

Incommensurate twisted bilayer graphene: Emerging quasiperiodicity and stability

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We consider a lattice model of twisted bilayer graphene (TBG) for incommensurate twist angles, focusing on the role of large-momentum-transfer umklapp terms. These terms, which nearly connect the Fermi points of different layers, are typically neglected in effective continuum descriptions but could, in principle, destroy the Dirac cones; they are indeed closely analogous to those appearing in fermions within quasiperiodic potentials, where they play a crucial role. We prove that, for small but finite interlayer coupling, the semimetallic phase is stable provided the angles belong to a fractal set of large measure (which decreases with the hopping strength) characterized by a number-theoretic Diophantine condition. In particular, this set excludes the (zero measure) commensurate angles. Our method combines a renormalization group (RG) analysis of the imaginary-time, zero-temperature Green’s functions, with number theoretic properties, and it is similar to the technique used in the Lindstedt series approach to Kolmogorov-Arnold-Moser (KAM) theory. The convergence of the resulting series allows us to rule out nonperturbative effects. The result provides a partial justification of the effective continuum description of TBG in which such large-momentum interlayer hopping processes are neglected.

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I. INTRODUCTION

Twisted bilayer graphene (TBG) consists of two graphene layers, stacked and twisted by an angle θ . The electronic properties of graphene are accurately captured by a tight-binding model of electrons hopping from one site to a neighboring one of a honeycomb lattice; the corresponding energy bands can be explicitly computed and intersect at two Dirac points, close to which the effective dispersion relation is approximately conical. In the case of half-filling, low-energy excitations have an effective continuum description in terms of two-dimensional massless Dirac fermions and the system behaves as a Dirac semimetal.

Stacking and twisting two honeycomb lattices produce an intricate Moiré pattern [1], and twisted bilayer graphene behaves very differently from the single layer, with a strong sensitivity to the twist angle, ranging from nonconductive to superconductive [2], see, e.g., Refs. [3–5]. From the theoretical side, in contrast to the monolayer case, an exact computation of the dispersion relation is impossible, and the validity of effective models is less clear.

The model for TBG we will consider consists of two layers of graphene with a quadratic interlayer coupling λ smaller than the intralayer hopping (chosen equal to 1). We denote the twist angle by θ . One has to distinguish between *commensurate* angles, such that the TBG is periodic and which are a discrete zero measure set indexed by two integers [6–8]; and *incommensurate* angles where periodicity is lost, and Bloch theory cannot be applied [9].

The Dirac points of layer 1 are $K_{\pm} = \frac{2\pi}{3}(1, \pm \frac{1}{\sqrt{3}})$ and the reciprocal lattice vectors are b_1, b_2 ; correspondingly, the Dirac points of layer 2 are $K'_{\pm} = R(\theta)K_{\pm}$ in which $R(\theta)$ is the rotation matrix of angle θ , and the reciprocal lattice vectors are $b'_i = R(\theta)b_i$, $i = 1, 2$. The interlayer hopping with coupling λ produces a sum of terms with an exchange of momenta given by multiples of the reciprocal lattice vectors; if k_a, k_b are the momenta of two fermions, the conservation rule is given by

$$k_a - k_b + G + G' = 0 \quad (1)$$

with $G = l_1 b_1 + l_2 b_2 = lb$, $G' = m_1 b'_1 + m_2 b'_2 = mb'$, $l = (l_1, l_2)$, $m = (m_1, m_2)$ with l_i, m_i integers. The processes involving nonvanishing $G + G'$ are called *umklapp terms*. The interlayer hopping decreases with a large exchange of momentum.

The processes connecting the Dirac points are of particular importance, that is, the processes such that $\Delta = 0$, where

$$\Delta = |G + G' + K_{\sigma}^i - K_{\sigma'}^j| \equiv |M + K_{\sigma}^i - K_{\sigma'}^j| \quad (2)$$

with $K_{\pm}^1 = K^{\pm}$, $K_{\pm}^2 = K'_{\pm}$ and $M = G + G'$. Among such terms, one can distinguish between the ones connecting the same Dirac point (that is $i = j, \sigma = \sigma'$) from those

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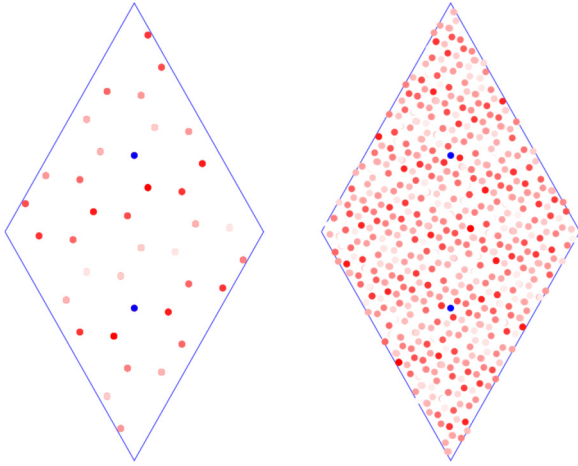


FIG. 1. (Left) The Dirac points in the commensurate case $\cos \theta = 37/38$. (Right) The Dirac points in the approximately incommensurate case $\theta = 0.16\pi$. The Dirac points $K_{\pm}$ of one layer are in blue, and those of the other layer $K'_{\pm} + G + G'$ are in red. In the incommensurate case, $K'_{\pm} + G + G'$ get arbitrarily close to $K_{\pm}$.

connecting different Dirac points. The form of the former is determined by the lattice symmetries [10,11] and their effect is to renormalize the parameters of the single-layer Hamiltonian, possibly enlarged to include new terms allowed by the breaking of symmetries produced by the interlayer hopping. The latter, that is, the ones involving different Dirac points, are present only for *commensurate* angles and are expected to be relevant and able to destroy the Dirac cones [12]. One can distinguish between the intralayer terms ($i = j, \sigma \neq \sigma'$) and interlayer terms ($i \neq j, \sigma \neq \sigma'$); the first case corresponds to the opening of a gap and the other to quadratic bands. The gap is expected to decrease with the momentum transferred due to the decay in momentum of the hopping. Due to periodicity, there is only a finite set of exchanged umklapp momenta, see Fig. 1. Therefore, in general, in the commensurate case, the terms that do not connect the Dirac points exactly have a large mismatch and are expected to be irrelevant.

Also the terms *almost* connecting the Dirac points can produce an important modification of the physical properties with respect to graphene. Terms almost connecting the Dirac points can be of two kinds. A first possibility is for angles θ close to zero, where one can see that Δ can be $O(\theta)$; they are present both in the commensurate or incommensurate case. An effective model was proposed in [13–15] keeping only a few terms ($G = G' = 0$, or $G = b_1, G' = -b'_1$ or $G = b_2, G' = -b'_2$) corresponding to $\Delta = O(\theta)$ and passing to the continuum. It was found that the effective hopping is greatly enlarged: it becomes of order $O(\lambda/\theta)$; therefore the conical dispersion relation can persist but with a strongly decreased velocity; for certain “magic” angles the velocity even vanishes and, remarkably, superconductivity has been found [2]. Effective models of TBG in which not only the velocity vanishes but the bands are completely flat have been also proposed [16] and mathematical derivations of such effective models in suitable limits have been provided [17].

The other possibility is present only in the incommensurate case and involves a large momentum transfer. Indeed in this case there are infinitely many umklapp terms *almost* connect-

ing the Dirac points, that is such that the distance between $K_{\pm}$ and $K'_{\pm} + G' + G$ can be arbitrarily small, that is Δ can be arbitrarily small, see Fig. 1. These terms, which nearly connect the Fermi points of different layers, are typically neglected in effective continuum descriptions but could, in principle, destroy the Dirac cones; they are indeed closely analogous to those appearing in the case of fermions in quasiperiodic potentials, where they play a crucial role. This can be easily realized comparing TBG for incommensurate angles with fermions on a chain with step 1 and with an Aubry quasiperiodic potential $\cos 2\pi\omega x$ [18], whose single-particle properties are described by the Harper equation. In that case, the conservation of momentum reads $k_a - k_b + 2l\pi + 2\pi\omega m = 0$, and the mismatch between Fermi points is

$$\Delta = |2\varepsilon K + 2l\pi + 2\pi\omega m| \quad \varepsilon = 0, \pm 1, \quad (3)$$

if K is the Fermi momentum. If $\omega/2\pi$ is rational either the above quantity is zero or is bounded while if it is irrational, there are almost relevant processes that almost connect the Fermi points. The quasiperiodicity acts as a disorder, and in the strong coupling regime, there is an insulating localized behavior while in the weak coupling a metallic behavior is found, as proved in [19–21]. The spectrum has remarkable fractal properties, which were observed in experiments with magnetic fields [22].

The prototypical problem involving quasiperiodicity is the Kolmogorov-Arnold-Moser (KAM) theorem for the persistence of Hamiltonian invariant tori under perturbation, and, in fact, results on the Aubry potential use KAM methods. The Renormalization Group can be used to prove the convergence of the KAM Lindstedt series [23] or to compute the Green’s function of fermions in a quasiperiodic potential [24]. It is also known that the interplay of many body interaction and quasiperiodic potentials leads to interesting phenomena, studied in 1d fermions in the delocalized regime in [25,26] and in the localized one in [27,28]. Quasi-periodic potentials have also been investigated in interacting Weyl semimetals [29,30]. Note also that the large-momentum umklapp terms in incommensurate TBG are also present when considering a hopping with a finite number of harmonics; they would be generated at higher order in perturbation theory. This fact is not captured by the effective models, as the lattice is replaced by the continuum and umklapp terms are therefore absent. These large-momentum umklapp terms are depressed by the decay of the hopping in momentum space, but there is no guarantee that this makes them irrelevant; in the case of the Aubry potential, they indeed produce an Anderson localized phase.

In order to evaluate the effect of the umklapp large-momentum terms on the physical properties of the system, we need, as is the case in KAM theory or for quasiperiodic systems, to use number theoretic properties of irrationals. Such properties are typically established in the context of measure theory. The typical number theoretic condition which is assumed in the case of problems involving quasiperiodicity is a *Diophantine condition* which consists in restricting ourselves to values of θ for which, for any ω, ω', i, j ,

$$|K_{\omega}^i - K_{\omega'}^j + lb + mb'| \geq \frac{C_0}{|y|^{\tau}}, \quad (4)$$

where y is either $y = l$ or $y = m$ and $y \neq 0$; this relates the size of the momentum G, G' with the precision with which the Dirac points are connected. A similar Diophantine condition is assumed in the case of Aubry-André systems, where the condition is on the frequency ω , and in KAM theory, where the condition is on the frequency of the tori. C_0 is called the Diophantine constant and it plays a crucial role.

