

Operator space transport in collective spin Lindbladians

Dominik Nemeth ^{*}, Ahsan Nazir , Robert-Jan Slager , and Alessandro Principi 

Department of Physics and Astronomy, The University of Manchester, Oxford Road, Manchester M13 9PL, United Kingdom



(Received 4 June 2026; revised 4 September 2026; accepted 21 September 2026; published 14 October 2026)

Boundary time crystals (BTCs) are paradigmatic examples of continuous time crystals that emerge in collective, dissipative quantum systems. Their microscopic origin, however, remains obscured by the complexity of Lindbladian many-body dynamics and has largely been investigated through semiclassical and numerical approaches. Here, we develop an exact operator space framework for collective spin Lindbladians based on irreducible tensor representations. We map the Lindbladian dynamics onto a local non-Hermitian hopping problem, where weak symmetries impose selection rules on operator transport, while the hopping amplitudes and their relative strengths determine the resulting dynamical regimes. We identify the microscopic ingredients that distinguish models supporting conventional collective precession from those supporting BTC dynamics. Our approach further reveals how nonunitary dissipation can provide a source for the traceless operator subspace, revealing a microscopic mechanism for the reduced sensitivity of BTC dynamics to initial conditions beyond mean-field descriptions. More broadly, our results establish operator space transport as a general perspective for understanding dissipative many-body dynamics and its connections to non-Hermitian phenomena.

DOI: [10.1103/x51f-y16f](https://doi.org/10.1103/x51f-y16f)

I. INTRODUCTION

Open quantum many-body dynamics has become a central arena for uncovering novel phases of matter beyond equilibrium, a particular example of which is a time crystal phase. While time crystals were originally proposed in closed, periodically driven systems [1–15], their extension to open quantum systems has revealed new mechanisms by which persistent oscillations can arise under dissipation in macroscopic systems [16–19]. A paradigmatic example is the boundary time crystal (BTC) [20], which emerges in macroscopically large collective spin systems forming the boundary of a much larger, weakly coupled Markovian environment.

Early studies of boundary time crystals have largely relied on semiclassical and mean-field descriptions [20,21], where the dynamics is reduced to nonlinear equations of motion and time crystal behavior is identified through linear stability analysis. However, the validity of such mean-field approximations in open many-body systems has been actively debated [22–25]. Recent works that go beyond mean-field theory [25–29] have revealed that BTCs possess genuinely many-body features that are not fully captured within semiclassical descriptions.

These developments highlight the need for a microscopic, fully quantum framework capable of describing collective spin Lindbladians beyond mean-field approximations. In

particular, such a framework should allow one to systematically classify the resulting dynamical regimes and provide a natural basis for analytical calculations, thereby offering deeper insight into the origin and properties of BTC behavior.

In problems involving spin systems or angular momentum degrees of freedom, it is customary to work in the angular momentum eigenbasis, expanding operators in terms of matrix elements between states $|jm\rangle$, where j and m denote the total angular momentum and its projection along the z axis, respectively. However, an alternative and often more physically transparent approach—ubiquitous in nuclear magnetic resonance (NMR) [30–32] and atomic, molecular, and optical (AMO) physics [33–35]—makes use of the spherical tensor operator formalism. In this representation, operators are expanded in terms of irreducible tensor operators T_q^k , characterized by tensor rank k and component q , which transform under rotations via the adjoint action of the $SU(2)$ group. For fixed k , these operators form a spin- k multiplet [36,37].

The dynamics of collective spin systems can be organized through couplings between angular momentum sectors [38,39]. At the same time, several classes of open quantum systems admit a description directly in operator space. For example, in quadratic fermionic systems, the Liouvillian can be viewed as acting on a Fock space of operators and described through ladder superoperators [40,41]. This suggests the possibility of a similar framework for collective spin systems, motivating the question of whether Lindbladian collective spin dynamics can likewise be understood through structured couplings between irreducible operator subspaces.

In this work, we develop a microscopic quantum framework for classifying collective spin Lindbladians according to their weak symmetries, defined by superoperators that commute with the Lindbladian [42,43]. Unlike semiclassical approaches based on truncations of the hierarchy of

^{*}Contact author: dominik.nemeth@manchester.ac.uk

correlations [20,21,25], our formulation retains the full operator space structure of the dynamics. We construct the framework using the irreducible decomposition of Liouville space, expanding the density operator in spherical tensor operators labeled by their rank k and component q [35–37]. In this representation, the Lindblad master equation maps onto a local non-Hermitian hopping problem in operator space. Weak symmetries constrain the allowed directions of transport, while the corresponding microscopic hopping amplitudes determine the resulting dynamical regime. This provides a direct microscopic route to identifying the features that distinguish conventional collective spin precession from BTC dynamics.

The paper is organized as follows. In Sec. II, we introduce the class of collective spin Lindbladians considered throughout. In Sec. III, we review the representation theory of collective spins and show how the relevant irreducible representations emerge in operator space. In Sec. IV, we discuss collective spin models based on their weak symmetry constraints by expanding the density operator in the spherical tensor basis. We illustrate the utility of our framework by focusing on driven collective spin models that experience either collective dephasing or collective decay. In Sec. V, we discuss how nonunitary dissipation can lead to a source term in operator space, which we use to explain the reduced initial state sensitivity of BTCs. We compare our method with semiclassical approaches in Sec. VI. In Sec. VII, we highlight how our framework can be used to study the extreme time crystal limit, which occurs in the strong-driving regime. In Sec. VIII, we examine how increasing dissipation drives the hybridization of spherical tensor modes, which can be interpreted as a form of delocalization in operator space. Finally, we comment on the thermodynamic limit of the BTC model in Sec. IX.

Throughout this work, we employ natural units. The terms irreducible tensor operators and spherical tensor operators, as well as irreducible representations (irreps) and spherical tensor sectors, are used interchangeably.

II. COLLECTIVE SPIN DYNAMICS UNDER MARKOVIAN DISSIPATION

We consider open quantum systems governed by a Lindblad master equation for the density operator $\rho(t)$ ($\hbar = 1$),

$$\begin{aligned} \frac{d\rho}{dt} &= \mathcal{L}[\rho] = -i[H, \rho] + \mathcal{D}[\rho], \\ \mathcal{D}[\rho] &= \frac{\Gamma}{2j} \sum_{\mu} \left(L_{\mu} \rho L_{\mu}^{\dagger} - \frac{1}{2} \{L_{\mu}^{\dagger} L_{\mu}, \rho\} \right), \end{aligned} \quad (1)$$

where $\mathcal{L}$ is the Liouvillian superoperator generating Markovian evolution and $\mathcal{D}$ is the Lindblad dissipator. We consider collective spin models in which the Hamiltonian H and jump operators L_{μ} are constructed from spin- j operators J_{α} ($\alpha = x, y, z$), with a uniform collective rate $\Gamma/2j$. The factor $1/2j$ constitutes a Kac scaling and guarantees that the collective dissipative dynamics remain extensive as $j \rightarrow \infty$. These operators satisfy the $\mathfrak{su}(2)$ commutation relations $[J_{\alpha}, J_{\beta}] = i\epsilon_{\alpha\beta\gamma} J_{\gamma}$, with ladder operators $J_{\pm} = J_x \pm iJ_y$.

For the class of models considered here, the total spin J^2 is conserved, allowing each fixed- j sector to be analyzed independently. Such models arise naturally in cavity systems

with cooperative emission [44–46], where N spin- $\frac{1}{2}$ degrees of freedom behave collectively as a single spin with $j = N/2$.

In this work, we focus on the collective spin models defined by $H = \Omega J_x$ with either $L_{\mu} = J_z$ or $L_{\mu} = J_{-}$, which serve as minimal representatives of models in which collective precession and BTC dynamics can emerge. In the precession case, the spin components J_y and J_z undergo single-frequency oscillations that generally depend on the initial state [27,28]. This behavior arises from a simple Liouvillian structure dominated by a single oscillatory mode. In contrast, BTC dynamics exhibit qualitatively richer behavior: the oscillations display multiple frequency components and, crucially, exist independently of the initial state [20,21,25]. In the thermodynamic limit ($j \rightarrow \infty$), the corresponding decay rate vanishes and these oscillations persist indefinitely, giving rise to the BTC phase [20,21,27,28].

The operators J_{α} generate a $\mathfrak{su}(2)$ Lie algebra, and their representation theory leads to the well-known basis of angular momentum eigenstates $\{|j m\rangle\}$, with $m = -j, \dots, j$. The dynamics can then be formulated in terms of a density matrix in the $\{|j m\rangle\}$ basis as

$$\rho = \sum_{m, m'} \rho_{mm'} |j m\rangle \langle j m'|, \quad (2)$$

for a fixed j . This usual approach has been employed in Ref. [27], which performed a perturbative treatment of collective spin Lindbladians.

However, for open quantum systems governed by a Liouvillian superoperator, this state-based representation is no longer the most symmetry-adapted description. In the following section, we introduce an operator-based formulation using the adjoint representation of $\mathfrak{su}(2)$.

III. REPRESENTATION THEORY FOR COLLECTIVE SPINS

In closed quantum systems, the fundamental dynamical object is the state vector $|\psi\rangle \in \mathcal{H}$. For systems whose degrees of freedom furnish a representation of the $\mathfrak{su}(2)$ Lie algebra, which in the case of collective spins is generated by the operators J_{α} , it is natural to adopt a state-based representation in which these generators act on $\mathcal{H}$. Representation theory then organizes the Hilbert space into irreps. Concretely, one diagonalizes the Casimir operator $J^2 = J_x^2 + J_y^2 + J_z^2$ (together with J_z), obtaining sectors labeled by total spin j and magnetic quantum numbers $m = -j, \dots, j$, such that

$$\mathcal{H} = \bigoplus_j \mathcal{H}_j, \quad \dim \mathcal{H}_j = 2j + 1. \quad (3)$$

This decomposition provides the conventional state-based quantum numbers typically used to describe spin systems.

Conversely, in open quantum systems the fundamental dynamical object is the density operator ρ . In the collective spin case, ρ belongs to the operator space $\mathcal{B}(\mathcal{H}_j)$, which is isomorphic to a tensor-product space. This can be seen from the fact that any operator may be constructed from the outer products of kets $\{|j m\rangle\}$ and bras $\{\langle j m'|\}$. Following the Choi–Jamiołkowski isomorphism (vectorization) [47–49], one applies the mapping $|j m\rangle \langle j m'| \mapsto |j m\rangle \otimes |j m'\rangle^*$, so that operators are represented as vectors in the tensor-product

space $\mathcal{H}_j \otimes \mathcal{H}_j^*$. Since tensor-product spaces are generally reducible, it is natural to instead decompose $\mathcal{H}_j \otimes \mathcal{H}_j^*$ into the irreps of $\mathcal{B}(\mathcal{H}_j)$. This decomposition introduces operator space quantum numbers and yields a symmetry-adapted basis particularly suited to describing dissipative dynamics.

For collective spins, this reducibility takes the explicit form

$$\mathcal{B}(\mathcal{H}_j) \cong \mathcal{H}_j \otimes \mathcal{H}_j^* \cong \bigoplus_{k=0}^{2j} \mathcal{B}_k, \quad (4)$$

where each $\mathcal{B}_k$ transforms irreducibly under $\text{SU}(2)$. The allowed values of k follow from the Clebsch–Gordan decomposition of two spin- j representations, $j \otimes j = \bigoplus_{k=0}^{2j} k$. Physically, this tensor-product structure may be viewed as the combination of two independent spin spaces. In the present case, however, the second space corresponds to the dual Hilbert space $\mathcal{H}_j^*$, which has important consequences for perturbative calculations (see Sec. VII). This decomposition highlights a key structural feature of open quantum dynamics: operator space splits into symmetry-resolved sectors that are not visible in a state-based description.

The generators of collective rotations in operator space are the superoperators implementing infinitesimal conjugation,

$$\mathcal{K}_\alpha[A] \equiv [J_\alpha, A], \quad \alpha = x, y, z, \quad (5)$$

commonly referred to as the adjoint generators (see Appendix A). These generate rotations while acting on operators rather than states. The adjoint generators obey the same $\text{su}(2)$ algebra as the collective spin operators,

$$[\mathcal{K}_\alpha, \mathcal{K}_\beta] = i\epsilon_{\alpha\beta\gamma} \mathcal{K}_\gamma, \quad (6)$$

and define the Casimir superoperator

$$\mathcal{K}^2 = \sum_{\alpha=x,y,z} \mathcal{K}_\alpha^2. \quad (7)$$

Within each irreducible sector $\mathcal{B}_k$, the Casimir superoperator acts diagonally with eigenvalue $k(k+1)$, providing the quantum number that labels the operator irreps.

The operator space analogues of angular momentum eigenstates are the spherical tensor operators T_q^k [35–37]. These form a basis of simultaneous eigenoperators of both $\mathcal{K}^2$ and $\mathcal{K}_z$,

$$\begin{aligned} \mathcal{K}^2[T_q^k] &= \sum_{\alpha=x,y,z} [J_\alpha, [J_\alpha, T_q^k]] = k(k+1) T_q^k, \\ \mathcal{K}_z[T_q^k] &= [J_z, T_q^k] = q T_q^k. \end{aligned} \quad (8)$$

The quantum numbers k and q correspond to the spherical tensor rank k and its magnetic component q . For a fixed k , the allowed values of q form a multiplet with $q = -k, -k+1, \dots, +k$, giving $(2k+1)$ operators in each sector.

