

Exploring entropic orders: High-temperature continuous symmetry breaking and topological order

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High temperature is usually expected to destroy order: as the Gibbs state approaches the infinite-temperature limit, it becomes an equal-weight ensemble over all states and the system is generically disordered. Recent works showed that entropic order can violate this expectation through coupling to bosons in classical lattice models and quantum field theories, where the ordered states have higher entropy. Here we present new analytic methods for constructing quantum lattice models that exhibit entropic orders. In particular, we construct quantum lattice models with continuous symmetry breaking at high temperature in $1 + 1$ dimensions and clarify how entropic order can evade the Hohenberg-Mermin-Wagner theorems. We also construct high-temperature entropic $p + ip$ chiral topological superconducting states in $2 + 1$ dimensions with temperature-independent anyon correlation functions. In addition, we obtain a broad family of high-temperature entropic nonchiral topological orders. We show that the entropic topological orders have strong higher-form symmetries at high temperature unlike the conventional topological orders, and the symmetry is spontaneously broken. These results follow from two general constructions that couple a given lattice model with a low-temperature ordered phase either to ordered bosons, or for local commuting-projector Hamiltonians, to more general bosonic degrees of freedom.

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I. INTRODUCTION

Traditions says at sufficiently high temperature a quantum system belongs to phase without any order. This is substantiated by the property that the Gibbs state for a system with bounded energy will approach the maximally mixed state at sufficiently high temperature $T \rightarrow \infty$, i.e., $\beta = 1/T \rightarrow 0$ (we will adopt the units where $k_B = 1$) [1,2]

$$(\beta \rightarrow 0) \quad e^{-\beta H} \rightarrow \mathbf{1} \quad \text{for } |H| < \mathcal{E}. \quad (1)$$

More precisely, the authors of [2] showed that for the tensor product Hilbert space with finite local Hilbert space, and the Hamiltonian was a sum of local term $H = \sum H_i$ such that $|H_i| \leq c$ was uniformly bounded by some constant c , then there was a critical temperature above which the Gibbs state became trivial. This is called “sudden death of thermal entanglement.”

This property of the Gibbs state can be violated if the local Hamiltonian is not bounded. Indeed, it was discovered in recent literature [3–6] that in a system with bosons whose particle number can be arbitrarily high $n = 0, 1, 2, \dots$, there can be entropic order, that is, the Gibbs state does not approach

the maximally mixed state, but instead the ordered one. A similar phenomenon was also discovered in quantum field theory [7–12]. The ordered states at arbitrarily high temperature were enabled by higher entropy compared to the disordered states, and they were called *entropic orders* in [3]. While there exist many constructions for entropic orders including classical statistical lattice models and quantum field theories, more general mechanisms for entropic quantum lattice models are desirable to explore the landscape of entropic orders.

In this article, we present two general methods for constructing quantum lattice models that exhibit entropic orders at high temperature from a given quantum lattice model with low-temperature order: a frustration-free Hamiltonian coupled to ordered bosons (Theorem 1), and a commuting projector Hamiltonian coupled to general bosons (Theorem 2). Both methods apply to any space dimensions and any lattice. We illustrate our method with various explicit models, including high-temperature Heisenberg ferromagnetism and high-temperature chiral topological orders, as well as high-temperature stable entropic topological ordered phases described by local commuting projector models. In particular, we show that entropic models enables us to bypass the Hohenberg-Mermin-Wagner theorems, which are no-go theorems for continuous symmetry breaking in spin chain models at low space dimensions [13,14]. The entropic orders discussed here are distinct from related phenomena in which a quantum system becomes ordered at finite, nonzero temperature, such as inverse melting [15,16]; the Pomeranchuk effect in ^3He , where the solid phase is favored over the liquid phase at higher temperature under suitable pressure [17,18]; the order-by-disorder mechanism [19]; and self-correcting

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quantum memories [20–23]. In all of these cases, the order disappears once the temperature becomes sufficiently high. This is fundamentally different from the entropic orders which persist to arbitrarily high temperature.

