Letter

Probing tensor monopoles and gerbe invariants in three-dimensional topological matter

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We show that momentum-space tensor monopoles corresponding to nontrivial vector bundle generalizations, known as bundle gerbes, can be realized in bands of three-dimensional topological matter with nontrivial Hopf invariants. We provide a universal construction of tensor Berry connections in these topological phases, demonstrating how obstructions therein lead to $\mathbb{Z}$-quantized bulk magnetoelectric and nonlinear optical phenomena in the presence of spacetime inversion ($\mathcal{PT}$) symmetry. We then pinpoint that these quantum effects are supported by intraband and interband torsion leading to nontrivial Dixmier-Douady classes in most known Hopf phases and in more general topological insulators realizing gerbe invariants falling beyond the tenfold classification of topological phases of matter. We furthermore provide an interacting generalization upon introducing many-body gerbe invariants by employing twisted boundary conditions. This opens an avenue to study gerbe invariants realized through higher-dimensional charge fractionalizations that can be electromagnetically probed.

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Introduction. A key manifestation of topology in physics is embodied by quantization conditions [1]. Fundamentally, physical topological quantizations can be understood in terms of monopole structures. Examples range from the quantization of charge in particle physics in the form of Abelian Dirac monopoles [2] to the quantization of the Hall conductivity in condensed matter physics [3,4]. Monopoles are generally inherently related to topological invariants in quantum field theory, in which there exist different kinds such as the Yang monopole in five dimensions [5] and the 't Hooft-Polyakov monopole associated with non-Abelian gauge theories [6,7]. Besides Abelian and non-Abelian vector gauge fields, tensorial gauge fields play a central role in high-energy physics, such as in gravity and string theory with relevant applications in condensed matter physics. In particular, the tensorial character of gauge fields allows for a coupling to extended and inhomogeneous force fields, such as strings [8,9], which provides for an emergence of exotic physical phenomena inaccessible to pointlike objects. For instance, suitable symmetric tensor gauge fields have been shown to describe fractons, which are dual to elasticity supporting topological defects in the form of disclinations or dislocations [10,11], and higher-spin field theory [12], a lower-dimensional version of which has recently been employed to study the p-atic phases of the fractional quantum Hall effect [13].

In string theory, antisymmetric tensor gauge fields known as Kalb-Ramond fields $B_{\mu\nu}$ [8] allow for an existence of monopoles sourcing a higher-order field strength $\mathcal{H}_{\mu\nu\rho}$, known as tensor monopoles [14,15], which have been experimentally realized in several synthetic matter systems [16–18]. At the same time, the higher flux $\mathcal{H}_{\mu\nu\rho}$ is also manifested via a kinematic term in the Kalb-Ramond Lagrangian, $\mathcal{L} = -\frac{1}{4}\mathcal{H}_{\mu\nu\rho}\mathcal{H}^{\mu\nu\rho}$, analogously to the Maxwell Lagrangian encompassing electrodynamics [19] and in the equations of motion associated to the topological BF theories [20–23]. Furthermore, in addition to its natural relation to Abelian gauge theories, tensorial flux $\mathcal{H}_{\mu\nu\rho}$ has also been shown to be directly related to the fully antisymmetric components of the torsion tensor in torsionful gravity [9,24].

Importantly, the mathematical roots of antisymmetric tensorial gauge theories are encoded in gerbe bundles [25,26], generalizations of vector bundles that naturally deal with cohomology of higher forms and differential topology in higher dimensions. The relevance of gerbes acquired a recent interest in topological band theory [27–31], and a particularly intensified focus within tensor network constructions for characterizing interacting topological phases of matter [32–37]. One of the main results of this approach is an identification of novel topological classes and invariants known as Dixmier-Douady classes and numbers, respectively [38,39]. These Dixmier-Douady numbers are defined as integers in odd dimensions, differently from Chern numbers defined in even dimensions.