These quantum numbers admit an interpretation in terms of multipole moments, where k denotes the order of the multipole [35]. In this sense, k labels the effective complexity of many-body operators. For example, the sector $k=1$ contains the dipole operators J_x , J_y , and J_z , which are captured by mean-field descriptions. Higher values of k correspond to operators encoding progressively higher-order spin correlations. Expanding the density operator in the spherical tensor basis therefore amounts to decomposing it into contributions of

increasing multipole order. In semiclassical approximations, this hierarchy is truncated by retaining only polynomial expectation values of $k=1$ operators (see Sec. VII).

Thus, this framework introduces a set of operator space quantum numbers that label operators independently of a state-based description. It also provides a symmetry-adapted operator basis that can be used for practical calculations. In the following section, we demonstrate the utility of this framework.

Throughout our work, calligraphic symbols ($\mathcal{A}$) are used for superoperators acting on the Hilbert space of operators $\mathcal{B}(\mathcal{H}_j)$, whereas double-lined symbols ($\mathbb{A}$) are used for superoperators acting on the vectorized, Liouville space $\mathcal{H}_j \otimes \mathcal{H}_j^*$ [49]. In addition, the vectorized form of an operator O is denoted via $|O\rangle\rangle$.

IV. WEAK SYMMETRIES AND OPERATOR SPACE TRANSPORT

In this section, we show how the adjoint generators introduced in Sec. III, together with their Casimir superoperator, can define weak symmetries of the Liouvillian [42,43]. Commutation with $\mathcal{K}^2$ or $\mathcal{K}_z$ conserves the tensor rank k or component q , respectively, thereby imposing selection rules on transport in operator space. These weak symmetries therefore provide a simple way to distinguish collective spin models according to their allowed transport structures. We first present a general overview of this framework and then apply it to two explicit examples, demonstrating how the preservation or breaking of these weak symmetries modifies the connectivity of an “operator lattice.” We emphasize that these symmetry constraints determine which transport processes are allowed, but do not by themselves determine the realized dynamical regime, which also depends on the Liouvillian matrix elements, model parameters, and spectral properties.

We analyze these systems by expanding ρ in the spherical tensor basis $\{T_q^k\}$ as

$$\rho(t) = \sum_{k=0}^{2j} \sum_{q=-k}^{+k} a_{k,q}(t) T_q^k, \quad (9)$$

as shown in Fig. 1(a).

The master equation $\frac{d}{dt}\rho = \mathcal{L}\rho$ can be expressed in terms of the expansion coefficients $\{a_{k,q}\}$ as

$$\frac{d}{dt}a_{k,q}(t) = \sum_{k'=0}^{2j} \sum_{q'=-k'}^{+k'} \mathcal{L}_{(k,q)(k',q')} a_{k',q'}(t), \quad (10)$$

with matrix elements

$$\mathcal{L}_{(k,q)(k',q')} \equiv \text{Tr}[(T_q^k)^\dagger \mathcal{L} T_{q'}^{k'}]. \quad (11)$$

Introducing a composite index $\alpha \equiv (k, q)$, this can be written compactly as

$$\frac{d}{dt}a_\alpha(t) = \sum_\beta \mathcal{L}_{\alpha\beta} a_\beta(t), \quad (12)$$

which has the form of a non-Hermitian tight-binding model. In this representation, the coefficients a_α play the role of

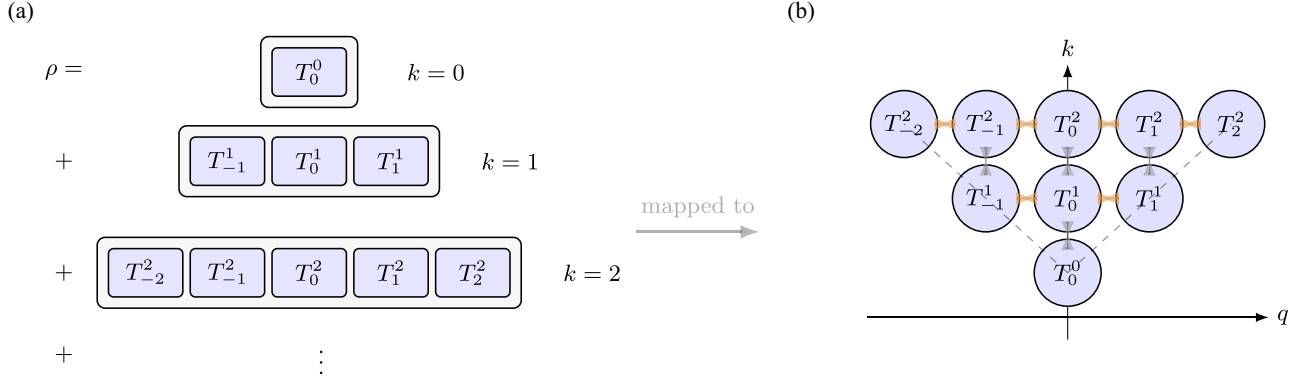


FIG. 1. Schematic representation of the spherical tensor decomposition of the density operator. (a) The density operator expanded into irreducible tensor multiplets of rank k , where each multiplet contains $2k + 1$ components labeled by $q = -k, \dots, +k$. (b) The same decomposition may be viewed as a triangular lattice in operator space, where each node corresponds to a spherical tensor mode T_q^k . Orange horizontal arrows indicate intrarank (q) mixing, while gray arrows indicate interranks (k) mixing.

amplitudes on an emergent operator space lattice, whose sites are labeled by the spherical tensor modes. The matrix elements $\mathcal{L}_{\alpha\beta}$ therefore act as effective hopping amplitudes from site β to site α on the lattice. As we show below, the structure of the Liouvillian leads to short-range couplings in (k, q) , giving rise to a local hopping model on the operator lattice [see Fig. 1(b)]. For a derivation of Eq. (10), see Appendix C.

We now illustrate the utility of this framework using two explicit models that share the same coherent drive but differ in their choice of collective jump operator. This difference changes the weak symmetry structure of the Liouvillian and hence the allowed transport channels on the (k, q) lattice.

A. Driven collective spins with dephasing: Collective precession

We first consider the case in which tensor rank k is conserved, while the component q is not,

$$[\mathcal{L}, \mathcal{K}^2] = 0, \quad [\mathcal{L}, \mathcal{K}_z] \neq 0. \quad (13)$$

The first condition makes each fixed- k subspace invariant under the Liouvillian evolution, whereas the second permits off-diagonal matrix elements that couple different q components within these subspaces. Consequently, operator weight may be redistributed within each multiplet but cannot be transported between different tensor ranks.

Therefore, sectors with different k evolve independently, and Eq. (10) reduces to a block-diagonal form,

$$\frac{d}{dt} \mathbf{a}^{(k)}(t) = \mathcal{L}^{(k)} \mathbf{a}^{(k)}(t), \quad (14)$$

where $\mathbf{a}^{(k)} = (a_{k,-k}, \dots, a_{k,k})^T$, and $\mathcal{L}^{(k)}$ denotes the restriction of $\mathcal{L}_{(k,q)(k',q')}$ to the subspace with fixed k , i.e., the $(2k+1)$ -dimensional block with matrix elements $(\mathcal{L}^{(k)})_{qq'} = \mathcal{L}_{(k,q)(k,q')}$.

A minimal realization of coherent q mixing at fixed k is provided by a transverse collective field, $H = \Omega J_x$, with a jump operator along the quantization axis,

$$\mathcal{L}[\rho] = -i\Omega[J_x, \rho] + \frac{\Gamma}{2j} \left(J_z \rho J_z - \frac{1}{2} \{J_z^2, \rho\} \right). \quad (15)$$

The Hamiltonian term generates rotations about the x axis and explicitly breaks conservation of q with respect to J_z . As a result, k remains a good quantum number, but q is dynamically mixed within each fixed- k sector. In the $\{T_q^k\}$ basis, the dynamics maps onto a local, non-Hermitian hopping model,

$$\frac{d}{dt} a_{k,q} = -i\Omega \mathcal{W}_{k,q} - \frac{\Gamma}{4j} q^2 a_{k,q}, \quad (16)$$

with

$$\mathcal{W}_{k,q} = w_+(k, q-1) a_{k,q-1} + w_-(k, q+1) a_{k,q+1}. \quad (17)$$

Here, $w_{\pm}(k, q) \equiv \frac{1}{2} \sqrt{k(k+1) - q(q \pm 1)}$ are position-dependent couplings generated by $\mathcal{K}_x$. The terms in $\mathcal{W}_{k,q}$ admit a hopping interpretation: the first term describes transport from $(k, q-1)$ to (k, q) with amplitude $a_{k,q-1}$ and coupling strength $w_+(k, q-1)$, while the second term describes transport from $(k, q+1)$ to (k, q) with amplitude $a_{k,q+1}$ and coupling $w_-(k, q+1)$. These couplings are reciprocal in q , satisfying $w_+(k, q) = w_-(k, q+1)$. Consequently, within each fixed- k sector the dynamics corresponds to nearest-neighbor hopping along the q direction, with q -dependent onsite contributions from the Lindbladian dissipator $\mathcal{D}$. For a detailed derivation of this mapping, see Appendix D.

The operators J_x , J_y , and J_z form the $k=1$ operator multiplet, so their dynamics is entirely determined by the $q \in \{-1, 0, +1\}$ components of this sector. The corresponding coefficients $(a_{1,-1}, a_{1,0}, a_{1,1})$ obey

$$\begin{aligned} \frac{d}{dt} a_{1,0}(t) &= -\frac{i\Omega}{\sqrt{2}} (a_{1,1}(t) + a_{1,-1}(t)), \\ \frac{d}{dt} a_{1,1}(t) &= -\frac{i\Omega}{\sqrt{2}} a_{1,0}(t) - \frac{\Gamma}{4j} a_{1,1}(t), \\ \frac{d}{dt} a_{1,-1}(t) &= -\frac{i\Omega}{\sqrt{2}} a_{1,0}(t) - \frac{\Gamma}{4j} a_{1,-1}(t). \end{aligned} \quad (18)$$

For a complete discussion of the eigenvalues and eigenvectors of this system, see Appendix E. When viewed in terms of the observables J_x , J_y , and J_z , this results in a dynamics where J_x is decoupled and exhibits exponential decay. (J_y , J_z)

are coupled together and exhibit decaying oscillations with frequency $\Omega\sqrt{1 - (\kappa/2)^2}$, where $\kappa \equiv \Gamma/(4j\Omega)$. The decay factor is given by $\Gamma/(8j)$, which vanishes in the thermodynamic limit ($j \rightarrow \infty$). In this limit, both J_y and J_z exhibit persistent oscillations, leading to collective precession. An exceptional point emerges at $\kappa = 2$, where oscillatory eigenmodes coalesce and the dynamics becomes pure relaxation [27,28]. This example explicitly demonstrates that the weak symmetry structure fixes the allowed transport channels but does not by itself determine whether the resulting dynamics is oscillatory or relaxational.

The main consequence of the weak symmetry generated by $\mathcal{K}^2$ is that operator weight cannot be transferred between different tensor ranks. The initial distribution across k sectors therefore cannot be dynamically redistributed, and observables in the $k = 1$ multiplet cannot acquire contributions from higher-rank modes. This restricted connectivity is the transport property predicted by the symmetry analysis; the frequencies, decay rates, and exceptional points within each block must instead be determined from the corresponding matrix elements of $\mathcal{L}^{(k)}$.

In the following subsection, we consider a model for which $[\mathcal{L}, \mathcal{K}^2] \neq 0$. Breaking this weak symmetry permits interranks transport and converts the independent one-dimensional q chains into a coupled two-dimensional operator space lattice. Whether this enlarged connectivity produces BTC or overdamped behavior remains a dynamical question governed by the model parameters and Liouvillian spectrum.

B. Driven collective spins with decay: BTC model

So far, we have considered a Liouvillian that commutes with $\mathcal{K}^2$, such that each tensor-rank sector evolves independently. We now turn to a model in which $\mathcal{K}^2$ is not conserved. Breaking this weak symmetry removes the fixed- k block structure and permits transport between different tensor ranks.

The canonical form of the BTC model is given by [20]

$$\mathcal{L}[\rho] = -i\Omega[J_x, \rho] + \frac{\Gamma}{2j} \left(J_- \rho J_+ - \frac{1}{2} \{J_+, J_-, \rho\} \right), \quad (19)$$

where the jump operator J_- describes collective decay. In contrast to the J_z dissipator considered above, its action does not preserve the adjoint Casimir superoperator,

$$[\mathcal{D}, \mathcal{K}^2] \neq 0. \quad (20)$$

Consequently, for $\Gamma \neq 0$,

$$[\mathcal{L}, \mathcal{K}^2] \neq 0. \quad (21)$$

The coherent drive generated by J_x also breaks conservation of the tensor component with respect to the z axis. Thus, for $\Gamma, \Omega \neq 0$,

$$[\mathcal{L}, \mathcal{K}^2] \neq 0, \quad [\mathcal{L}, \mathcal{K}_z] \neq 0. \quad (22)$$

Therefore, neither k nor q is conserved, and transport along both directions of the operator space lattice is allowed. In the limiting case $\Omega = 0$, conservation of q is restored and the dynamics reduces to dissipative transport at fixed q .