II. ENTROPIC ORDERS FROM ORDERED BOSONS

Let us consider a local Hamiltonian in the tensor product Hilbert space with finite-dimensional local Hilbert spaces, $H_0 = \sum H_i$. We will shift each local Hamiltonian by a constant to make sure that the spectrum of energies is non-negative, $E_i \geq 0$. The Gibbs state at temperature $T = 1/\beta$ is $\rho_0 = \sum_I e^{-\beta E_I} |E_I\rangle\langle E_I|$, where $|E_I\rangle$ is the energy eigenstate with energy E_I .

To simplify the discussion, we will focus on the original model H_0 that is frustration free, i.e., $H_0 = \sum_i H_i$ such that the ground states minimize each term H_i . We will also add a constant to the Hamiltonian such that the energy eigenvalues are nonnegative $H_i \geq 0$, hence the ground state has energy $E_0 = 0$ [24].

In the following, we will couple the above Hamiltonian to two types of bosons with local boson number operators $n_i, m_i \in \{0, 1, 2, \dots\}$, such that n_i, m_i mutually commute and also commute with all operators in the original Hilbert space for H_0 . The resulting model realizes an entropic order, such that a ground state of the original Hamiltonian H_0 is realized by a Gibbs state at arbitrary high finite temperature.

The ordered boson entropic models is $H = H_1 + H_2$:

$$\begin{aligned} H_1 &= \sum_i (a + bH_i)(1 + n_i), \\ H_2 &= \sum_i (a' - b'\delta_{(n_i - n_{i+1}), 0})(1 + m_i), \end{aligned} \quad (2)$$

where a, b, a', b' are real constants such that the $a \geq 0, b > 0$, and $a' > b' > 0$. The energy spectrum is bounded below since $H_i \geq 0$ by the assumption described above.

Theorem 1. Suppose that the Hamiltonian $H_0 = \sum_i H_i$ is frustration free, and $H_i \geq 0$ and has a unique ground state $|E_0\rangle$ with the energy $E_0 = 0$. Consider the Gibbs state of Eq. (2) with $a' \rightarrow b'^+$, $a = 0$. At any nonzero temperature, the Gibbs state after tracing out the bosons $|n_i\rangle, |m_j\rangle$ produces the ground state of the original model H_0 .

See the Supplemental Material (SM) [25] for the proof. In the proof, the ordered bosons reduce the coupling to the noncommuting local terms of the frustration-free Hamiltonian to a coupling between the full Hamiltonian H and a single collective bosonic occupation number, thereby making the partition function exactly tractable.

When the original model has degenerate ground states $|0\rangle_\alpha$ labeled by α , in the entropic model the Gibbs state will produce a mixed state given by equal weight sum of these degenerate ground states $\bigoplus_\alpha |0\rangle_\alpha$ with the limit $\lim_{L \rightarrow \infty} \lim_{a \rightarrow 0^+} \lim_{a' \rightarrow b'^+}$. We remark that the entropic model with ordered bosons in Theorem 1 is reminiscent of the entropic order described by continuum field theory with two bosons discussed in [3].

III. SPONTANEOUSLY BROKEN CONTINUOUS SYMMETRY AT HIGH TEMPERATURE

We construct quantum lattice models in $1 + 1d$ that spontaneously break continuous symmetries at arbitrary high temperature. The symmetry breaking can be detected by first turning on a small external field h that breaks the symmetry explicitly (and thus lifts the vacua degeneracy), taking the thermodynamic limit, and then taking the limit $h \rightarrow 0$.

Consider the ferromagnetic Heisenberg model whose ground state spontaneously breaks $SO(3)$ symmetry, coupled to a boson that enables an entropic order at high temperature. At any fixed finite temperature, for any nonzero h the Gibbs state realizes an exact ground state of the original Heisenberg model breaking the symmetry, therefore the order parameter of the continuous symmetry has a nonzero expectation value in the limit $h \rightarrow 0$ in the entropic model at any fixed temperature. Thus we conclude that the continuous symmetry is spontaneously broken in the entropic model at finite temperature. For spin s , the Heisenberg model is

$$H_0 = J \sum_i [2s(2s + 1) - (S_i^x + S_{i+1}^x)(S_i^y + S_{i+1}^y)],$$

where $J > 0$ is a constant and $S_i^\pm := S_i^x \pm iS_i^y$. The model has $SO(3)$ symmetry, with generator $\tau^\alpha = \sum_i S_i^\alpha$ for $\alpha = x, y, z$, as well as time-reversal symmetry. The last line shows that the Hamiltonian is a sum of nonnegative terms. Moreover, the ground state minimizes each Hamiltonian term, and is hence frustration free.