The main goal of this Letter is twofold: (i) We provide a construction of tensor Berry connections [27–31],

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i.e., momentum-space Kalb-Ramond fields, for fermionic condensed matter systems that could support not only stable topological invariants, but also delicate topology [40]. Most interestingly, we show that this also leads to interacting generalizations. In addition, (ii) we identify their observable manifestations provided by tensor monopole quantizations embodied by nontrivial Dixmier-Douady classes in topological gapped phases, such as Hopf insulators [41–46] with underlying quaternionic structures [47], which we show to translate into their linear and nonlinear electromagnetic responses in the presence of spacetime inversion ($\mathcal{PT}$) symmetry. Importantly, we retrieve these quantizations in delicate topological insulators, i.e., falling beyond the tenfold classification based on K-theory [48–50], and unstable against the hybridizations with additional trivial bands [40], showing that the tensorial gauge structures are intrinsic to homotopy-classified phases of matter, i.e., phases with topological invariants defined only under finite multiband partitionings [51]. As such, we propose distinct ways to probe nontrivial bundle gerbe structures and tensorial monopoles in topological phases with nontrivial Dixmier-Douady classes, such as Hopf insulators that realize delicate topology [40]. In this Letter, we thus demonstrate physically observable manifestations of tensor connections in a class of magnetic topological insulators with broken time-reversal symmetry, and with particle conservation symmetry.

Bundle gerbe invariants. We now detail the construction of tensorial Berry connections amounting to gerbe bundle invariants, which we further retrieve as observable in certain topological insulators beyond the tenfold classification [48]. We begin with standard vector Berry connections central to Hall phenomena. In condensed matter physics, the Hall phenomena in two dimensions are induced by the Berry curvature flux in the space of momenta [3,52],

$$F^n_{xy} = \partial_x A^n_y - \partial_y A^n_x, \tag{1}$$

where $A^n_i = \mathrm{i}\langle u_{n\mathbf{k}}|\partial_i u_{n\mathbf{k}}\rangle$ is the vectorial Berry connection, $\mathbf{k}$ is a single-particle momentum, $|u_{n\mathbf{k}}\rangle$ is the nth Bloch band eigenvector, and we use $\partial_i \equiv \partial_{k_i}$ for brevity. Formally, A^n_i is a connection one-form over a line bundle of eigenvectors $|u_{n\mathbf{k}}\rangle$ over Brillouin zone (BZ) in k space, and can be seen as the momentum-space analog of the electromagnetic potential. In a similar way, an existence of a momentum-space version of the Kalb-Ramond field, dubbed a tensor Berry connection B_{ij}, with $i, j = x, y, z$, has been proposed [27–31], which generates the following fluxes of the momentum-space tensorial gauge fields in three dimensions,

$$\mathcal{H}_{xyz} = \partial_x B_{yz} + \partial_y B_{zx} + \partial_z B_{xy}. \tag{2}$$

More formally, B_{ij} can be thought of as a connection of the bundle gerbe (see Sec. I of the Supplemental Material (SM) for details [53]), which we here retrieve on gauging fluxes F_{ij} with pseudoscalar fields ϕ [28]. As a central result, we will construct the tensor connection B_{ij} from the Wilson lines of the Berry connections of Bloch states.

Crucially, the monopoles in tensor Berry connection fields defined over bundle gerbes provide for nontrivial Dixmier-Douady ($\mathcal{DD}$) invariants, as an obstruction to Gauss-Ostrogradsky divergence theorem,

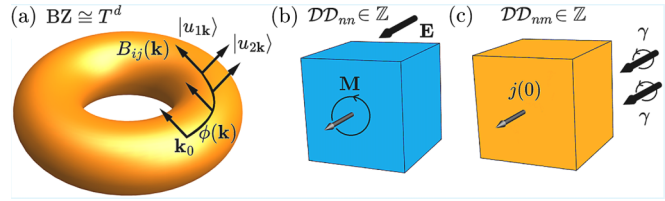


FIG. 1. Responses from nontrivial gerbe bundles in topological insulators. (a) Construction of the pseudoscalar $\phi(\mathbf{k})$ and tensor connection $B_{ij}(\mathbf{k})$ from Bloch states $|u_{1\mathbf{k}}\rangle$, $|u_{2\mathbf{k}}\rangle$ and $\phi(\mathbf{k})$ over d-dimensional Brillouin zone (BZ) realizing topology of a torus (T^d). (b) Magnetoelectric response linear in electric field $\mathbf{E}$, probing intraband $\mathcal{DD}_{nn} \in \mathbb{Z}$ invariants. (c) Nonlinear shift current $j(0)$ response to a pair of circularly polarized photons γ, quadratic in electric field, probing interband $\mathcal{DD}_{nm} \in \mathbb{Z}$ ($n \neq m$) gerbe invariants.