We emphasize that Eq. (22) establishes only that $\mathcal{K}^2$ and $\mathcal{K}_z$ are not weak symmetries of this Liouvillian; it does not exclude the existence of other weak or strong symmetries. For

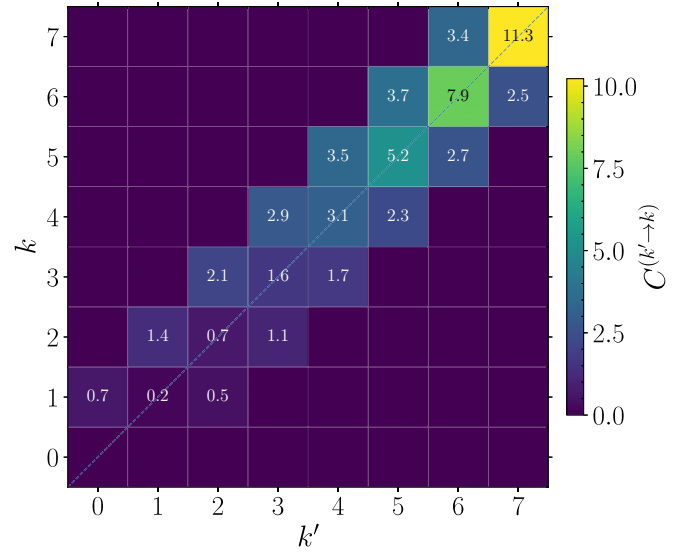


FIG. 2. Rank coupling matrix. A numerical visualization of the coupling strengths between irreducible sectors $k' \rightarrow k$, quantified by $C^{(k' \rightarrow k)}$ [Eq. (27)] and represented by the color scale. Results are shown for total spin $j = 7/2$ ($k = 0, 1, \dots, 7$) with $\Gamma = \Omega = 1$. Numerical values are displayed for nonzero entries to highlight the allowed transitions between sectors.

example, the total spin operator J^2 remains a strong symmetry because it commutes separately with the Hamiltonian and the jump operator. Since we work within a fixed- j sector this strong symmetry does not constrain transport between operator sectors labeled by k and q .

The action of $\mathcal{D}$ couples neighboring tensor-rank sectors, thereby permitting interranks transport. Together with the q mixing generated by the coherent drive, this produces a two-dimensional connectivity structure on the (k, q) lattice. Specifically, $\mathcal{D}$ generates nearest-neighbor hopping along the k direction together with local onsite contributions (see Appendix F). Note that nearest-neighbor hopping in k is a consequence of linear jump operators. More general jumps, such as J_+^2 , may generate longer-range couplings in k , with the allowed hopping range determined by the irreducible tensor addition rules discussed in Appendix F.

The Lindbladian dynamics can therefore be represented as a nearest-neighbor non-Hermitian hopping problem, where k and q act as effective coordinates on an emergent two-dimensional operator lattice. In this representation, Eq. (19) maps to

$$\frac{d}{dt} a_{k,q} = -i\Omega \mathcal{W}_{k,q} + \frac{\Gamma}{2j} \mathcal{T}_{k,q}, \quad (23)$$

with

$$\begin{aligned} \mathcal{T}_{k,q} = & t_+(k-1, q) a_{k-1,q} + t_-(k+1, q) a_{k+1,q} \\ & - \gamma(k, q) a_{k,q}. \end{aligned} \quad (24)$$

Here, $\gamma(k, q)$ denotes an onsite term at the coordinate (k, q) , while $t_{\pm}(k, q)$ describe hopping between (k, q) and $(k \pm 1, q)$. In the present case, hopping along the k direction is non-reciprocal, $t_+(k, q) \neq t_-(k+1, q)$ (see Fig. 2), giving rise to an effective non-Hermitian transport problem in k . The

hopping amplitudes $t_{\pm}(k, q)$ generally depend on the total spin j . Since our analysis is performed within a fixed- j sector, for qualitative discussions this may be suppressed for notational simplicity. The full expressions $t_{\pm}(k, q, j)$ together with their derivation, are presented in Appendix F. On the other hand, $\gamma(k, q)$ is independent of the total spin j , yielding

$$\gamma(k, q) = \frac{1}{2}[k(k+1) - q^2]. \quad (25)$$

This corresponds to an inhomogeneous onsite decay in operator space: the damping increases with rank k , while for fixed k it is largest at the center of a multiplet ($q = 0$) and weakest at its boundaries ($q = \pm k$).

In contrast, the coherent contribution remains identical to the collective precession case, with $w_{\pm}(k, q)$ generating reciprocal hopping along the q direction between (k, q) and $(k, q \pm 1)$. The corresponding hopping amplitudes can also be extracted numerically from the Liouvillian matrix elements in the $\{T_q^k\}$ basis, as shown in Fig. 2.

A numerical visualization of the coupling strengths between different ranks k' and k , $C^{(k' \rightarrow k)}$, is shown in Fig. 2. To construct $C^{(k' \rightarrow k)}$, we first define the projectors onto the eigenspace of $\mathcal{K}^2$ with eigenvalues $k(k+1)$ as $\mathbb{P}^{(k)}$. Then, by introducing

$$\mathbb{D}^{(k' \rightarrow k)} = \mathbb{P}^{(k)} \mathbb{D} \mathbb{P}^{(k')} \quad (26)$$

as the flow of operator weight from the sector k' to k generated by the Lindbladian dissipator $\mathbb{D}$ (acting on the vectorized $|\rho\rangle\rangle$), we define $C^{(k' \rightarrow k)}$ as

$$C^{(k' \rightarrow k)} = \sqrt{\sum_{qq'} |(\mathbb{D}^{(k' \rightarrow k)})_{qq'}|^2} \quad (27)$$

such that we sum over all possible q channels to go from k' to k . Therefore, this represents a measure of the overall coupling strength between the two sectors. For further details on the construction of the projectors, see Appendix G.

In Fig. 2, the first off-diagonal elements encode effective, nearest-neighbor couplings between sectors. These couplings are asymmetric, which is because of the nonreciprocal hopping amplitudes, $t_+(k, q) \neq t_-(k+1, q)$. The diagonal elements correspond to dissipative contributions that preserve k , captured by the onsite decay rates $\gamma(k, q)$ [cf. Eq.(25)].

Therefore, we have demonstrated how the dynamics of the canonical BTC model [Eq. (19)] can be formulated as a nearest-neighbor transport problem on the two-dimensional (k, q) lattice. The symmetry analysis predicts that operator weight is no longer confined to fixed-rank multiplets and that low-rank observables may hybridize with higher-rank operator sectors; see Sec. VIII for more detail on the hybridization. These symmetries therefore constrain the allowed forms of collective spin dynamics, whose specifics are determined by the ratio Γ/Ω . We expand on this in Sec. VII by discussing how our approach can be used to study the extreme time crystal regime, which is characterized by $\Gamma \ll \Omega$.

V. INITIAL STATE INDEPENDENT DYNAMICS: BTC MODEL

The example considered in Sec. IV A illustrates how oscillatory dynamics can arise in the strong-driving regime,

$\Gamma \ll \Omega$. The coherent term generated by J_x produces imaginary couplings between different q components within each fixed- k multiplet. For $k = 1$, these couplings give the familiar collective precession of J_y and J_z about the x axis. The same mechanism applies more generally to every tensor-rank sector. Higher-rank multiplets, however, contain several distinct nonzero values of $|q|$, and can therefore support multiple oscillation frequencies, beating, and higher harmonics; see Ref. [50] for further discussion.

When k is conserved, these oscillations are necessarily initial state dependent. Each multiplet evolves independently according to $\mathbf{a}^{(k)}(t) = (a_{k,-k}(t), \dots, a_{k,k}(t))^T$, so a sector can evolve only if it is populated initially. For example, since the $k = 1$ multiplet is spanned by J_x, J_y , and J_z , an initial state satisfying

$$\text{Tr}(\rho(0)J_{\alpha}) = 0, \quad \alpha = x, y, z, \quad (28)$$

has no initial weight in this sector. If $[\mathcal{L}, \mathcal{K}^2] = 0$, no other rank sector can subsequently populate it, and the $k = 1$ contribution therefore remains zero.

The situation changes qualitatively once the weak symmetry generated by $\mathcal{K}^2$ is broken. When $[\mathcal{L}, \mathcal{K}^2] \neq 0$, operator weight can be transported between different tensor ranks. For the BTC model, the Lindbladian dissipator with a linear collective jump operator produces nearest-neighbor hopping in k , so the dynamics of a given multiplet is no longer determined solely by its initial amplitudes: it can also be fed dynamically by neighboring sectors.

Crucially, such feeding contains both initial state dependent and initial state independent contributions. To distinguish them, we decompose the density operator into its identity and traceless parts [51–54],

$$\rho(t) = \frac{1}{d} + \delta\rho(t), \quad (29)$$

where $d = 2j + 1$ and

$$\delta\rho(t) = \sum_{k \geq 1} \sum_{q=-k}^k a_{k,q}(t) T_q^k. \quad (30)$$

The coefficient of the identity is fixed by normalization and is therefore the same for every physical initial state.

Trace preservation implies $\text{Tr}(\mathcal{L}[\rho]) = 0$ [55]. In particular, if $\delta\rho$ is traceless, then $\mathcal{L}[\delta\rho]$ is also traceless, so the traceless operator subspace is invariant under the evolution. The identity, however, need not itself be invariant: for a nonunital Lindbladian, $\mathcal{L}[\mathbf{1}] \neq 0$. Because $\mathcal{L}[\mathbf{1}]$ is necessarily traceless, the identity sector may act as a source for the traceless subspace, whereas trace preservation forbids the reverse process.

In the decomposition into identity and traceless sectors, the Liouvillian therefore takes the block-triangular form [56]

$$\mathcal{L} = \begin{bmatrix} 0 & 0 \\ S & \mathcal{M} \end{bmatrix}, \quad (31)$$

where $\mathcal{M}$ governs evolution entirely within the traceless subspace, while $S[\mathbf{1}] = \mathcal{L}[\mathbf{1}]$ encodes the nonunital source.

The traceless component consequently obeys the inhomogeneous equation [57–59]

$$\frac{d}{dt}\delta\rho(t) = \mathcal{M}\delta\rho(t) + \frac{1}{d}\mathcal{S}[\mathbf{1}]. \quad (32)$$

Its formal solution is

$$\delta\rho(t) = e^{\mathcal{M}t}\delta\rho(0) + \frac{1}{d}\int_0^t ds e^{\mathcal{M}(t-s)}\mathcal{S}[\mathbf{1}]. \quad (33)$$

This expression makes the distinction transparent. The first term is the homogeneous contribution and retains information about the initial traceless components of the density operator. The second is an inhomogeneous contribution generated entirely by the nonunital source. Since the identity component $\mathbf{1}/d$ is fixed by normalization, this contribution is common to all physical initial states.

For the BTC model, Appendix F gives

$$\mathcal{S}[\mathbf{1}] = -\frac{\Gamma}{j}J_z. \quad (34)$$

Thus, collective decay continuously injects operator weight from the $k = 0$ identity sector into the $k = 1$, $q = 0$ component, since $T_0^1 \propto J_z$. The coherent J_x drive then mixes this component with $q = \pm 1$ within the same multiplet, generating oscillatory J_y and J_z dynamics. Importantly, this mechanism does not require any initial population of the $k = 1$ sector.

The same sourced weight can subsequently propagate to higher tensor ranks through dissipative hopping in k . Consequently, higher-rank oscillatory components can also be generated dynamically even when they are absent from the initial state. The maximally mixed state, $\rho(0) = \mathbf{1}/d$, provides the clearest example: initially, every traceless amplitude $a_{k,q}(0)$ vanishes, yet nonunital dissipation first generates the $k = 1$ component and the combined k and q transport can then spread this weight throughout operator space [50].

Nonunitality therefore provides a microscopic mechanism by which oscillatory modes can be populated independently of the initial traceless state $\delta\rho(0)$. In the operator space picture, the identity sector acts as a universal source, while coherent transport in q can convert the sourced weight into oscillatory dynamics and dissipative transport in k distributes it between different multipole sectors. Further details of these source-driven contributions and their dynamical behavior are given in Ref. [50].

Finally, we emphasize that this is a purely dissipative effect and cannot arise from unitary Hamiltonian dynamics. Indeed, for any Hermitian Hamiltonian H , one has $[H, \mathbf{1}] = 0$, so the identity sector remains invariant under coherent evolution. Consequently, the nonreciprocal transport discussed here originates entirely from the dissipative (Lindbladian) contribution.

VI. RELATION TO SEMICLASSICAL APPROACHES: BTC MODEL

The time crystal phase of the canonical BTC model [Eq. (19)] was originally identified through the emergence of stable limit-cycle dynamics [20,21]. These approaches begin from the Heisenberg equations of motion for the collective spin operators J_α . Since the resulting equations couple first

moments to higher-order spin correlations, an approximation is required to obtain a closed set of equations.

