

Fractionalized superconductivity and spin stiffnesses in the SU(2) gauge theory of the two-dimensional Hubbard model

Demetrio Vilardi¹ and Pietro M. Bonetti^{1,2}

¹*Max Planck Institute for Solid State Research, D-70569 Stuttgart, Germany*

²*Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA*



(Received 11 November 2025; revised 4 March 2026; accepted 2 June 2026; published 7 July 2026)

We consider the SU(2) gauge theory for spin fluctuations in the two-dimensional Hubbard model, where the electron field is fractionalized in terms of spinons and chargons. In this theory, spinons are described by a nonlinear sigma model, while chargons are treated as fermions at a mean-field level. We investigate the instability to a superconducting state (SC*), arising from a fractionalized Fermi liquid (FL*) where pairing between chargons occurs. Consistent with previous studies, our analysis reveals a coexisting phase characterized by both magnetic and superconducting order for the chargons. The central contribution of this work is the calculation of the feedback of superconductivity on spatial and temporal spin stiffnesses, thereby quantifying its impact on spin fluctuations. Our key finding is that superconductivity significantly suppresses these spin stiffnesses, enhancing quantum spin fluctuations. This enhancement suggests that superconductivity can play a crucial role in stabilizing quantum disorder against long-range magnetic ordering.

DOI: [10.1103/xcm5-c81y](https://doi.org/10.1103/xcm5-c81y)

I. INTRODUCTION

In high- T_c superconductors, the understanding of the pseudogap phase still poses many challenges [1]. Many experiments exhibit features that are compatible with some sort of magnetic order. Examples include the rapid change in the Hall number across the pseudogap critical doping [2] and angle-dependent magnetoresistance measurements in Nd-LSCO [3]. In La-based cuprates, these observations are broadly consistent with the persistence of spin order up to p^* when superconductivity is sufficiently weakened [4,5], while in other cuprate families, such as YBCO, direct evidence for magnetic order remains more limited.

A possible strategy to understand the pseudogap phase has been proposed in Refs. [6,7] where the electron field has been fractionalized in terms of a *spinon* and a *chargon* field with the appearance of an emergent SU(2) gauge symmetry. The spinon field is an SU(2) matrix which rotates the spin reference frame of the chargon field locally in space and time. The chargon field instead represents a spinless fermion, carrying the electromagnetic charge of the electron, as well as an SU(2) gauge charge. Within such a framework, a Higgs field can be introduced [7–9], *Higgsing down* the gauge group to U(1) or even $\mathbb{Z}_2$, depending on the specific form of the condensate. As long as the spinons remain uncondensed, such a Higgs phase represents a topological state with no magnetic order and emerging gauge fields, in other words a *spin liquid*

[10] coexisting with spinless fermionic degrees of freedom that will form a Fermi surface. The latter is reconstructed by the condensation of the Higgs field, leading to a violation of Luttinger theorem, which is made possible by the emergent fractionalized degrees of freedom [11–13]. A quantity that serves as a proxy of the strength of magnetic fluctuations is the *spin stiffness*, a large value of which implies strong tendencies towards long range magnetic ordering, whereas small values enhance quantum fluctuations, thereby leaving the open possibility for topological order to remain stable.

Moreover, (doped) topological phases can be unstable to other kinds of perturbations, the most relevant of them all being superconductivity. There are essentially two routes to induce superconductivity in a spin liquid. It could occur as a *confining* transition [14] that completely Higgses out the emergent gauge field, leaving behind a trivial superconducting state that smoothly connects to its analog that one would obtain as an instability of a Fermi liquid [15]. Alternatively, in our language, it could occur as pairing of *chargons*, realizing a so-called SC* phase, where superconductivity coexists with fractionalized excitations. Such a superconducting state is separated from a conventional one by a deconfined quantum critical point, describing a nontrivial phase transition. Such an SC* phase has also been studied in the context of Kondo lattice systems in Refs. [16,17].

In this manuscript, we consider an SC* phase emerging out of the topologically ordered phase of the SU(2) gauge theory for the pseudogap phase described above. While confining superconducting phases are probably more suited to describing the phenomenology of cuprate superconductors [15], within the present formalism, simultaneously confining the gauge fields and obtaining Cooper pairing is a nontrivial task. On the other hand, an SC* state can simply emerge as chargon pairing. In this work, we aim to address how the presence

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Open access publication funded by Max Planck Society.

of superconductivity influences the spin stiffnesses. These stiffnesses are crucial as they parametrize the low-energy dispersion of spinon degrees of freedom. Furthermore, they determine the strength of quantum fluctuations, which are required to keep the ground state stable against magnetic order. A useful by-product of our calculation is that our formulas for the spin stiffnesses also apply to more conventional systems with coexisting magnetic and superconducting orders, as in that case chargons can be interpreted as conventional electrons. States with such coexisting orders have recently been observed in clean samples of cuprate superconductors at low doping [18]. A calculation complementary to our work has been performed in Ref. [19], where the superconducting phase stiffness has been computed in such a coexisting order.

We calculate microscopically the spin stiffnesses starting from the square lattice Hubbard model, which has been proposed as a prototypical minimal model for electrons in the copper-oxide planes. The pseudogap phase and superconductivity have been found in this model, using both numerical calculations [20,21] and unbiased diagrammatic techniques [22–24]. Similarly to Ref. [25], where superconductivity was not considered, we fix the parameters of the gauge theory employing the functional renormalization group (fRG) [26] in combination with mean-field theory [27–29]. We find that the presence of superconductivity typically diminishes the value of the spin stiffnesses, thereby enhancing spin fluctuations, which in turn could stabilize topological order with no long range magnetism. This effect on the spin stiffness is expected to persist even in the presence of strong superconducting fluctuations, which exist in the range between the mean-field critical temperature T_{MF} and the Berezinskii-Kosterlitz-Thouless (BKT) [30,31] transition temperature T_{BKT} .

The paper is structured as follows. In Sec. II, we introduce the SU(2) gauge theory and derive an effective low energy model for the low-energy degrees of freedom. In Sec. III, we derive formulas for the spin stiffnesses in the SC* state. In Sec. IV we present our numerical results and, finally, our conclusions are presented in Sec. V.

II. LOW ENERGY MODEL

A. Spiral and superconducting orders

Before introducing the low energy theory for spin fluctuations, we briefly discuss the underlying mean-field order considered here, constituted by the coexistence of incommensurate spiral magnetism and superconductivity in a two-dimensional square lattice.

The spiral order is characterized by a distribution of the average spin operator lying in the xy plane as

$$\langle \mathbf{S}_j \rangle = m[\cos(\mathbf{Q} \cdot \mathbf{R}_j)\mathbf{e}_x + \sin(\mathbf{Q} \cdot \mathbf{R}_j)\mathbf{e}_y], \quad (1)$$

where m is the magnetization, $\mathbf{R}_j$ the position vector of the j th lattice site, $\mathbf{e}_\alpha$ are unit vectors in real space, with $\alpha = 1, 2, 3$, and $\mathbf{Q}$ is a fixed wave vector. The choice of the plane for the spiral order is arbitrary, since it is randomly selected once the SU(2) spin symmetry is broken; here, the xy plane is chosen

for convenience. We define the spin-charge operator as

$$S_j^a = \frac{1}{2} \sum_{s,s'} c_{j,s}^\dagger \sigma_{ss'}^a c_{j,s'}, \quad (2)$$

where σ^a ($a = 0, 1, 2, 3$) are the Pauli matrices (for $a = 1, 2, 3$), σ^0 is the 2×2 identity matrix, and $c_{j,s}$ ($c_j^\dagger$) is the annihilation (creation) operator in real space. For $a = 1, 2, 3$ Eq. (2) represents the components of the spin operator appearing in Eq. (1), while for $a = 0$ it reflects the charge operator. After Fourier transforming Eqs. (1) and (2), we obtain the order parameter

$$m = \int_{\mathbf{k}} \langle c_{\mathbf{k},\uparrow}^\dagger c_{\mathbf{k}+\mathbf{Q},\downarrow} \rangle, \quad (3)$$

where $c_{\mathbf{k},s}$ and $c_{\mathbf{k},s}^\dagger$ are creation and annihilation operators in momentum space. Here, we use a short-hand notation for the momentum integration as $\int_{\mathbf{k}} = \int \frac{d^2\mathbf{k}}{(2\pi)^2}$.

The mean-field Hamiltonian for the state with coexisting spiral magnetism and superconductivity takes the form [19]

$$H = \int_{\mathbf{k}} \sum_{\sigma} \epsilon_{\mathbf{k}} c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} + \int_{\mathbf{k}} \Delta_m c_{\mathbf{k}+\mathbf{Q},\downarrow}^\dagger c_{\mathbf{k}\uparrow} + \int_{\mathbf{k}} \Delta_{\text{p},\mathbf{k}} c_{-\mathbf{k},\downarrow}^\dagger c_{\mathbf{k}\uparrow}^\dagger + \text{H.c.}, \quad (4)$$

where $\Delta_{\text{p},\mathbf{k}}$ and Δ_m are the magnetic and superconducting gap, respectively. Here, we assume the magnetic gap Δ_m to be momentum independent.

This state breaks both the spin SU(2) and charge U(1). In the special case where $\mathbf{Q} = (\pi, \pi)$, i.e., a Néel antiferromagnet, the system has a residual U(1) spin symmetry, referring to the rotations around the magnetization axis. For an incommensurate spiral order with $\mathbf{Q} = (\pi - 2\pi\eta, \pi)$ or $\mathbf{Q} = (\pi, \pi - 2\pi\eta)$, the SU(2) spin group is broken to $\mathbb{Z}_2$.