$$\mathcal{DD}_{nm} = -\frac{1}{4\pi^2}\int_{\mathrm{BZ}} \mathrm{d}^3\mathbf{k}\, \mathcal{H}^{nm}_{xyz}, \tag{3}$$

with n, m the band indices, which is a higher-dimensional analog of the Chern class quantization as an obstruction to Stokes's theorem,

$$C_n = \frac{1}{2\pi}\int_{\mathrm{BZ}} \mathrm{d}^2\mathbf{k}\, F^n_{xy}. \tag{4}$$

It is well known that the Chern class underlies the quantization of the Hall conductivity $\sigma_{xy} = \frac{e^2}{h}\sum_n C_n$ supported by individual occupied bands n [3]. On the contrary, here, we construct $\mathcal{H}^{nm}_{xyz}$ from B^{nm}_{ij}, following Eq. (2), with $B^{nm}_{ij} \equiv \phi_{nm}\mathcal{F}^{mn}_{ij}$, where $\mathcal{F}^{nm}_{ij} = \partial_i A^{nm}_j - \partial_j A^{nm}_i$, with $A^{nm}_i = \langle u_{n\mathbf{k}}|\partial_{k_i} u_{m\mathbf{k}}\rangle$ the derivative part of the non-Abelian Berry curvature flux. As such, by construction, $\partial_k \mathcal{F}^{nm}_{ij} + \partial_i \mathcal{F}^{nm}_{jk} + \partial_j \mathcal{F}^{nm}_{ki} = 0$, which shows that tensor monopoles require singular features in $\phi_{nm}(\mathbf{k})$. However, differently from the previous gerbe constructions [27–31,54], and centrally to this work, here we define the pseudoscalar fields as follows,

$$\phi_{nm}(\mathbf{k}) \equiv \int_{\mathbf{k}_0}^{\mathbf{k}} dk'^i A^{nm}_i(\mathbf{k}'), \tag{5}$$

which formally is equivalent to a Wilson line from point $\mathbf{k}_0 = \mathbf{0}$ over the momentum space [see Fig. 1(a)]. In the above, we assume Einstein summation convention, as in the rest of this work. By default, we choose the shortest path, i.e., geodesic, over the parameter space as an integration contour [see Fig. 1(a)]. We demand that the pseudoscalar fields are real by construction, which for $n \neq m$ must be enforced with an additional symmetry, e.g., spacetime inversion symmetry ($\mathcal{PT}$), such that the Hamiltonian and the eigenvectors can be written in real gauge [51,55]. We note that the above construction resembles the dimensional reduction of one-form gauge fields that allows obtaining axion fields in quantum field theory [56,57]. In fact, such pseudoscalar fields can be seen as momentum-space versions of real-space axions. The pseudoscalar phases $\phi_{nm}(\mathbf{k})$ play a key role in realizing the nonvanishing $\mathcal{DD} \in \mathbb{Z}$ invariants. In particular, if $\phi_{nm}(\mathbf{k}) = \mathrm{const}$, then $\mathcal{H}_{xyz} = 0$, identically. The physical interpretation of $\phi_{nm}(\mathbf{k})$ as accumulated geometric phases amounts to positional displacements of the wave packets

formed in bands n and m, upon changing their center-of-mass momentum from $\mathbf{k}_0$ to momentum $\mathbf{k}$, consistently with the formalism of Ref. [58].

Notably, shifting the origin $\mathbf{k}_0 = \mathbf{0}$ to another base point $\mathbf{k}_0'$ collinear with $\mathbf{k}$ and $\mathbf{k}_0$, transforms the pseudoscalars as $\phi_{nm}(\mathbf{k}) \to \phi_{nm}(\mathbf{k}) + \phi^0_{nm}$, with $\phi^0_{nm} \equiv \int_{\mathbf{k}_0}^{\mathbf{k}_0'} dk'^i A_i^{nm}(\mathbf{k}')$. Here, the path between $\mathbf{k}_0$ and $\mathbf{k}_0'$ is the shortest path across the BZ. These can be viewed as gauge transformations of the gerbe two-form connections,

$$B^{nm}_{ij} \to B^{nm}_{ij} + \partial_i \xi^{nm}_j - \partial_j \xi^{nm}_i \equiv B^{nm}_{ij} + \Lambda^{nm}_{ij}, \tag{6}$$