The simplest closure is provided by the mean-field approximation [60–62],

$$\langle J_\alpha J_\beta \rangle \approx \langle J_\alpha \rangle \langle J_\beta \rangle, \quad (35)$$

where $\alpha, \beta = x, y, z$. Introducing the normalized magnetizations

$$m_\alpha = \frac{\langle J_\alpha \rangle}{j}, \quad (36)$$

this factorization closes the hierarchy at the level of first moments and yields the nonlinear equations [20,21,50]

$$\begin{aligned} \frac{d}{dt}m_x &= \kappa m_x m_z, \\ \frac{d}{dt}m_y &= -\Omega m_z + \kappa m_y m_z, \\ \frac{d}{dt}m_z &= \Omega m_y - \kappa(m_x^2 + m_y^2), \end{aligned} \quad (37)$$

with $\kappa = \Gamma/2$. In the thermodynamic limit, this approximation becomes exact for the class of initially uncorrelated states [22–24]. It therefore provides a particularly useful description of the symmetry-breaking and limit-cycle structure of the BTC phase. Note that any nonzero initial magnetization vector $\mathbf{m}(0) = (m_x(0), m_y(0), m_z(0))^T$ can, in principle, develop oscillatory dynamics through the nonlinear coupling between its components. For example, for an initial state polarized along the x direction,

$$\dot{m}_z(0) = -\kappa m_x^2(0) \neq 0, \quad (38)$$

so a nonzero m_z component can be generated dynamically even when $m_y(0) = m_z(0) = 0$. This differs from collective precession [cf. Eq. (15)], where m_x decouples from m_y and m_z , and an initially x -polarized state remains stationary in the thermodynamic limit (see Appendix E for more detail).

Our operator space representation provides a complementary way to understand what information is retained by this approximation. The first moments $m_{x,y,z}$ are completely determined by the $k = 1$ spherical tensor multiplet, $\{a_{1,-1}, a_{1,0}, a_{1,1}\}$. By contrast, the sectors $k \geq 2$ contain higher-order multipole moments and correlations. Mean-field factorization does not retain these higher-rank amplitudes as independent dynamical variables. Instead, their influence on the equations of motion is approximated by products of the $k = 1$ expectation values.

This also clarifies its dependence on the initial state. Two states with identical first moments but different higher-order correlations are indistinguishable within a mean-field description, despite occupying different regions of the full operator space. In the particularly important case $m_x(0) = m_y(0) = m_z(0) = 0$, the mean-field variables contain no information about higher-rank components of the initial density operator. Consequently, dynamics originating from these correlations, or from their subsequent transport into the $k = 1$ sector, cannot in general be reconstructed from the first-moment equations alone. This provides an operator space interpretation of the initial state restrictions identified in Ref. [25].

Namely, initial states satisfying

$$m_x(0) = m_y(0) = m_z(0) = 0, \quad (39)$$

equivalently $a_{1,q}(0) = 0$ for $q = 0, \pm 1$, have no initial support in the $k = 1$ sector.

A systematic improvement is obtained by extending the closure to second order. As demonstrated in Ref. [25], retaining second-order cumulants allows BTC oscillations to be recovered for a broader class of initial states [62–65]. In our accompanying work [50], we show explicitly how the nonunitary sourcing of the $k = 1$ multiplet appears through the second-order cumulants, and that it is absent from the first-order mean-field description.

The cumulant hierarchy may therefore be viewed as a sequence of progressively refined closures of the full operator space dynamics: mean-field theory retains only first moments, second-order cumulant expansions additionally capture quadrupolar ($k = 2$) correlations, and higher-order closures progressively restore further multipole sectors. By contrast, the spherical tensor formulation employed here retains the complete hierarchy of k sectors from the outset. This is particularly useful for obtaining analytical results at finite system size, where mean-field theory is not generally exact and becomes asymptotically controlled only in the thermodynamic limit, subject to assumptions on the class of initial states [23–25]. Moreover, perturbative approaches formulated directly in the angular momentum basis can still require numerical diagonalization [27]. In Sec. VII, we instead exploit the irreducible tensor structure to derive an effective Lindbladian that provides an analytical description of the extreme time crystal regime. Retaining the full operator space structure also makes the weak symmetries of $\mathcal{L}$ explicit and provides a framework for identifying properties that emerge upon approaching the thermodynamic limit (see Sec. IX for more detail).

VII. PERTURBATIVE APPROACH IN THE EXTREME TIME CRYSTAL PHASE

The above framework can be extended to a perturbative description of the dynamics. In this section, we outline how the spherical tensor representation can be used to construct an effective Liouvillian valid deep in the time crystal regime. To this end, we perform a perturbative expansion, treating the dissipative contribution to first order in Γ/j .

Following the Choi–Jamiołkowski isomorphism [47,48], we vectorize the density operator as $\rho \mapsto |\rho\rangle\rangle$ [49]. Under this mapping, the adjoint generators transform as

$$\mathcal{K}_\alpha \mapsto \mathbb{S}_\alpha \equiv J_\alpha \otimes \mathbf{1} - \mathbf{1} \otimes J_\alpha^T, \quad (40)$$

where $\mathbb{S}_\alpha$ ($\alpha = x, y, z$) define the components of an effective “superspin” representation, as introduced in Ref. [28].

In that work, the notion of a *superspin* was introduced by observing that the tensor-product space $\mathcal{H}_j \otimes \mathcal{H}_j^*$ can be interpreted as describing two coupled subsystems. The second subsystem corresponds to the dual Hilbert space, such that spin operators act as $J_\alpha \otimes \mathbf{1}$ on the original space and $-\mathbf{1} \otimes J_\alpha^T$ on the dual space.

In the limit $\Gamma \rightarrow 0$, Eq. (19) reduces to purely unitary evolution generated by these two contributions, which can

be combined into a single effective degree of freedom. This emergent object is referred to as the superspin [28].

For $\Gamma \neq 0$, the dissipator induces an effective coupling between these subsystems and renormalizes their energy levels. Nevertheless, as shown in Ref. [28], the dynamics can still be captured by an effective description in terms of the superspin to first order in Γ/j ,

$$\mathbb{L}_{\text{eff}} = -i\Omega \mathbb{S}_x - \frac{\Gamma}{8j} [\mathbb{S}_x^2 + \mathbb{S}^2]. \quad (41)$$

This framework has since been used to derive precision bounds for frequency estimation in continuously monitored BTCs and to show that these systems can surpass the Heisenberg limit [66,67].

Here, we show that the superspin method emerges directly from the adjoint representation and working in the spherical tensor basis. The superspin components shown in Eq. (40) are equivalent to the adjoint generators, expressed in a form that acts on vectorized Liouville space. Consequently, Eq. (41) can be written as

$$\mathcal{L}_{\text{eff}} = -i\Omega \mathcal{K}_x - \frac{\Gamma}{8j} [\mathcal{K}_x^2 + \mathcal{K}^2]. \quad (42)$$

Its eigenoperators are given by $T_{q_x}^k$ with eigenvalues

$$\lambda_{k,q_x} = -i\Omega q_x - \frac{\Gamma}{8j} [q_x^2 + k(k+1)], \quad (43)$$

where the quantization axis has been rotated along the x axis. To first order in Γ , the eigenvalues therefore form multiplets labeled by $k = 0, 1, \dots, 2j$ and $q_x = -k, \dots, +k$. This structure explains the eigenspectrum of $\mathcal{L}$ observed in numerics [20,27,28]. In addition, the superspin basis $\{|s, s_x\rangle\rangle\}$ is equivalent to the vectorized spherical tensor basis such that $T_{q_x}^k \mapsto |k, q_x\rangle\rangle$.

Therefore, in the extreme time crystal limit, the leading-order BTC dynamics [Eq. (42)] reduces to the same qualitative form as Eq. (15) [28]. The key distinction is that Eq. (42) features anisotropic rather than isotropic dephasing, as discussed in detail in Ref. [28].

Next, we demonstrate how dissipation modifies the dynamics beyond the extreme time crystal limit.

VIII. DISSIPATION-DRIVEN HYBRIDIZATION IN OPERATOR SPACE

In Sec. IV B, it was demonstrated how the BTC model can be mapped to a local hopping problem, which describes the transport of operator weight across a two-dimensional operator lattice. Eq. (23) highlights that the dissipator drives transport across k ; therefore, in general, we expect the Liouvillian eigenmodes to be a superposition of many spherical tensor modes. We demonstrate this numerically in Fig. 3.

We quantify how delocalized each eigenmode is in k via the k -resolved participation ratio $\text{PR}_k^{(n)}$, which we define as

$$\text{PR}_k^{(n)} = \frac{1}{\sum_k (w_k^{(n)})^2}, \quad w_k^{(n)} = \sum_q |c_{kq}^{(n)}|^2. \quad (44)$$

This measures the effective number of rank sectors contributing to the n th right eigenvector of $\mathbb{L}$. Here, $c_{kq} = \langle\langle k|q|r^{(n)}\rangle\rangle$

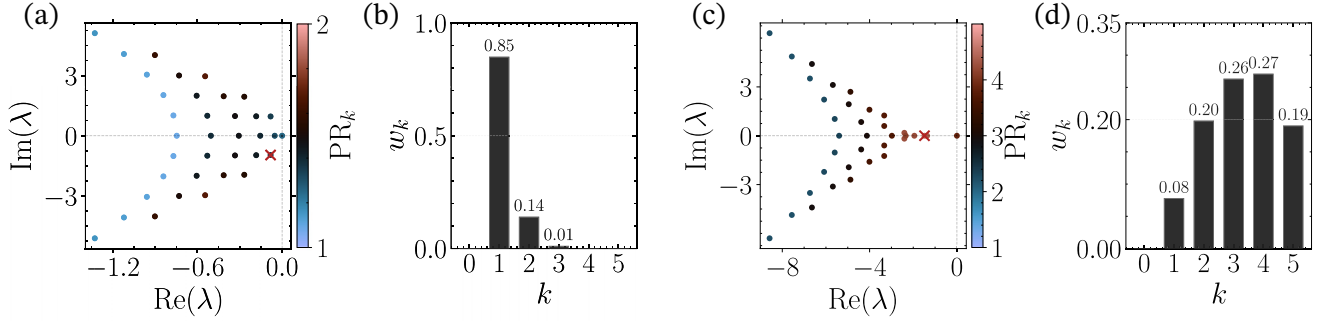


FIG. 3. Hybridization of spherical tensor modes. Results are shown for $j = 5/2$ with $\Gamma/\Omega = 0.5$ (a), (b) and $\Gamma/\Omega = 4$ (c), (d). Panels (a) and (c) display the Liouvillian eigenvalues (filled circles) in the $(\text{Re}(\lambda), \text{Im}(\lambda))$ plane, where the color indicates the k -resolved participation ratio PR_k of each eigenmode. The eigenmode analyzed in (b) and (d) is marked by a red cross. Panels (b) and (d) show the corresponding distribution of weights across rank sectors, w_k , as a function of k . The bar heights and accompanying numerical labels indicate the total weight in each sector, with labels shown only for weights larger than 0.01.

is an expansion coefficient of this right eigenvector in the vectorized spherical tensor basis and q is defined with respect to the quantization axis rotated along the x direction, such that $q \equiv q_x$. If $\text{PR}_k = 1$, then only a single sector contributes and the eigenmode is localized. Alternatively, if an eigenmode is highly delocalized and has equal weights across all sectors then $\text{PR}_k = 2j + 1$.

To further characterize this hybridization, we examine the distributions w_k and $w_{kq} = |c_{kq}|^2$ for an eigenmode belonging to the slowest-decaying oscillatory pair. These are shown in Figs. 3(b), 3(d), and 4, respectively.

The dynamics of the model is governed by two competing mechanisms that dominate in opposite limits. The unitary evolution generated by the Hamiltonian localizes the dynamics in tensor rank k while permitting transport in q . Consequently, the unperturbed eigenmodes are the vectorized spherical tensors $|k q_x\rangle$, and the dynamics correspond to collective precession within a fixed- k multiplet. In contrast, the Lindblad dissipator induces nonreciprocal hopping between different tensor ranks, promoting transport and delocalization along the k direction.

The BTC phase emerges in the regime where transport along both directions is present. Here, dissipation is sufficiently strong to hybridize different k sectors, yet weak enough that coherent transport in q remains the dominant

effect. As a result, operator weight spreads over a broad region of operator space while retaining the coherent dynamics necessary for persistent oscillations.

When $\Gamma \gg \Omega$, transport along k dominates with coherent transport in q becoming perturbatively suppressed. The multiplets therefore cease to undergo collective precession, leading to the disappearance of the BTC oscillations. This transport-based picture is fully consistent with finite-size spectral analysis: as the dissipation strength increases, complex conjugate eigenmode pairs [Figs. 3(a) and 3(c)] merge at exceptional points, signaling the breakdown of oscillatory dynamics and the onset of an overdamped regime [27,28].

Furthermore, delocalization in k hybridizes $\langle J_z(t) \rangle$ with higher-rank multiplets containing larger values of q . Since q determines the oscillation frequencies of the Liouvillian eigenmodes [Eqs. (23) and (43)], this hybridization causes $\langle J_z(t) \rangle$ to acquire frequency components originating from higher-order sectors, as observed in Ref. [20].