We consider a lattice model of TBG and we perform an exact Renormalization Group analysis of the two-point zero temperature imaginary time Green function in order to find conditions for the persistence of the Dirac cones and the stability of the semimetallic phase. Such zero temperature correlations provide information on the ground state and, in particular, on the persistence or not of the semimetallic behavior. Note that, as we are not considering a many-body interaction, information on the cones could also be obtained, in principle, by studying the corresponding one-particle Schrödinger operator. We find, however, more suitable the functional integral formalism, considering the hopping as a perturbation and expanding the correlation as a convergent series.

The stability is a nonperturbative phenomenon which cannot be understood on the basis of lowest order computations but it depends on the convergence or divergence of the whole series. This is a common feature in problems involving incommensurability; the KAM series are convergent and correspond to stable motions, while the Birkhoff series are diverging and correspond to chaos but both have a small divisor problem.

We choose an interlayer hopping with coupling λ that decays exponentially for large momenta and satisfies suitable lattice symmetries so that the terms connecting the same Dirac points just renormalize the single-layer parameters except for a modification of the location of Dirac points. Such a choice of symmetries is done for definiteness, and it could be avoided by including other terms (counterterms) in the single-layer Hamiltonian that are compatible with the symmetries, while the fast decay of the hopping in momentum space seems essential.

Our main result is the following. In a lattice model of twisted bilayer graphene with an angle θ verifying the Diophantine condition (4) for $|\lambda| \leq \varepsilon_0(C_0)$ with $\varepsilon_0(C_0)$ proportional to C_0^β for some positive power β , the Dirac cones persist and the Dirac semimetallic phase is stable. This result ensures that the large momentum umklapp terms, even if they almost connect the Dirac points, are not able to destroy the semimetallic phase at weak coupling. The proof is based on a renormalization group analysis for the zero temperature imaginary time Green functions and the convergence of the series allows to exclude nonperturbative effects.

The stability of the cones is established with a smallness condition which depends on a power of the Diophantine constant C_0 ; given an angle, one can, in principle, compute it but it is expected to depend wildly on the exact value of θ . It is, therefore, convenient to give a generic statement for angles in a certain set, as it is done for disordered systems where the properties hold for generic (not all) configurations of disorder. Given an angle in a certain interval of size $2\delta\theta$ around θ , we can prove that the angles verifying (4) have almost a full measure: the complement of the set has a measure that is bounded by $C_{\theta,\delta\theta} C_0^\alpha (2\delta\theta)$, with some positive power α and a constant $C_{\theta,\delta\theta}$ (that depends on derivatives of $K_i^\pm, b_j$); the

explicit value of the power α and the form of $C_{\theta,\delta\theta}$ are written explicitly below. One can check that $C_{\theta,\delta\theta}$ is finite apart from a finite set of special, pathological angles (there are 20 such pathological angles, which arise from technical issues). Therefore choosing a hopping λ fixes an admissible value for C_0 , hence choosing, say, $\lambda = \varepsilon_0(C_0)/2$ the measure of the angles for which the Dirac cones persist has a complement that is bounded by $C_{\theta,\delta\theta} C_0^\alpha (2\delta\theta)$. In addition $\delta\theta$ also has to be chosen small with respect to C_0 but this is probably just a technical assumption which could be removed by a lengthier analysis. The set for which stability is ensured is, of course, rather complex and has a fractal nature, and its relative measure decreases with the hopping. The situation is therefore entirely analogous to the breaking of invariant tori in KAM theory, see, e.g., Refs. [31,32]. Note that the excluded set contains the commensurate angles, but experimentally to see an instability can be difficult as the possible gap could be very small if the corresponding umklapp momentum is very large. Another equivalent way to state the result is that for any angle (except the pathological ones) one can find an interval of angles near it such that, with probability 1, there is a nonzero value of the hopping for which the semimetallic behavior persists.

Our result therefore ensures that for weak hopping for most (in the sense specified above) angles the semimetallic behavior persists even in the presence of the large momentum umklapp terms which almost connect the Dirac points. The result provides a partial justification of the effective continuum description of TBG in which terms are neglected. It would be, of course, a very interesting problem to consider hopping outside the regime we are considering to see if there can be some transition to a different phase due to such terms, as it happens for fermions in a 1D quasiperiodic Aubry potential; or to detect in the weak coupling regime some sign of fractality following from the above number theoretical considerations.

The paper is organized in the following way. In Sec. II, the lattice model of TBG is presented. In Sec. III, a perturbative expansion for the correlations is derived. The main result is stated in Sec. IV. In Sec. V, the emerging quasiperiodicity and the small divisor problem is described, together with the required (number theoretic) Diophantine conditions. In Sec. VI, the renormalization group derivation is presented. Appendixes detail the more technical aspects of the analysis.

II. INCOMMENSURATE TBG

We consider the lattice TBG model introduced in Refs. [13,14]. We focus on this model for the sake of definiteness but our methods could be applied more generally. We consider two graphene layers rotated with respect to one another by an angle θ around a point $\xi = (0, 1/2)$ (that is, the point between an a and b atom, chosen so that the twisted model preserves the C_2T symmetry in Appendix D 1). The Hamiltonian of the system will be written as

$$H = H_1 + H_2 + V, \quad (5)$$

where H_1 and H_2 are hopping Hamiltonians within the layers 1 and 2 respectively and V is an interlayer hopping term. The first graphene layer is defined on the lattice $\mathcal{L}_1 := \{n_1 A_1 + n_2 A_2, n_1, n_2 \in \mathbb{Z}\}$ with $A_1 = \frac{1}{2}(3, \sqrt{3})$, $A_2 = \frac{1}{2}(3, -\sqrt{3})$.

We introduce the nearest-neighbor vectors: $\delta_1 = (1, 0)$, $\delta_2 = \frac{1}{2}(-1, \sqrt{3})$, $\delta_3 = \frac{1}{2}(-1, -\sqrt{3})$. We will write the Hamiltonian in second quantized form: for $x \in \mathcal{L}_1$, we introduce *annihilation operators* $c_{1,x,a}$ and $c_{1,x,b}$ corresponding respectively to annihilating a fermion located at x and $x + \delta_1$. The nearest neighbor hopping Hamiltonian is ($t = 1$ from now on)

$$H_1 = -t \sum_{x \in \mathcal{L}_1} \sum_{i=0}^2 (c_{1,x,a}^\dagger c_{1,x+A_i,b} + c_{1,x+A_i,b}^\dagger c_{1,x,a}), \quad (6)$$

where $A_0 := 0$ (note that $\delta_1 - \delta_2 = A_2$, $\delta_2 - \delta_3 = A_3$). We will do much of the computation in Fourier space, and we here introduce the Fourier transform $\hat{c}_{1,k,\alpha}$ of $c_{1,x,\alpha}^\pm$ in such a way that, for $\alpha \in \{a, b\}$,

$$c_{1,x,\alpha} = \frac{1}{|\hat{\mathcal{L}}_1|} \int_{\hat{\mathcal{L}}_1} dk e^{-ik(x-\xi)} \hat{c}_{1,k,\alpha} \quad (7)$$

with $|\hat{\mathcal{L}}_1| = 8\pi^2/3\sqrt{3}$, and $\hat{\mathcal{L}}_1 := \mathbb{R}^2/(b_1\mathbb{Z} + b_2\mathbb{Z})$ in which $b_1 = \frac{2\pi}{3}(1, \sqrt{3})$, $b_2 = \frac{2\pi}{3}(1, -\sqrt{3})$. In Fourier space,

$$H_1 = \frac{t}{|\hat{\mathcal{L}}_1|} \int_{\hat{\mathcal{L}}_1} dk (\Omega(k) \hat{c}_{1,k,a}^\dagger \hat{c}_{1,k,b} + \Omega^*(k) \hat{c}_{1,k,b}^\dagger \hat{c}_{1,k,a}) \quad (8)$$

with $\Omega(k_x, k_y) := 1 + 2e^{-i\frac{3}{2}k_x} \cos(\frac{\sqrt{3}}{2}k_y)$. Note that, for small θ , $|K_\pm - K'_\pm| = O(\theta)$.

The second graphene layer is rotated by an angle θ around the point $\xi = (0, 1/2)$, that is, it is defined on the lattice

$$\mathcal{L}_2 = \xi + R(\theta)(\mathcal{L}_1 - \xi), \quad R(\theta) = \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix} \quad (9)$$

(we use the shorthand throughout this paper that $c_\theta \equiv \cos \theta$, $s_\theta \equiv \sin \theta$). The annihilation operators in the second layer are denoted by $c_{2,x,a}$ and $c_{2,x,b}$. The hopping Hamiltonian of this second layer is

$$H_2 = -t \sum_{x \in \mathcal{L}_2} \sum_{i=0}^2 (c_{2,x,a}^\dagger c_{2,x+RA_i,b} + c_{2,x+RA_i,b}^\dagger c_{2,x,a}), \quad (10)$$

where $R \equiv R(\theta)$. We define the Fourier transform in the second layer: if $b'_1 := Rb_1$, $b'_2 := Rb_2$ and

$$c_{2,x,\alpha} = \frac{1}{|\hat{\mathcal{L}}_2|} \int_{\hat{\mathcal{L}}_2} dk e^{-ik(x-\xi)} \hat{c}_{2,k,\alpha}, \quad (11)$$

we find

$$H_2 = \frac{t}{|\hat{\mathcal{L}}_2|} \int_{\hat{\mathcal{L}}_2} dk (\Omega(R^T k) \hat{c}_{2,k,a}^\dagger \hat{c}_{2,k,b} + \Omega^*(R^T k) \hat{c}_{2,k,b}^\dagger \hat{c}_{2,k,a}). \quad (12)$$

In the absence of interlayer coupling, the two graphene layers are decoupled; the single particle spectrum for layer 1 is $\pm|\Omega(k)|$ and the Fermi points are given by the relation $\Omega(K_\pm^1) = 0$ with

$$K_\pm^1 = \frac{2\pi}{3} \left(1, \pm \frac{1}{\sqrt{3}}\right) \quad (13)$$

for momenta close to such points one has $|\Omega(k)| \sim \frac{3}{2}t|k - K_\pm^1|$, that is, the dispersion relation is almost linear (relativistic) up to quadratic corrections, forming approximate *Dirac cones*. In the same way the dispersion relation for layer 2 is

$\pm|\Omega(R^T k)|$; the Fermi points are $\Omega(R^T K_\pm^2) = 0$ with $K_\pm^2 = R(K_\pm^1)$ and $|\Omega(R^T k)| \sim \frac{3}{2}t|k - K_\pm^2|$. We are interested in understanding how these four Dirac cones are modified in the presence of the interlayer hopping.