To lift the degeneracy, we add an order parameter

$$H'_0 = H_0 + h \sum_i (s + S_i^z), \quad (3)$$

with $h > 0$. This leaves a unique ground state.

Now, we start with the frustration-free model H'_0 and couple it to the bosons with number operators n_i, m_i . Denote $H_0 = \sum_i H_i$. The entropic Heisenberg model is

$$\begin{aligned} H &= \sum_i [a + bH_i + bh(s + S_i^z)](1 + n_i) \\ &+ \sum_i [a' - b'\delta_{(n_i - n_{i+1}), 0}](1 + m_i). \end{aligned} \quad (4)$$

For $a \rightarrow 0, a' \rightarrow b'^+$, the Gibbs state at any nonzero temperature for the above $1 + 1d$ model with external field h exhibits continuous symmetry breaking: for system size L , $\lim_{h \rightarrow 0^+} \lim_{L \rightarrow \infty} \lim_{a \rightarrow 0, a' \rightarrow b'^+} \langle S_i^z \rangle = -s \neq 0$.

IV. HOW ENTROPIC MODELS BYPASS HOHENBERG-MERMIN-WAGNER THEOREMS

The original Hohenberg-Mermin-Wagner (HMW) theorems are no-go theorems for symmetry breaking on certain spin-chain models [13, 14] at $T > 0$, which is later generalized to the cases $T = 0$ in, e.g., [26, 27] (see [28] for a review). The theorems are derived by using the Bogoliubov inequality that originates from the Cauchy-Schwarz inequality for correlation functions. Here we will review the arguments following [29] and discuss how the conclusion is modified in the entropic models due to the bosons.

We first review the logic of the original HMW theorem. Consider a $1d$ spin chain model with the Hamiltonian H with the system size L . The Hamiltonian has a spontaneously broken continuous symmetry whose charge is $Q = \sum_x Q_x$. The order parameter for the spontaneous symmetry breaking (SSB) is given by a local Hermitian operator O_x at the site x , with $O := \sum_x O_x$. We assume the translation invariance of the operators H, Q, O . We define the Hamiltonian with the Zeeman field as $H(h) = H - hO$. We also define a Hermitian operator R_i at each site x , defined through the equation $O_x = [iQ_x, R_x]$. The HMW theorem is then derived by the use of the Bogoliubov inequality valid at finite temperature $T > 0$ [13,14,30]

$$\frac{1}{L^2} \sum_k \langle R_k^\dagger R_k + R_k R_k^\dagger \rangle \geq \frac{1}{L^2} \sum_k \frac{2T |\langle [iQ_k^\dagger, R_k] \rangle|^2}{\langle [Q_k, [H(h), Q_k^\dagger]] \rangle}, \quad (5)$$

where O_k is the Fourier transform of the local operators, $O_k := \sum_x e^{ikx} O_x$. Here $\langle \rangle$ denotes the evaluation in the Gibbs ensemble of $H(h)$: $\langle O \rangle = \text{Tr}(e^{-\beta H(h)} O) / \text{Tr}(e^{-\beta H(h)})$. Let us also write the denominator of the right-hand side (rhs) as

$$\frac{1}{L} [Q_k, [H(h), Q_k^\dagger]] = A_k + hB_k. \quad (6)$$

The left-hand side (lhs) of Eq. (5) is equal to $\frac{2}{L} \sum_x \langle R_x^2 \rangle$, which remains $O(1)$ in the limit of $L \rightarrow \infty$. On the rhs of Eq. (5), the numerator $\langle [iQ_k^\dagger, R_k] \rangle / L$ approaches the order parameter $\langle O_x \rangle$ in the limit $k \rightarrow 0$. Meanwhile, as long as the eigenvalues of local Hamiltonian H_i and local charge Q_i are bounded from above, the denominator of the rhs has vanishing asymptotic behavior $A_k \propto k^2$ as $k \rightarrow 0$, as shown in the SM [25].