IX. THERMODYNAMIC LIMIT OF THE BTC MODEL

The $1/j$ Kac scaling of the Lindblad dissipator suggests that dissipative effects vanish in the thermodynamic limit, $j \rightarrow \infty$. However, the operator space representation reveals a more subtle behavior: different contributions generated by

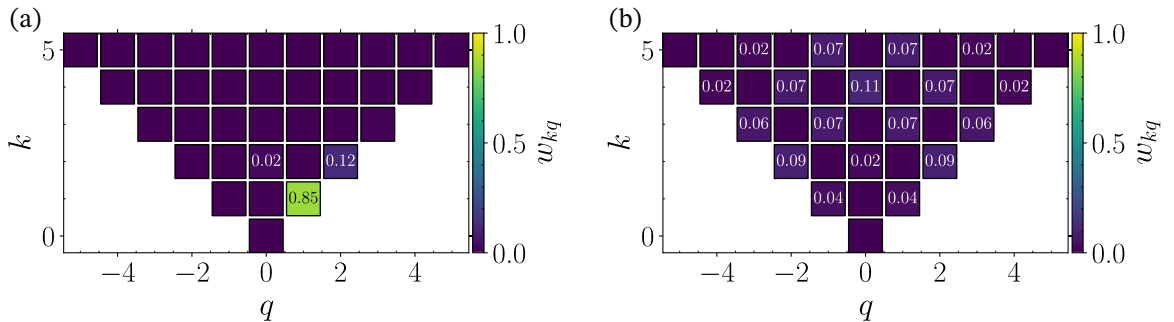


FIG. 4. Heatmap on the (k, q) lattice. Results are shown for $j = 5/2$ and $\Gamma/\Omega = 0.5$ (a) and $\Gamma/\Omega = 4$ (b). Weights w_{kq} are shown for each site on the (k, q) lattice, illustrating the spherical tensor modes contributing to a given eigenmode and their relative amplitudes. The eigenmode analyzed is marked by a red cross in Figs. 3(a) and 3(c), respectively. Numerical labels are shown only for weights larger than 0.01. The quantum number q is defined with respect to a quantization axis rotated along the x direction, such that $q \equiv q_x$.

the dissipator scale differently with j , such that local damping vanishes while dissipative hopping remains finite. As we highlight below, the surviving transport suggests a microscopic relation between the mean-field transition point [20] and the bulk transport properties of the operator lattice.

For $j \rightarrow \infty$, the interramp hopping amplitudes scale as $t_{\pm}(k, q, j) \sim O(j)$, whereas the local damping terms scale as $\gamma(k, q) \sim O(1)$. After including the Kac scaling of the dissipator, the onsite decay vanishes, $\gamma(k, q)/j \rightarrow 0$, while the hopping amplitudes remain finite, $t_{\pm}(k, q, j)/j \sim O(1)$. Thus, in the thermodynamic limit, local damping vanishes but dissipative transport remains finite.

More explicitly, the BTC model retains the local hopping structure of Eq. (23), with $\gamma(k, q)/j \rightarrow 0$ and

$$\begin{aligned} t_{-}(k, q, j) &= [(k-1)f(k, q)]j + O(1), \\ t_{+}(k, q, j) &= -[(k+2)f(k+1, q)]j + O(1), \end{aligned} \quad (45)$$

where

$$f(k, q) = \sqrt{\frac{(k-q)(k+q)}{(2k-1)(2k+1)}}. \quad (46)$$

The long lifetime of BTC oscillations is typically diagnosed through finite-size scaling of the Liouvillian gap [20,21,27]. Our operator space representation reveals the microscopic origin of this behavior: in the thermodynamic limit, the local damping contributions generated by the dissipator vanish as $j \rightarrow \infty$. Consequently, the thermodynamic BTC dynamics becomes a transport-dominated process in operator space, where operator weight propagates along the k and q directions. Note that this is fundamentally different from the model presented in Eq. (15), in which case only the Hamiltonian-induced coherent transport remains, and each k sector evolves independently.

The same operator space picture provides a microscopic connection to the BTC phase transition. In the thermodynamic limit, the (k, q) lattice becomes macroscopic, allowing the bulk transport properties to be analyzed for $1 \ll k \ll j$. The coherent hopping generated by the Hamiltonian is

$$-i\Omega w_{\pm}(k, q) = -\frac{i\Omega}{2} \sqrt{k^2 - q^2} \sqrt{1 + \frac{1}{k \pm q}}, \quad (47)$$

which away from the lattice boundaries, $q = \pm k$, reduces to

$$-i\Omega w_{\pm}(k, q) \approx -\frac{i\Omega}{2} \sqrt{k^2 - q^2}. \quad (48)$$

Similarly, the dissipative hopping amplitudes become

$$\frac{\Gamma}{2j} t_{\pm}(k, q) \approx \mp \frac{\Gamma}{4} \sqrt{k^2 - q^2}. \quad (49)$$

Hence, the bulk dynamics is governed by

$$\begin{aligned} \frac{d}{dt} a_{k,q} &= [-i\Omega(a_{k,q-1} + a_{k,q+1}) \\ &\quad - \frac{\Gamma}{2}(a_{k-1,q} - a_{k+1,q})]g(k, q), \end{aligned} \quad (50)$$

with

$$g(k, q) = \frac{1}{2} \sqrt{k^2 - q^2}. \quad (51)$$

To determine the corresponding bulk spectrum, we consider a locally uniform region around (k_0, q_0) and introduce the plane-wave ansatz [68–70]

$$a_{k,q}(t) = A(k_0, q_0) e^{i\lambda t} e^{ip_k(k-k_0)} e^{ip_q(q-q_0)}, \quad (52)$$

where p_k and p_q are conjugate momenta to the two operator space directions. Substitution into Eq. (50) yields

$$\lambda(p_k, p_q) = 2i\Omega g(k_0, q_0)(r \sin(p_k) - \cos(p_q)), \quad (53)$$

where $r = \frac{\Gamma}{2\Omega}$.

Therefore, the mean-field critical point $\Gamma_c = 2\Omega$ emerges directly from the bulk operator transport, marking the balance between coherent- and dissipative-dominated propagation.

X. DISCUSSION

The operator space transport picture developed here suggests several extensions beyond the canonical collective spin models considered in this work. One immediate direction is to consider more general reservoir-engineered dissipators [71,72]. In the spherical tensor representation, the irreducible tensor content of the jump operators constrains which tensor-rank sectors can be coupled. Higher-rank jump operators may therefore generate longer-ranged transport along the k direction of the operator lattice, with the precise hopping structure determined by the irreducible tensor coupling rules and by cancellations between the different contributions to the Lindblad dissipator; see Appendix F. This raises the possibility of engineering the connectivity of the operator space lattice directly through the choice of dissipative channels.

A broader question is whether a similar transport-based description can be formulated for systems without permutation invariance. For generic dissipative spin chains, the spherical tensor basis could be replaced by a Pauli-string basis, which has been widely used to characterize operator spreading and growth in quantum circuits and many-body systems [73–78]. In such a representation, the relevant notion of transport would no longer be between collective tensor ranks, but between operators of different spatial support and complexity. Analogously, fermionic systems admit simple operator space descriptions in terms of Majorana operators, as exploited in Refs. [40,79]. It would therefore be interesting to investigate whether the symmetry-constrained transport picture introduced here admits a useful analog in spatially resolved open many-body systems.

Such an extension would be particularly relevant for dissipative time crystals away from the idealized limit of perfect collectivity. Several works have investigated BTC-like behavior in systems with spatial structure and long-range interactions [19,80,81], but their descriptions have so far been limited to mean-field theory and cumulant expansions.

The role of nonunitarity also raises several open questions. In the canonical BTC model, nonunital dissipation provides a source for traceless operator components and can dynamically populate oscillatory modes even when these are absent from the initial state. This offers a microscopic explanation for the initial state insensitivity observed in Refs. [20,25,50], which is further discussed in Ref. [50]. It would be interesting to determine how general this mechanism is across other

realizations of dissipative time crystals, and in particular whether the so-called quantum BTC regimes discussed in Refs. [80,81] admit a similar source-driven interpretation. More generally, this raises the question of whether nonunitarity can be used as a diagnostic for identifying dissipative models capable of generating oscillatory dynamics independently of their initial traceless components.

In the thermodynamic limit, the BTC model can be further investigated as a bulk transport problem in operator space, in which the critical threshold $\Gamma_c = 2\Omega$ emerges directly from the competition between coherent and dissipative propagation. This raises the broader question of whether operator space transport can provide a useful microscopic framework for describing dissipative phase transitions. In particular, it would be interesting to determine whether changes in the dynamical phase can be related to qualitative changes in the transport properties of operator space, such as transitions between localized and delocalized operator space eigenmodes. Establishing whether such a correspondence exists represents a natural direction for future work.

Finally, since our framework maps the collective spin dynamics directly onto a lattice model in operator space, it suggests connections with topological lattice systems, including free-fermion models. Given the progress in categorizing topology in such systems from both a Hermitian [82–87] as well as non-Hermitian Hamiltonian point of view [88,88–94], this provides a direct area of research in topological features, for example, to address the robustness of BTC oscillations [50].

XI. CONCLUSIONS

In this work, we have developed a microscopic operator space framework for collective spin Lindbladians based on their weak symmetries and spherical tensor structure. By expressing the density operator in this tensor operator basis, we map the Lindbladian evolution onto a non-Hermitian hopping problem on an emergent operator space lattice. Within this representation, weak symmetries impose kinematic constraints on the allowed classes of operator space dynamics, which we interpret as selection rules on the permitted transport processes. The realized dynamical regime is then determined by the competition between the corresponding microscopic processes, in particular coherent intrarank and dissipative interrang transport, including whether BTC oscillations emerge.

Beyond mean-field descriptions, our framework reveals how operator sectors beyond the collective spin triplet (J_x , J_y , and J_z) remain dynamically relevant and provides a microscopic explanation for the initial state insensitivity of BTC oscillations through nonunitary operator space transport. Furthermore, the spherical tensor representation enables systematic analytical approaches, including perturbative descriptions of the extreme time crystal regime explored in Ref. [28].

In the thermodynamic limit, we have shown that the BTC dynamics can be related to a bulk transport problem in operator space, where the mean-field phase transition threshold emerges from the competition between coherent and dissipative propagation.

More broadly, our results suggest that dissipative many-body dynamics can be understood through the language of transport in operator space. This perspective provides a framework for studying nonequilibrium phases of open quantum systems and opens connections to broader concepts including operator spreading, non-Hermitian dynamics, and topology in quantum many-body systems.

ACKNOWLEDGMENTS

D.N. acknowledges support from The University of Manchester under the Dean’s Doctoral Scholarship. A.P. acknowledges support from the Leverhulme Trust under the Grant Agreement No. RPG-2023-253. R.-J.S. acknowledges funding from an EPSRC ERC underwrite Grant No. EP/X025829/1.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: ADJOINT GENERATORS

Under a symmetry transformation generated by J_α , the density operator transforms as

$$\rho \mapsto U \rho U^\dagger, \quad (\text{A1})$$

where

$$U = e^{-i\theta J_\alpha} \in \text{SU}(2) \quad (\text{A2})$$

in the spin- j representation. Expanding for infinitesimal θ ,

$$\begin{aligned} U \rho U^\dagger &= (\mathbf{1} - i\theta J_\alpha) \rho (\mathbf{1} + i\theta J_\alpha) \\ &= \rho - i\theta [J_\alpha, \rho] + \mathcal{O}(\theta^2), \end{aligned} \quad (\text{A3})$$

which shows that the induced symmetry action on operators is generated by the adjoint (commutator) action,

$$\mathcal{K}_\alpha[O] \equiv \text{ad}_{J_\alpha}[O] = [J_\alpha, O]. \quad (\text{A4})$$

The operators $\{\mathcal{K}_\alpha\}$ therefore define a representation of the $\text{su}(2)$ algebra on the operator space $\mathcal{B}(\mathcal{H}_j)$. In direct analogy with the closed-system case, one may construct the corresponding Casimir superoperator,

$$\mathcal{K}^2 = \mathcal{K}_x^2 + \mathcal{K}_y^2 + \mathcal{K}_z^2. \quad (\text{A5})$$

Simultaneous diagonalization of $\mathcal{K}^2$ and $\mathcal{K}_z$ yields the operator space quantum numbers (k, q) , which label irreducible multiplets. For each fixed k , the allowed values of q are

$$q = -k, -k+1, \dots, k, \quad (\text{A6})$$

forming a $(2k+1)$ -dimensional irrep.

APPENDIX B: SPHERICAL TENSOR OPERATORS

In this appendix, we summarize standard results on spherical tensor operators used throughout the main text.

A spherical tensor operator of rank k and component q , denoted T_q^k , is defined through its transformation under the

adjoint action of the collective spin generators [36,37],

$$\begin{aligned}\mathcal{K}_z[T_q^k] &\equiv [J_z, T_q^k] = q T_q^k, \\ \mathcal{K}_\pm[T_q^k] &\equiv [J_\pm, T_q^k] = \sqrt{k(k+1) - q(q \pm 1)} T_{q\pm 1}^k.\end{aligned}\quad (\text{B1})$$

For a collective spin system with fixed total spin j , the operators T_q^k furnish an explicit basis for the irreducible representations appearing in the decomposition

$$j \otimes j^* = \bigoplus_{k=0}^{2j} k. \quad (\text{B2})$$

An explicit construction in terms of angular momentum eigenstates $\{|j m\rangle\}_{m=-j}^j$ is given by

$$T_q^k = \sum_{m, m'=-j}^j (-1)^{j-m} \langle j m'; j - m | k q \rangle |j m'\rangle \langle j m|, \quad (\text{B3})$$

where $\langle j m'; j - m | k q \rangle$ are Clebsch–Gordan coefficients [35].