B. Nonlinear sigma model

The nonlinear sigma model (NL σ M) is a well-established effective field theory that describes the low-energy fluctuations of an order parameter. It is widely used to capture magnetic fluctuations around ordered spin textures [25,32,33]. Its derivation from microscopic models like the Hubbard Hamiltonian, which forms the basis of our study, has been established in Refs. [25,33]. This can be done via fractionalization of the electrons, described in this context by the Grassmann field c_j ($\bar{c}_j$), into

$$c_j = R_j \psi_j, \quad \bar{c}_j = \bar{\psi}_j R_j^\dagger, \quad (5)$$

where R_j is a SU(2) matrix and represents the *spinons*. Such a matrix acts on the fermionic spinor field $\psi_{j,\sigma}$ representing the *chargons* and carrying the pseudospin index σ . From the symmetry point of view, the spinon field R_j carries the SU(2) physical spin while the chargin does not transform under spin transformations. The fractionalization introduces an emergent SU(2) gauge symmetry due to the invariance under the new local transformations

$$\psi_j \rightarrow V_j \psi_j, \quad \bar{\psi}_j \rightarrow \bar{\psi}_j V_j^\dagger, \quad (6)$$

$$R_j \rightarrow R_j V_j^\dagger, \quad R^\dagger \rightarrow V_j R_j^\dagger. \quad (7)$$

The spinon matrix R_j carries the SU(2) gauge charge through its second index while the chargon ψ_j carries the same gauge charge through its spinor index. The physical U(1) charge transformation acts only on the chargon field ψ_j as

$$\psi_j \rightarrow e^{i\theta_j} \psi_j, \quad \bar{\psi}_j \rightarrow e^{-i\theta_j} \bar{\psi}_j. \quad (8)$$

The fractionalization in Eqs. (6) can be applied to the Hubbard action, leading to an equivalent formulation of the original problem by the new degrees of freedom, ψ_j and R_j , that are treated separately [25,33]. In particular, the chargon field can be integrated out via a (renormalized) mean field calculation, leading to an effective action for the spinon field, after a long-wavelength approximation, as

$$S_{\text{NL}\sigma\text{M}}[\mathcal{R}] = \int dx \left\{ \frac{1}{2} \text{Tr}[\mathcal{P}_{\mu\nu}(\partial_\mu \mathcal{R}^T)(\partial_\nu \mathcal{R})] \right\}, \quad (9)$$

where $\mathcal{R}$ is the adjoint representation of the SU(2) matrix R_j

$$\mathcal{R}^{ab} \sigma^b = R^\dagger \sigma^a R. \quad (10)$$

σ^a are the Pauli matrices ($a = 1, 2, 3$). The matrix $\mathcal{P}_{\mu\nu}$ can be written in terms of the more commonly known stiffness matrix $\mathcal{J}_{\mu\nu} = \text{Tr}[\mathcal{P}_{\mu\nu}] \mathbb{1} - \mathcal{P}_{\mu\nu}$ and is derived from the integration of the chargon field. All the details can be found in Ref. [25].

The structure of the matrix $\mathcal{P}_{\mu\nu}$ depends on the specific ordering state of the chargon. For instance, the Néel antiferromagnetic order, where the spin texture can be written as $\langle \vec{S}_j \rangle \sim (-1)^j \mathbf{e}_x$, has two degenerate Goldstone modes and a residual O(2) symmetry for rotations around the axis defined by $\mathbf{e}_x$. The stiffness matrix becomes

$$\mathcal{J}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & J_{\mu\nu} & 0 \\ 0 & 0 & J_{\mu\nu} \end{pmatrix}, \quad (11)$$

with $J_{\mu\nu} = \text{diag}(-Z, J, J)$. In this case, the first term of the action (9) simplifies to the known O(3)/O(2) $\simeq S^2 \text{NL}\sigma\text{M}$ [34,35]

$$S_{\text{NL}\sigma\text{M}}[\mathbf{\hat{\Omega}}] = \int dx [Z |\partial_\tau \mathbf{\hat{\Omega}}|^2 + J |\nabla \mathbf{\hat{\Omega}}|^2], \quad (12)$$

where the vector field $\mathbf{\hat{\Omega}}$ has components $\hat{\Omega}^a = \mathcal{R}^{a1}$ and obeys $|\mathbf{\hat{\Omega}}| = 1$.

Another state considered here is the spiral antiferromagnet, characterized by three Goldstone modes, one for in-plane and two for out-of-plane fluctuations. In this case the stiffness matrix has the form

$$\mathcal{J}_{\mu\nu} = \begin{pmatrix} \frac{1}{2} J_{\mu\nu}^\perp & 0 & 0 \\ 0 & \frac{1}{2} J_{\mu\nu}^\perp & 0 \\ 0 & 0 & J_{\mu\nu}^\square \end{pmatrix}, \quad (13)$$

with

$$J_{\mu\nu}^\perp = \begin{pmatrix} -Z^\perp & 0 & 0 \\ 0 & J_{xx}^\perp & J_{xy}^\perp \\ 0 & J_{yx}^\perp & J_{yy}^\perp \end{pmatrix} \quad (14)$$

for the out-of-plane fluctuations and

$$J_{\mu\nu}^\square = \begin{pmatrix} -Z^\square & 0 & 0 \\ 0 & J_{xx}^\square & J_{xy}^\square \\ 0 & J_{yx}^\square & J_{yy}^\square \end{pmatrix} \quad (15)$$

for the in-plane ones.

III. STIFFNESS CALCULATION

In this section, we compute the stiffness matrix as defined in Eq. (13) from the mean-field Hamiltonian (4). While the spin stiffness in the presence of purely spiral order was previously studied in Ref. [36], the case of coexisting magnetic and superconducting orders remains unexplored. Consequently, the calculation of the spin stiffness in this coexisting phase will be the primary focus of the following sections. We follow Ref. [36], where the stiffnesses have been connected to the zero momentum and frequency limits of the response function to an external SU(2) gauge field. More specifically, as also explained in Refs. [36] and [37], such a connection requires the application of a symmetry breaking field ϕ_0 that needs to be sent to zero only at the end of the calculation. In other words, the limits as the frequency, momentum, and the field ϕ_0 go to zero are highly nontrivial and the correct stiffness is recovered only when the zero field ϕ_0 limit is performed after the momentum and frequency limits.

The superfluid stiffness has been computed in Ref. [19] for the coexisting phase, while the magnetic stiffness has not been calculated yet and will be the topic of the next sections.

A. Gauge response function

In this section, we derive the formula of the spin stiffness for a generic action $S[\psi, \bar{\psi}, A]$ by applying an external local SU(2) gauge field $A_\mu^a(x)$. We start by defining the generating functional

$$\mathcal{W}[A, \bar{h}] = -\ln \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S[\psi, \bar{\psi}, A] + \int \bar{h} \cdot \bar{\psi} \vec{\sigma} \psi}, \quad (16)$$

which depends explicitly on the external gauge field A_μ^a and on the source field $\bar{h}$. By considering functional derivatives, one can compute the system's response to an external SU(2) gauge field. At the lowest order in the gauge field, the induced spin current can be expressed as

$$j_\mu^a(q) = \int_{q'} K_{\mu\nu}^{h,ab}(q, q') A_\nu^b(q'), \quad (17)$$

where we have defined the SU(2) gauge kernel as

$$K_{\mu\nu}^{h,ab}(q, q') = - \frac{\delta^2 \mathcal{W}[A]}{\delta A_\nu^b(q') \delta A_\mu^a(-q)} \Big|_{A=0}, \quad (18)$$

where we used the shorthand notation $q = (\mathbf{q}, \omega)$. The kernel is not computed at vanishing source field, but instead at a finite local symmetry breaking field $\bar{h}(x)$, whose explicit form depends, in general, on the specific symmetry broken state. In the case of the spiral order, also if coexisting with superconductivity, this field assumes the form $\bar{h}(x) = h(\cos(\mathbf{Q} \cdot \mathbf{x}), \sin(\mathbf{Q} \cdot \mathbf{x}), 0)$. Spiral order breaks also

translational symmetry, so that the gauge kernel in (18) is expected to be nonzero also for $q \neq q'$.

The spin stiffness, defined as the static limit of the momentum diagonal component of the SU(2) gauge kernel [36,37]

$$J_{\alpha\beta}^{ab} = -\lim_{h \rightarrow 0} \lim_{\mathbf{q} \rightarrow \mathbf{0}} K_{\alpha\beta}^{h,ab}(\mathbf{q}, 0), \quad (19)$$

equals the spatial component of Eq. (13), i.e.,

$$J_{\alpha\beta}^{ab} = \mathcal{J}_{\alpha\beta}^{ab}. \quad (20)$$

Note that in the above formulas, and from this point on, we have used the convention that the indices labeled as μ, ν can be either spatial or temporal, that is, $\mu = (0, x, y)$, whereas α, β are only spatial, $\alpha = (x, y)$.

Here we defined

$$K_{\mu\nu}^{h,ab}(\mathbf{q}, \mathbf{q}, \omega) = \delta_{\mathbf{q},\mathbf{q}} K_{\mu\nu}^{h,ab}(\mathbf{q}, \omega). \quad (21)$$

In the low energy theory for magnetic fluctuations, an important role is also played by the *temporal stiffnesses*, given by

$$\chi_{\text{dyn}}^{ab} = \mathcal{J}_{00}^{ab}, \quad (22)$$

which equals the *dynamical limit* of the gauge kernel [36]

$$\chi_{\text{dyn}}^{ab} = \lim_{h \rightarrow 0} \lim_{\omega \rightarrow 0} K_{00}^{ab}(\mathbf{0}, \omega). \quad (23)$$

Since the magnetic order we are focusing on is of spiral type (with Néel order being a limiting case), the spin stiffness matrix $J_{\mu\nu}^{ab}$ can be written explicitly in terms of the out-of-plane and in-plane quantities, $J_{\mu\nu}^{\perp}$ and $J_{\mu\nu}^{\square}$, as given in Eq. (13). The temporal stiffness takes the following form [25,36]:

$$\chi_{\text{dyn}} = \begin{pmatrix} \frac{1}{2}Z^{\perp} & 0 & 0 \\ 0 & \frac{1}{2}Z^{\perp} & 0 \\ 0 & 0 & Z^{\square} \end{pmatrix}, \quad (24)$$

where $Z^{\perp}$ and $Z^{\square}$ have been introduced in Eqs. (14) and (15).