We couple the two layers by an interlayer hopping Hamiltonian, which couples atoms of type a to atoms of type b:

$$V = \lambda \sum_{x_1 \in \mathcal{L}_1} \sum_{x'_2 \in \mathcal{L}_2} \sum_{\alpha \in \{a,b\}} \varsigma(x_1 + d_\alpha - x'_2 - Rd_\alpha) \times (c_{1,x_1,\alpha}^\dagger c_{2,x'_2,\alpha} + c_{2,x'_2,\alpha}^\dagger c_{1,x_1,\alpha}), \quad (14)$$

$d_a = (0, 0)$, $d_b = \delta_1$, and $\varsigma(x) = \varsigma(-x)$,

$$\varsigma(x_1 - x_2) = \int_{\mathbb{R}^2} \frac{dq}{4\pi^2} e^{iq(x_1 - x_2)} \hat{\varsigma}(q), \quad |\hat{\varsigma}(q)| \leq e^{-\kappa|q|}. \quad (15)$$

We restrict the interlayer term to hoppings between atoms of type a to atoms of type a and type b to b so that the “inversion” symmetry in Appendix D 1 is satisfied. Note that, whereas the Fourier transform for c is defined on $\hat{\mathcal{L}}_i$, the Fourier transform of ς is defined on all $\mathbb{R}^2$. We write V in Fourier space: we get, see Appendix A

$$V = \frac{\lambda}{4\pi^2 |\hat{\mathcal{L}}_1|} \sum_{\alpha} \left(\sum_{l \in \mathbb{Z}^2} \int_{\hat{\mathcal{L}}_1} dk \tau_{l,\alpha}^{(1)}(k + lb) \hat{c}_{1,k,\alpha}^\dagger \hat{c}_{2,k+lb,\alpha} + \sum_{m \in \mathbb{Z}^2} \int_{\hat{\mathcal{L}}_2} dk \tau_{m,\alpha}^{(2)}(k + mb') \hat{c}_{2,k,\alpha}^\dagger \hat{c}_{1,k+mb',\alpha} \right), \quad (16)$$

where we use the notation $lb \equiv l_1 b_1 + l_2 b_2$, $mb' \equiv m_1 b'_1 + m_2 b'_2$, and

$$\tau_{l,\alpha}^{(1)}(k) := e^{i\xi lb} e^{-ik(d_\alpha - Rd_\alpha)} e^{-i\xi \sigma_{k,1} b'} \hat{\varsigma}^*(k), \quad (17)$$

$$\tau_{m,\alpha}^{(2)}(k) := e^{i\xi mb'} e^{-ik(d_\alpha - Rd_\alpha)} e^{-i\xi \sigma_{k,2} b} \hat{\varsigma}(k), \quad (18)$$

in which $\sigma_{k,i} \in \mathbb{Z}^2$ is the unique integer vector such that $k - \sigma_{k,1} b' \in \hat{\mathcal{L}}_2$, $k - \sigma_{k,2} b \in \hat{\mathcal{L}}_1$. Note that the difference of the momenta of the two fermions is given by $lb + mb'$.

The position of the Dirac points is, in general, modified (renormalized) by the interlayer hopping. It is convenient to fix the values of the renormalized Dirac points by properly choosing the bare ones. This can be achieved by replacing $\Omega(k)$ with $\Omega(k) + v_{i,\omega}$ close to each Dirac point, that is, adding a counterterm has the form

$$M = \sum_{\omega \in \{+, -\}} \sum_{i=1,2} \int_{\hat{\mathcal{L}}_i} dk \chi_{\omega,i}(k) (v_{i,\omega} \hat{c}_{i,k,a}^\dagger \hat{c}_{i,k,b} + v_{i,\omega}^* \hat{c}_{i,k,b}^\dagger \hat{c}_{i,k,a}), \quad (19)$$

where $\chi_{\omega,i}(k)$ is a smooth compactly supported function that is nonvanishing for $\|k - K_\omega^i\|_i \leq 1/\gamma$, for some $\gamma > 1$, in which $\|\cdot\|_i$ is the norm on the torus $\hat{\mathcal{L}}_i$.

Our main result concerns the Matsubara imaginary time Green functions, which we define as follows. We first introduce a Euclidean time component: given an inverse temperature $\beta > 0$, we define for $x_0 \in [0, \beta)$,

$$c_{j,x,\alpha}(x_0) := e^{-x_0 \tilde{H}} c_{j,x,\alpha} e^{x_0 \tilde{H}} \quad (20)$$

and $\bar{H} = H + M$. Combining the Euclidean time component with the spatial one, we define $\Lambda_i := [0, \infty) \times \mathcal{L}_i$. The corresponding Fourier-space operators are

$$\hat{c}_{j,k,\alpha}(k_0) = \int_0^\beta dx_0 e^{-ix_0 k_0} \sum_{x \in \mathcal{L}_j} e^{-i(x-\xi)k} c_{j,x,\alpha}(x_0), \quad (21)$$

which is defined for $(k_0, k) \in \hat{\Lambda}_j := \frac{2\pi}{\beta}(\mathbb{Z} + 1/2) \times \hat{\mathcal{L}}_j$.

Now, given $j, j' \in \{1, 2\}$, $\mathbf{k} = (k_0, k) \in \Lambda_j$, the two-point function is defined as the 2×2 matrix $\hat{S}_{j,j'}(\mathbf{k})$ whose components are indexed by $\alpha, \alpha' \in \{a, b\}$:

$$(\hat{S}_{j,j'}(\mathbf{k}))_{\alpha,\alpha'} := \frac{\text{Tr}(e^{-\beta \bar{H}} T(\hat{c}_{j,k,\alpha}(k_0), \hat{c}_{j',k,\alpha'}^\dagger(k_0)))}{\text{Tr}(e^{-\beta \bar{H}})}, \quad (22)$$

where T is the time ordering operator, which is bilinear, and is defined in real-space

$$\begin{aligned} & T(c_{j,x,\alpha}(x_0), c_{j',y,\alpha'}^\dagger(y_0)) \\ &= \begin{cases} c_{j,x,\alpha}(x_0) c_{j',y,\alpha'}^\dagger(y_0) & \text{if } x_0 < y_0, \\ -c_{j',y,\alpha'}^\dagger(y_0) c_{j,x,\alpha}(x_0) & \text{if } x_0 \geq y_0. \end{cases} \end{aligned} \quad (23)$$

To compute the two-point functions, we will use the Grassmann integral formalism. To do so, we add an imaginary time component to all position vectors: we define $\mathbf{A}_i := (0, A_i)$. Given $\mathbf{x} = (x_0, x) \in \Lambda_j \equiv [0, \beta) \times \mathcal{L}_j$, we introduce Grassmann variables $\psi_{\mathbf{x},a,j}^\pm, \psi_{\mathbf{x},b,j}^\pm$ and their Fourier transforms

$$\psi_{j,\mathbf{x},\alpha}^\pm = \frac{1}{|\hat{\Lambda}_j|} \int_{\hat{\Lambda}_j} d\mathbf{k} e^{\pm i\mathbf{k}(\mathbf{x}-\xi)} \hat{\psi}_{j,\mathbf{k},\alpha}^\pm \quad (24)$$

$$V(\psi) = \frac{\lambda}{4\pi^2 |\hat{\Lambda}_1|} \sum_{\alpha} \left(\sum_{l \in \mathbb{Z}^2} \int_{\hat{\Lambda}_1} d\mathbf{k} \tau_{l,\alpha}^{(1)}(k + lb) \hat{\psi}_{1,\mathbf{k},\alpha}^+ \hat{\psi}_{2,\mathbf{k}+l\mathbf{b},\alpha}^- + \sum_{m \in \mathbb{Z}^2} \int_{\hat{\Lambda}_2} d\mathbf{k} \tau_{m,\alpha}^{(2)}(k + mb') \hat{\psi}_{2,\mathbf{k},\alpha}^+ \hat{\psi}_{1,\mathbf{k}+m\mathbf{b}',\alpha}^- \right), \quad (29)$$

$$M(\psi) = \sum_{\omega,1} \int d\mathbf{k} \chi_{\omega,i}(k) (v_{i,\omega} \psi_{i,\mathbf{k},\alpha}^+ \psi_{i,\mathbf{k},\beta}^- + v_{i,\omega}^* \psi_{i,\mathbf{k},\beta}^+ \psi_{i,\mathbf{k},\alpha}^-). \quad (30)$$

III. PERTURBATIVE EXPANSION AND SMALL DIVISORS

To compute the two-point function $\hat{S}(\mathbf{k})$, we will use Feynman graph expansions. The graphs for this model are chain graphs of the form depicted in Fig. 2.

Let us give the rule first for the graphs with $v = 0$. Each line s is associated a layer label $i_s \in \{1, 2\}$ and two indices $\alpha_s, \alpha'_s \in \{a, b\}$. Each vertex has one entering line s_1 and an exiting line s_2 , and we impose that these lines have different

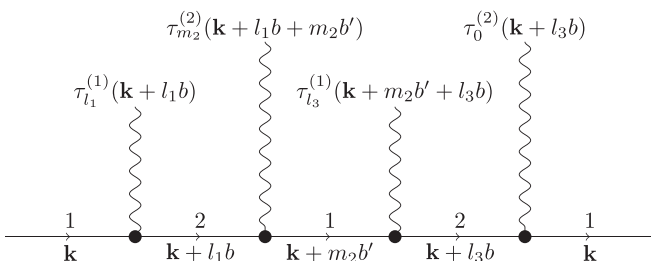


FIG. 2. Example of a Feynman diagram for $S_{1,1}(\mathbf{k})$ with four vertices.

with $|\hat{\Lambda}_j| = 8\pi^2/3\sqrt{3}$, $\xi = (0, \xi)$. The two-point function can be written as

$$(S_{j,j'}(\mathbf{x}, \mathbf{y}))_{\alpha,\alpha'} := \frac{\int P(d\psi) e^{-V(\psi) - M(\psi)} \psi_{j,\mathbf{x},\alpha}^- \psi_{j',\mathbf{y},\alpha'}^+}{\int P(d\psi) e^{-V(\psi) - M(\psi)}} \quad (25)$$

where $P(d\psi) = P(d\psi_1)P(d\psi_2)$, where $P(d\psi_1)$ is the Grassmann integration with propagator

$$\hat{g}_1(\mathbf{k}) = \begin{pmatrix} -ik_0 & \Omega(k) \\ \Omega^*(k) & -ik_0 \end{pmatrix}^{-1}, \quad (26)$$

$$g_1(\mathbf{x}, \mathbf{y}) = \frac{t}{|\hat{\mathcal{L}}_1|} \int_{\hat{\Lambda}_1} d\mathbf{k} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} \hat{g}_1(\mathbf{k}), \text{ and}$$

$$\hat{g}_1(\mathbf{k} + K^\pm) \sim \begin{pmatrix} -ik_0 & -v_F(-ik_1 \pm k_2) \\ -v_F(ik_1 \pm k_2) & -ik_0 \end{pmatrix}^{-1}. \quad (27)$$

$P(d\psi_2)$ has propagator

$$g_2(\mathbf{x}, \mathbf{y}) = \frac{1}{|\hat{\mathcal{L}}_1|} \int_{\hat{\Lambda}_2} d\mathbf{k} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} \hat{g}_2(\mathbf{k}) \quad (28)$$

with $\hat{g}_2(\mathbf{k}) = \begin{pmatrix} -ik_0 & \Omega(R^T k) \\ \Omega^*(R^T k) & -ik_0 \end{pmatrix}^{-1} \equiv \hat{g}_1(R^T \mathbf{k})$, which is singular at $K_\pm^2 := RK_\pm^1$ and periodic in k with period b'_1, b'_2 . The interaction and counterterm are rewritten formally in terms of Grassmann variables:

indices: $i_{s_1} = 3 - i_{s_2}$, and the same indices $\alpha_{s_1} = \alpha_{s_2}$. To each vertex r that has an entering line s_1 and an exiting line s_2 , we associate an index, which if $i_{s_1} = 1$ we denote by $l_s \in \mathbb{Z}^2$, and if $i_{s_2} = 2$ we denote by $m_s \in \mathbb{Z}^2$.