Now, in the limit of $L \rightarrow \infty$, the sum over k in Eq. (5) is converted into the integral over $-\pi \leq k \leq \pi$. Equation (5) is then satisfied in the limit $\lim_{h \rightarrow 0^+} \lim_{L \rightarrow \infty}$ only if

$$\int_{-\pi}^{\pi} \frac{dk}{2\pi} \frac{2T |m|^2}{A_k} = O(1), \quad (7)$$

where $m := \langle O_x \rangle_{h \rightarrow 0^+}$ is the order parameter. Since $A_k \propto k^2$ as $k \rightarrow 0$, one needs to satisfy $m = 0$, therefore the symmetry cannot be broken at finite temperature.

Let us now consider the entropic Heisenberg model at $s = 1/2$: the Hamiltonian is $H = H_1 + H_2$,

$$\begin{aligned} H_1 &= \sum_i [a + b(1 - X_i X_{i+1} - Y_i Y_{i+1} - Z_i Z_{i+1})](1 + n_i), \\ H_2 &= \sum_i (a' - b' \delta_{(n_i - n_{i+1}), 0})(1 + m_i), \end{aligned} \quad (8)$$

where the $U(1)$ symmetry is generated by $Q = \frac{1}{2}X$, with the order parameter $O = \frac{1}{2}Z, R = -\frac{1}{2}Y$. In this entropic model, Eq. (5) is compatible with the SSB due to unbounded local Hamiltonian. Now, let us consider the limit of $a' \rightarrow b'^+, a \rightarrow 0^+, L \rightarrow \infty$ while fixing h to be finite. After tracing out the bosons $\{m_i, n_i\}$ with this limit, only the polarized ground state of the Heisenberg model satisfying $H_i = Z_i = 1$ contributes to the expression

$$\lim_{a \rightarrow 0^+} \lim_{a' \rightarrow b'^+} A_k = 2b(1 - \cos k) \lim_{a \rightarrow 0^+} \left(\frac{e^{-\beta a}}{1 - e^{-\beta a}} \right) \rightarrow \infty.$$

See the SM [25] for detailed derivations of this divergent behavior. Hence, the sum over k in the rhs of Eq. (5) with $k = 2\pi j/L, j \in \mathbb{Z}_L$ becomes zero in the limit $\lim_{a \rightarrow 0^+}$ due to the diverging denominator, even if the numerator remains nonzero. As a result, Eq. (5) is trivially satisfied in the entropic model in the limit $\lim_{h \rightarrow 0^+} \lim_{L \rightarrow \infty} \lim_{a \rightarrow 0^+} \lim_{a' \rightarrow b'^+}$, even if the order parameter in the numerator $\langle O_x \rangle_{h \rightarrow 0^+}$ remains finite. Therefore, the Bogoliubov inequality is compatible with SSB in the entropic model. A possible field-theory description for the entropic model can be obtained from the standard field-theory description of the Heisenberg spin chain using compact bosons (see, e.g., [31]) and replace the Heisenberg coupling by the boson current for another complex boson that has a quartic potential similar to [3] so the boson is ordered. We leave the detailed analysis of such field theory to future study.

The example also illustrates that ordered bosons are important for the construction. The goal is to entropically favor the ferromagnetic ground state selected by the field. However, if the bosonic variables independently fluctuate from site to site, they act analogously to a disordered local field or locally varying couplings, which does not coherently favor a uniform ferromagnetic state. The ordered bosonic sector avoids this problem: the bosons are forced into a spatially ordered configuration, so that tracing them out produces an entropic weight depending on the global energy of the original frustration-free Hamiltonian. In the appropriate coupling limit, this selectively enhances the exact ground-state sector of H_0 .

V. CHIRAL ENTROPIC STATES

Let us apply the construction of the entropic model to chiral topological states. It is known that any local Hamiltonian $H_0 = \sum_i H_i$ with gapped ground states admits decomposition into frustration-free quasilocal Hamiltonians $H_0 = \sum_i \tilde{H}_i$, where $\tilde{H}_i$ has tails decaying faster than any polynomials [32]. See also [33] for explicit constructions of frustration free Hamiltonians of gapped free complex fermions with exponentially decaying tails.