B. Mean-field action

We are now ready to explicitly compute the gauge kernel for the mean-field order introduced in Sec. II A. For this scope, we couple the Hamiltonian (4) to an external SU(2) gauge field, yielding

$$S[\psi, \bar{\psi}, A] = S_0[\psi, \bar{\psi}, A] + S_m[\psi, \bar{\psi}] + S_p[\psi, \bar{\psi}, A], \quad (25)$$

where we have defined

$$S_0[\psi, \bar{\psi}, A] = \int_{\tau} \sum_{jj'} \bar{\psi}_j [(\partial_{\tau} - \mu - A_{0,j})\delta_{jj'} + t_{jj'} e^{-\mathbf{r}_{jj'} \cdot (\nabla + i\mathbf{A}_j)}] \psi_{j'}, \quad (26)$$

$$S_p[\psi, \bar{\psi}, A] = \int_{\tau} \sum_{jj'} \Delta_{p,jj'} \bar{\psi}_j [i\sigma^2 e^{-\mathbf{r}_{jj'} \cdot (\nabla + i\mathbf{A}_j)}] \bar{\psi}_{j'} + \text{c.c.}, \quad (27)$$

and

$$S_m[\psi, \bar{\psi}] = \int_{\mathbf{k}} \Delta_m [\bar{\psi}_{\mathbf{k}+Q\downarrow} \psi_{\mathbf{k}\uparrow} + \bar{\psi}_{\mathbf{k}\uparrow} \psi_{\mathbf{k}+Q\downarrow}]. \quad (28)$$

We stress that also the superconducting gap $\Delta_{p,jj'}$ couples to the gauge field A . In fact, since the former is nonlocal in space, coupling the system with an external gauge field produces a change in the phase when going from the lattice point j to j' [38]. In other words, the coupling (27) ensures the invariance of the action under the gauge transformation

$$\psi_j \rightarrow \psi'_j = \mathcal{V}_j \psi_j, \quad (29)$$

$$A_j \rightarrow A'_{\mu,j} = \mathcal{V}_j^{\dagger} A_{\mu,j} \mathcal{V}_j + i\mathcal{V}_j^{\dagger} \partial_{\mu} \mathcal{V}_j. \quad (30)$$

C. Evaluation of the spatial spin stiffness

We are now ready to compute the spin stiffness by using the formula (20), together with the functional differentiation, as in Eq. (18), with the action in Eq. (25).

We expand the exponential in Eqs. (26) and (27) to second order in the gauge field A_j , giving rise to

$$S_0[\psi, \bar{\psi}, A] = \frac{1}{2} \int_{k,q} A_{\mu}^a(q) \gamma_{\mathbf{k}}^{\mu} \bar{\psi}_{\mathbf{k}+q} \sigma^a \psi_{\mathbf{k}} - \frac{1}{8} \int_{k,q,q'} A_{\mu}^a(q - q') A_{\nu}^a(q') \gamma_{\mathbf{k}}^{\mu\nu} \bar{\psi}_{\mathbf{k}+q} \psi_{\mathbf{k}} \quad (31)$$

and, similarly,

$$S_p[\psi, \bar{\psi}, A] = \frac{1}{2} \int_{k,q} A_{\mu}^a(q) \Delta_{p,\mathbf{k}}^{\mu} \bar{\psi}_{\mathbf{k}+q} (i\sigma^2 \sigma^a) \bar{\psi}_{-\mathbf{k}} - \frac{1}{8} \int_{k,q,q'} A_{\mu}^a(q - q') A_{\nu}^a(q') \Delta_{p,\mathbf{k}}^{\mu\nu} \bar{\psi}_{\mathbf{k}+q} (i\sigma^2) \times \bar{\psi}_{-\mathbf{k}} + \text{c.c.} \quad (32)$$

Here, we have also defined

$$\gamma_{\mathbf{k}}^{\mu} = (1 \mid \partial_{k_{\alpha}} \xi_{\mathbf{k}}), \quad (33a)$$

$$\gamma_{\mathbf{k}}^{\mu\nu} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \partial_{k_{\alpha}} \partial_{k_{\beta}} \xi_{\mathbf{k}} \end{array} \right), \quad (33b)$$

$$\Delta_{\mathbf{k}}^{\mu} = (0 \mid \partial_{k_{\alpha}} \Delta_{p,\mathbf{k}}), \quad (33c)$$

$$\Delta_{\mathbf{k}}^{\mu\nu} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \partial_{k_{\alpha}} \partial_{k_{\beta}} \Delta_{p,\mathbf{k}} \end{array} \right), \quad (33d)$$

with $\alpha = x, y$ and $\partial_{k_{\alpha}}$ a shortcut for $\partial/\partial k_{\alpha}$.

We perform our calculations at finite temperature T using the Matsubara formalism. Unless otherwise specified, we adopt a consistent notation where $k = (\mathbf{k}, i\nu_n)$ denotes the internal vector with $\nu_n = \pi T(2n+1)$ being the fermionic Matsubara frequencies. The vector $q = (\mathbf{q}, i\Omega_n)$ is reserved for the external transfer momentum and bosonic Matsubara frequency $\Omega_n = 2\pi Tn$. Only after performing the Matsubara summations do we perform the analytic continuation to real frequencies, $i\Omega_n \rightarrow \omega + i\eta$, to evaluate the physical response functions such as those in Eqs. (19) and (23).

It is convenient to introduce a four-component spinor $\Psi_{\mathbf{k}} = (\psi_{\mathbf{k}\uparrow}, \psi_{\mathbf{k}\downarrow}, \bar{\psi}_{-\mathbf{k}\uparrow}, \bar{\psi}_{-\mathbf{k}\downarrow})^T$. In this basis, the full action can be

rewritten as

$$S[\psi, \bar{\psi}, A] = \int_{k,k'} \bar{\Psi}_k V_{kk'}[A] \Psi_{k'} + \frac{1}{16T} \int_{q,k} \gamma_k^{\mu\nu} A_\mu^a(-q) A_\nu^a(q) + \frac{1}{2T} \int_k \xi_k, \quad (34)$$

where $V_{kk'}[A]$ reads

$$V_{kk'}[A] = -\frac{1}{2} \mathcal{G}_{\mathbf{k}\mathbf{k}'}^{-1}(\nu) \delta_{\nu,\nu'} + \frac{1}{4} A_\mu^a(k-k') P_{\mathbf{k}\mathbf{k}'}^{\mu,a} - \frac{1}{16} \int_q A_\mu^a(k-k'-q) A_\nu^a(q) P_{\mathbf{k}\mathbf{k}'}^{\mu\nu}. \quad (35)$$

Here we defined $P_{\mathbf{k}\mathbf{k}'}^{\mu,a} = \Gamma_{\mathbf{k}\mathbf{k}'}^{\mu,a} + D_{\mathbf{k}\mathbf{k}'}^{\mu,a}$, $P_{\mathbf{k}\mathbf{k}'}^{\mu\nu} = \Gamma_{\mathbf{k}\mathbf{k}'}^{\mu\nu} + D_{\mathbf{k}\mathbf{k}'}^{\mu\nu}$, with

$$\Gamma_{\mathbf{k}\mathbf{k}'}^{\mu,a} = \begin{pmatrix} \gamma_{\mathbf{k}'}^\mu \sigma^a & 0 \\ 0 & \gamma_{\mathbf{k}}^\mu (\sigma^a)^T \end{pmatrix}, \quad (36a)$$

$$\Gamma_{\mathbf{k}\mathbf{k}'}^{\mu\nu} = \begin{pmatrix} \gamma_{\mathbf{k}'}^{\mu\nu} \mathbb{1} & 0 \\ 0 & -\gamma_{\mathbf{k}}^{\mu\nu} \mathbb{1} \end{pmatrix} \quad (36b)$$

and

$$D_{\mathbf{k}\mathbf{k}'}^{\mu,a} = \begin{pmatrix} 0 & t^a \Delta_{\mathbf{k}'}^\mu \\ t^{a\dagger} \Delta_{\mathbf{k}}^\mu & 0 \end{pmatrix}, \quad (37a)$$

$$D_{\mathbf{k}\mathbf{k}'}^{\mu\nu} = \begin{pmatrix} 0 & i\sigma^2 \Delta_{\mathbf{k}'}^{\mu\nu} \\ -i\sigma^2 \Delta_{\mathbf{k}}^{\mu\nu} & 0 \end{pmatrix}. \quad (37b)$$

In the formulas above, $\mathcal{G}_{\mathbf{k}\mathbf{k}'}(\nu_n)$ is the Green's function, which, in this basis, can be written as

$$\mathcal{G}_{\mathbf{k}\mathbf{k}'}^{-1}(\nu_n) = \begin{pmatrix} (i\nu_n - \xi_{\mathbf{k}}) \delta_{\mathbf{k},\mathbf{k}'} & -\Delta_m \delta_{\mathbf{k}+\mathbf{Q},\mathbf{k}'} & 0 & -\Delta_{p,\mathbf{k}} \delta_{\mathbf{k},\mathbf{k}'} \\ -\Delta_m \delta_{\mathbf{k}-\mathbf{Q},\mathbf{k}'} & (i\nu_n - \xi_{\mathbf{k}}) \delta_{\mathbf{k},\mathbf{k}'} & \Delta_{p,\mathbf{k}} \delta_{\mathbf{k},\mathbf{k}'} & 0 \\ 0 & \Delta_{p,\mathbf{k}}^* \delta_{\mathbf{k},\mathbf{k}'} & (i\nu_n + \xi_{\mathbf{k}}) \delta_{\mathbf{k},\mathbf{k}'} & \Delta_m \delta_{\mathbf{k}',\mathbf{k}+\mathbf{Q}} \\ -\Delta_{p,\mathbf{k}}^* \delta_{\mathbf{k},\mathbf{k}'} & 0 & \Delta_m \delta_{\mathbf{k}',\mathbf{k}-\mathbf{Q}} & (i\nu_n + \xi_{\mathbf{k}}) \delta_{\mathbf{k},\mathbf{k}'} \end{pmatrix}. \quad (38)$$

The full $\mathcal{G}_{\mathbf{k}\mathbf{k}'}(\nu_n)$ is then obtained inverting Eq. (38) both in the spinor and momentum indices.

We now evaluate the gauge kernel in Eq. (18). Inspecting the structure of the actions (31) and (32), one can deduce that it consists of a paramagnetic and a diamagnetic term

$$K_{\mu\nu}^{ab}(\mathbf{q}, \mathbf{q}', \omega) = K_{\mu\nu}^{p,ab}(\mathbf{q}, \mathbf{q}', \omega) + \delta_{ab} \delta_{\mathbf{q},\mathbf{0}} \delta_{\mathbf{q}',\mathbf{0}} \delta_{\omega,0} K_{\mu\nu}^d. \quad (39)$$

The diamagnetic contribution reads as

$$K_{\mu\nu}^d = -\frac{1}{8} \int_{\mathbf{k},\mathbf{k}'} T \sum_{\nu_n} \text{Tr}[\mathbf{P}_{\mathbf{k}\mathbf{k}'}^{\mu\nu} \mathcal{G}_{\mathbf{k}\mathbf{k}'}(\nu_n)]. \quad (40)$$

The paramagnetic contribution is given by

$$K_{\mu\nu}^{p,ab}(\mathbf{q}, \mathbf{q}', \omega) = -\frac{1}{4} \int_{\mathbf{k},\mathbf{k}'} T \sum_{\nu_n} \text{Tr}[\mathbf{P}_{\mathbf{k}+\mathbf{q},\mathbf{k}}^{\mu,b} \mathcal{G}_{\mathbf{k}\mathbf{k}'}(\nu_n) \times P_{\mathbf{k}',\mathbf{k}'+\mathbf{q}'}^{\nu,a} \mathcal{G}_{\mathbf{k}'+\mathbf{q}',\mathbf{k}+\mathbf{q}}(\nu_n + \Omega_m)] \Big|_{i\Omega_m \rightarrow \omega + i0^+}. \quad (41)$$

In Eq. (39), the frequency ω is real and its dependence can be formally obtained via analytic continuation in Eq. (41).