(1) Each internal line s with layer label 1 coming from a vertex with index m_s corresponds to the propagator $g_{1;\alpha_s,\alpha'_s}(\mathbf{k} + m_r b')$ (where $\mathbf{k}$ is the momentum at which S is being evaluated). Each internal line s with layer label 2 coming from a vertex with index l_s corresponds to the propagator $g_{2;\alpha_s,\alpha'_s}(\mathbf{k} + l_r b)$.

(2) Each external line s corresponds to the propagator $g_{is;\alpha_s,\alpha'_s}(\mathbf{k})$.

Each vertex r that has an entering line s_1 and exiting line s_2 with $i_{s_1} = 1$ and $i_{s_2} = 2$, if s_1 comes from a vertex with index m_{r-1} , the vertex r corresponds to an interaction term $\lambda \tau_{l_r, \alpha_{s_2}, \alpha'_{s_1}}^{(1)}(\mathbf{k} + m_{r-1} b' + l_r b)$.

(3) Each vertex r that has an entering line s_1 and exiting line s_2 with $i_{s_1} = 2$ and $i_{s_2} = 1$, if s_2 comes from a vertex with index l_{r-1} , the vertex r corresponds to an interaction term $\lambda \tau_{m_r, \alpha_{s_2}, \alpha'_{s_1}}^{(2)}(\mathbf{k} + l_{r-1} b + m_r b')$.

The value of a graph is obtained by taking the product over the lines of the corresponding propagators and multiplying them by the product over the vertices of the interaction terms. The *two-point function* is then obtained by taking a sum over all possible graphs with the correct external labels. A more explicit expression for the *two-point function* is given in Appendix C. The graphs with $v \neq 0$ are obtained by the above ones in the standard way adding vertices such that the momentum of the external lines is the same.

The persistence or not of the semimetallic behavior depends on the convergence of the expansion. One of the more important and difficult reasons for which convergence could, in principle, not occur is that small divisors could accumulate, as we will now explain. Note that if $\mathbf{k}_1$ and $\mathbf{k}_2$ are 2 neighboring terms, their difference is given by $k_1 - k_2 + mb' + lb$; moreover, the propagator is singular at the Dirac points K_ω^i . Consider therefore $g_1(\mathbf{k})\tau_l^{(1)}(k + lb)g_2(\mathbf{k} + lb)$ and suppose that k is in the first Brillouin zone and is close to K_ω^1 , that is $k = K_\omega^1 + r_1$ with $r_1 = O(\varepsilon)$ and ε is a small parameter (so that the propagator is $O(\varepsilon^{-1})$); suppose now that $k + lb + mb'$ is also in the first Brillouin zone (by periodicity we can always add mb' to achieve this) and close to $K_{\omega'}^2$, that is $k + lb + mb' = K_{\omega'}^2 + r_2$ with $r_2 = O(\varepsilon)$. In principle, this would produce an $O(\varepsilon^{-2})$ contribution, that is, an accumulation of small divisors which could produce large contributions, which can destroy convergence. Note however that

$$O(\varepsilon) = |r_1| + |r_2| \geq |r_1 - r_2| = |lb + mb' - K_{\omega'}^2 + K_\omega^1| \quad (31)$$

so that this accumulation of small divisors is possible only when the quantity in the r.h.s. is small. The effect of such terms is rather delicate to be understood. Indeed, similar terms in the case of random potential produce a localization phase in 1d, while an extended phase in higher dimension, while a quasiperiodic disorder in 1d produces an extended phase.

IV. MAIN RESULT

As we noted above, the accumulation of small divisors can only occur when $|lb + mb' + K_\omega^i - K_{\omega'}^j|$ is small. For generic values of θ (for which s_θ or c_θ is irrational), this happens when l, m are large enough, and the basic issue for stability is whether small divisors are balanced by the fact that the interlayer hopping is weak for large values of l, m . We therefore need to quantify the relation between the size of $lb + mb' + K_\omega^i - K_{\omega'}^j$ and that of l, m , which we do using number-theoretical considerations. One such consideration is imposing on the angles a *Diophantine condition* (4). We will prove in Lemma 1 below that this condition holds for *most* angles, in a sense made precise below.

If this is satisfied, then when $|K_\omega^i - K_{\omega'}^j + lb + mb'| = O(\varepsilon)$, we have

$$O(\varepsilon) \geq C_0 |l|^{-\tau}. \quad (32)$$

Therefore $|l| \geq \varepsilon^{-1/\tau}$, and due to the exponential decay of $\hat{\zeta}$ in $\tau^{(1)}$ [see (15)], we get $|\tau_l^{(1)}(k + lb)| \leq e^{-\kappa \varepsilon^{-1/\tau}}$, which compensates the small divisors ε^{-2} . This simple argument shows that small divisors due to adjacent propagators do not accumulate. Of course, this argument is not sufficient to prove

the convergence of the series, and thus the persistence of cones; it only says that two adjacent propagators cannot be simultaneously small, but it says nothing about nonadjacent ones, which is the generic case. This requires a renormalization group analysis, as we will discuss below. We first present our main result.

Theorem 1. For any θ chosen to satisfy the Diophantine condition (4), there exists an $\varepsilon_0(C_0)$ such that, for any $|\lambda| \leq \varepsilon_0$, there exists $v_{j,\omega}$ such that, in the limit $\beta \rightarrow \infty$, the two-point function verifies

$$\begin{aligned} \hat{S}_{j,j}(\mathbf{k} + \mathbf{K}_\omega^j) &= \begin{pmatrix} -iZ_{j,\omega}k_0 & (iv_{j,\omega}k_1 - w_{j,\omega}\omega k_2) \\ (-iv_{j,\omega}^*k_1 - w_{j,\omega}^*\omega k_2) & -iZ_{j,\omega}k_0 \end{pmatrix}^{-1} \\ &\times (1 + O(|\mathbf{k}|^\alpha)) \end{aligned} \quad (33)$$

with $0 \leq \alpha \leq 1$, $Z_{j,\omega} = 1 + O(\lambda)$ real and $v_{j,\omega} = 3t/2 + O(\lambda)$, $w_{j,\omega} = 3t/2 + O(\lambda)$, $v_{j,\omega} = O(\lambda)$.

The interlayer coupling modifies the position of the Dirac points; we have properly chosen the bare Dirac points $K_\pm^j(\lambda)$ in the absence of interlayer coupling [given by $\Omega(K_\pm^j(\lambda)) + v_{j,\pm} = 0$] so that their renormalized physical value is $K_\pm^i$ given by (13). This is equivalent to say that the position of the Dirac points generically moves depending on the angle, the layer, and the coupling.

The velocities $w_{j,\omega}$, $v_{j,\omega}$ and the wave-function normalization $Z_{j,\omega}$ are renormalized in a way generically dependent on the layer and the angle. Note that a priori several other relevant terms could be present, but they are excluded by symmetry. The singularity of the two-point function is given by $Z_{j,\omega}^2 k_0^2 + R(k)$ with $R(k) \sim (|v_{j,\omega}|^2 k_1^2 + |w_{j,\omega}|^2 k_2^2)$; the singularity of the two-point function is therefore the same as in the absence of interlayer coupling at weak interaction, proving the stability of the semimetallic phase.

V. THE DIOPHANTINE CONDITION

One has, of course, to check the validity of (4). As the value of the Diophantine constant C_0 is expected to depend strongly on the exact value of θ , it is convenient to give a generic statement for angles in a certain set. As in the case of disordered systems, we give a statement in measure (or probability) sense. We want to estimate the relative measure of the set for which (4) holds. This requires some care as the $K_\pm$, b , b' are not independent parameters but given functions of the angles.

As a starting point, let us consider the case $i = j$ and $\omega = \omega'$. Let $M := lb + mb'$, and we wish to obtain a lower bound on $|M|$. To do so, we use a simple inequality: $\forall x, y \in \mathbb{R}$,

$$\sqrt{x^2 + y^2} \geq \frac{\sqrt{3}}{2}x - \omega \frac{1}{2}y \quad (34)$$

and so, since

$$\begin{aligned} M &= \frac{2\pi}{3}(l_1 + l_2 + m_1\varphi_1 + m_2\varphi_2, \\ &\quad \sqrt{3}(l_1 - l_2 + m_1\varphi_3 - m_2\varphi_4)) \end{aligned} \quad (35)$$

with $\varphi_1 = c_\theta - s_\theta\sqrt{3}$, $\varphi_2 = c_\theta + s_\theta\sqrt{3}$, $\varphi_3 = c_\theta + s_\theta/\sqrt{3}$, $\varphi_4 = c_\theta - s_\theta/\sqrt{3}$, we have

$$|M| \geq \frac{2\pi}{\sqrt{3}}(l_\omega + m \cdot f_\omega) \quad (36)$$

with $l_+ \equiv l_1$ and $l_- \equiv l_2$, and

$$f_\omega(\theta) := \left(\frac{\varphi_1 - \omega\varphi_3}{2}, \frac{\varphi_2 + \omega\varphi_4}{2} \right). \quad (37)$$

Thus we wish to impose a condition on θ such that $g_{l_\omega, m}(\theta) := |l_\omega + m \cdot f_\omega(\theta)| \geq C_1|m|^{-\tau}$. The measure of the complement of the set where this is true is bounded by

$$\sum_{m, l_\omega}^* \int_{-C_1|m|^{-\tau}}^{C_1|m|^{-\tau}} \frac{1}{g'_{l_\omega, m}} dg_{l_\omega, m}, \quad (38)$$

where $g'_{l_\omega, m}$ is the derivative of $g_{l_\omega, m}$, and $\sum_{k, l}^*$ has the constraint that $\exists \theta \in [\theta_0, \theta_1]$ such that $g_{l_\omega, m}(\theta) \in [-C_1|k|^{-\tau}, C_1|k|^{-\tau}]$. Therefore we get the bound, taking into account the sum over l_ω ,

$$\sum_m^* 2C_1 \frac{|m|^{1-\tau}}{\min_\theta |m \cdot f'_\omega(\theta)|}. \quad (39)$$

Now, in order for this bound to be useful, we need a good bound on $m \cdot f'_\omega$. Now $m \cdot f'_\omega(\theta)$ can be small, but only if m is large enough. This suggests we should control it using another Diophantine condition, but we would run into an infinite problem: we would need mf''_ω to be bounded below, for which we would impose an extra Diophantine condition, which would itself require a Diophantine condition on the third derivatives, et cetera.