For instance, consider a frustration-free realization of the $p + ip$ superconductor of free Majorana fermions in $2 + 1d$: $H_0^{(p+ip)} = \sum_i \tilde{H}_i^{(p+ip)}$, where $\tilde{H}_i^{(p+ip)}$ is a Hermitian quadratic operator of Majorana fermions with exponentially decaying tails. The entropic model (2) with this $p + ip$ Hamiltonian $H_0 = H_0^{(p+ip)}$ then produces an entropic version of chiral topological superconductors.

After gauging the fermion parity symmetry of the above entropic model with ν copies of $p + ip$ superconductors, the entropic models describe entropic chiral $\mathbb{Z}_2$ topological orders given by the $\text{Spin}(\nu)_1$ Chern-Simons theory realized at any finite temperature. In particular, for $\nu = 2$ this is the Abelian anyon $\text{Spin}(2)_1 = U(1)_4$, for $\nu = 4$ this is the Abelian anyon $\text{Spin}(4)_1 = SU(2)_1 \times SU(2)_1 = U(1)_2 \times U(1)_2$, for $\nu = 8$ this is the three-fermion theory $\text{Spin}(8)_1 = \text{FFF}$.

Consider the correlation functions of the anyon string operators $\mathcal{W}_i$ such as the vortices in topological superconductors. The operators are independent of the boson operator n_i , and thus the correlation functions do not depend on the

temperature in the entropic model for $a = 0$, $a' \rightarrow b'^+$:

$$\text{Tr}\left(e^{-\beta H} \prod_i \mathcal{W}_i\right) = \langle 0 | \prod_i \mathcal{W}_i | 0 \rangle, \quad (9)$$

where $|0\rangle$ is the ground state of the original model. In particular, when the correlation function contains a string operator $\mathcal{W}_i$ that creates excitations, then the thermal correlation function vanishes, i.e., no anyons are created from thermal fluctuations. This is in drastic contrast to the thermal behavior of the p -wave topological superconductors (e.g., [34]).

VI. EXACT HIGHER-FORM SYMMETRIES AT HIGH TEMPERATURES

In the above chiral topological order state since the string operators that create nontrivial excitations have vanishing correlation functions, the Gibbs state has *exact (strong) one-form symmetry* generated by closed loop operators of anyons: $\mathcal{W}_{\text{loop}}\rho = \alpha\rho$, $\rho = e^{-\beta H}$, where α is a number. For comparison, in the ordinary chiral topological superconductors, the anyon loop symmetry is only a weak (average) symmetry where the symmetry operator commutes with the density matrix, but not a exact (strong) symmetry [35]. Moreover, just as the ground state of the original model has spontaneously broken strong one-form symmetry down to nothing since the anyon loops have the same correlation functions in the Gibbs state of the entropic model as the original ground states, the anyon loop strong one-form symmetry is also spontaneously broken in the Gibbs state of the entropic model. This is true for the Gibbs state of the entropic model at arbitrarily high temperature.

More generally, a finite-temperature Gibbs state on a tensor-product Hilbert space cannot realize an exact strong symmetry whenever the theory admits charged excitations of finite energy. Indeed, at any nonzero temperature, such excitations carry nonzero Boltzmann weight, so the thermal state is no longer confined to the neutral sector required by the strong symmetry. Entropic models evade this obstruction through the divergent entropy of the ground-state sector due to the bosons, which overwhelmingly enhances the statistical weight of strongly symmetric states and correspondingly suppresses the contribution of charged excitations.

We remark that the (approximate) strong two-form symmetry of finite-temperature Gibbs state is also discussed in, e.g., [36]. Here, approximate strong symmetry means that the Gibbs state ρ is mapped by a local quantum channel $\mathcal{C}$ into a state with the exact strong symmetry. This approximate strong symmetry is then used to show that the low-temperature thermal state must exhibit long-range entanglement. By contrast, in the entropic models the Gibbs state itself enjoys an *exact* strong higher-form symmetry without the need of a local recovery channel. This exact strong symmetry persists even at arbitrarily high temperature, rather than only below a finite-temperature threshold.