with gauge vectors $\xi^{nm}_i = \phi^0_{nm} A^{mn}_i$, defining an exact two-form Λ^{nm}_{ij}, which is divergenceless and cannot contribute to the flux $\mathcal{H}^{nm}_{ijk}$, nor to the $\mathcal{DD}$ invariants. Moreover, specifically for $\phi_{nn} \equiv \int_{\mathbf{k}_0}^{\mathbf{k}} dk'^i A^n_i(\mathbf{k}')$, we admit a local transformation $A^n_i(\mathbf{k}) \to A^n_i(\mathbf{k}) + \partial_i \alpha^n(\mathbf{k})$ on transforming the Bloch states $|u_{n\mathbf{k}}\rangle \to e^{i\alpha_n(\mathbf{k})}|u_{n\mathbf{k}}\rangle$, $\phi_{nn}(\mathbf{k}) \to \phi_{nn}(\mathbf{k}) + \alpha_n(\mathbf{k})$. Under the non-Abelian transformations with matrices U, $|u_{m\mathbf{k}}\rangle \to [U]_{mn}|u_{n\mathbf{k}}\rangle$, the pseudoscalar matrix, $[\Phi]_{nm} \equiv \phi_{nm}$, transforms as $\Phi \to U\Phi U^\dagger$, consistently with the vector of non-Abelian Berry connection matrices, $[\mathbf{A}]_{nm} \equiv (A^{nm}_x, A^{nm}_y, A^{nm}_z)$, $\mathbf{A} \to U\mathbf{A}U^\dagger$ [31]. U is a general unitary transformation that can depend on $\mathbf{k}$. By construction, we can choose a gauge: $A^{nm}_i(\mathbf{k}) = \partial_i \phi_{nm}(\mathbf{k})$. Upon such particular gauge fixing for non-Abelian Berry connection, $\mathcal{H}^{nn}_{xyz} = \varepsilon^{abc} A^n_a \partial_b A^n_c$, or $\mathcal{H}^{nm}_{xyz} = \varepsilon^{abc} A^{nm}_a \partial_b A^{mn}_c$, which are equal to Abelian Chern-Simons forms amounting to quantized Hopf and real Hopf indices [41,45–47] in topological insulators, when integrated over single-particle eigenstates across BZ. Importantly, although the numerical value of $\mathcal{H}_{xyz}$ can coincide with the Abelian Chern-Simons upon gauge fixing, $\mathcal{H}_{xyz}$ is an independent gauge invariant observable quantity, which can be viewed as a contribution of individual quasiparticle with momentum $\mathbf{k}$ to the magnetoelectric gyration, analogously to the Hall drift contributions from the Berry curvature F^n_{ij} in the Hall current. Notably, the gauge-dependent Chern-Simons form gives rise to a gauge-invariant tensorial flux only when taken in a special, real gauge. Appealing to the real gauges allows isolating interband coherences underpinned by the non-Abelian Berry connections, which physically underlie the introduced constructions of tensor Berry connections and fluxes.

In the following, we show that the $\mathcal{DD}$ invariants, including intraband gerbe invariants $\mathcal{DD}_{nn}$, quantize geometric contributions to magnetoelectric responses, providing for $\mathbb{Z}$ quantizations in realistic three-dimensional topological crystallites [see Fig. 1(b)]. Furthermore, we show that the more general interband $\mathcal{DD}$ gerbe invariants $\mathcal{DD}_{nm}$ quantize nonlinear second-order integrated shift responses to circularly polarized light [see Fig. 1(c)].

Gerbes and magnetoelectric effects. Here, we retrieve the gerbe quantization by the $\mathcal{DD}$ invariants in magnetoelectric responses [59–63]. The magnetoelectric tensor α_{ij} yielding orbital magnetization currents (M_i) from electric fields E_j, $M_i = \alpha_{ij} E_j$, consists of topological and nontopological terms (see Sec. II of SM [53]), where the topological term dominates in tenfold-classified topological materials [61]. We find that the trace of topological magnetoelectric coupling amounts to