We can avoid this problem as follows. Let us fix for definiteness θ in a small interval around $\bar{\theta}$. If $f'_\omega(\bar{\theta}) = |f'_\omega(\bar{\theta})|(\cos \beta, \sin \beta)$ is nonvanishing and $\theta - \bar{\theta}$ is small, then $(m \cdot f'_\omega/|f'_\omega|) \sim |m| \cos(\theta_m)$ where θ_m is the angle between β and m . We can distinguish in the sum (39) the sum over m such that $|\theta_m|, |\theta_m - \pi| < \pi/4$ and the complementary set $|\theta_m - \pi/2|, |\theta_m - 3\pi/2| < \pi/4$. In the first sum, $(m \cdot f'_\omega/|f'_\omega|)$ will be greater than $|m|$ up to some constant; we impose a Diophantine condition only for the second term: for $|\theta_m - \pi/2|, |\theta_m - 3\pi/2| < \pi/4$, we assume that $|m \cdot f'_\omega| \leq C_1|m|^{-\tau}$. Again we will end up with a condition like (39) involving $m \cdot (f'_\omega/|f'_\omega|)'$ which in principle could be arbitrarily small. However $(f'/|f'|)'$ is orthogonal to $f'_\omega/|f'_\omega|$, hence $m \cdot (f'_\omega/|f'_\omega|)' \sim |m| \cos(\theta_m + \pi/2)$, which in this region, is greater than $|m|$. Note that the derivatives $f'(\bar{\theta})$ must be finite.

Thus, we construct a large-measure set of θ 's that satisfy $|l_\omega + m \cdot f_\omega(\theta)| \geq C_1|m|^{-\tau}$. More exactly, in Appendix B 2, we prove the following lemma (this proof is computer-assisted).

Lemma 1. For any $\theta \in [0, 2\pi) \setminus (\mathcal{B}_1 \cup \mathcal{B}_2)$ where $\mathcal{B}_1 \cup \mathcal{B}_2$ is a finite set of special angles [see (B34) and (B35) for an explicit expression], and any $\delta\theta$ small enough, for almost every angle in $[\theta - \delta\theta, \theta + \delta\theta]$, there exists C_0 such that, for all ω, ω', i, j , and for all $l, m \in \mathbb{Z}^2 \setminus \{0\}$,

$$\begin{aligned} |K_\omega^i - K_{\omega'}^j + lb + mb'| &\geq \frac{C_0}{|m|^\tau}, \\ |K_\omega^i - K_{\omega'}^j + lb + mb'| &\geq \frac{C_0}{|l|^\tau}. \end{aligned} \quad (40)$$

VI. THE RENORMALIZED EXPANSION

A. Multiscale decomposition

We introduce smooth cutoff functions: for $i = 1, 2$, $\omega = \pm$, $h \in \{-\infty, \dots, 0\}$, let $\chi_{h,i,\omega}(\mathbf{k})$ be a smooth function that vanishes outside the region $\|\mathbf{k} - \mathbf{K}_\omega^i\| \leq \gamma^{h-1}$ and that is equal to 1 for $\|\mathbf{k} - \mathbf{K}_\omega^i\| \geq \gamma^{h-2}$. The constant $\gamma > 1$ will be chosen below to be large enough. Note that in this way, the supports of $\chi_{0,i,+}$ and $\chi_{0,i,-}$ do not overlap. We define $\hat{g}_{i,\omega}^{(\leq 0)}(\mathbf{k}) = \chi_{0,i,\omega}(\mathbf{k})\hat{g}_i(\mathbf{k})$ and

$$\hat{g}_i(\mathbf{k}) = g_i^{(1)}(\mathbf{k}) + \sum_{\omega=\pm} \hat{g}_{i,\omega}^{(\leq 0)}(\mathbf{k}) \quad (41)$$

with $\hat{g}^{(1)}(\mathbf{k}) = (1 - \sum_\omega \chi_{0,i,\omega}(k))\hat{g}_i(\mathbf{k})$; this induces the Grassmann variable decomposition $\hat{\psi}_{i,\mathbf{k},\alpha} = \hat{\psi}_{i,\mathbf{k},\alpha}^{(1)} + \sum_{\omega=\pm} \hat{\psi}_{i,\mathbf{k},\alpha,\omega}^{(\leq 0)}$ with propagators given by $\hat{g}_i^{(1)}$ and $\hat{g}_{i,\omega}^{(\leq 0)}$ respectively. Note that $\hat{\psi}^{(1)}$ correspond to fermions with momenta far from the Fermi points, while $\hat{\psi}^{(\leq 0)}$ with momenta around $K_\pm$.

We further decompose

$$\hat{g}_{i,\omega}^{(\leq 0)}(\mathbf{k}) = \sum_{h=-\infty}^0 \hat{g}_{i,\omega}^{(h)}(\mathbf{k}), \quad (42)$$

where $\hat{g}_{i,\omega}^{(h)}(\mathbf{k}) := f_{h,i,\omega}(\mathbf{k})\hat{g}_{i,\omega}^{(\leq 0)}$ in which $f_{h,i,\omega} := \chi_{h,i,\omega} - \chi_{h-1,i,\omega}$ is a smooth cutoff function supported in $\gamma^{h-3} \leq \|\mathbf{k} - K_\omega^i\| \leq \gamma^{h-1}$ such that $\sum_{h=-\infty}^0 f_{h,i,\omega} = \chi_{0,i,\omega}$. The integration is done recursively in the following way: assume that we have integrated the fields $\psi^{(1)}, \dots, \psi^{(h-1)}$ obtaining

$$e^{W(\phi)} = \int \bar{P}(d\psi^{(\leq h)}) e^{V^{(h)}(\psi, \phi)}, \quad (43)$$

where $\bar{P}(d\psi^{(\leq h)})$ is Gaussian integration with propagator $\hat{g}_{i,\omega}^{(\leq h)}$, which will be defined inductively in (52), and

$$\begin{aligned} V^{(h)}(\psi, 0) &= \sum_{i,\omega,\omega',l,\alpha,\alpha'} \int_{\hat{\Lambda}_i} d\mathbf{k} W_{i,2,l,\alpha,\alpha'}^{(h,\omega,\omega')}(\mathbf{k}) \psi_{i,\mathbf{k},\alpha,\omega}^+ \psi_{2,\mathbf{k}+lb,\alpha',\omega'}^- \\ &+ \sum_{i,\omega,\omega',m,\alpha,\alpha'} \int_{\hat{\Lambda}_i} d\mathbf{k} W_{i,1,m,\alpha,\alpha'}^{(h,\omega,\omega')}(\mathbf{k}) \psi_{i,\mathbf{k},\alpha,\omega}^+ \psi_{1,\mathbf{k}+mb',\alpha',\omega'}^-. \end{aligned} \quad (44)$$

According to the RG procedure, we renormalize the relevant and marginal terms; we will see below that the terms with l or m nonzero are actually irrelevant, due to improvements in the estimates due to the Diophantine condition. We therefore define a localization operation in the following way:

$$\begin{aligned} \mathcal{L}W_{i,j,l}^{(h,\omega,\omega')}(\mathbf{k}) &= \delta_{\omega,\omega'} \delta_{i,j} \delta_{l,0} [W_{i,i,0}^{(h,\omega,\omega)}(0, K_\omega^i) \\ &+ k_0 \partial_0 W_{i,i,0}^{(h,\omega,\omega)}(0, K_\omega^i) \\ &+ (k - K_\omega^i) \partial W_{i,i,0}^{(h,\omega,\omega)}(0, K_\omega^i)]. \end{aligned} \quad (45)$$

The terms for which $\mathcal{L} = 0$ are called *nonresonant* terms and the ones for which $\mathcal{L} \neq 0$ *resonant* terms. The terms containing derivatives are marginal ones and produce wave function or velocity renormalizations, while the terms without

derivatives are the relevant terms. The action of $\mathcal{L}$ on the effective potential $V^{(h)}$ is

$$\mathcal{L}V^{(h)} = \mathcal{L}_1V^{(h)} + \mathcal{L}_2V^{(h)} \quad (46)$$

with

$$\mathcal{L}_1V^{(h)} := \sum_{i,\omega,\alpha,\alpha'} \int_{\hat{\Lambda}_i} d\mathbf{k} \gamma^h v_{h,\omega,\alpha,\alpha',i} \psi_{\mathbf{k},\omega,i,\alpha}^+ \psi_{\mathbf{k},\omega,i,\alpha'}^- \quad (47)$$

and

$$\mathcal{L}_2V^{(h)} := \sum_{i,j,\omega,\alpha,\alpha'} \int_{\hat{\Lambda}_i} d\mathbf{k} z_{h,\omega,\alpha,\alpha',i,j} (\mathbf{k} - \mathbf{K}_\omega^i)_j \psi_{\mathbf{k},\omega,i,\alpha}^+ \psi_{\mathbf{k},\omega,i,\alpha'}^- \quad (48)$$

with

$$v_{h,\omega,\alpha,\alpha',i} := \gamma^{-h} W_{i,i,0,\alpha,\alpha'}^{h,\omega,\omega}(0, K_\omega^i), \quad (49)$$

$$z_{h,\omega,\alpha,\alpha',i,j} := -\partial_{\mathbf{k}_j} W_{i,i,0,\alpha,\alpha'}^{h,\omega,\omega}(0, K_\omega^i). \quad (50)$$

