $J_{\parallel} = 0$ because the quantum connection is proportional to $J_{\parallel}$ as in Eq. (82). Explicit calculations show that they are zero even for $J_z \neq 0$ and $J_{\parallel} \neq 0$.

Higher-order shift currents are calculated:

$$\frac{\sigma_{\text{shift}}^{xx^3}}{\sigma_{\text{shift}}^{(3)}} = \theta(\omega - \omega_{\text{cr}}) \frac{64\hbar B^2 J_{\parallel}^2 \sqrt{J^2 \hbar^2 \omega^2 - 4B^2 J_{\parallel}^2}}{(\hbar\omega)^7}, \quad (108)$$

$$\begin{aligned} \frac{\sigma_{\text{shift}}^{xx^5}}{\sigma_{\text{shift}}^{(5)}} &= \theta(\omega - \omega_{\text{cr}}) \frac{3072\hbar B^2 J_{\parallel}^2 \sqrt{J^2 \hbar^2 \omega^2 - 4B^2 J_{\parallel}^2}}{(\hbar\omega)^{11}} \\ &\times (J^2 \hbar^2 \omega^2 - 8B^2 J_{\parallel}^2). \end{aligned} \quad (109)$$

The ω dependence of $\log_{10}|\sigma_{\text{shift}}^{xx^{l'}}/\sigma_{\text{shift}}^{(\ell)}|$ is shown in Fig. 2(b). The asymptotic behavior for large ω is given by

$$\frac{\sigma_{\text{shift}}^{xx^{2\ell'+1}}}{\sigma_{\text{shift}}^{(2\ell'+1)}} \propto \theta(\omega - \omega_{\text{cr}}) \frac{\hbar B^2 J_{\parallel}^2 J^{2\ell'-1}}{(\hbar\omega)^{4\ell'+3}} \quad (110)$$

for $\ell' \geq 1$ with Eq. (51). There are several dips in Fig. 2(b), where $\sigma_{\text{shift}}^{xx^{2\ell'+1}} = 0$. In general, the conductivity $\sigma_{\text{shift}}^{xx^{2\ell'+1}}$ is a polynomial of ω and B , which has several zeros as a function of ω . It leads to the negative divergence of $\log_{10}|\sigma_{\text{shift}}^{xx^{l'}}/\sigma_{\text{shift}}^{(\ell)}|$. There are ℓ dips in the $(2\ell + 1)$ th-order shift current.

E. Finite temperature effects

Finally, we study effects on finite temperature. The jerk current at finite temperature is shown as a function of ω in

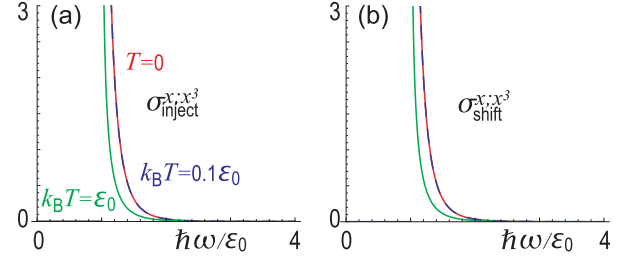


FIG. 3. (a) Jerk current at finite temperature. (b) Third-order shift current at finite temperature. Red dashed curves indicate the absolute zero temperature $T = 0$, blue dashed curves indicate $k_B T = 0.1\varepsilon_0$, and green curves indicate $k_B T = \varepsilon_0$. We have set $\Theta = \pi/2$, $J = \varepsilon_0/k_0$, $\hbar^2 k_0^2/(2m) = \varepsilon_0/2$, and $B = 0.5\varepsilon_0$.

Fig. 3(a). The jerk current decreases for high temperature. The third-order shift current at finite temperature is shown as a function of ω in Fig. 3(b). It also decreases for high temperature.