VII. HIGH-TEMPERATURE HALL CONDUCTIVITY IN ENTROPIC MODELS

We start with an integer quantum Hall system of Chern number ν and construct a fractional quantum Hall (FQH)

system by gauging the fermion parity. The well-defined charge operator is $Q' := \sum_i 2Q_i$, i.e., the faithful $U(1)$ symmetry is the $\mathbb{Z}_2$ quotient of the previous $U(1)$. The gauged model has Hall conductance $\sigma_{xy} = \frac{e^2}{4h}\nu$.

When the original Chern insulator is realized by a frustration-free Hamiltonian, then the resulting bosonic FQH Hamiltonian H_0 is also frustration free. We can couple this system to an ordered boson to obtain an entropic model. Let us compute the fractional Hall conductivity of the entropic model in the Gibbs state in the limit of couplings where the high-temperature Gibbs state reduces to a pure state. To do so, we can put the system on torus with twisted boundary conditions in the x, y direction by the $U(1)$ transformation θ_x, θ_y . We denote the twisted FQH Hamiltonian on a torus $H_0(\theta_x, \theta_y)$; this is a frustration-free parent Hamiltonian of the FQH state $|\theta_x, \theta_y\rangle$, where the $U(1)$ twist θ_x, θ_y is inserted adiabatically starting from the untwisted ground state $|0, 0\rangle$. Here, any choice of ground state will give the same Hall conductance.

The twisted boundary condition can be implemented by the $U(1)$ symmetry defects V_{θ_x, θ_y} . Consider the following path in the state space: starting from the ground state without twisted boundary condition $(\theta_x, \theta_y) = (0, 0)$, we continuously increase the twist to $(\theta_x = 2\pi, \theta_y = 0)$, and then $(\theta_x = 2\pi, \theta_y = 2\pi)$, and then decrease it to $(\theta_x = 0, \theta_y = 2\pi)$, and finally to $(\theta_x = 0, \theta_y = 0)$. Then the Hall conductance is obtained from the Berry phase of the state along the path [37]

$$\langle V_{0,2\pi}^\dagger V_{2\pi,0}^\dagger V_{0,2\pi} V_{2\pi,0} \rangle = \text{Tr}(V_{0,2\pi}^\dagger V_{2\pi,0}^\dagger V_{0,2\pi} V_{2\pi,0} \rho) = e^{\frac{2\pi i \nu}{4}}, \quad (10)$$

where the Berry phase reduces to the commutator of the $U(1)$ symmetry defects along the x, y directions. In the limit $a' \rightarrow b'$, $a = 0$, the density matrix becomes the ground state of the original model, and the above equation gives the same fractional Hall conductivity as the original model. The Hall conductance discussed above should be understood as a response protected by the strong $U(1)$ symmetry of the entropic Gibbs state. In this setting, it is meaningful to consider the adiabatic insertions of $U(1)$ fluxes, or equivalently, $U(1)$ twisted boundary conditions, and to define the Hall response by following Laughlin's argument. We note that the system here has strong $U(1)$ symmetry without additional weak symmetry. For more general mixed states in $2 + 1D$ that also have additional weak symmetry such as time reversal, see, e.g., [38–40]. The symmetry properties of mixed states can also be treated using symmetry TFT methods as in, e.g., [41–43]. We note that the correlation functions in the Gibbs state is given by averaging over the degenerate ground states in the original model; since every ground state in the original model gives the same Hall conductivity, the average does not do anything.

VIII. ENTROPIC ORDERS FROM LOCAL COMMUTING PROJECTOR MODELS

We begin with a local commuting projector model on tensor product Hilbert space

$$H_0 = - \sum_i P_i, \quad P_i^2 = P_i, \quad P_i P_j = P_j P_i. \quad (11)$$

Let us enlarge the Hilbert space by including environment: the new Hilbert space is the original one tensored with an infinite dimensional Hilbert space labeled by arbitrary boson numbers $|n_i\rangle$ for $n_i = 0, 1, 2, \dots$. We modify the original Hamiltonian H_0 to couple the system with the environment by the new Hamiltonian

$$H = \sum (a - bP_i)(1 + n_i), \quad (12)$$

where $a > b > 0$. The boson number operators n_i commute with all operators in the original Hilbert space, $n_i P_j = P_j n_i$. The ground states satisfy $P_i = 1$, $n_i = 0$ for all i , i.e., the ground states are the same as the original model, with the environment bosons absent.