The form of the resonant terms is severely constrained by symmetries: as is proved in Appendix D,

$$\begin{aligned} v_{h,\omega,a,a,i} &= v_{h,\omega,b,b,i} = 0 & v_{h,\omega,a,b,i} &= v_{h,\omega,b,a,i}^* \\ z_{h,\omega,b,a,i,1} &= z_{h,\omega,a,b,i,1}^* & z_{h,\omega,b,a,i,2} &= z_{h,\omega,a,b,i,2}^* \\ z_{h,\omega,a,a,i,1} &= z_{h,\omega,b,b,i,1} = z_{h,\omega,a,a,i,2} = z_{h,\omega,b,b,i,2} = 0 \\ z_{h,\omega,b,a,i,0} &= z_{h,\omega,a,b,i,0} = 0 & z_{h,\omega,a,a,i,0} &= z_{h,\omega,b,b,i,0} \in i\mathbb{R} \end{aligned} \quad (51)$$

The contributions from $\mathcal{L}_2V$ are marginal, and are absorbed into the propagator at every step of the integration:

$$\begin{aligned} \bar{g}_{i,\omega}^{(\leq h)}(\mathbf{k}) &:= \chi_{h,i,\omega}(\mathbf{k}) \cdot \left((\bar{g}_{i,\omega}^{(\leq h+1)}(\mathbf{k}))^{-1} \right. \\ &\quad \left. - \sum_j z_{h,\omega,\cdot,\cdot,i,j} (\mathbf{k} - \mathbf{K}_\omega^i)_j \right)^{-1}. \end{aligned} \quad (52)$$

Thus

$$\begin{aligned} \bar{g}_{i,\omega}^{(\leq h)}(\mathbf{k} + \mathbf{K}_\omega^i) &= \chi_{h,i,\omega}(\mathbf{k})(1 + O(\mathbf{k})) \\ &\cdot \left(\begin{pmatrix} -iZ_{i,\omega,h}k_0 & (iv_{i,\omega,h}k_1 - w_{i,\omega,h}k_2) \\ (-iv_{i,\omega,h}^*k_1 - w_{i,\omega,h}^*k_2) & -iZ_{i,\omega,h}k_0 \end{pmatrix} \right)^{-1} \end{aligned} \quad (53)$$

with

$$\begin{aligned} Z_{i,\omega,h} &= Z_{i,\omega,h+1} - iz_{h,\omega,a,a,i,0}, \\ v_{i,\omega,h} &= v_{i,\omega,h+1} + iz_{h,\omega,a,b,i,1}, \\ w_{i,\omega,h} &= w_{i,\omega,h+1} + \omega z_{h,\omega,a,b,i,2}. \end{aligned} \quad (54)$$

After absorbing $\mathcal{L}_2V^{(h)}$ into the propagator, we are left with integrating $\mathcal{L}_1V^{(h)}$ and

$$\mathcal{R}V^{(h)} := (1 - \mathcal{L})V^{(h)} \quad (55)$$

so

$$e^W = \int \bar{P}(d\psi^{(\leq h)}) e^{\mathcal{L}_1V^{(h)}(\psi) + \mathcal{R}V^{(h)}(\psi)}. \quad (56)$$

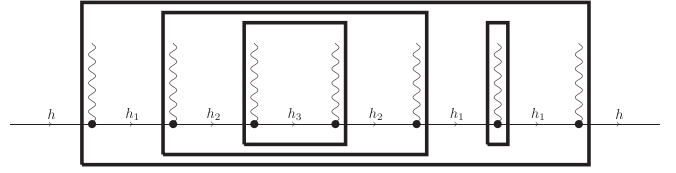


FIG. 3. An example of a graph of order λ^7 with the associated clusters, denoted by thick rectangles. In this example, $h < h_1 < h_2 < h_3$.

B. Feynman rules for the renormalized expansion

The renormalized expansion described above has a graphical representation that is similar to the Feynman diagram expansion from Sec. III. There are two main differences: first, there are two different types of vertices: “ τ -vertices,” coming from $\mathcal{R}V^{(h)}$ in (56), and “ ν -vertices,” coming from $\mathcal{L}_1V^{(h)}$. Second, every line has a *scale label* h , corresponding to a propagator on scale h :

$$\bar{g}_{i,\omega}^{(h)}(\mathbf{k}) := f_{h,i,\omega}(\mathbf{k}) \bar{g}_{i,\omega}^{(\leq h)}(\mathbf{k}). \quad (57)$$

The scale labels induce an important structure: given a diagram, we group vertices together into nested *clusters*, which are connected subgraphs in which the scales of the lines leaving the cluster are all smaller than the scales of the lines inside the cluster, see Fig. 3. A cluster that is such that $\mathcal{L}$ applied to the cluster yields 0 is called *nonresonant*, otherwise it is called *resonant*. In other words, the action of $\mathcal{R} = 1 - \mathcal{L}$ is trivial on nonresonant clusters, and nontrivial on resonant ones.

Some clusters are single vertices (either ν or τ) and are called *trivial clusters*. The clusters that contain internal lines are called *nontrivial clusters*. As per the construction above, if h_T^{ext} is the largest of the scales of the external lines of a nontrivial cluster T , all its internal lines have a scale $h > h_T^{ext}$; h_T is the largest scale of the propagators internal to the cluster T . A nontrivial cluster T contains subclusters $\tilde{T} \subset T$. We call a cluster $\tilde{T} \subset T$ a *maximal cluster* if there is no other cluster $\bar{T}$ such that $\tilde{T} \subset \bar{T} \subset T$. See e.g. [24,25,28] for more details.

For each cluster, there are two external lines that connect the cluster to other ones. In a nonresonant cluster T with external lines of type $i_1 = i_2 = 1$ and momenta k_1, k_2 with k_1 in the first Brillouin zone and $k_2 = k_1 + \hat{m}_T b + lb$, where lb is chosen so that k_2 is in the first Brillouin zone, if $A_0, \dots, A_N$ are the momenta associated to the τ vertices contained in T (the ν vertices do not change momentum) one has $A_0 = k_1 + l_0 b$, $A_1 = k_1 + l_0 b + m_1 b'$, $A_2 = k_1 + m_1 b' + l_2 b$, $A_3 = k_1 + m_3 b' + l_2 b, \dots, A_N = k_1 + m_N b' + l_{N-1} b$ and $m_N = \hat{m}_T$ with N odd. In the same way if the nonresonant cluster T has one external line of type $i_1 = 1$ and momentum k_1 , and one of type $i_2 = 2$ and momentum k_2 ; assume that k_1 is in the first Brillouin zone and $k_2 = k_1 + \hat{l}_T b + mb'$, where mb' is chosen such that k_2 is in the first Brillouin zone. Now with N even the momenta associated to the τ vertices in T are

$$\begin{aligned} A_0 &= k_1 + l_0 b, & A_1 &= k_1 + l_0 b + m_1 b', \\ A_2 &= k_1 + m_1 b' + l_2 b, & A_3 &= k_1 + m_3 b' + l_2 b, \dots, \\ A_{N-1} &= k_1 + m_{N-2} b' + l_{N-1} b, & l_{N-1} &= \hat{l}_T. \end{aligned}$$

The value associated to a graph Γ is denoted by $W_\Gamma(\mathbf{k})$ and is given by the product of the propagators and ν, τ factors associated to the vertices, with the $\mathcal{R}$ operation acting on each nonresonant cluster. The effect of the $\mathcal{R}$ operation on the nonresonant clusters can be written as

$$\mathcal{R}\hat{W}^h(\mathbf{k} + \mathbf{K}_\omega^i) = k^2 \int_0^1 \partial^2 \hat{W}(t\mathbf{k}). \quad (58)$$

C. Convergence of the series and persistence of the cones

In this section, we want to prove the convergence of the series, extending methods previously used [24,25,28] for fermions in a quasiperiodic potential. In order to bound the contribution of a Feynman graph, we note that $|\bar{g}^{(h)}(k)| \leq C\gamma^{-h}$ (by (53)); therefore the product of propagators in the Feynman graph is bounded by

$$\prod_{T \text{ n.t.}} \gamma^{-h_T(M_T + R_T - 1)}, \quad (59)$$

where M_T is the number of maximal nonresonant clusters contained in T , R_T is the number of maximal resonant clusters contained in T , and $\prod_{T \text{ n.t.}}$ is a product over nontrivial clusters. Note that the trivial clusters do not contribute, as they contain no internal lines. The effect of the $\mathcal{R} := \mathbb{1} - \mathcal{L}$ [recall (45)] operation on the nontrivial resonant clusters produces an extra $\gamma^{2(h_T^{\text{ext}} - h_T)}$ (the gain h_T^{ext} comes from the extra $(k - K_\omega^i)^2$ terms (which are on the scale of the external lines) and the h_T loss from the extra second derivative (which are on the scale of the internal lines)). Therefore we get an extra factor:

$$\prod_{T \text{ res n.t.}} \gamma^{2(h_T^{\text{ext}} - h_T)}, \quad (60)$$

where $\prod_{T \text{ res n.t.}}$ is the product over the nontrivial resonant clusters. In addition, each maximal resonant cluster corresponds to a weight $\gamma^h \nu_h$, so, if $|\nu_h| \leq C|\lambda|$, we get an extra factor:

$$\prod_{T \text{ n.t.}} \gamma^{h_T M_T^v} C|\lambda|, \quad (61)$$

where M_T^v is the number of maximal resonant trivial clusters in T . Thus we get the following bound for the sum over all the labels of a renormalized graph with q vertices:

$$\begin{aligned} \sum_{\underline{h}, \underline{l}, \underline{m}}^* [L(\underline{l}, \underline{m})] |\lambda|^q C^q & \left(\prod_{T \text{ n.t.}} \gamma^{-h_T(M_T + R_T - 1)} \right) \\ & \times \left(\prod_{T \text{ res n.t.}} \gamma^{2(h_T^{\text{ext}} - h_T)} \right) \prod_{T \text{ n.t.}} \gamma^{h_T M_T^v}, \end{aligned} \quad (62)$$

where $L(\underline{l}, \underline{m})$ is the norm of the product of the τ functions:

$$L(\underline{l}, \underline{m}) := \prod_i |\tau(A_i)|, \quad (63)$$

where A_i are the momenta, which depend on the structure of the graph, and $\sum_{\underline{h}, \underline{l}, \underline{m}}^*$ is the sum over the scales and momenta associated to the vertices, as explained in Sec. VIB; these sums are not independent, as they are constrained by the compact support properties of the propagators and of the Diophantine condition. Now defining $\mathbb{1}_{\Gamma \text{ res}}$ is equal to 1 if the maximal cluster T_m is resonant and 0 otherwise,

we have

$$\begin{aligned} \prod_{T \text{ n.t.}} \gamma^{-h_T R_T} \prod_{T \text{ res n.t.}, T \neq T_m} \gamma^{h_T^{\text{ext}}} \prod_{T \text{ n.t.}} \gamma^{h_T M_T^v} & \leq 1 \\ \prod_{T \text{ n.t.}} \gamma^{h_T} \prod_{T \text{ res n.t.}, T \neq T_m} \gamma^{-h_T} & \leq \gamma^{h_{T_m} \mathbb{1}_{\Gamma \text{ res}}} \end{aligned} \quad (64)$$

and then using that if T_m is resonant and $\mathcal{L}$ is applied, then $h_{T_m} = h + 1$ and we get

$$\prod_{T \text{ n.t.}} \gamma^{-h_T(R_T - 1)} \gamma^{h_T M_T^v} \prod_{T \text{ res n.t.}} \gamma^{(h_T^{\text{ext}} - h_T)} \leq \gamma^{(h+1) \mathbb{1}_{\Gamma \text{ res}}}. \quad (65)$$

Therefore (62) is bounded by

$$\begin{aligned} \sum_{\underline{h}, \underline{l}, \underline{m}}^* [L(\underline{l}, \underline{m})] |\lambda|^q C^q \gamma^{h \mathbb{1}_{\Gamma \text{ res}}} \\ \times \left(\prod_{T \text{ n.t.}} \gamma^{-h_T R_T} \right) \left(\prod_{T \text{ res n.t.}} \gamma^{(h_T^{\text{ext}} - h_T)} \right). \end{aligned} \quad (66)$$

As the graphs are chains, there is no problematic combinatorial factor; the main issue is the sum over $\underline{h}$; if we neglect the constraint in $\sum_{\underline{h}, \underline{l}, \underline{m}}^*$ then we will get a factor $\prod_{T \text{ n.t.}} (\sum_{h_T = -\infty}^0 \gamma^{-h_T})$ which is divergent.