In order to gain further insight, it is useful to cast the 4×4 Green's function as a block matrix

$$\mathcal{G}_{\mathbf{k}\mathbf{k}'}(\nu_n) = \begin{pmatrix} \mathbf{G}_{\mathbf{k}\mathbf{k}'}(\nu_n) & \mathbf{F}_{\mathbf{k}\mathbf{k}'}^p(\nu_n) \\ [\mathbf{F}_{\mathbf{k}\mathbf{k}'}^p(\nu_n)]^\dagger & -\mathbf{G}_{-\mathbf{k}',-\mathbf{k}}^T(-\nu_n) \end{pmatrix}, \quad (42)$$

where $\mathbf{G}_{\mathbf{k}\mathbf{k}'}(\nu_n)$ and $\mathbf{F}_{\mathbf{k}\mathbf{k}'}^p(\nu_n)$ are 2×2 matrices, representing the spin resolved normal and anomalous propagators,

respectively. The normal propagator is parametrized as

$$\mathbf{G}_{\mathbf{k}\mathbf{k}'}(\nu_n) = \begin{pmatrix} G_{\mathbf{k}}(\nu_n) \delta_{\mathbf{k},\mathbf{k}'} & F_{\mathbf{k}}^m(\nu_n) \delta_{\mathbf{k},\mathbf{k}'-\mathbf{Q}} \\ F_{\mathbf{k}-\mathbf{Q}}^m(\nu_n) \delta_{\mathbf{k},\mathbf{k}'+\mathbf{Q}} & G_{-\mathbf{k}}(\nu_n) \delta_{\mathbf{k},\mathbf{k}'} \end{pmatrix}. \quad (43)$$

The anomalous one reads as

$$\mathbf{F}_{\mathbf{k}\mathbf{k}'}^p(\nu_n) = \begin{pmatrix} F_{\mathbf{k}}^t(\nu_n) \delta_{\mathbf{k}+\mathbf{Q},\mathbf{k}'} & F_{\mathbf{k}}^s(\nu_n) \delta_{\mathbf{k},\mathbf{k}'} \\ -F_{-\mathbf{k}'}^s(-\nu_n) \delta_{\mathbf{k},\mathbf{k}'} & -F_{-\mathbf{k}}^t(-\nu_n) \delta_{\mathbf{k},\mathbf{k}'+\mathbf{Q}} \end{pmatrix} \quad (44)$$

and obeys the relation $\mathbf{F}_{\mathbf{k},\mathbf{k}'}^p(\nu_n) = -[\mathbf{F}_{-\mathbf{k}',-\mathbf{k}}^p(-\nu_n)]^T$. Note that the specific form of the matrix Eq. (44) relies on the absence of a spin-triplet superconducting gap.

In general, after selecting the $\mathbf{q} = \mathbf{q}'$ component in Eq. (41), and taking the $\mathbf{q} \rightarrow \mathbf{0}$ limit, after having set $\omega = 0$, one obtains expressions for the spin stiffnesses. Decomposing the paramagnetic current vertex as $P_{\mathbf{k}\mathbf{k}'}^{a,\mu} = \Gamma_{\mathbf{k}\mathbf{k}'}^{\mu,a} + D_{\mathbf{k}\mathbf{k}'}^{\mu,a}$ [cf. Eqs. (36a) and (37a)], we have

$$J_{\alpha\beta}^X = J_{\alpha\beta}^d + J_{\alpha\beta}^{X,\Gamma\Gamma} + J_{\alpha\beta}^{X,DD} + J_{\alpha\beta}^{X,\Gamma D} + J_{\alpha\beta}^{X,D\Gamma}, \quad (45)$$

with $X = \perp, \square$. The first term in the equation above represents the diamagnetic contribution to the spin stiffness and it is the same for the in- and out-of-plane components. The $J_{\alpha\beta}^{X,\Gamma\Gamma}$ and $J_{\alpha\beta}^{X,DD}$ terms are obtained by setting $P_{\mathbf{k}\mathbf{k}'}^{a,\mu} = \Gamma_{\mathbf{k}\mathbf{k}'}^{a,\mu}$ and $P_{\mathbf{k}\mathbf{k}'}^{a,\mu} = D_{\mathbf{k}\mathbf{k}'}^{a,\mu}$ in Eq. (41), respectively. Finally, in the term

$J_{\alpha\beta}^{X,\Gamma D}$ ($J_{\alpha\beta}^{X,D\Gamma}$) we have set one the first (second) current vertex in Eq. (41) equal to $\Gamma_{\mathbf{k}\mathbf{k}'}^{a,\mu}$ and the other one $D_{\mathbf{k}\mathbf{k}'}^{a,\mu}$. Note that it can be shown that $J_{\alpha\beta}^{X,\Gamma D} = J_{\beta\alpha}^{X,D\Gamma}$, so that only the evaluation of one of the last two terms in Eq. (45) is required.

$$J_{\alpha\beta}^{\square} = \int_{\mathbf{k}} T \sum_{v_n} \{ -\gamma_{\mathbf{k}}^{\alpha} \gamma_{\mathbf{k}+\mathbf{Q}}^{\beta} ([F_{\mathbf{k}}^m(v_n)]^2 - [F_{\mathbf{k}}^t(v_n)]^2) - \Delta_{\mathbf{k}}^{\alpha} \Delta_{\mathbf{k}+\mathbf{Q}}^{\beta} (|F_{\mathbf{k}}^m(v_n)|^2 - |F_{\mathbf{k}}^t(v_n)|^2) + 2(\gamma_{\mathbf{k}}^{\alpha} \Delta_{\mathbf{k}+\mathbf{Q}}^{\beta} + \gamma_{\mathbf{k}}^{\beta} \Delta_{\mathbf{k}+\mathbf{Q}}^{\alpha}) F_{\mathbf{k}}^m(v_n) F_{\mathbf{k}}^t(v_n) \} \quad (46)$$

and for the out-of-plane one

$$J_{\alpha\beta}^{\perp} = \frac{1}{2} J_{\alpha\beta}^{\square} + \frac{1}{2} \int_{\mathbf{k}} T \sum_{v_n} \{ \gamma_{\mathbf{k}}^{\alpha} \gamma_{\mathbf{k}}^{\beta} [G_{\mathbf{k}}(v_n)(G_{-\mathbf{k}}(v_n) - G_{\mathbf{k}}(v_n)) - F_{\mathbf{k}}^s(v_n)(F_{-\mathbf{k}}^s(v_n) - F_{\mathbf{k}}^s(v_n))] - \Delta_{\mathbf{k}}^{\alpha} \Delta_{\mathbf{k}}^{\beta} [G_{\mathbf{k}}(v_n)(G_{-\mathbf{k}}^*(v_n) - G_{\mathbf{k}}^*(v_n)) - F_{\mathbf{k}}^s(v_n)(F_{-\mathbf{k}}^s(v_n) - F_{\mathbf{k}}^s(v_n))] + 2(\gamma_{\mathbf{k}}^{\alpha} \Delta_{\mathbf{k}}^{\beta} + \gamma_{\mathbf{k}}^{\beta} \Delta_{\mathbf{k}}^{\alpha}) G_{\mathbf{k}}(v_n)(F_{-\mathbf{k}}^s(v_n) - F_{\mathbf{k}}^s(v_n)) \}. \quad (47)$$

It is possible to analytically perform the Matsubara summation, as shown in Appendix B.

Here, we simplify the momentum dependence of the gap function via a simple form-factor projection: s -wave sector for the magnetic gap and d -wave for the superconducting gap.

D. Temporal stiffness

In this section, we compute the temporal stiffness, defined as the dynamical limit of the temporal component of the gauge response kernel [see Eq. (23)]. Analyzing the coupling of the electrons to the gauge field [Eqs. (31) and (32)], one can convince himself that the $\mu = \nu = 0$ components of the gauge kernel correspond to the spin susceptibilities,

$$K_{00}^{ab}(\mathbf{q}, \mathbf{q}', \omega) = \chi^{ab}(\mathbf{q}, \mathbf{q}', \omega) \sim \langle S_{-\mathbf{q},-\omega}^a S_{\mathbf{q}',\omega}^b \rangle, \quad (48)$$

with $S_{q=(\mathbf{q},\omega)}^a = (1/2) \int_{\mathbf{k}} \bar{\psi}_{\mathbf{k}+\mathbf{q}} \sigma^a \psi_{\mathbf{k}}$ being the spin bilinear. As discussed in Refs. [39,40], it is convenient to work in a rotated spin reference frame in which translational invariance is restored. Such a basis is defined by rotating the fermion field ψ_i by an angle $\theta_j = \mathbf{Q} \cdot \mathbf{R}_j$ in the xy plane

$$\tilde{\psi}_j = e^{-i\frac{\theta_j}{2}} e^{i\frac{\sigma_3}{2}\theta_j} \psi_j. \quad (49)$$

In this basis, the real-time retarded susceptibility is defined as

$$\tilde{\chi}^{ab}(j-j', t) = -i\Theta(t) \langle [\tilde{S}_j^a(t), \tilde{S}_{j'}^b(0)] \rangle, \quad (50)$$

with

$$\tilde{S}_j^a(t) = \bar{\tilde{\psi}}_j(t) \frac{\sigma^a}{2} \tilde{\psi}_j(t). \quad (51)$$

The susceptibilities in the physical reference frame are obtained from the matrix relation

$$\chi(j, j', t) = M_j^T \tilde{\chi}(j-j', t) M_{j'}. \quad (52)$$

The transformation matrix is given by

$$M_j = \begin{pmatrix} \cos(\mathbf{Q} \cdot \mathbf{R}_j) & \sin(\mathbf{Q} \cdot \mathbf{R}_j) & 0 \\ -\sin(\mathbf{Q} \cdot \mathbf{R}_j) & \cos(\mathbf{Q} \cdot \mathbf{R}_j) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (53)$$

In Appendix A, we show the explicit expressions for each of the terms in Eq. (45). Integrating the diamagnetic term by parts and summing all terms together, we find for the in-plane stiffness

In Fourier space, the momentum- and spin-diagonal physical susceptibilities read as

$$\chi^{11}(\mathbf{q}, \mathbf{q}, \omega) = \chi^{22}(\mathbf{q}, \mathbf{q}, \omega) \quad (54a)$$

$$= \tilde{\chi}^{-+}(\mathbf{q} + \mathbf{Q}, \omega) + \tilde{\chi}^{+-}(\mathbf{q} - \mathbf{Q}, \omega),$$

$$\chi^{33}(\mathbf{q}, \mathbf{q}, \omega) = \tilde{\chi}^{33}(\mathbf{q}, \omega), \quad (54b)$$

with $\tilde{S}_q^{\pm} = (\tilde{S}_q^1 \pm i\tilde{S}_q^2)/2$.