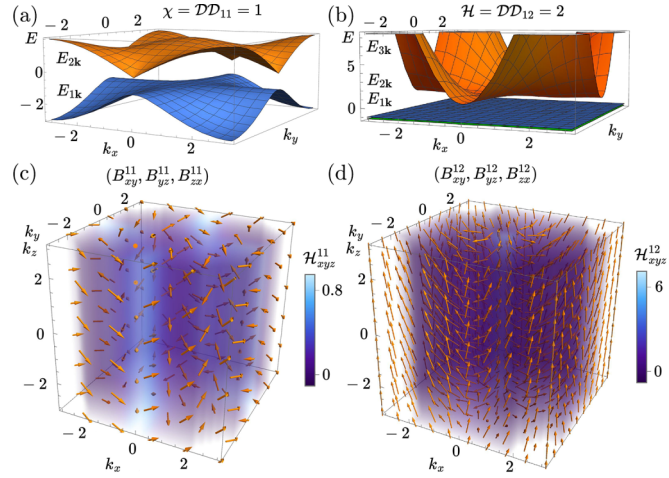


FIG. 2. Tensor monopoles in topological insulators with Hopf indices. (a) Band structure of a two-band Hopf insulator realizing intraband $\mathcal{DD}_{11} \in \mathbb{Z}$ invariant at $M = 1$ and (b) a three-band real Hopf insulator realizing interband $\mathcal{DD}_{12} \in \mathbb{Z}$ invariant at $M = 3/2$. The bands are shown at momentum $k_z = 0$. (c) Tensor monopole of momentum-space Kalb-Ramond field B^{11}_{ij} yielding flux $\mathcal{H}^{11}_{ijk}$ in a Hopf insulator with $\chi = 1$. (d) Tensor monopole of Kalb-Ramond field B^{12}_{ij} introducing an interband torsional flux $\mathcal{H}^{12}_{ijk}$ in a real Hopf insulator with topological index $\mathcal{H} = 2$.

the higher Berry curvature and $\mathcal{DD}$ invariants,

$$\mathrm{Tr}\,\alpha^{\mathrm{top}}_{ij} = -\frac{e^2}{8\pi^2 hc}\int_{\mathrm{BZ}} \mathrm{d}^3\mathbf{k} \sum_{n,n'} \mathcal{H}^{nn'}_{xyz} = \frac{e^2}{2hc}\sum_{n,n'} \mathcal{DD}_{nn'}. \tag{7}$$

Here, the Kalb-Ramond flux over occupied bands, $\mathcal{H}^{nn'}_{xyz} = \varepsilon^{ijk}\mathcal{T}^{nn'}_{ijk}$, admits intraband torsion $\mathcal{T}^{nn}_{ijk}$, when $n = n'$, which we introduce as a generalization of the interband torsion $\mathcal{T}^{nn'}_{ijk}$ [46,64] (see Sec. III of SM [53]). In particular, in a Hopf insulator [41],

$$\mathrm{Tr}\,\alpha^{\mathrm{top}}_{ij} = \frac{e^2}{2hc}\mathcal{DD}_{11} = \frac{e^2}{2hc}\chi, \tag{8}$$

with $\chi \in \mathbb{Z}$ the Hopf invariant, consistently with the $\mathbb{Z}$-quantized Chern-Simons form, $\theta_{\mathrm{CS}} = \pi\chi$ [65]. We identify that irrespectively of the value of θ_{CS}, defined up to 2π as an angle [65], the gauge-invariant gerbe flux $\mathcal{H}^{nn'}_{xyz}$ yields a $\mathbb{Z}$-magnetoelectric effect in the Hopf insulator. Analogously, the $\mathbb{Z}$-magnetoelectric effect arises in the chiral three-dimensional insulators with winding numbers in the chiral symmetry class AIII [63,66]. In fact, in these strong topological insulators, the $\mathbb{Z}$ quantization of the magnetoelectric effect can be also formally explained by employing the gerbe formalism, as the three-dimensional winding number coincides with the $\mathcal{DD}$ invariant [28,67].

In two-band Hopf insulator realizations [41,42], with an occupied topological band $n = 1$, the invariant reads $\chi = -\frac{1}{4\pi^2}\int_{\mathrm{BZ}} \mathrm{d}^3\mathbf{k}\,[\mathbf{A}_{11} \cdot (\boldsymbol{\nabla}_{\mathbf{k}} \times \mathbf{A}_{11})] \in \mathbb{Z}$, which represents the nontrivial homotopy elements $\pi_3[S^2] \cong \mathbb{Z}$, mathematically corresponding to distinct Hopf fibrations: $S^3 \to S^2$ [41]. We show the corresponding Hopf bands in Fig. 2(a), tensor monopole [Fig. 2(c)], and topological magnetoelectric response quantized by the gerbe invariant against the corrections from nontopological responses, as a function of