However, we have still not taken advantage of the Diophantine condition. In order to do that we note that $|\tau^{(i)}(k)| \leq C e^{-\kappa|k|}$ from the exponential decay or $\hat{\zeta}(k)$, see (15), (17), (18); we can write

$$e^{-\kappa/2|k|} = \prod_{h=-\infty}^0 e^{-\kappa 2^h |k|}. \quad (67)$$

Consider the product of τ factors:

$$L(\underline{l}, \underline{m}) \equiv \prod_i |\tau(A_i)| \leq \prod_i e^{-\kappa|A_i|/2} \prod_{h=-\infty}^1 e^{-\kappa 2^h |A_i|}, \quad (68)$$

which we estimate as

$$L(\underline{l}, \underline{m}) \leq \prod_i e^{-\kappa|A_i|/2} \prod_{T \text{ nonres} \ni i} e^{-\kappa 2^{h_T} |A_i|}, \quad (69)$$

where in the last product i is the label of the points τ contained in T . We then exchange the products:

$$L(\underline{l}, \underline{m}) \leq \left(\prod_i e^{-\kappa|A_i|/2} \right) \prod_{T \text{ nonres}} \prod_{i \in T} e^{-\kappa 2^{h_T} |A_i|}. \quad (70)$$

Let us consider a nonresonant cluster with external lines of type 1; we have

$$|A_0| + |A_1| + \dots \geq |A_0 - A_1 + A_2 - A_3 \dots| = |\hat{m}_T b'| \quad (71)$$

(see Sec. VIB) so that

$$\prod_{i \in T} e^{-\kappa 2^{h_T} |A_i|} \leq e^{-\kappa 2^{h_T} |\hat{m}_T b'|}. \quad (72)$$

A similar analysis holds for $i_1 = i_2 = 2$ with $\hat{l} \neq 0$. In the same way if the cluster T has $i_1 = 1$ and $i_2 = 2$ and k_1, k_2

are the momenta of the external lines using that

$$|A_0| + |A_1| + \dots \geq |A_0 - A_1 + A_2 - A_3 \dots| \geq |k_1 + \hat{l}b| \geq |\hat{l}b| - \frac{4\pi}{3} \quad (73)$$

(where we used that $|k_1| \leq \frac{4\pi}{3}$) as k_1 is in the first Brillouin zone

$$\prod_{i \in T} e^{-\kappa 2^{h_T} |A_i|} \leq e^{-\kappa 2^{h_T} (|\hat{l}b| - \frac{4\pi}{3})}. \quad (74)$$

In addition, the Diophantine condition imposes a constraint between the momenta of the external lines of a cluster T and the l, m labels associated to its internal vertices. Consider a nonresonant cluster with the following indices:

(1) If $i_1 = i_2 = 1$ and k_1, k_2 are the momenta of the external lines; assume that k_1 is in the first Brillouin zone and $k_1 =: \bar{k}_1 + K_{\omega_1}^1$. Moreover $k_2 := k_1 + \hat{m}_T b' + lb =: \bar{k}_2 + K_{\omega_2}^1$, with l chosen such that k_2 is in the first Brillouin zone; then, if $\hat{m}_T \neq 0$, by (4),

$$2\gamma^{h_T^{\text{ext}}} \geq |\bar{k}_1| + |\bar{k}_2| \geq |\bar{k}_1 - \bar{k}_2| = |K_{\omega_1}^1 - K_{\omega_2}^1 + \hat{m}_T b' + lb| \geq \frac{C_0}{|\hat{m}_T|^\tau} \quad (75)$$

so that, if $\hat{m}_T \neq 0$

$$|\hat{m}_T| \geq \left(\frac{1}{2} C_0 \gamma^{-h_T^{\text{ext}}}\right)^{\frac{1}{\tau}} \quad (76)$$

and so

$$|\hat{m}_T b'| \geq c_1 \gamma^{-h_T^{\text{ext}}/\tau} \quad (77)$$

for some constant $c_1 > 0$.

On the other hand, if $\hat{m}_T = 0$ but $\omega_1 \neq \omega_2$ then $l = 0$ so $\gamma^{h_T^{\text{ext}}} \geq \frac{1}{2} |K_{\omega_1}^1 - K_{\omega_2}^1| = \frac{2\pi}{3\sqrt{3}}$ [recalling (13)]. Thus this eventuality does not occur provided γ is large enough.

(2) In the case $i_1 = 1$ and $i_2 = 2$ and k_1, k_2 are the momenta of the external lines; assume that k_1 is in the first Brillouin zone and $k_1 =: \bar{k}_1 + K_{\omega_1}^1$. Moreover $k_2 := k_1 + \bar{l}b + mb' =: \bar{k}_2 + K_{\omega_2}^2$, with m chosen in such a way that k_2 is in the first Brillouin zone; then, if $\bar{l} \neq 0$, by (4),

$$2\gamma^{h_T^{\text{ext}}} \geq |\bar{k}_1| + |\bar{k}_2| \geq |\bar{k}_1 - \bar{k}_2| = |K_{\omega_1}^1 - K_{\omega_2}^2 + \bar{l}b + mb'| \geq \frac{C_0}{|\bar{l}b|^\tau} \quad (78)$$

so that

$$|\bar{l}b| \geq \left(\frac{1}{2} C_0 \gamma^{-h_T^{\text{ext}}}\right)^{\frac{1}{\tau}} \quad (79)$$

and so

$$|\hat{l}_T b| \geq c_1 \gamma^{-h_T^{\text{ext}}/\tau}. \quad (80)$$

If $\hat{l}_T = 0$ then $2\gamma^{h_T^{\text{ext}}} \geq O(\theta)$ for $\omega_1 = \omega_2$ and $2\gamma^{h_T^{\text{ext}}} \geq O(1)$ for $\omega_1 = -\omega_2$. Thus, provided $\gamma \gg \theta^{-1}$, these eventualities do not present themselves provided γ is large enough.

(3) A similar analysis holds for $i_1 = 2, i_2 = 1$, and $i_1 = i_2 = 2$.

Thus

$$\prod_{i \in T} e^{-\kappa 2^{h_T} |A_i|} \leq e^{-\kappa 2^{h_T} (c_1 \gamma^{-h_T^{\text{ext}}/\tau} - \frac{4\pi}{3})}, \quad (81)$$

which provided γ is large enough, yields

$$\prod_{i \in T} e^{-\kappa 2^{h_T} |A_i|} \leq e^{-c_2 \gamma^{-h_T^{\text{ext}}/\tau}} \quad (82)$$

for some constant c_2 ; note that such a constant is proportional, up to some numerical constant, to C_0^τ , as follows from (76). Therefore

$$L(\underline{l}, \underline{m}) \leq e^{-c_2 \gamma^{-h/\tau} \mathbb{1}_{\Gamma \text{ nonres}}} \prod_i e^{-\kappa |A_i|/2} \prod_{T \text{ n.t.}} e^{-c_2 M_T \gamma^{-h_T/\tau}}, \quad (83)$$

where $\mathbb{1}_{\Gamma \text{ nonres}}$ is equal to 1 if the maximal cluster is nonresonant and 0 otherwise. Note that, provided γ is chosen to be large enough, $e^{-c_2 \gamma^{-h/\tau} \mathbb{1}_{\Gamma \text{ nonres}}} \leq \gamma^{3h \mathbb{1}_{\Gamma \text{ nonres}}}$, so

$$L(\underline{l}, \underline{m}) \leq \gamma^{3h \mathbb{1}_{\Gamma \text{ nonres}}} \prod_i e^{-\kappa |A_i|/2} \prod_{T \text{ n.t.}} e^{-c_2 M_T \gamma^{-h_T/\tau}}. \quad (84)$$

Furthermore, using the bound $e^{-\alpha x} \leq (\frac{\beta}{\alpha})^\beta e^{-\beta x^{-\beta}}$ with $\beta = 3\tau M_T$, we find

$$e^{-c_2 M_T \gamma^{-h_T/\tau}} \leq \left(\frac{c_2 e^1}{3\tau}\right)^{-3\tau M_T} \gamma^{3M_T h_T}. \quad (85)$$

In addition, $\sum_{T \text{ n.t.}} M_T \leq q$, since the clusters are nested in each other and for two clusters to be different they must differ by at least one vertex.