The in- and out-of-plane temporal stiffnesses are then obtained from [see Eq. (24)]

$$Z^{\square} = \lim_{\omega \rightarrow 0} \chi^{33}(\mathbf{0}, \mathbf{0}, \omega), \quad (55a)$$

$$Z^{\perp} = \lim_{\omega \rightarrow 0} 2\chi^{22}(\mathbf{0}, \mathbf{0}, \omega). \quad (55b)$$

Before discussing how to compute the susceptibility, we first write the Green's function in Eq. (42) within the rotated basis. Defining an extended rotated basis as $\zeta_k = (\psi_{k\uparrow}, \psi_{k+Q,\downarrow}, \bar{\psi}_{-k-Q,\uparrow}, \bar{\psi}_{-k,\downarrow})^T$, we get

$$\tilde{\mathcal{G}}_{\mathbf{k}}^{-1}(v_n) = \begin{pmatrix} iv_n - \xi_{\mathbf{k}} & -\Delta_m & 0 & -\Delta_{p,\mathbf{k}} \\ -\Delta_m & iv_n - \xi_{\mathbf{k}+\mathbf{Q}} & \Delta_{p,-\mathbf{k}-\mathbf{Q}} & 0 \\ 0 & \Delta_{p,-\mathbf{k}-\mathbf{Q}} & iv_n + \xi_{-\mathbf{k}-\mathbf{Q}} & \Delta_m \\ -\Delta_{p,\mathbf{k}} & 0 & \Delta_m & iv_n + \xi_{-\mathbf{k}} \end{pmatrix}. \quad (56)$$

In this basis, 16 different bilinears of the form $\bar{\zeta}_{k+q,a} \zeta_{k,b}$ ($a, b = 1, \dots, 4$) can be constructed, leading to $16 \times 16 = 256$ different susceptibilities of the form

$$\tilde{\chi}^{abcd}(q) \sim \int_{k,k'} \langle (\bar{\zeta}_{k+q,a} \zeta_{k,b}) (\bar{\zeta}_{k',c} \zeta_{k'+q,d}) \rangle. \quad (57)$$

To simplify the treatment and help physical intuition, we introduce a set of 16 linearly independent 4×4 matrices, Υ^ℓ , obeying

$$\frac{1}{4} \text{Tr}[(\Upsilon^\ell)^\dagger \Upsilon^{\ell'}] = \delta_{\ell,\ell'}. \quad (58)$$

These matrices can be arbitrarily chosen and we define them such that (at least some of) the bilinears $(1/4)\bar{\zeta}_{k+q}\Upsilon^\ell\zeta_k$ acquire a physical meaning. We define the 16 Hermitian 4×4 matrices $\Upsilon^\ell = \{\Gamma^a, A^a, \Phi^a, B^a\}$ with $\ell = 0, \dots, 15$ and

$$\Gamma^a = \begin{pmatrix} \sigma^a & 0 \\ 0 & -(\sigma^a)^T \end{pmatrix}, \quad (59a)$$

$$A^a = \begin{pmatrix} 0 & t^a \\ -t^a & 0 \end{pmatrix}, \quad (59b)$$

$$\Phi^a = \begin{pmatrix} 0 & -it^a \\ -it^a & 0 \end{pmatrix}, \quad (59c)$$

$$B^a = \begin{pmatrix} \sigma^a & 0 \\ 0 & (\sigma^a)^T \end{pmatrix}, \quad (59d)$$

with $a = 0, 1, 2, 3$ and $t^a = i\sigma^2\sigma^a$. With this choice, the $(1/4)\bar{\zeta}_{k+q}\Gamma^a\zeta_k$ takes the meaning of charge ($a = 0$) and spin ($a = 1, 2, 3$) fluctuations, while $(1/4)\bar{\zeta}_{k+q}A^a\zeta_k$ and $(1/4)\bar{\zeta}_{k+q}\Phi^a\zeta_k$ represent, in the case of singlet component ($a = 0$), superconducting amplitude and phase fluctuations.

The susceptibilities (57) can be conveniently reexpressed as

$$\tilde{\chi}^{\ell\ell'}(q) = \frac{1}{4} \sum_{abcd} \Upsilon_{ab}^\ell \tilde{\chi}^{abcd}(q) \Upsilon_{cd}^{\ell'}. \quad (60)$$

Within the RPA, $\tilde{\chi}(q)$ as a matrix in ℓ, ℓ' is computed as

$$\tilde{\chi}(q) = \tilde{\chi}_0(q)[\mathbb{1} - \tilde{V}(\mathbf{q})\tilde{\chi}_0(q)]^{-1}. \quad (61)$$

In the above equation, we have assumed the effective interaction $U_{\mathbf{k}\mathbf{k}'}^{\ell\ell'}(\mathbf{q})$ to be factorizable, namely that

$$U_{\mathbf{k}\mathbf{k}'}^{\ell\ell'}(\mathbf{q}) = \tilde{V}^{\ell\ell'}(\mathbf{q})f_{\mathbf{k}}^\ell f_{\mathbf{k}'}^{\ell'}, \quad (62)$$

with $f_{\mathbf{k}}^\ell$ some model-dependent form factors. For the specific case under study, we have

$$f_{\mathbf{k}}^\ell = \begin{cases} 1 & \text{if } \ell \in [0, 3], \\ \cos k_x - \cos k_y & \text{if } \ell = 4 \text{ or } \ell = 8, \\ 0 & \text{otherwise,} \end{cases} \quad (63)$$

while the functions $\tilde{V}^{\ell\ell'}(\mathbf{q})$ read as

$$\tilde{V}(\mathbf{q}) = \text{diag}(-U_c(\mathbf{q}), \quad U_m(\mathbf{q} - \mathbf{Q}), \quad U_m(\mathbf{q} - \mathbf{Q}), \\ U_m(\mathbf{q}), \quad U_p(\mathbf{q}), \quad U_p(\mathbf{q})), \quad (64)$$

for $l, l' = 0, \dots, 6$, while all the other components are zero. The bare susceptibilities or bubbles $\tilde{\chi}_0^{\ell\ell'}(q)$ appearing in Eq. (61) are given by

$$\tilde{\chi}_0^{\ell\ell'}(\mathbf{q}, \omega) = -\frac{1}{4} \int_{\mathbf{k}} f_{\mathbf{k}}^\ell f_{\mathbf{k}}^{\ell'} T \sum_{v_n} \text{Tr}[\Upsilon^{\ell'} \tilde{\mathcal{G}}_{\mathbf{k}}(v_n) \\ \times \Upsilon^\ell \tilde{\mathcal{G}}_{\mathbf{k}+\mathbf{q}}(v_n + \Omega_m)]|_{i\Omega_m \rightarrow \omega + i0^+}. \quad (65)$$

The Matsubara summation above can be carried out analytically as described in Appendix B.

To summarize, the temporal stiffnesses, as in Eq. (55), are calculated by combining Eqs. (61) and (54a) for the susceptibility.

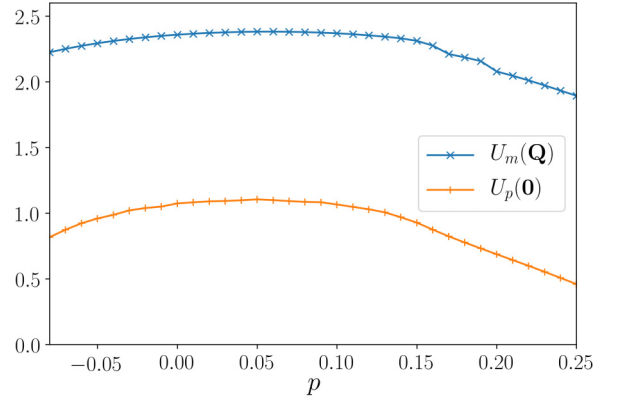


FIG. 1. Magnetic and pairing effective interactions as a function of the doping.

E. Gap equations

The gaps are computed via a renormalized set of mean-field gap equations [28,29]

$$\Delta_m = -U_m(\mathbf{Q}) \int_{\mathbf{k}} T \sum_v F_{\mathbf{k}}^m(v), \quad (66)$$

$$\Delta_p = U_p(\mathbf{0}) \int_{\mathbf{k}} T \sum_v d_{\mathbf{k}} F_{\mathbf{k}}^p(v), \quad (67)$$

where the full superconducting gap $\Delta_p(\mathbf{k}) = d_{\mathbf{k}}\Delta_p$, Fourier transform of $\Delta_{p,jj'}$, has been decomposed into the d -wave form factor $d_{\mathbf{k}} = \cos k_x - \cos k_y$ as a consequence of the decomposition (63). The interactions U_m and U_p will be computed via some functional renormalization group method.

Here, we simplify the momentum dependence of the gap function via a simple form-factor projection: s -wave sector for the magnetic gap and d -wave for the superconducting gap.

The wave vector $\mathbf{Q}$ is determined by minimizing the free energy, which is given by

$$F(\mathbf{Q}) = -\frac{1}{2} T \int_{\mathbf{k}} \sum_l \ln(1 + e^{-E_{\mathbf{k}}^l/T}) + \frac{1}{2} \int_{\mathbf{k}} (\epsilon_{\mathbf{k}} + \epsilon_{\mathbf{k}+\mathbf{Q}}) \\ + \frac{\Delta_m^2}{U_m(\mathbf{Q})} + \frac{\Delta_p^2}{U_p(\mathbf{0})} + \mu n. \quad (68)$$

$E_{\mathbf{k}}^l$ are the four eigenvalues of the Hamiltonian (4). An exact analytical formula for $E_{\mathbf{k}}^l$ can be found in Ref. [19].