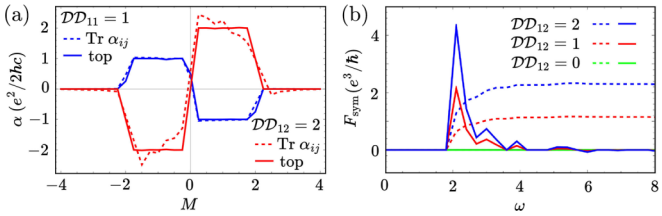


FIG. 3. Quantized responses and gerbe invariants. (a) Topological (top) magnetoelectric susceptibility $\mathrm{Tr}\,\alpha_{ij}^{\rm top}$ (solid) and total ($\mathrm{Tr}\,\alpha_{ij}$) magnetoelectric susceptibility (dashed) in Hopf and real Hopf insulators, as a function of topological mass parameter M. Topological phases with $\mathcal{DD}_{11} = 1$, $\mathcal{DD}_{12} = 2$ correspond to $|M| < 2$. The nontopological magnetoelectric response is dominated by the topological contribution. (b) Integrated shift response against cutoff frequency $\omega = \omega_{\rm max}$ (dashed) for different interband gerbe invariants $\mathcal{DD}_{nm}$ in three-band real Hopf insulators with the lower two bands occupied. The frequency resolution (solid) of the integrated response indicates dominant contributions from the Kalb-Ramond flux $\mathcal{H}_{xyz}^{nm}$ due to fermions photoexcited at frequencies directly above the band gap.

topological mass parameter M in Fig. 3(a). For further details on the models, topological classifications, and underlying gerbe structures, see SM [53].

Moreover, we find that the topological magnetoelectric response is quantized by an *interband* $\mathcal{DD}$ invariant in three-band real Hopf insulators with an occupied pair of topological bands $\{|u_{1\mathbf{k}}\rangle, |u_{2\mathbf{k}}\rangle\}$,

$$\mathrm{Tr}\,\alpha_{ij}^{\rm top} = \frac{e^2}{2hc}\mathcal{DD}_{12} = \frac{e^2}{2hc}\mathcal{H}, \tag{9}$$

where we define $\mathcal{H} = -\frac{1}{4\pi^2}\int_{\rm BZ} \mathrm{d}^3\mathbf{k}\,[\mathbf{A}_{12}\cdot\mathbf{Eu}_{12}] \in \mathbb{Z}$ which is quantized under $\mathcal{PT}$ spacetime inversion symmetry [46], with $\mathbf{Eu}_{12} \equiv \boldsymbol{\nabla}_{\mathbf{k}} \times \mathbf{A}_{12}$ the Euler curvature [68–70]. Under the quantizing symmetry, $\mathcal{H} \in \mathbb{Z}$ furthermore represents distinct nontrivial homotopy elements $\pi_3[S^2] \cong \mathbb{Z}$ [46]. We stress that this quantized magnetoelectric response, which we here show to arise from the gerbe invariant, was not identified in Ref. [46], which introduced the three-band Hopf-Euler band topologies, where a trivial θ angle, $\theta_{\rm CS} = 0\ (\mathrm{mod}\ 2\pi)$, was only recognized. We show the corresponding real Hopf bands in Fig. 2(b), the tensor monopole [Fig. 2(d)], and the magnetoelectric response against the topological mass M for phases with a distinct topological index $\mathcal{DD}_{12}$ [Fig. 3(a)]; see SM (Sec. I) [53] for details on the models. The identification of the Hopf insulator as a condensed matter realization of a momentum-space tensor monopole,

$$\chi = -\frac{1}{4\pi^2}\int_{\rm BZ} \mathrm{d}^3\mathbf{k}\ \mathcal{H}_{xyz} \in \mathbb{Z}, \tag{10}$$

is a central result of this Letter. The integer character of $\chi = \mathcal{DD}_{11}$ can be magnetoelectrically probed, e.g., within the previously proposed general experimental setup [63]. Furthermore, the N-band generalization of a Hopf insulator [71] also realizes a tensor monopole, upon a reformulation of the invariant as a tensor flux, which we explicitly demonstrate in the SM [53].