The spherical tensor operators are chosen to be orthonormal with respect to the Hilbert–Schmidt inner product [35],

$$\text{Tr}[(T_q^k)^\dagger T_{q'}^{k'}] = \delta_{kk'} \delta_{qq'}, \quad (\text{B4})$$

and therefore form a complete orthonormal basis for operators acting within the spin- j sector.

Using the orthonormality of spherical tensor operators, one can show that all tensors with $(k \geq 1)$ are traceless. Recall that the identity operator corresponds to the unique rank-zero

tensor $T_0^0 = \frac{1}{\sqrt{2j+1}} \mathbf{1}$. Then,

$$\begin{aligned}\text{Tr}(T_q^k) &= \text{Tr}[(\mathbf{1})^\dagger T_q^k] \\ &= \sqrt{2j+1} \text{Tr}[(T_0^0)^\dagger T_q^k] \\ &= \sqrt{2j+1} \delta_{k0} \delta_{q0} \\ &= 0 \quad \text{for } k \geq 1.\end{aligned}\quad (\text{B5})$$

Next, we will outline the coupling rules obeyed by irreducible tensors with arbitrary normalizations and later specialize to the case where both tensors describe the state of a spin- j object.

Irreducible tensor operators obey angular momentum coupling rules analogous to those for state vectors. In general, the direct product of two irreducible tensor operators decomposes as

$$V_{q_1}^{k_1} U_{q_2}^{k_2} = \sum_{k=|k_1-k_2|}^{k_1+k_2} \langle k_1 q_1; k_2 q_2 | k q \rangle W_q^k, \quad (\text{B6})$$

where the allowed values of k satisfy the triangle rule

$$k = |k_1 - k_2|, |k_1 - k_2| + 1, \dots, k_1 + k_2, \quad q = q_1 + q_2.$$

Thus, the composition of irreducible tensor operators follows the same algebra as the addition of angular momenta. This construction extends straightforwardly to products of three or more tensors by successive application of Eq. (B6).

For the irreducible tensors considered here, $T_{q_1}^{k_1}(j)$ and $T_{q_2}^{k_2}(j)$, both irreducible tensors describe the polarization of an underlying spin- j particle [36] and satisfy the following relations:

$$T_{q_1}^{k_1}(j) T_{q_2}^{k_2}(j) = \sum_K (-1)^{2j+K} \sqrt{(2k_1+1)(2k_2+1)} \left\{ \begin{matrix} k_1 & k_2 & K \\ j & j & j \end{matrix} \right\} \langle k_1 q_1; k_2 q_2 | K Q \rangle T_Q^K(j), \quad (\text{B7})$$

and

$$\{T_{q_1}^{k_1}(j), T_{q_2}^{k_2}(j)\} = \sum_K (-1)^{2j+K} [1 + (-1)^{k_1+k_2+K}] \sqrt{(2k_1+1)(2k_2+1)} \left\{ \begin{matrix} k_1 & k_2 & K \\ j & j & j \end{matrix} \right\} \langle k_1 q_1; k_2 q_2 | K Q \rangle T_Q^K(j), \quad (\text{B8})$$

where

$$\left\{ \begin{matrix} k_1 & k_2 & K \\ j & j & j \end{matrix} \right\} \quad (\text{B9})$$

is a $6j$ symbol that enforces the triangle rule $K = |k_1 - k_2|, \dots, k_1 + k_2$. Here, $\langle k_1 q_1; k_2 q_2 | K Q \rangle$ is a Clebsch–Gordan coefficient. Importantly, the $6j$ symbols possess a high degree of permutation symmetry: they are invariant under the interchange of any two columns, as well as under the simultaneous interchange of upper and lower entries in any two columns. Clebsch–Gordan coefficients, however, are sensitive to the ordering of the coupled tensors. Permuting their arguments generally introduces phase factors, and therefore the chosen coupling convention must be applied consistently throughout the derivation to preserve the correct sign structure. See Ref. [36] for further details.

APPENDIX C: GENERAL FORM OF EVOLUTION EQUATION IN THE SPHERICAL TENSOR BASIS

Equation (10) follows directly by expanding the density operator in the spherical tensor basis [Eq. (9)] and inserting this into the master equation $\frac{d}{dt} \rho = \mathcal{L} \rho$. Explicitly,

$$\begin{aligned}\frac{d}{dt} \left(\sum_{k'q'} a_{k',q'}(t) T_{q'}^{k'} \right) &= \mathcal{L} \left(\sum_{k'q'} a_{k',q'}(t) T_{q'}^{k'} \right) \\ &\Rightarrow \sum_{k'q'} \left(\frac{d}{dt} a_{k',q'}(t) \right) T_{q'}^{k'} = \sum_{k'q'} a_{k',q'}(t) \mathcal{L} T_{q'}^{k'}.\end{aligned}\quad (\text{C1})$$

Projecting onto a given spherical tensor mode by multiplying from the left with $(T_q^k)^\dagger$ and taking the trace

yields

$$\begin{aligned} & \sum_{k'q'} \left(\frac{d}{dt} a_{k',q'}(t) \right) \text{Tr}((T_q^k)^\dagger T_{q'}^{k'}) \\ &= \sum_{k'q'} a_{k',q'}(t) \text{Tr}((T_q^k)^\dagger \mathcal{L} T_{q'}^{k'}). \end{aligned} \quad (\text{C2})$$

Using the orthonormality relation [Eq. (B4)], we obtain

$$\frac{d}{dt} a_{k,q}(t) = \sum_{k'q'} \mathcal{L}_{(k,q)(k',q')} a_{k',q'}(t), \quad (\text{C3})$$

where the matrix elements of the Liouvillian in the spherical tensor basis are given by

$$\mathcal{L}_{(k,q)(k',q')} \equiv \text{Tr}[(T_q^k)^\dagger \mathcal{L} T_{q'}^{k'}]. \quad (\text{C4})$$

APPENDIX D: MAPPING TO A NON-HERMITIAN HOPPING MODEL (COLLECTIVE PRECESSION)

In this appendix, we derive Eq. (16) by expressing the Liouvillian $\mathcal{L}$ in the spherical tensor basis $\{T_q^k\}$ [cf. Eq. (11)].

The unitary contribution may be written in terms of the adjoint generators as $\mathcal{K}_x[\cdot] = [J_x, \cdot]$, using Eq. (5). For Hermitian jump operators, the dissipator admits the double-commutator form

$$J_z T_q^k J_z - \frac{1}{2} \{J_z^2, T_q^k\} = -\frac{1}{2} [J_z, [J_z, T_q^k]]. \quad (\text{D1})$$

Using the defining relations of spherical tensors [Eq. (B1)], we obtain

$$[J_z, [J_z, T_q^k]] = \mathcal{K}_z[\mathcal{K}_z[T_q^k]] = q^2 T_q^k, \quad (\text{D2})$$

which shows that the dissipator contributes a diagonal term proportional to q^2 .

The Liouvillian therefore takes the compact form

$$\mathcal{L} = -i\Omega \mathcal{K}_x - \frac{\Gamma}{4j} \mathcal{K}_z^2. \quad (\text{D3})$$

We now evaluate the matrix elements in Eq. (11). The unitary contribution reads

$$\begin{aligned} \text{Tr}[(T_q^k)^\dagger \mathcal{K}_x T_{q'}^{k'}] &= \frac{1}{2} \text{Tr}[(T_q^k)^\dagger [J_+, T_{q'}^{k'}]] \\ &+ \frac{1}{2} \text{Tr}[(T_q^k)^\dagger [J_-, T_{q'}^{k'}]], \end{aligned} \quad (\text{D4})$$

where we used $J_x = \frac{1}{2}(J_+ + J_-)$. Applying Eq. (B1), yields

$$\begin{aligned} & \frac{1}{2} \sqrt{k'(k'+1) - q'(q'+1)} \text{Tr}[(T_q^k)^\dagger T_{q'+1}^{k'}] \\ &+ \frac{1}{2} \sqrt{k'(k'+1) - q'(q'-1)} \text{Tr}[(T_q^k)^\dagger T_{q'-1}^{k'}]. \end{aligned} \quad (\text{D5})$$

Using the orthonormality relation [Eq. (B4)], this reduces to

$$w_+(k', q') \delta_{k,k'} \delta_{q,q'+1} + w_-(k', q') \delta_{k,k'} \delta_{q,q'-1}, \quad (\text{D6})$$

where we defined

$$w_{\pm}(k, q) = \frac{1}{2} \sqrt{k(k+1) - q(q \pm 1)}.$$

Including the dissipative contribution, we obtain the Liouvillian matrix elements

$$\begin{aligned} \mathcal{L}_{(k,q)(k',q')} &= -i\Omega [w_+(k', q') \delta_{k,k'} \delta_{q,q'+1} \\ &+ w_-(k', q') \delta_{k,k'} \delta_{q,q'-1}] \\ &- \frac{\Gamma}{4j} q^2 \delta_{k,k'} \delta_{q,q'}. \end{aligned} \quad (\text{D7})$$

Substituting this result into Eq. (10) yields the hopping model in Eq. (16).

APPENDIX E: EIGENSYSTEM IN THE $k = 1$ SUBSPACE (COLLECTIVE PRECESSION)

We now derive the equations of motion for the expansion coefficients $a_{k,q}(t)$ [cf. Eq. (18)]. Substituting into the master equation yields

$$\begin{aligned} \frac{d}{dt} a_{k,q}(t) &= -\frac{i\Omega}{2} [\sqrt{k(k+1) - (q-1)q} a_{k,q-1}(t) \\ &+ \sqrt{k(k+1) - q(q+1)} a_{k,q+1}(t)] \\ &- \frac{\Gamma}{4j} q^2 a_{k,q}(t). \end{aligned} \quad (\text{E1})$$

To diagonalize this system, we focus on the $k = 1$ sector and introduce symmetric and antisymmetric combinations,

$$a_{\pm}(t) = \frac{1}{\sqrt{2}} (a_{1,1}(t) \pm a_{1,-1}(t)). \quad (\text{E2})$$

In this basis, the antisymmetric mode decouples,

$$\frac{d}{dt} a_-(t) = -\frac{\Gamma}{4j} a_-(t), \quad (\text{E3})$$

with solution

$$a_-(t) = a_-(0) e^{-\frac{\Gamma}{4j} t}. \quad (\text{E4})$$

The remaining components $(a_{1,0}, a_+)$ satisfy

$$\frac{d}{dt} \begin{pmatrix} a_{1,0}(t) \\ a_+(t) \end{pmatrix} = \begin{pmatrix} 0 & -i\Omega \\ -i\Omega & -\frac{\Gamma}{4j} \end{pmatrix} \begin{pmatrix} a_{1,0}(t) \\ a_+(t) \end{pmatrix}. \quad (\text{E5})$$

The corresponding eigenvalues are

$$\lambda_{\pm} = -\frac{\Gamma}{8j} \pm i\Omega \sqrt{1 - \left(\frac{\kappa}{2}\right)^2}, \quad (\text{E6})$$

where $\kappa \equiv \Gamma/(4j\Omega)$.

To connect to physical observables, we use the relations between spherical tensors and spin operators (following Ref. [35]),

$$T_0^1 = N_1(j) J_z, \quad T_{\pm 1}^1 = \mp \frac{N_1(j)}{\sqrt{2}} J_{\pm}, \quad (\text{E7})$$

with normalization

$$N_1(j) = \left[\frac{3}{(2j+1)(j+1)j} \right]^{1/2}. \quad (\text{E8})$$

This implies that the symmetric and antisymmetric combinations correspond to

$$T_+ = -\sqrt{2} N_1(j) iJ_y, \quad T_- = -\sqrt{2} N_1(j) J_x. \quad (\text{E9})$$

We therefore conclude that J_x decouples and exhibits purely exponential decay with rate $\Gamma/(4j)$, while J_y and J_z

form a coupled pair undergoing damped oscillations with decay rate $\Gamma/(8j)$ and frequency $\Omega\sqrt{1 - (\kappa/2)^2}$. This matches the results presented in Ref. [28].

APPENDIX F: DISSIPATOR-INDUCED k -HOPPING (BTC)

Here, we show that the dissipator in Eq. (19) couples spherical tensor operators with different tensor rank k . A simple way to see this is to evaluate its action on low-rank tensors.

The $k = 0$ sector is one-dimensional, with $T_0^0 = 1/\sqrt{2j+1}$. Acting with the dissipator yields

$$\begin{aligned}\mathcal{D}[\mathbf{1}] &= \frac{\Gamma}{2j} \left(J_- \mathbf{1} J_+ - \frac{1}{2} (J_+ J_- \mathbf{1} + \mathbf{1} J_+ J_-) \right) \\ &= \frac{\Gamma}{2j} (J_- J_+ - J_+ J_-) \\ &= \frac{\Gamma}{2j} [J_-, J_+] \\ &= -\frac{\Gamma}{j} J_z \propto T_0^1,\end{aligned}\quad (\text{F1})$$

showing that the identity operator (rank $k = 0$) is mapped to a rank-1 tensor.