IV. NUMERICAL RESULTS

In this section, we show the spatial and temporal stiffnesses computed with nearest and next-to-nearest hopping amplitudes, respectively t and $t' = -0.2t$. We fix the energy units by imposing $t = 1$. The gap functions defined in the Hamiltonian (4) are computed through Eqs. (66) and (67). The effective couplings are calculated via a functional renormalization group [25,28], where the high energy modes are integrated out and effective low energy couplings can be obtained in all interacting channels. As in Ref. [25], we use a temperature flow [41] starting with a bare coupling $U = 4t$. More details can be found in [25,28]. In Fig. 1 we show both the magnetic and pairing effective interactions as a function of the doping $p = 1 - n$.

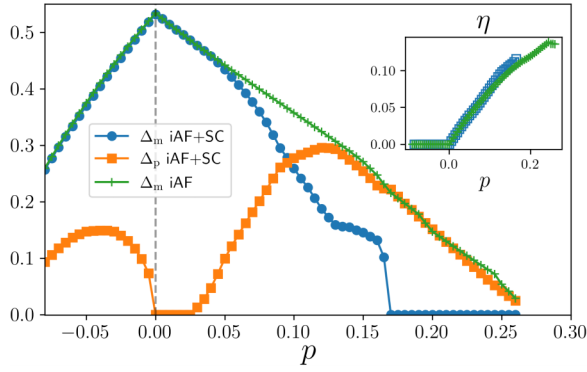


FIG. 2. Superconducting and magnetic gaps as functions of the doping $p = 1 - n$ at temperature $T = 0.001t$. Δ_m and Δ_p in blue and orange, respectively, refer to the solution with coexisting magnetism and superconductivity. Conversely, the green data refer to the magnetic solution without the superconducting phase. Inset: incommensurability η as a function of the doping for the magnetic solution (in green) and for the superconducting magnet one (in blue).

A. Gaps

In Fig. 2 we show the magnetic and superconducting gaps as a function of the doping computed at finite but small temperature $T = 0.001t$. We confirm the already found coexisting order for a wide doping regime, as also found in Refs. [19,28,29]. For comparison, we also plot the antiferromagnetic solution (without superconductivity), which is always found to be higher in energy.

The magnetic gap peaks at half filling and vanishes around $n = 0.83$ at the van Hove singularity. Consistent with Ref. [28], a coexisting solution was also observed for $n < 0.83$; however, we exclusively show here only the superconducting state. This is because the energy difference between the coexisting and purely superconducting states is negligible ($10^{-6}t$), indicating a fragile magnetic order and that the energy gain is primarily due to superconductivity.

The superconducting gap behavior at small doping is different for the hole-doped and the electron-doped cases, as also found in [19]. For hole doping, once the superconducting gap becomes sizable, it reduces the magnetization. In contrast, the superconducting gap emerges rapidly upon the introduction of electron doping, while the magnetic gap remains practically unaffected by the presence of the superconducting order itself.

Regarding the specific type of magnetic ordering, we found the Néel state in the electron-doped case, while for hole doping the magnetic order is an incommensurate spiral with momentum vector $\mathbf{Q} = (\pi, \pi - 2\pi\eta)$. The incommensurability η , shown in the inset of Fig. 2, is finite for any doping value $p > 0$.

The correct spiral momentum vector $\mathbf{Q}$ is determined via the minimization of the free energy. In practice, we find the zero of the function $\frac{dF}{d\eta} = -2\pi \frac{dF}{dQ_y}$, which is shown in Fig. 3 as a function of η . It is worthwhile to remark that the second derivative $\frac{d^2F}{dQ_y^2}$ equals the in-plane stiffness $J_{yy}^\square$ [36], which is also shown in the lower plot. In fact, when the derivative $\frac{dF}{d\eta}$, in the upper plot, changes slope, at around $\eta \approx 0.62$, the in-plane stiffness drops to very small values. In this scenario, the Néel

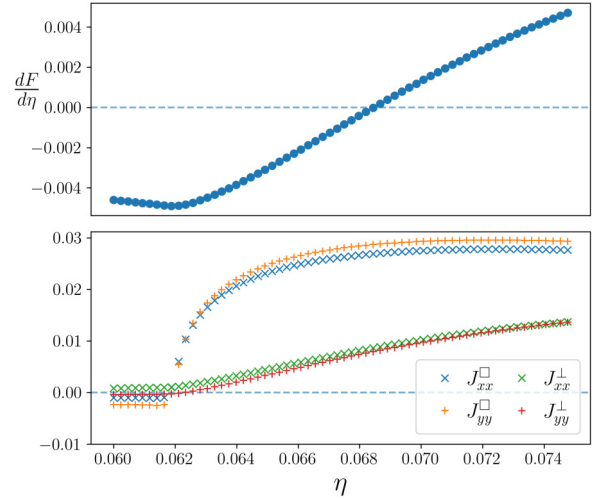


FIG. 3. Upper panel: derivative of the free energy shown as a function of the incommensurability factor η for $n = 0.91$. Lower panel: out-of-plane $J^\perp$ and in-plane $J^\square$ stiffnesses as a function of η .

state ($\eta = 0$) constitutes a maximum of the free energy, or $\frac{dF}{d\eta}(\eta = 0) = 0$; hence the derivative $\frac{dF}{d\eta}$ needs to be negative as $\eta \rightarrow 0$.

B. Stiffnesses

Figure 4 presents a comprehensive scan of the out-of-plane spin stiffness as a function of the electron and hole doping. At small hole doping, where the superconducting gap is still small, the stiffness of the superconducting state closely matches that of the pure magnetic state, indicating that superconductivity has a minimal initial impact on the magnetic correlations. However, for larger dopings, as the superconducting order strengthens, the spin stiffness of the superconducting state proves significantly smaller than that of the pure magnetic state. This suppression continues as doping increases, a trend that persists until the magnetic order completely stops at the van Hove filling ($n = 0.83$).

This pronounced reduction in stiffness at larger hole doping is attributed to an interplay of three principal effects. First, and most significantly, the emergence of the

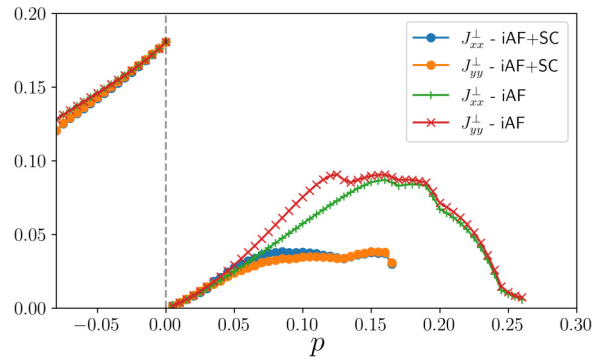


FIG. 4. Out-of-plane stiffnesses as a function of the doping $p = 1 - n$. In blue and orange are the stiffnesses for the coexistence and in red and green for the purely magnetic state.

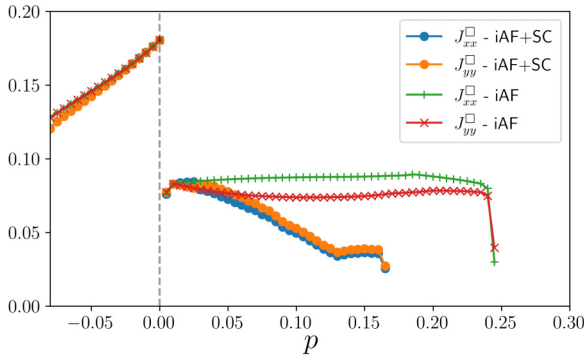


FIG. 5. In-plane stiffness as a function of the doping $p = 1 - n$. In blue and orange are the stiffnesses for the coexistence and in red and green for the purely magnetic state.

superconducting state directly suppresses the underlying magnetic gap. This suppression, in turn, leads to a smaller contribution from the terms in Eqs. (47) and (46) which are proportional to the normal Green's function $G_{\mathbf{k}}(\nu)$ and the magnetic anomalous Green's function $F_{\mathbf{k}}^m(\nu)$. These terms, representing contributions also present in a purely magnetic state, are diminished not only due to the reduced magnetic gap but also because the finite superconducting gap further suppresses the values of both $G_{\mathbf{k}}(\nu)$ and $F_{\mathbf{k}}^m(\nu)$. Secondly, novel terms arise that are proportional to the superconducting anomalous Green's functions $F_{\mathbf{k}}^s(\nu)$ and $F_{\mathbf{k}}^t(\nu)$. These additional terms directly contribute to and further reduce the value of the final stiffness. Thirdly, while potentially contributing to stiffness suppression, superconducting vertex corrections, defined as $\Delta_{\mathbf{k}}^{\alpha} = \partial_{\mathbf{k}} \Delta_{\mathbf{k}}$, were numerically checked and confirmed to be marginal (less than 10^{-5}), suggesting their impact is negligible. Interestingly, on the electron-doped side, this difference in stiffness between the superconducting and pure magnetic states becomes extremely small, almost negligible, despite the rapid emergence of the superconducting gap with electron doping.

Figure 5 presents the in-plane stiffness, which qualitatively replicates the behavior of the out-of-plane stiffness. It exhibits a similar suppression by the superconducting gap for hole doping, confirming that the impact of superconductivity on magnetic correlations is generally strong in this regime. Conversely, also the in-plane stiffness remains largely unaffected for electron doping.

The temporal stiffness, shown in Fig. 6, is a crucial indicator of the dynamics of spin fluctuations and the overall stability of magnetic order. The substantial distinction with the purely magnetic solution underscores how the interplay with superconductivity significantly modifies the underlying electronic structure and its influence on magnetic stability, potentially leading to a quantum disordered state in the absence of long-range magnetic order.

V. CONCLUSIONS

In this study, within the framework of an SU(2) gauge theory for the pseudogap phase, we use the fractionalization of the electron field into spinons and chargons [8] to include superconductivity. Specifically, we considered the

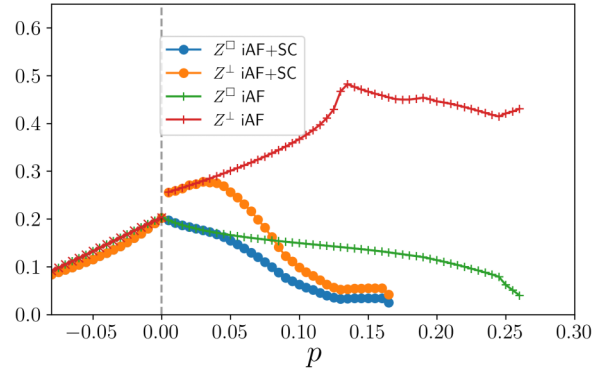


FIG. 6. In-plane and out-of-plane temporal stiffnesses as a function of doping.

superconducting instability, named the SC* phase, arising from the pairing between chargons. Our analysis reveals that the saddle-point solution for the chargon naturally accommodates a coexisting phase with both magnetic and superconducting orders, a finding consistent with prior observations from functional renormalization group improved gap equations [28,29]. Our primary objective was to address how the superconducting state influences the strength of quantum fluctuations, a property quantitatively assessed via the spin stiffnesses. This work is complementary to previous studies that have focused on the effect of magnetic order on the superconducting phase stiffness [19].