Gerbes in quantized shift responses. In the following, we show how the gerbe structure associated with interband $\mathcal{DD}_{nm}$ invariants ($n \neq m$) can be probed with nonlinear second-order optical responses. Inspired by high-energy physics results in real spacetime [24], we here employ the momentum-space correspondence between the gerbe flux and the totally antisymmetric part of the torsion, namely $\mathcal{H}_{ijk}^{mn} = \mathcal{T}_{[ijk]}^{mn}$, with $\mathcal{T}_{ijk}^{mn}$ interband torsion, which captures virtual transitions between bands n, m through additional bands p [72–74]. We first recast the torsional sum rule of Ref. [74] for circular shift photoconductivities $\sigma_{ijk}^{\rm shift,C}(\omega)$ [72] that determine geometric dc photovoltaic current density responses $j_i(0) = \sigma_{ijk}^{\rm shift,C}(\omega)E_j(\omega)E_k(-\omega)$ from electronic positional shifts [74–79] induced by optical electric fields $E_j(\omega)$, which we further detail and comment in Sec. III of SM [53],

$$\begin{aligned} F_{\rm sym} &\equiv \int_0^{\omega_{\rm max}} \mathrm{d}\omega\,\left[\sigma_{xyz}^{\rm shift,C}(\omega) + \sigma_{yzx}^{\rm shift,C}(\omega) + \sigma_{zxy}^{\rm shift,C}(\omega)\right] \\ &= -\frac{e^3}{4\pi^2\hbar^2}\int_{\rm BZ} \mathrm{d}^3\mathbf{k}\sum_{m,n} f_{nm}\mathcal{H}_{xyz}^{mn}. \end{aligned} \tag{11}$$

$\omega_{\rm max}$ is an energy cutoff targeting optical transitions between all single-particle states in occupied/unoccupied bands n/m. The filling factor differences $f_{nm} \equiv f_n - f_m$ are fixed on setting a zero-temperature limit, with $f_n = 1$ for occupied and $f_m = 0$ for unoccupied bands, which results in a general torsional quantization from fully occupied bands

$$F_{\rm sym} = \frac{e^3}{\hbar^2}\sum_{n\neq m} \mathcal{DD}_{nm}. \tag{12}$$

We show the integrated shift response against the cutoff frequency $\omega_{\rm max}$ for phases with distinct topological index $\mathcal{DD}_{12}$ in Fig. 3(b); see SM (Sec. I) [53] for details on the models. While a similar physical quantization condition was retrieved in $\mathcal{PT}$-symmetric three-band and four-band models [46,74], the more general gerbe structures providing the quantization as tensor monopoles related to momentum-space tensor Berry connections, as well as their additive character underpinned by Eq. (12), have not been uncovered before. This moreover provides a generalization to arbitrary N-band topologies realizing $\mathcal{DD}$ invariants, beyond the previous four-band and three-band realizations of topological insulators with Hopf indices [45,46,74,80], as the sum over n, m with n occupied and m unoccupied bands is otherwise unconstrained.

Many-body generalization. Importantly, the discussed linear and nonlinear electromagnetic responses can also be utilized to study interacting systems. For example, optical probes, such as circular dichroism, have been employed to study interacting two-dimensional topological systems, such as fractional Chern insulators [81]. Generalizing our results to interactions in three-dimensional topological systems central to this work, necessarily involves many-body $\mathcal{DD}$ invariants. Hence, as an outlook, we briefly detail how the $\mathcal{DD}$ invariants can be generalized to interacting systems with twisted boundary conditions analogous to the construction of many-body Chern invariants [3]. We impose twisted boundary on a many-body wave function $|\psi(\boldsymbol{\theta})\rangle$ with twist angles $\boldsymbol{\theta} = (\theta_x, \theta_y, \theta_z)$ acting as twisted boundary conditions across linear system size lengths L_i [52], $\psi(\{x_i + L_i\}) = e^{i\theta_i}\psi(\{x_i\})$, where $\{x_i\}$ are coordinates of the single particles constituting the

many-body wave function in many-body particle coordinate representation, $\psi(\{x_i\}) = \langle\{x_i\}|\psi(\boldsymbol{\theta})\rangle$. We define the many-body connection, $A_i(\boldsymbol{\theta}) \equiv \mathrm{i}\langle\psi(\boldsymbol{\theta})|\partial_{\theta_i}\psi(\boldsymbol{\theta})\rangle$, and many-body flux, $F_{ij}(\boldsymbol{\theta}) = \partial_{\theta_i}A_j(\boldsymbol{\theta}) - \partial_{\theta_j}A_i(\boldsymbol{\theta})$, alongside a pseudoscalar,