They are understood as follows. The Fermi distribution function difference is given by

$$f_{-+} = \frac{1}{\left(\exp \frac{\hbar^2 k_{\perp}^2}{2m} + \frac{\omega}{2} - \mu\right) + 1} - \frac{1}{\left(\exp \frac{\hbar^2 k_{\perp}^2}{2m} - \frac{\omega}{2} - \mu\right) + 1}. \quad (111)$$

We plot f_{-+} as a function of ω in Fig. 4(a). It significantly decreases at the optical band edge. We also plot f_{-+} at the optical band edge in Fig. 4(b). f_{-+} is analytically given by

$$f_{-+} = \tanh \frac{|B|}{2k_B T} \quad (112)$$

at the optical band edge $\omega = 2|B|$, where we used $k = 0$ and $\omega = 2|B|$. It monotonically decreases as a function of T . The asymptotic behavior is analytically given as

$$\lim_{\omega \rightarrow \infty} f_{-+} = \exp \left[\frac{-\omega^2}{8J^2 m k_B T} \right] \quad (113)$$

for large ω .

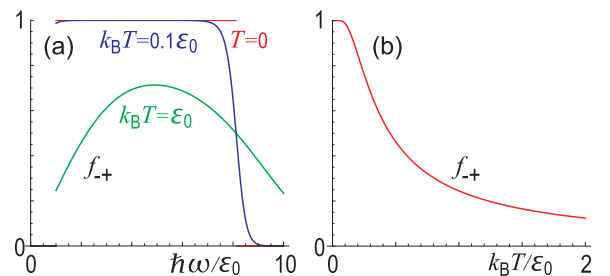


FIG. 4. (a) ω dependence of the Fermi distribution difference f_{-+} . (b) Temperature dependence of the Fermi distribution difference f_{-+} at the optical band edge $\omega = 2B$. We have set $J = \varepsilon_0/k_0$, $\hbar^2 k_0^2/(2m) = \varepsilon_0/2$, and $B = 0.5\varepsilon_0$.

V. CONCLUSION

We have generalized the second-order photovoltaic currents to the ℓ th-order photovoltaic currents for an arbitrary ℓ . Explicitly, the ℓ th-order injection conductivity is given by

$$\sigma_{\text{inject}}^{x;x^\ell} \propto \tau^{\ell-1} \sum_{\mathbf{k}} f_{-+} \frac{\partial^{\ell-1} \omega_{+-}}{\partial k_x^{\ell-1}} r_{-+}^x r_{+-}^x \delta(\omega_{+-} - \omega). \quad (114)$$

The ℓ th-order shift conductivity is given by

$$\sigma_{\text{shift}}^{x;x^\ell} \propto \sum_{\mathbf{k}} f_{-+} r_{-+;x}^x \nabla_{k_x}^{\ell-2} r_{+-}^x \delta(\omega_{+-} - \omega). \quad (115)$$

Then, we have calculated them in p -wave magnets and obtained various analytic results.

We have found that the injection current and the shift current are zero based on the two-band Hamiltonian. This Hamiltonian well describes the p -wave magnet since it is derived [49] from the four-band Hamiltonian describing a realistic material [41].

We estimate the magnitude of the higher-order injection current and the higher-order shift current.

The order of the jerk current (105) is estimated as

$$J_{\text{inject}}^{(3)} \sim \frac{e^4 \tau^2}{\hbar^3} \frac{B^4 J^3}{(\hbar\omega)^7} E_x^3 \approx 1 \mu\text{A}, \quad (116)$$

where we have set $\hbar\omega \approx 2B = 10$ meV, and used [22] $E_x = 10^6$ [V/m], $\tau = 100$ [fs]. We have also used [41,48] $J = 1.6$ eV Å. Then, third-order shift current is estimated from Eq. (108) as

$$J_{\text{shift}}^{(3)} \sim \frac{e^4 \tau^2}{\hbar^3} \frac{B^4 J^3}{(\hbar\omega)^7} E_x^3 \approx 1 \mu\text{A}. \quad (117)$$

The ratio between the ℓ th-order injection current and the $(\ell + 1)$ th-order injection current is estimated from Eq. (107) as

$$\frac{eJ}{\hbar^2 \omega} E_x \tau \approx 0.01. \quad (118)$$

Namely, the crackle current is estimated to be ≈ 1 nA. The ratio between the ℓ th-order shift current and the $(\ell + 1)$ th-order shift current is estimated from Eq. (110) as

$$\frac{eE_x J}{(\hbar\omega)^2} \approx 1. \quad (119)$$

Namely, the fifth-order shift current is estimated to be ≈ 0.1 μA . They are experimentally feasible in current technology [59,60].

These results will be useful for future experiments.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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