We have rigorously derived both the spatial and temporal spin stiffnesses for a state characterized by coexisting magnetic and superconducting order. These were obtained as response functions to an external SU(2) gauge field. A crucial aspect of our theoretical formulation is the explicit inclusion of the coupling between the gauge field and the nonlocal superconducting gap [38]. This coupling introduces, in addition to the standard current vertex, a term proportional to the gradient of the gap function, which proved essential for ensuring the restoration of gauge symmetry within our formalism.

Our calculations, for the specific coupling values employed, demonstrate that the system exhibits an incommensurate spiral order at any finite hole doping, while preserving Néel order at electron dopings and at half filling. We have reproduced the previously reported jump in the stiffnesses at low hole doping, as noted in Refs. [25,42]. A significant outcome of this work is the finding that, at large hole doping, the presence of the superconducting gap actively enhances spin fluctuations by suppressing the spin stiffnesses. This reduction in spin stiffnesses suggests that superconductivity can play a role in destabilizing long-range magnetic order, thereby promoting and potentially stabilizing the topological order.

Our framework thus provides a valuable theoretical laboratory for exploring the complex interplay of competing orders in systems that might host fractionalized phases. The insights gained into the reduction of spin stiffness by superconductivity contribute to the broader understanding of quantum critical phenomena and the conditions under which topological order can persist. Furthermore, a useful by-product of our calculation is that the derived formulas for the spin stiffnesses

are generally applicable to more conventional systems where magnetic and superconducting orders coexist, allowing for a broader application of our results beyond the specific fractionalized context.

ACKNOWLEDGMENTS

We are thankful to W. Metzner, A. Chubukov, M.-H. Julien, and S. Sachdev for enlightening discussions. We are also grateful to W. Metzner for carefully reading our

manuscript. P.M.B. acknowledges support by the German National Academy of Sciences Leopoldina through Grant No. LPDS 2023-06.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: EXPRESSIONS FOR THE PARAMAGNETIC AND DIAMAGNETIC TERMS

Here we report the complete formulas for the in-plane and out-of-plane stiffnesses.

1. In-plane mode

For the in-plane spin stiffness, we find the following contributions:

$$J_{\alpha\beta}^{\square,\Gamma\Gamma} = \frac{1}{2} \int_{\mathbf{k}} T \sum_{v_n} \{ \gamma_{\mathbf{k}}^{\alpha} \gamma_{\mathbf{k}}^{\beta} G_{\mathbf{k}}^2(v_n) - \gamma_{\mathbf{k}}^{\alpha} \gamma_{\mathbf{k}}^{\beta} [F_{\mathbf{k}}^s(v_n)]^2 - \gamma_{\mathbf{k}}^{\alpha} \gamma_{\mathbf{k}+\mathbf{Q}}^{\beta} [F_{\mathbf{k}}^m(v_n)]^2 + \gamma_{\mathbf{k}+\mathbf{Q}}^{\alpha} \gamma_{\mathbf{k}}^{\beta} [F_{\mathbf{k}}^t(v_n)]^2 \}, \quad (\text{A1a})$$

$$J_{\alpha\beta}^{\square,DD} = \frac{1}{2} \int_{\mathbf{k}} T \sum_{v_n} \{ -\Delta_{\mathbf{k}}^{\alpha} \Delta_{\mathbf{k}}^{\beta} |G_{\mathbf{k}}(v_n)|^2 + \Delta_{\mathbf{k}}^{\alpha} \Delta_{\mathbf{k}}^{\beta} [F_{\mathbf{k}}^s(v_n)]^2 - \Delta_{\mathbf{k}}^{\alpha} \Delta_{\mathbf{k}+\mathbf{Q}}^{\beta} |F_{\mathbf{k}}^m(v_n)|^2 + \Delta_{\mathbf{k}}^{\alpha} \Delta_{\mathbf{k}+\mathbf{Q}}^{\beta} |F_{\mathbf{k}}^t(v_n)|^2 \}, \quad (\text{A1b})$$

$$J_{\alpha\beta}^{\square,\Gamma D} = \int_{\mathbf{k}} T \sum_{v_n} \{ \gamma_{\mathbf{k}}^{\alpha} \Delta_{\mathbf{k}}^{\beta} G_{\mathbf{k}}(v_n) F_{\mathbf{k}}^s(v_n) + \gamma_{\mathbf{k}}^{\alpha} \Delta_{\mathbf{k}+\mathbf{Q}}^{\beta} F_{\mathbf{k}}^m(v_n) F_{\mathbf{k}}^t(v_n) \}. \quad (\text{A1c})$$

The diamagnetic term reads as

$$J_{\alpha\beta}^d = \frac{1}{2} \int_{\mathbf{k}} T \sum_{v_n} \{ \gamma_{\mathbf{k}}^{\alpha\beta} G_{\mathbf{k}}(v_n) + \Delta_{\mathbf{k}}^{\alpha\beta} F_{\mathbf{k}}^s(v_n) \}. \quad (\text{A2})$$

2. Out-of-plane stiffness

For the out-of-plane stiffness, we find instead

$$J_{\alpha\beta}^{\perp,\Gamma\Gamma} = \frac{1}{2} \int_{\mathbf{k}} T \sum_{v_n} \{ \gamma_{\mathbf{k}}^{\alpha} \gamma_{\mathbf{k}}^{\beta} G_{\mathbf{k}}(v_n) G_{-\mathbf{k}}(v_n) - \gamma_{\mathbf{k}}^{\alpha} \gamma_{\mathbf{k}}^{\beta} F_{\mathbf{k}}^s(v_n) F_{-\mathbf{k}}^s(v_n) \}, \quad (\text{A3a})$$

$$J_{\alpha\beta}^{\perp,DD} = \frac{1}{2} \int_{\mathbf{k}} T \sum_{v_n} \{ -\Delta_{\mathbf{k}}^{\alpha} \Delta_{\mathbf{k}}^{\beta} G_{\mathbf{k}}(v_n) G_{-\mathbf{k}}^*(v_n) + \Delta_{\mathbf{k}}^{\alpha} \Delta_{\mathbf{k}}^{\beta} F_{\mathbf{k}}^s(v_n) F_{-\mathbf{k}}^s(v_n) \}, \quad (\text{A3b})$$

$$J_{\alpha\beta}^{\perp,\Gamma D} = \int_{\mathbf{k}} T \sum_{v_n} \{ \gamma_{\mathbf{k}}^{\alpha} \Delta_{\mathbf{k}}^{\beta} G_{-\mathbf{k}}(v_n) F_{\mathbf{k}}^s(v_n) \}. \quad (\text{A3c})$$

APPENDIX B: MATSUBARA SUMMATION

Here, we show how we perform the Matsubara summation for the stiffnesses in Eqs. (47) and (46), as well as for the bubble (65). We start by considering the latter case, which involves the Green's function in the rotated basis. Due to its translational invariance, it can be simply written as

$$\tilde{G}_{\mathbf{k}}(v) = U_{\mathbf{k}}^{\dagger} \text{diag} \left(\frac{1}{i\nu - E_{\mathbf{k}}^l} \right) U_{\mathbf{k}}, \quad (\text{B1})$$

where $E_{\mathbf{k}}^l$ are the four eigenvalues of the Hamiltonian, which, in this context, is $\hat{H}_{\mathbf{k}} = i\nu \mathbb{1} - \tilde{G}_{\mathbf{k}}$. $U_{\mathbf{k}}$ is the unitary transformation matrix that diagonalizes the Green's function. By using (B1), the Matsubara summation in the bubble (65) can

be carried giving rise to

$$\begin{aligned} \tilde{\chi}_0^{ab}(\mathbf{q}, \omega) &= -\frac{1}{16} \int_{\mathbf{k}} \sum_{l,l'} \text{Tr}[\Gamma^a \mathbf{u}_{l,\mathbf{k}} \Gamma^b \mathbf{u}_{l',\mathbf{k}+\mathbf{q}}] \\ &\times \frac{f(E_{\mathbf{k}}^l) - f(E_{\mathbf{k}+\mathbf{q}}^{l'})}{\omega + i0^+ + E_{\mathbf{k}}^l - E_{\mathbf{k}+\mathbf{q}}^{l'}}, \end{aligned} \quad (\text{B2})$$

where we defined $u_{l,\mathbf{k}}^{\alpha\beta} = U_{l,\mathbf{k}}^{*\alpha} U_{l,\mathbf{k}}^{\beta}$ and $U_{l,\mathbf{k}}^{\beta} = (U_{\mathbf{k}})_{l\beta}$ are the matrix elements of the transformation matrix $U_{\mathbf{k}}$.

Regarding the Matsubara summation in the calculation of the stiffness, Eqs. (46) and (47), we first note the relations between the rotated and unrotated Green's function components;

for instance,

$$\begin{aligned} F_{\mathbf{k}}^{\text{m}}(\nu) &= \tilde{F}_{\mathbf{k}}^{\text{m}}(\nu), & \bar{F}_{\mathbf{k}}^{\text{m}}(\nu) &= \tilde{\bar{F}}_{\mathbf{k}}^{\text{m}}(\nu), \\ G_{\mathbf{k}}(\nu) &= \tilde{G}_{\mathbf{k}}(\nu), & \bar{G}_{\mathbf{k}-\mathbf{Q}}(\nu) &= \tilde{\bar{G}}_{\mathbf{k}-\mathbf{Q}}(\nu), \\ F_{\mathbf{k}}^{\uparrow\downarrow}(\nu) &= \tilde{F}_{\mathbf{k}-\mathbf{Q}}^{\uparrow\downarrow}(\nu), & F_{\mathbf{k}}^{\uparrow}(\nu) &= \tilde{F}_{\mathbf{k}}^{\uparrow}(\nu). \end{aligned}$$

We can then use decomposition (B1) to write

$$F_{\mathbf{k}}^{\text{m}}(\nu) = \tilde{F}_{\mathbf{k}}^{\text{m}}(\nu) = \sum_l \frac{u_{l,\mathbf{k}}^{12}}{i\nu - E_{\mathbf{k}}^l} \quad (\text{B3})$$

for each Green's function component in (46) and (47), making the Matsubara summation analytically doable.