$$\phi(\boldsymbol{\theta}) \equiv \int_{\boldsymbol{\theta}_0}^{\boldsymbol{\theta}} \mathrm{d}\theta_i' \, A_i(\boldsymbol{\theta}'), \tag{13}$$

with a gauge freedom of choosing $\boldsymbol{\theta}_0$. The combination of flux and pseudoscalar defines a tensor connection $B_{ij}(\boldsymbol{\theta}) = \phi(\boldsymbol{\theta})F_{ij}(\boldsymbol{\theta})$. Analogously to the single-particle case, the many-body tensor connection $B_{ij}(\boldsymbol{\theta})$ transforms as $B_{ij}(\boldsymbol{\theta}) \to B_{ij}(\boldsymbol{\theta}) + \Lambda_{ij}(\boldsymbol{\theta})$, with divergenceless two-form $\Lambda_{ij}(\boldsymbol{\theta})$, under the shift of the initial integration limit specifying the zero of twisted boundary conditions ($\boldsymbol{\theta} = 0$) within the many-body pseudoscalar definition. The many-body tensor connection determines the tensorial flux,

$$\mathcal{H}_{xyz}(\boldsymbol{\theta}) = \partial_{\theta_x}B_{yz}(\boldsymbol{\theta}) + \partial_{\theta_y}B_{zx}(\boldsymbol{\theta}) + \partial_{\theta_z}B_{xy}(\boldsymbol{\theta}), \tag{14}$$

and integrates to a many-body $\mathcal{DD}_{\mathrm{MB}}$ invariant,

$$\mathcal{DD}_{\mathrm{MB}} \equiv -\frac{1}{4\pi^2}\iiint \mathrm{d}\theta_x \mathrm{d}\theta_y \mathrm{d}\theta_z \, \mathcal{H}_{xyz}(\boldsymbol{\theta}) \in \mathbb{Z}. \tag{15}$$

On one hand, in the presence of short-range interactions, and no ground-state degeneracy, we expect that this invariant remains quantized, similarly to the quantization of the first Chern number in two-dimensional interacting systems [82,83]. On the other hand, in the case of a q-degenerate ($q \in \mathbb{N}$) ground state, we expect the $\mathcal{DD}_{\mathrm{MB}}$ invariant to be fractionalizable ($\mathcal{DD}_{\mathrm{MB}} = 1/q$), analogously to the many-body Chern invariant [52,82]. The fractionalization could possibly be probed with the discussed linear and nonlinear electromagnetic effects that target the gerbe invariants. The realizations and detailed calculations of fractional $\mathcal{DD}_{\mathrm{MB}}$ invariant in specific three-dimensional interacting and topologically ordered phases will be addressed in future work.

Discussion and conclusion. We demonstrate that gerbe bundle invariants provided by tensor gauge fields associated with tensor monopoles can be probed in three-dimensional fermionic phases of matter such as Hopf insulators and other topological phases with nontrivial $\mathcal{DD}$ invariants. We retrieve probes of these structures in integer magnetoelectric responses targeting the intraband gerbe invariants, as well as nonlinear optical responses probing interband torsion and gauge-invariant Kalb-Ramond fluxes $\mathcal{H}^{nm}_{ijk}$ selectively at the momentum-space sections where the transitions at the photon frequency ω occur (see Fig. 3). In doing so, we provide a unique construction of tensor Berry connections in topological insulators, beyond the conventional Berry connections, showing that momentum-space Kalb-Ramond fields can also be manifestly realized by free fermionic topological phases.

Crucially, in this Letter, we have shown $\mathbb{Z}$-magnetoelectric phenomena in condensed matter systems supported by tensor Berry connections and $\mathcal{DD}$ invariants, apart from significantly generalizing the quantized shift effects beyond the realms of previous works.

Our findings are applicable to three-dimensional magnetic materials, as well as to quantum simulation setups, which proved successful in simulating matter with nontrivial Hopf indices [84]. As in the case of interactions fractionalizing Chern invariants, we derive the many-body generalization and predict the fractionalization of the $\mathcal{DD}$ invariant, here defined under twisted boundary conditions, to arise from many-body ground-state degeneracies. The provided electromagnetic probes could allow one to pinpoint a possible existence of fractional topological insulators characterized by the many-body $\mathcal{DD}$ invariant via magnetoelectric and optical phenomena originally retrieved here for integer quantizations under the noninteracting setups.

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Data availability. The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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