Theorem 2. The Gibbs state of the entropic model (12) at temperature T with the environment bosons $|n_i\rangle$ traced out is equivalent to the Gibbs state of the clean model (11) at temperature T_{eff} , where T_{eff} is related to T by their inverse temperatures $\beta = 1/T$, $\beta_{\text{eff}} = 1/T_{\text{eff}}$ as

$$\beta_{\text{eff}} = \beta b + \ln \left(\frac{1 - e^{-\beta a}}{1 - e^{-\beta(a-b)}} \right). \quad (13)$$

In particular, even when β is finite, $\beta_{\text{eff}} \rightarrow \infty$ (i.e., approaching zero effective temperature for the original model H_0) for $a \rightarrow b^+$.

The proof of Theorem 2 is found in the SM [25]. We remark that a special case of the entropic models for local commuting projectors is discussed in [3,5], which only consider Calderbank-Shor-Steane (CSS) codes. Note that the entropic versions of these stabilizer codes are not error correcting quantum codes as ordinary stabilizer codes due to zero code distance; instead, one should consider them as subsystem codes, where the bosons correspond to gauge qudits. Since we want to engineer the ground-state wave function of the original model using the Gibbs state of the entropic model, the question is how to prepare the Gibbs states rather than the ground states in a fault-tolerant way such as quantum Gibbs sampling [44–46].

IX. ENTROPIC NONCHIRAL TOPOLOGICAL ORDERS AND TOPOLOGICAL PHASE TRANSITIONS

Using the above construction of entropic models from local commuting projector models, we can construct entropic versions of the Levin-Wen lattice models [47], Walker-Wang lattice models [48,49], quantum double models [50,51], and their twisted versions [52], fractonic models such as [53–55], as well as lattice models for group cohomology symmetry protected topological (SPT) phases [56] and beyond group cohomology SPT phases [57–59].

Of particular interest are the thermally stable topological orders such as the $4 + 1d$ or $6 + 1d$ loop toric codes [20–22]. The model exhibits quantum memories at finite temperature $0 < T < T_c$. Since the original loop toric code models have electric and magnetic excitations, different excitations have different critical temperatures depending on the tension of the

excitations, called T_A, T_B in [60]. The critical temperature for quantum memory is $T_c = \min(T_A, T_B)$. For temperature in between $T_c < T < \max(T_A, T_B) := T'_c$, one type of excitation is suppressed, while the other type of excitation still proliferates and we have a classical memory with nonzero topological entanglement entropy as shown in [60].

In the corresponding entropic model, where the effective inverse temperature is Eq. (13) this means that there are following phase transitions from dialing the couplings a, b : (1) the entropic model has stable topological order; (2) the entropic model does not have topological order, but it has classical memory; and (3) no topological order and no classical memory. The detailed couplings at these phase separations are given in the SM [25].

We remark that the topological phase transitions from dialing the couplings a, b at fixed temperature are described by local commuting projector models. This is in contrast to the conventional transverse field Ising model, where the phase transitions are described by noncommuting Hamiltonians.

X. CONCLUSION

In this work, we substantially enlarge the scope of entropic orders by introducing two analytic constructions that realize new forms of entropic order, including continuous symmetry breaking, chiral topological states, and nonchiral stable topologically ordered phases at arbitrarily high temperature.

Several directions remain open. It would be interesting to study their low free-energy excited states when the continuous symmetry is broken, including possible Goldstone modes in models with ordered bosonic sectors. Our constructions mainly realize high-temperature states that reduce, in the appropriate sector, to pure-state orders. It would be valuable to generalize the framework to genuinely mixed-state orders, including mixed-state topological orders and strong-to-weak symmetry-breaking phenomena. Finally, it would be useful to identify the physical observables that sharply diagnose these entropic orders, such as thermal and thermoelectric transport coefficients in chiral entropic states. These questions may help clarify how symmetry breaking and topology can persist in thermal quantum matter.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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