-
- [1] B. Keimer, S. A. Kivelson, M. R. Norman, S. Uchida, and J. Zaanen, From quantum matter to high-temperature superconductivity in copper oxides, *Nature (London)* **518**, 179 (2015).
- [2] C. Collignon, S. Badoux, S. A. A. Afshar, B. Michon, F. Laliberté, O. Cyr-Choinière, J.-S. Zhou, S. Licciardello, S. Wiedmann, N. Doiron-Leyraud, and L. Taillefer, Fermi-surface transformation across the pseudogap critical point of the cuprate superconductor $\text{La}_{1.6-x}\text{Nd}_{0.4}\text{Sr}_x\text{CuO}_4$, *Phys. Rev. B* **95**, 224517 (2017).
- [3] Y. Fang, G. Grissonnanche, A. Legros, S. Verret, F. Laliberté, C. Collignon, A. Ataei, M. Dion, J. Zhou, D. Graf, M. J. Lawler, P. A. Goddard, L. Taillefer, and B. J. Ramshaw, Fermi surface transformation at the pseudogap critical point of a cuprate superconductor, *Nat. Phys.* **18**, 558 (2022).
- [4] M. Frachet, I. Vinograd, R. Zhou, S. Benhabib, S. Wu, H. Mayaffre, S. Krämer, S. K. Ramakrishna, A. P. Reyes, J. Debray, T. Kurosawa, N. Momono, M. Oda, S. Komiya, S. Ono, M. Horio, J. Chang, C. Proust, D. LeBoeuf, and M.-H. Julien, Hidden magnetism at the pseudogap critical point of a cuprate superconductor, *Nat. Phys.* **16**, 1064 (2020).
- [5] I. Vinograd, R. Zhou, H. Mayaffre, S. Krämer, S. K. Ramakrishna, A. P. Reyes, T. Kurosawa, N. Momono, M. Oda, S. Komiya, S. Ono, M. Horio, J. Chang, and M.-H. Julien, Competition between spin ordering and superconductivity near the pseudogap boundary in $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$: Insights from NMR, *Phys. Rev. B* **106**, 054522 (2022).
- [6] S. Sachdev and D. Chowdhury, The novel metallic states of the cuprates: Topological Fermi liquids and strange metals, *Prog. Theor. Exp. Phys.* **2016**, 12C102 (2016).
- [7] S. Chatterjee, S. Sachdev, and M. S. Scheurer, Intertwining topological order and broken symmetry in a theory of fluctuating spin-density waves, *Phys. Rev. Lett.* **119**, 227002 (2017).
- [8] M. S. Scheurer, S. Chatterjee, W. Wu, M. Ferrero, A. Georges, and S. Sachdev, Topological order in the pseudogap metal, *Proc. Natl. Acad. Sci. USA* **115**, E3665 (2018).
- [9] S. Sachdev, H. D. Scammell, M. S. Scheurer, and G. Tarnopolsky, Gauge theory for the cuprates near optimal doping, *Phys. Rev. B* **99**, 054516 (2019).
- [10] S. Sachdev, Topological order, emergent gauge fields, and Fermi surface reconstruction, *Rep. Prog. Phys.* **82**, 014001 (2019).
- [11] M. Oshikawa, Topological approach to Luttinger's theorem and the Fermi surface of a Kondo lattice, *Phys. Rev. Lett.* **84**, 3370 (2000).
- [12] T. Senthil, S. Sachdev, and M. Vojta, Fractionalized Fermi liquids, *Phys. Rev. Lett.* **90**, 216403 (2003).
- [13] T. Senthil, M. Vojta, and S. Sachdev, Weak magnetism and non-Fermi liquids near heavy-fermion critical points, *Phys. Rev. B* **69**, 035111 (2004).
- [14] M. Christos, Z.-X. Luo, H. Shackleton, Y.-H. Zhang, M. S. Scheurer, and S. Sachdev, A model of d -wave superconductivity, antiferromagnetism, and charge order on the square lattice, *Proc. Natl. Acad. Sci. USA* **120**, e2302701120 (2023).
- [15] M. Christos and S. Sachdev, Emergence of nodal Bogoliubov quasiparticles across the transition from the pseudogap metal to the d -wave superconductor, *npj Quantum Mater.* **9**, 4 (2024).
- [16] M. Bunney, U. F. P. Seifert, S. Rachel, and M. Vojta, Fractionalized superconductivity mediated by Majorana fermions in the Kitaev-Kondo lattice, *Phys. Rev. Lett.* **134**, 206602 (2025).
- [17] S. D. Lundemo and A. Sudbø, Topological superconductivity induced by a Kitaev spin liquid, *Phys. Rev. B* **109**, 184508 (2024).
- [18] K. Kurokawa, S. Isono, Y. Kohama, S. Kunisada, S. Sakai, R. Sekine, M. Okubo, M. D. Watson, T. K. Kim, C. Cacho, S. Shin, T. Tohyama, K. Tokiwa, and T. Kondo, Unveiling phase diagram of the lightly doped high- T_c cuprate superconductors with disorder removed, *Nat. Commun.* **14**, 4064 (2023).
- [19] W. Metzner and H. Yamase, Phase stiffness in an antiferromagnetic superconductor, *Phys. Rev. B* **100**, 014504 (2019).
- [20] M. Qin, T. Schäfer, S. Andergassen, P. Corboz, and E. Gull, The Hubbard model: A computational perspective, *Annu. Rev. Condens. Matter Phys.* **13**, 275 (2022).
- [21] E. Gull, O. Parcollet, and A. J. Millis, Superconductivity and the pseudogap in the two-dimensional Hubbard model, *Phys. Rev. Lett.* **110**, 216405 (2013).
- [22] C. J. Halboth and W. Metzner, d -wave superconductivity and Pomeranchuk instability in the two-dimensional Hubbard model, *Phys. Rev. Lett.* **85**, 5162 (2000).
- [23] D. Vilardi, C. Taranto, and W. Metzner, Dynamically enhanced magnetic incommensurability: Effects of local dynamics on nonlocal spin correlations in a strongly correlated metal, *Phys. Rev. B* **97**, 235110 (2018).
- [24] O. Gunnarsson, T. Schäfer, J. P. F. LeBlanc, E. Gull, J. Merino, G. Sangiovanni, G. Rohringer, and A. Toschi, Fluctuation diagnostics of the electron self-energy: Origin of the pseudogap physics, *Phys. Rev. Lett.* **114**, 236402 (2015).
- [25] P. M. Bonetti and W. Metzner, $\text{SU}(2)$ gauge theory of the pseudogap phase in the two-dimensional Hubbard model, *Phys. Rev. B* **106**, 205152 (2022).
- [26] N. Dupuis, L. Canet, A. Eichhorn, W. Metzner, J. Pawłowski, M. Tissier, and N. Wschebor, The nonperturbative functional renormalization group and its applications, *Phys. Rep.* **910**, 1 (2021).
- [27] J. Wang, A. Eberlein, and W. Metzner, Competing order in correlated electron systems made simple: Consistent fusion of functional renormalization and mean-field theory, *Phys. Rev. B* **89**, 121116(R) (2014).
- [28] H. Yamase, A. Eberlein, and W. Metzner, Coexistence of incommensurate magnetism and superconductivity in the

- two-dimensional Hubbard model, *Phys. Rev. Lett.* **116**, 096402 (2016).
- [29] D. Vilardi, P. M. Bonetti, and W. Metzner, Dynamical functional renormalization group computation of order parameters and critical temperatures in the two-dimensional Hubbard model, *Phys. Rev. B* **102**, 245128 (2020).
- [30] V. L. Berezinskii, Destruction of long-range order in one-dimensional and two-dimensional systems having a continuous symmetry group. I. Classical systems, *Zh. Eksp. Teor. Fiz.* **59**, 907 (1971) [*Sov. Phys. JETP* **32**, 493 (1971)].
- [31] J. M. Kosterlitz and D. J. Thouless, Ordering, metastability and phase transitions in two-dimensional systems, *J. Phys. C* **6**, 1181 (1973).
- [32] A. Auerbach, *Interacting Electrons and Quantum Magnetism* (Springer-Verlag, Berlin, 1994).
- [33] N. Dupuis, Spin fluctuations and pseudogap in the two-dimensional half-filled Hubbard model at weak coupling, *Phys. Rev. B* **65**, 245118 (2002).
- [34] F. D. M. Haldane, Nonlinear field theory of large-spin Heisenberg antiferromagnets: Semiclassically quantized solitons of the one-dimensional easy-axis Néel state, *Phys. Rev. Lett.* **50**, 1153 (1983).
- [35] F. Haldane, Continuum dynamics of the 1-d Heisenberg antiferromagnet: Identification with the $O(3)$ nonlinear sigma model, *Phys. Lett. A* **93**, 464 (1983).
- [36] P. M. Bonetti, Local Ward identities for collective excitations in fermionic systems with spontaneously broken symmetries, *Phys. Rev. B* **106**, 155105 (2022); Erratum: Local Ward identities for collective excitations in fermionic systems with spontaneously broken symmetries, **110**, 079902(E) (2024).
- [37] I. A. Goremykin and A. A. Katanin, Antiferromagnetic and spin spiral correlations in the doped two-dimensional Hubbard model: Gauge symmetry, Ward identities, and dynamical mean-field theory analysis, *Phys. Rev. B* **110**, 085153 (2024).
- [38] N. Verma, T. Hazra, and M. Randeria, Optical spectral weight, phase stiffness, and T_c bounds for trivial and topological flat band superconductors, *Proc. Natl. Acad. Sci. USA* **118**, e2106744118 (2021).
- [39] A. P. Kampf, Collective excitations in itinerant spiral magnets, *Phys. Rev. B* **53**, 747 (1996).
- [40] P. M. Bonetti and W. Metzner, Spin stiffness, spectral weight, and Landau damping of magnons in metallic spiral magnets, *Phys. Rev. B* **105**, 134426 (2022).
- [41] C. Honerkamp and M. Salmhofer, Temperature-flow renormalization group and the competition between superconductivity and ferromagnetism, *Phys. Rev. B* **64**, 184516 (2001).
- [42] D. Vilardi, P. M. Bonetti, and W. Metzner, Spin stiffnesses and stability of magnetic order in the lightly doped two-dimensional Hubbard model, *Phys. Rev. B* **112**, 245149 (2025).