Next, consider the action on $J_z \propto T_0^1$. A straightforward calculation gives

$$\begin{aligned}\mathcal{D}[J_z] &= \frac{\Gamma}{2j} \left(J_- J_z J_+ - \frac{1}{2} (J_+ J_- J_z + J_z J_+ J_-) \right) \\ &= \frac{\Gamma}{2j} (J^2 - 3J_z^2 - J_z),\end{aligned}\quad (\text{F2})$$

where we have used $[J_z, J_\pm] = \pm J_\pm$ and $[J^2, J_z] = 0$. This expression can be decomposed into spherical tensor components as

$$\mathcal{D}[J_z] = \frac{\Gamma}{2j} (-J_z - (3J_z^2 - J^2)) \quad (\text{F3})$$

Here, $J_z \propto T_0^1$ and $(3J_z^2 - J^2) \propto T_0^2$. Thus, the dissipator maps a rank-1 tensor onto a superposition of tensors with ranks $k = 1$ and $k = 2$.

These examples demonstrate that the dissipator preserves the quantum number q , while it can induce nearest-neighbor coupling in k . In particular, it generates transitions $k \rightarrow k \pm 1$, implying that

$$[\mathcal{D}, \mathcal{K}^2] \neq 0. \quad (\text{F4})$$

This structure admits a representation-theoretic interpretation. Since the jump operators $J_\pm$ transform as rank-1 irreducible tensors, the product $J_- T_q^k$ corresponds to the coupling of tensors with ranks 1 and k . By angular momentum addition, this decomposes into irreducible tensors with ranks $K = k - 1, k, k + 1$. Acting subsequently with J_+ produces tensors with ranks $K' = k - 2, k - 1, k, k + 1, k + 2$, while the component remains fixed at q , since $Q' = -1 + q + 1 = q$.

However, for the specific Lindblad structure in Eq. (19), cancellations between the jump term and the anti-commutator remove the $K' = k \pm 2$ contributions. As a result, the dissipator couples only the neighboring ranks $k - 1, k$, and $k + 1$. We prove this explicitly below.

We first note that

$$J_- T_q^k J_+ - \frac{1}{2} \{J_+ J_-, T_q^k\} = \frac{1}{2} [J_-, T_q^k J_+] + \frac{1}{2} [J_- T_q^k, J_+], \quad (\text{F5})$$

which can be further simplified to

$$\frac{1}{2} [J_-, T_q^k] J_+ + \frac{1}{2} J_- [T_q^k, J_+] - \{J_z, T_q^k\} \quad (\text{F6})$$

using $[J_-, J_+] = -2J_z$. The commutator terms can be simplified via Eq. (B1) to obtain terms proportional to $T_{q\pm 1}^k$. Then, $T_{q-1}^k J_+$ and $J_- T_{q+1}^k$ may be evaluated using Eq. (B7) and the anticommutator via Eq. (B8). For each of these three terms, the triangle rule enforces $K = k - 1, k, k + 1$, which already demonstrates that only nearest-neighbor and onsite terms remain. We can evaluate the coefficients explicitly by noting that

$$\left\{ \begin{matrix} 1 & k & k \\ j & j & j \end{matrix} \right\} = (-1)^{-2j-k+1} \sqrt{\frac{k(k+1)}{4j(j+1)(2j+1)(2k+1)}} \quad (\text{F7})$$

for $1 \leq k \leq 2j$ and 0 for $k = 0$,

$$\left\{ \begin{matrix} 1 & k & k-1 \\ j & j & j \end{matrix} \right\} = (-1)^{-2j-k} \sqrt{\frac{k[(2j+1)^2 - k^2]}{4j(j+1)(2j+1)(2k-1)(2k+1)}}, \quad (\text{F8})$$

for $1 \leq k \leq 2j$ and 0 for $k = 0$ and

$$\left\{ \begin{matrix} 1 & k & k+1 \\ j & j & j \end{matrix} \right\} = (-1)^{-2j-k+1} \sqrt{\frac{(k+1)[(2j+1)^2 - (k+1)^2]}{4j(j+1)(2j+1)(2k+1)(2k+3)}}, \quad (\text{F9})$$

for $0 \leq k \leq 2j - 1$ and 0 for $k = 2j$.

The $6j$ symbols are implemented in various numerical software packages, for example, in Mathematica via the `SixJSymbol` function, or can be evaluated analytically from Racah coefficients given in Ref. [36]. These expressions appear in all three terms of Eq. (F6), which yield $K = k - 1, k, k + 1$ via the triangle rule.

Combining these results we obtain

$$\begin{aligned}\mathcal{D}[T_q^k] &= \frac{\Gamma}{2j} (t_-(k, q, j) T_q^{k-1} + t_+(k, q, j) T_q^{k+1} \\ &\quad - \gamma(k, q, j) T_q^k).\end{aligned}\quad (\text{F10})$$

Here, the onsite damping factors are independent of j such that

$$\gamma(k, q, j) = \gamma(k, q) = \frac{1}{2}[k(k+1) - q^2], \quad (\text{F11})$$

for $0 \leq k \leq 2j$ and $-k \leq q \leq +k$. The hopping amplitude $t_-(k, q, j)$ is given by

$$t_-(k, q, j) = \frac{(k-1)}{2} \sqrt{\frac{[(2j+1)^2 - k^2](k^2 - q^2)}{(2k-1)(2k+1)}}, \quad (\text{F12})$$

for $1 \leq k \leq 2j$ and $-k \leq q \leq +k$ and it vanishes for $k = 1$, reflecting trace preservation. Furthermore, $t_-(k, q, j) \propto \sqrt{k^2 - q^2}$, implying that hopping out of the extremal multiplet states $q = \pm k$ is forbidden. Geometrically, this is because the neighboring rank- $(k-1)$ multiplet only supports $|q| \leq k-1$, so no destination states $(k-1, \pm k)$ exist.

Similarly,

$$t_+(k, q, j) = -\frac{(k+2)}{2} S(k, q, j),$$

$$S(k, q, j) = \sqrt{\frac{(2j-k)(2j+k+2)[(k+1)^2 - q^2]}{(2k+1)(2k+3)}}, \quad (\text{F13})$$

is valid for $0 \leq k \leq 2j-1$ and $-k \leq q \leq +k$. This hopping amplitude vanishes for $k = 2j$ since it corresponds to the upper edge of the lattice along the k coordinate.

In practice, one can also extract these coefficients numerically from the matrix elements of the Liouvillian in the $\{T_q^k\}$ basis. This basis can either be constructed from angular momentum states or by finding the simultaneous eigenvectors of the vectorized superoperators $\mathbb{K}^2$ and $\mathbb{K}_z$.

In the thermodynamic limit $j \rightarrow \infty$, one can expand $t_\pm(k, q, j)$ in powers of j with the leading order contributions being $O(j)$, followed by $O(1)$, $O(j^{-1})$ etc.

We note that nearest-neighbor hoppings arising from dissipators with collective jump operators $J_\pm$ have also been identified in Ref. [39]. However, that work is restricted to models with a conserved charge, in contrast to the present setting.

APPENDIX G: PROJECTORS FOR $\mathcal{K}^2$ EIGENSAPCES

In this appendix, we provide further details on the projectors used in the main text.

In the vectorized Liouville space, the spherical tensor basis $\{|k q\rangle\rangle\}$ satisfies the completeness relation

$$\begin{aligned} \mathbb{I} &= \sum_{k=0}^{2j} \sum_{q=-k}^{+k} |k q\rangle\rangle \langle\langle k q| \\ &= \sum_{k=0}^{2j} \underbrace{\left(\sum_{q=-k}^{+k} |k q\rangle\rangle \langle\langle k q| \right)}_{\mathbb{P}^{(k)}} \\ &= \sum_{k=0}^{2j} \mathbb{P}^{(k)}, \end{aligned} \quad (\text{G1})$$

where $\mathbb{P}^{(k)} = (\mathbb{P}^{(k)})^\dagger$ projects onto the eigenspace of $\mathbb{K}^2$ with eigenvalue $k(k+1)$.

Using these projectors, the Liouvillian can be decomposed into blocks connecting different tensor sectors,

$$\begin{aligned} \mathbb{D}^{(k' \rightarrow k)} &\equiv \mathbb{P}^{(k)} \mathbb{D} \mathbb{P}^{(k')} \\ &= \sum_{q=-k}^{+k} \sum_{q'=-k'}^{+k'} |k q\rangle\rangle \langle\langle k q| \mathbb{D} |k' q'\rangle\rangle \langle\langle k' q'| \\ &= \sum_{q=-k}^{+k} \sum_{q'=-k'}^{+k'} |k q\rangle\rangle (\mathbb{D}^{(k' \rightarrow k)})_{qq'} \langle\langle k' q'|. \end{aligned} \quad (\text{G2})$$

This operator describes the transfer of operator weight from the k' sector to the k sector under the action of $\mathbb{D}$.

To quantify the strength of this coupling, we define

$$\begin{aligned} C^{(k' \rightarrow k)} &\equiv \sqrt{\text{Tr}[(\mathbb{D}^{(k' \rightarrow k)})^\dagger \mathbb{D}^{(k' \rightarrow k)}]} \\ &= \sqrt{\sum_{q, q'} |(\mathbb{D}^{(k' \rightarrow k)})_{qq'}|^2}, \end{aligned} \quad (\text{G3})$$

which corresponds to the Frobenius norm of the block $\mathbb{D}^{(k' \rightarrow k)}$.

-
- [1] F. Wilczek, Quantum time crystals, *Phys. Rev. Lett.* **109**, 160401 (2012).
 - [2] P. Bruno, Impossibility of spontaneously rotating time crystals: A No-Go theorem, *Phys. Rev. Lett.* **111**, 070402 (2013).
 - [3] H. Watanabe and M. Oshikawa, Absence of quantum time crystals, *Phys. Rev. Lett.* **114**, 251603 (2015).
 - [4] V. Khemani, A. Lazarides, R. Moessner, and S. L. Sondhi, Phase structure of driven quantum systems, *Phys. Rev. Lett.* **116**, 250401 (2016).
 - [5] V. Khemani, R. Moessner, and S. L. Sondhi, A brief history of time crystals, *arXiv:1910.10745*.
 - [6] K. Sacha, Modeling spontaneous breaking of time-translation symmetry, *Phys. Rev. A* **91**, 033617 (2015).
 - [7] K. Sacha and J. Zakrzewski, Time crystals: A review, *Rep. Prog. Phys.* **81**, 016401 (2018).
 - [8] D. V. Else, B. Bauer, and C. Nayak, Floquet time crystals, *Phys. Rev. Lett.* **117**, 090402 (2016).
 - [9] D. V. Else, B. Bauer, and C. Nayak, Prethermal phases of matter protected by time-translation symmetry, *Phys. Rev. X* **7**, 011026 (2017).
 - [10] D. V. Else, C. Monroe, C. Nayak, and N. Y. Yao, Discrete time crystals, *Annu. Rev. Condens. Matter Phys.* **11**, 467 (2020).
 - [11] C. W. von Keyserlingk, V. Khemani, and S. L. Sondhi, Absolute stability and spatiotemporal long-range order in Floquet systems, *Phys. Rev. B* **94**, 085112 (2016).
 - [12] N. Y. Yao, A. C. Potter, I.-D. Potirniche, and A. Vishwanath, Discrete time crystals: Rigidity, criticality, and Realizations, *Phys. Rev. Lett.* **118**, 030401 (2017).
 - [13] A. Lazarides and R. Moessner, Fate of a discrete time crystal in an open system, *Phys. Rev. B* **95**, 195135 (2017).

- [14] M. P. Zaletel, M. Lukin, C. Monroe, C. Nayak, F. Wilczek, and N. Y. Yao, Colloquium: Quantum and classical discrete time crystals, *Rev. Mod. Phys.* **95**, 031001 (2023).
- [15] J. Zhang, P. W. Hess, A. Kyprianidis, P. Becker, A. Lee, J. Smith, G. Pagano, I. D. Potirniche, A. C. Potter, A. Vishwanath, N. Y. Yao, and C. Monroe, Observation of a discrete time crystal, *Nature (London)* **543**, 217 (2017).
- [16] K. Seibold, R. Rota, and V. Savona, Dissipative time crystal in an asymmetric nonlinear photonic dimer, *Phys. Rev. A* **101**, 033839 (2020).
- [17] C. Lledó and M. H. Szymańska, A dissipative time crystal with or without Z2 symmetry breaking, *New J. Phys.* **22**, 075002 (2020).
- [18] C. Booker, B. Buča, and D. Jaksch, Non-stationarity and dissipative time crystals: Spectral properties and finite-size effects, *New J. Phys.* **22**, 085007 (2020).
- [19] G. Passarelli, P. Lucignano, R. Fazio, and A. Russomanno, Dissipative time crystals with long-range Lindbladians, *Phys. Rev. B* **106**, 224308 (2022).
- [20] F. Iemini, A. Russomanno, J. Keeling, M. Schirò, M. Dalmonte, and R. Fazio, Boundary time crystals, *Phys. Rev. Lett.* **121**, 035301 (2018).
- [21] G. Piccitto, M. Wauters, F. Nori, and N. Shammah, Symmetries and conserved quantities of boundary time crystals in generalized spin models, *Phys. Rev. B* **104**, 014307 (2021).
- [22] T. Mori, Exactness of the mean-field dynamics in optical cavity systems, *J. Stat. Mech.* (2013) P06005.
- [23] F. Carollo and I. Lesanovsky, Exactness of mean-field equations for open Dicke models with an application to pattern retrieval dynamics, *Phys. Rev. Lett.* **126**, 230601 (2021).
- [24] E. Fiorelli, M. Müller, I. Lesanovsky, and F. Carollo, Mean-field dynamics of open quantum systems with collective operator-valued rates: Validity and application, *New J. Phys.* **25**, 083010 (2023).
- [25] A. Mukherjee, Y. Ibrahim, M. Hajdušek, and S. Vinjanampathy, Symmetries and correlations in continuous time crystals, *Phys. Rev. A* **110**, 012220 (2024).
- [26] F. Carollo and I. Lesanovsky, Exact solution of a boundary time-crystal phase transition: Time-translation symmetry breaking and non-Markovian dynamics of correlations, *Phys. Rev. A* **105**, L040202 (2022).
- [27] Y. Nakanishi and T. Sasamoto, Dissipative time crystals originating from parity-time symmetry, *Phys. Rev. A* **107**, L010201 (2023).
- [28] D. Nemeth, A. Principi, and A. Nazir, Analytical solution of boundary time crystals via the superspin basis, [arXiv:2507.06998](https://arxiv.org/abs/2507.06998).
- [29] Z. Liu, Y. Li, Z. Fei, and X. Wang, Boundary time crystals beyond mean-field theory: A stroboscopic rotating-wave approximation, *Phys. Rev. A* **114**, 012211 (2026).
- [30] P. Callaghan, *Principles of Nuclear Magnetic Resonance Microscopy*, Oxford Science Publications (Clarendon Press, Oxford, 1993).
- [31] A. Abragam, *The Principles of Nuclear Magnetism*, International Series of Monographs on Physics (Clarendon Press, Oxford, 1961).
- [32] C. Slichter, *Principles of Magnetic Resonance*, Springer Series in Solid-State Sciences (Springer, Berlin, 1996).
- [33] U. Fano, Description of states in quantum mechanics by density matrix and operator techniques, *Rev. Mod. Phys.* **29**, 74 (1957).
- [34] W. Happer, Optical pumping, *Rev. Mod. Phys.* **44**, 169 (1972).
- [35] K. Blum, Irreducible components of the density matrix, in *Density Matrix Theory and Applications* (Springer, Berlin, 2012), pp. 115–163.
- [36] D. A. Varshalovich, A. N. Moskalev, and V. K. Khersonskii, *Quantum Theory of Angular Momentum* (World Scientific, Singapore, 1988).
- [37] K. T. Hecht, Spherical tensor operators, in *Quantum Mechanics* (Springer, New York, 2000), pp. 294–298.
- [38] C. J. Wood, T. W. Borneman, and D. G. Cory, Cavity cooling of an ensemble spin system, *Phys. Rev. Lett.* **112**, 050501 (2014).
- [39] P. Ribeiro and T. Prosen, Integrable quantum dynamics of open collective spin models, *Phys. Rev. Lett.* **122**, 010401 (2019).
- [40] T. Prosen, Third quantization: A general method to solve master equations for quadratic open Fermi systems, *New J. Phys.* **10**, 043026 (2008).
- [41] T. Prosen and T. H. Seligman, Quantization over boson operator spaces, *J. Phys. A: Math. Theor.* **43**, 392004 (2010).
- [42] B. Buča and T. Prosen, A note on symmetry reductions of the Lindblad equation: Transport in constrained open spin chains, *New J. Phys.* **14**, 073007 (2012).
- [43] V. V. Albert and L. Jiang, Symmetries and conserved quantities in Lindblad master equations, *Phys. Rev. A* **89**, 022118 (2014).
- [44] D. F. Walls, P. D. Drummond, S. S. Hassan, and H. J. Carmichael, Non-equilibrium phase transitions in Cooperative Atomic Systems, *Prog. Theor. Phys. Suppl.* **64**, 307 (1978).
- [45] P. Drummond and H. Carmichael, Volterra cycles and the cooperative fluorescence critical point, *Opt. Commun.* **27**, 160 (1978).
- [46] P. D. Drummond, Observables and moments of cooperative resonance fluorescence, *Phys. Rev. A* **22**, 1179 (1980).
- [47] M.-D. Choi, Completely positive linear maps on complex matrices, *Linear Algebra Appl.* **10**, 285 (1975).
- [48] A. Jamiołkowski, Linear transformations which preserve trace and positive semidefiniteness of operators, *Rep. Math. Phys.* **3**, 275 (1972).
- [49] J. A. Gyamfi, Fundamentals of quantum mechanics in Liouville space, *Eur. J. Phys.* **41**, 063002 (2020).
- [50] D. Nemeth, A. Nazir, A. Principi, and R.-J. Slager, Topological boundary time crystal oscillations, [arXiv:2602.17765](https://arxiv.org/abs/2602.17765).
- [51] M. S. Byrd and N. Khaneja, Characterization of the positivity of the density matrix in terms of the coherence vector representation, *Phys. Rev. A* **68**, 062322 (2003).
- [52] G. Kimura, The Bloch vector for N-level systems, *Phys. Lett. A* **314**, 339 (2003).
- [53] R. A. Bertlmann and P. Krammer, Bloch vectors for qudits, *J. Phys. A: Math. Theor.* **41**, 235303 (2008).
- [54] M. S. Byrd, C. A. Bishop, and Y.-C. Ou, General open-system quantum evolution in terms of affine maps of the polarization vector, *Phys. Rev. A* **83**, 012301 (2011).
- [55] A. Rivas and S. F. Huelga, Quantum Markov process: Mathematical structure, in *Open Quantum Systems: An Introduction* (Springer, Berlin, 2012), pp. 33–48.
- [56] E. R. Bittner, B. Tyagi, and K. E. Bassler, Graph symmetry organizes exceptional dynamics in open quantum systems, [arXiv:2603.10654](https://arxiv.org/abs/2603.10654).
- [57] S. G. Schirmer and X. Wang, Stabilizing open quantum systems by Markovian reservoir engineering, *Phys. Rev. A* **81**, 062306 (2010).

- [58] S. G. Schirmer, D. K. L. Oi, W. Zhou, E. Gong, and M. Zhang, Open quantum system identification, [arXiv:1205.6220](#).
- [59] H. Yuan, Reachable set of operators on controlled open quantum systems in a Markovian environment, *Syst. Control Lett.* **61**, 1085 (2012).
- [60] M. C. Gutzwiller, Effect of correlation on the ferromagnetism of transition metals, *Phys. Rev. Lett.* **10**, 159 (1963).
- [61] W. Krauth, M. Caffarel, and J.-P. Bouchaud, Gutzwiller wave function for a model of strongly interacting bosons, *Phys. Rev. B* **45**, 3137 (1992).
- [62] R. Fazio, J. Keeling, L. Mazza, and M. Schirò, Many-body open quantum systems, *SciPost Phys. Lect. Notes* **99** (2025).
- [63] R. Kubo, Generalized cumulant expansion method, *J. Phys. Soc. Jpn.* **17**, 1100 (1962).
- [64] K. Henschel, J. Majer, J. Schmiedmayer, and H. Ritsch, Cavity QED with an ultracold ensemble on a chip: Prospects for strong magnetic coupling at finite temperatures, *Phys. Rev. A* **82**, 033810 (2010).
- [65] P. Kirton and J. Keeling, Superradiant and lasing states in driven-dissipative Dicke models, *New J. Phys.* **20**, 015009 (2018).
- [66] E. O'Connor, V. Montenegro, F. Albarelli, M. G. A. Paris, A. Bayat, and M. G. Genoni, Attaining quantum sensing enhancement from monitored Dissipative time crystals, [arXiv:2508.15448](#).
- [67] M. Jirasek, I. Lekanovsky, and A. Cabot, The boundary time crystal as a light source for quantum enhanced sensing beyond the Heisenberg limit *Phys. Rev. A* **114**, 032611 (2026).
- [68] G. Sundaram and Q. Niu, Wave-packet dynamics in slowly perturbed crystals: Gradient corrections and Berry-phase effects, *Phys. Rev. B* **59**, 14915 (1999).
- [69] N. Silberstein, J. Behrends, M. Goldstein, and R. Ilan, Berry connection induced anomalous wave-packet dynamics in non-Hermitian systems, *Phys. Rev. B* **102**, 245147 (2020).
- [70] M. V. Berry and K. E. Mount, Semiclassical approximations in wave mechanics, *Rep. Prog. Phys.* **35**, 315 (1972).
- [71] S.-Y. Bai and J.-H. An, Generating stable spin squeezing by squeezed-reservoir engineering, *Phys. Rev. Lett.* **127**, 083602 (2021).
- [72] P. Groszkowski, M. Koppenhöfer, H.-K. Lau, and A. A. Clerk, Reservoir-engineered spin squeezing: Macroscopic even-odd effects and hybrid-systems implementations, *Phys. Rev. X* **12**, 011015 (2022).
- [73] A. Nahum, S. Vijay, and J. Haah, Operator spreading in random unitary circuits, *Phys. Rev. X* **8**, 021014 (2018).
- [74] C. W. von Keyserlingk, T. Rakovszky, F. Pollmann, and S. L. Sondhi, Operator hydrodynamics, OTOCs, and entanglement growth in systems without conservation laws, *Phys. Rev. X* **8**, 021013 (2018).
- [75] V. Khemani, A. Vishwanath, and D. A. Huse, Operator spreading and the emergence of dissipative hydrodynamics under unitary evolution with conservation laws, *Phys. Rev. X* **8**, 031057 (2018).
- [76] D. E. Parker, X. Cao, A. Avdoshkin, T. Scaffidi, and E. Altman, A universal operator growth hypothesis, *Phys. Rev. X* **9**, 041017 (2019).
- [77] Z. Qi, T. Scaffidi, and X. Cao, Surprises in the deep Hilbert space of all-to-all systems: From superexponential scrambling to slow entanglement growth, *Phys. Rev. B* **108**, 054301 (2023).
- [78] S. Omanakuttan, K. Chinni, P. D. Blocher, and P. M. Poggi, Scrambling and quantum chaos indicators from long-time properties of operator distributions, *Phys. Rev. A* **107**, 032418 (2023).
- [79] T. Prosen and I. Pižorn, Operator space entanglement entropy in a transverse Ising chain, *Phys. Rev. A* **76**, 032316 (2007).
- [80] Z. Wang, R. Gao, X. Wu, B. Buča, K. Mølmer, L. You, and F. Yang, Boundary time crystals induced by local dissipation and long-range interactions, *Phys. Rev. Lett.* **135**, 230401 (2025).
- [81] F. Russo and T. Pohl, Quantum dissipative continuous time crystals, *Phys. Rev. Lett.* **135**, 110404 (2025).
- [82] J. Kruthoff, J. de Boer, J. van Wezel, C. L. Kane, and R.-J. Slager, Topological classification of crystalline insulators through band structure combinatorics, *Phys. Rev. X* **7**, 041069 (2017).
- [83] B. Bradlyn, L. Elcoro, J. Cano, M. G. Vergniory, Z. Wang, C. Felser, M. I. Aroyo, and B. A. Bernevig, Topological quantum chemistry, *Nature (London)* **547**, 298 (2017).
- [84] R.-J. Slager, A. Mesaros, V. Juričić, and J. Zaanen, The space group classification of topological band-insulators, *Nat. Phys.* **9**, 98 (2013).
- [85] A. Bouhon, T. Bzdušek, and R.-J. Slager, Geometric approach to fragile topology beyond symmetry indicators, *Phys. Rev. B* **102**, 115135 (2020).
- [86] H. C. Po, A. Vishwanath, and H. Watanabe, Symmetry-based indicators of band topology in the 230 space groups, *Nat. Commun.* **8**, 50 (2017).
- [87] P. W. Brouwer and V. Dwivedi, Homotopic classification of band structures: Stable, fragile, delicate, and stable representation-protected topology, *Phys. Rev. B* **108**, 155137 (2023).
- [88] N. Hatano and D. R. Nelson, Localization transitions in non-Hermitian quantum mechanics, *Phys. Rev. Lett.* **77**, 570 (1996).
- [89] C. M. Bender, Making sense of non-Hermitian Hamiltonians, *Rep. Prog. Phys.* **70**, 947 (2007).
- [90] E. J. Bergholtz, J. C. Budich, and F. K. Kunst, Exceptional topology of non-Hermitian systems, *Rev. Mod. Phys.* **93**, 015005 (2021).
- [91] H. Zhou and J. Y. Lee, Periodic table for topological bands with non-Hermitian symmetries, *Phys. Rev. B* **99**, 235112 (2019).
- [92] K. Kawabata, K. Shiozaki, M. Ueda, and M. Sato, Symmetry and topology in non-Hermitian physics, *Phys. Rev. X* **9**, 041015 (2019).
- [93] D. S. Borgnia, A. J. Kruchkov, and R.-J. Slager, Non-Hermitian boundary modes and topology, *Phys. Rev. Lett.* **124**, 056802 (2020).
- [94] N. Okuma, K. Kawabata, K. Shiozaki, and M. Sato, Topological origin of non-Hermitian skin effects, *Phys. Rev. Lett.* **124**, 086801 (2020).