Now, let us introduce M_T^τ as the number of maximal nonresonant trivial clusters (i.e., maximal τ -vertices) contained in T , and use the trivial bound $3M_T \leq 2M_T + M_T^\tau$ along with (85) to obtain

$$\prod_{T \text{ n.t.}} e^{-c_2 M_T \gamma^{-h_T/\tau}} \leq C_3^q \cdot \prod_{T \text{ n.t.}} \gamma^{h_T (2M_T + M_T^\tau)}. \quad (86)$$

Thus, plugging this into (66), we find

$$\begin{aligned} & \gamma^h \sum_{\substack{\underline{l}, \underline{m} \\ *}} [L]^\frac{1}{2} |\lambda|^q (CC_3)^q \gamma^{h \mathbb{1}_{\Gamma \text{ res}}} \gamma^{3h \mathbb{1}_{\Gamma \text{ nonres}}} \\ & \times \left[\prod_{T \text{ res}} \gamma^{(h_T^{\text{ext}} - h_T)} \right] \prod_{T \text{ n.t.}} \gamma^{h_T (2M_T + M_T^\tau)}. \end{aligned} \quad (87)$$

In addition,

$$\prod_{T \text{ n.t.}} \gamma^{2h_T M_T} = \gamma^{-2h \mathbb{1}_{\Gamma \text{ nonres}}} \prod_{T \text{ nonres}} \gamma^{2h_T^{\text{ext}}}, \quad (88)$$

$$\gamma^h \sum_{\substack{\underline{l}, \underline{m} \\ *}} [L]^\frac{1}{2} |\lambda|^q (CC_3)^q \left[\prod_{T \text{ n.t.}} \gamma^{(h_T^{\text{ext}} - h_T)} \right] \prod_{T \text{ n.t.}} \gamma^{h_T M_T^\tau}. \quad (89)$$

The crucial point is that the sum over the scales h can be performed by summing over all the differences, taking into account that the scale h is fixed. Finally, the sum over the l, m is done using the factor $[L(\underline{l}, \underline{m})]^\frac{1}{2}$. (The gain term $\gamma^{h_T M_T^\tau}$ is dropped, as it does not lead to any significant gain.) In conclusion, the bound on a graph with q vertices is $C_4^q \gamma^h |\lambda|^q$ assuming that $|\nu_h|, |Z_h - 1|, |v_h - 1|, |w_h - 1| \leq C|\lambda|$. The constant C_4 is proportional to $C_0^{4\tau}$ as one can check by collecting the estimates above.

D. Beta function and imaginary time Green functions

We are left with checking our assumption on Z_h , v_h , w_h , v_h . We know that $v_{i,\omega,h} = v_{i,\omega,h+1} - iz_{h,\omega,a,b,i,1}$ with $z_{h,\omega,a,b,i,1}$ expressed by the sum of renormalized Feynman graphs Γ such that the maximal scale of the clusters is $h+1$, an extra derivative is applied (which costs a factor γ^{-h}) and the momenta of the external lines is fixed equal to K_ω . Moreover, by the compact support of the propagator, there is at least a τ vertex, as the $k=0$ value of a graph with only v vertices is zero; therefore, the analogue of (89) becomes

$$\sum_{\substack{h \\ L, m}}^* [L]^{\frac{1}{2}} |\lambda|^q (CC_3)^q \left[\prod_{T \text{ n.t.}} \gamma^{(h_T^{\text{ext}} - h_T)} \right] \gamma^{2h_T^*}, \quad (90)$$

where T^* is the nontrivial cluster containing a τ vertex whose scale is the largest possible (we now use the gain $\gamma^{h_T M_T^*}$ dropped in the bound (89)). In addition, summing the differences $h_T^{\text{ext}} - h_T$ along a sequence of clusters that goes from h to h_T^* and discarding the others, we bound

$$\sum_T (h_T^{\text{ext}} - h_T) \leq h - h_T^*, \quad (91)$$

and so (90) is bounded by

$$\gamma^{\frac{h}{2}} \sum_{\substack{h \\ L, m}}^* [L]^{\frac{1}{2}} |\lambda|^q (CC_3)^q \prod_{T \text{ n.t.}} \gamma^{\frac{1}{2}(h_T^{\text{ext}} - h_T)} r. \quad (92)$$

Estimating the sum as above, we find that

$$|z_{h,\omega,a,b,i,1}| \leq C_5 \lambda \gamma^{\frac{h}{2}} \quad v_{i,\omega,h} = v_{i,\omega,0} - i \sum_{h'} z_{h',\omega,a,b,i,1}, \quad (93)$$

hence $v_{i,\omega,h} = v_{i,\omega,0} + O(\lambda)$; moreover, the limiting value is reached exponentially fast $v_{i,\omega,h} = v_{i,\omega,-\infty} + O(\lambda \gamma^{h/2})$. At the end $v_{i,\omega,-\infty}$ is expressed by a sum over renormalized graphs; the lowest order contribution, after summing over scales, contains the derivative of (considering for definiteness the external lines with layer label 1) a term proportional to the sum of $\lambda^2 \tau_m^{(1)}(K_\omega^1 + lb) g_2(K_\omega^1 + lb + mb') \tau_{m,1}^{(2)}(K_\omega^1 + lb)$, where mb' is chosen so that $K_\omega^1 + lb + mb'$ is in the first Brillouin zone. In the case of small angles, we can separate in the sum the contributions: (a) $m_1 = m_2 = 0$, $l_1 = l_2 = 0$; (b) $m_1 = -l_1 = 1$ and $m_2 = l_2 = 0$; and (c) $m_2 = -l_2 = 1$ and $m_1 = l_1 = 0$. In the above three cases, one has $\partial g_2(K + mb' + lb) = O(\theta^{-2})$; actually one can decompose the propagator in a dominant part, with a result coinciding with the effective continuum model, and whose contribution is a constant times λ^2/θ^2 , plus a correction which is bounded by $C\lambda^2$. The contribution from the other moments is bounded as $\lambda^2 \sum_l CC_0 |l|^\tau e^{-\xi|l|}$ and the higher orders are $O(\lambda^3)$ by (89). A similar argument holds for Z_h , w_h .

It remains to discuss the flow of v_h ; we can write, see (49),

$$W_{i,i,0,\alpha,\alpha'}^{h,\omega,\omega}(0, K_\omega^i) = \gamma^{h+1} v_{h+1} + \tilde{W}_{i,i,0,\alpha,\alpha'}^{h,\omega,\omega}(0, K_\omega^i), \quad (94)$$

where $\tilde{W}$ is given by the sum of the terms with a number of vertices greater or equal to 2; therefore

$$v_{h,\omega,\alpha,\alpha',i} = \gamma v_{h+1,\omega,\alpha,\alpha',i} + \beta_v^h \quad (95)$$

with $\beta_v^h = \gamma^{-h} \tilde{W}_{i,i,0,\alpha,\alpha'}^{h,\omega,\omega}(0, K_\omega^i)$ is given by the sum with $q \geq 2$ of terms bounded by (92). We have to prove that it is possible to choose the counterterms $v_{\omega,\alpha,\alpha',i}$ so that $v_{h,\omega,\alpha,\alpha',i}$ is bounded by $C\lambda$ for any scale h . Indeed from (95), we get $h \leq -1$

$$v_{h,\omega,\alpha,\alpha',i} = \gamma^{-h} \left(v_{\omega,\alpha,\alpha',i} + \sum_{i=h}^{-1} \gamma^i \beta_v^i \right), \quad (96)$$

and choosing $v_{\omega,\alpha,\alpha',i}$ so that $v_{-\infty,\omega,\alpha,\alpha',i} = 0$, we get

$$v_{h,\omega,\alpha,\alpha',i} = -\gamma^{-h} \sum_{i=-\infty}^h \gamma^i \beta_v^i, \quad (97)$$

and by using a fixed point argument, we can show that there is a sequence such that $|v_{h,\omega,\alpha,\alpha',i}| \leq C\lambda \gamma^{\frac{h}{2}}$.

The application of the above bounds to the 2-point function, in order to derive (33), is straightforward. The two-point function can be written as

$$\hat{S}_{j,j}(\mathbf{k} + K_\omega^j) = \sum_{h=-\infty}^0 \hat{g}^{(h)}(\mathbf{k} + \mathbf{K}_\omega^j) + r^h(\mathbf{k}), \quad (98)$$

where $r^h(\mathbf{k})$ includes the contribution of a term with at least a vertex. We can replace in $g^{(h)}(\mathbf{k} + \mathbf{K}_\omega^j)$ the $v_{i,\omega,h}$, $w_{i,\omega,h}$, $Z_{i,\omega,h}$ with $v_{i,\omega,-\infty}$, $w_{i,\omega,-\infty}$, $Z_{i,\omega,-\infty}$ obtaining the dominant term in (33); the subdominant term is obtained both from the term containing the difference between $v_{i,\omega,h} - v_{i,\omega,-\infty}$, $z_{i,\omega,h} - w_{i,\omega,-\infty}$, $Z_{i,\omega,h} - Z_{i,\omega,-\infty}$, which have an extra factor $O(\lambda \gamma^{h/2})$, or the terms with at least a vertex which have at least a v or a nonresonant trivial vertex, with an extra $O(\gamma^{h/2})$ from the bounds after (90).

VII. CONCLUSIONS

We have considered the role of large-momentum-transfer umklapp terms. These terms, which nearly connect the Fermi points of different layers, are typically neglected in effective continuum descriptions but could destroy the Dirac cones. We rigorously find that, for finite interlayer coupling, the semimetallic phase is stable provided the angles belong to a fractal set of large relative measure (decreasing with the hopping) characterized by the validity of a Diophantine condition. Our result therefore ensures that for weak hopping for most (in the sense specified above) angles the semimetallic behavior persists even in the presence of the large momentum umklapp terms which almost connect the Dirac points. The result provides a partial justification to the effective continuum description of TBG in which terms are neglected.

It would be, of course, a very interesting problem to consider hopping outside the regime we are considering to see if there can be some transition to a different phase due to such terms, as it happens for fermions in a 1D quasiperiodic Aubry potential; or to detect in the weak coupling regime some sign of fractality following from the above number theoretical considerations. In addition, the present analysis paves the way to a more accurate evaluation of the velocities as functions of the angles, taking into account lattice or higher-order effects, and the effect of many-body interactions, whose interplay with the emerging quasiperiodicity could lead to interesting phases.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: FOURIER TRANSFORM OF THE INTERLAYER HOPPING

We write V in Fourier space: we get

$$V = \frac{\lambda}{|\hat{\mathcal{L}}_1|^2} \sum_{x_1 \in \mathcal{L}_1} \sum_{x'_2 \in \mathcal{L}_2} \sum_{\alpha \in \{a,b\}} \int_{\mathbb{R}^2} \frac{dq}{4\pi^2} \int_{\hat{\mathcal{L}}_1} dk_1 \int_{\hat{\mathcal{L}}_2} dk'_2 (e^{i(k_1 x_1 - k'_2 x'_2 + q(x_1 - x'_2))} e^{iq(d_\alpha - R d_\alpha)} e^{i\xi(k'_2 - k_1)} \hat{\zeta}(q) \hat{c}_{1,k_1,\alpha}^\dagger \hat{c}_{2,k'_2,\alpha} + e^{-i(k_1 x_1 - k'_2 x'_2 - q(x_1 - x'_2))} e^{iq(d_\alpha - R d_\alpha)} e^{-i\xi(k'_2 - k_1)} \hat{\zeta}(q) \hat{c}_{2,k'_2,\alpha}^\dagger \hat{c}_{1,k_1,\alpha}), \quad (\text{A1})$$

and using the Poisson summation formula

$$\sum_{x_1 \in \mathcal{L}_1} e^{i(k_1 + q)x_1} = |\hat{\mathcal{L}}_1| \sum_{l \in \mathbb{Z}^2} \delta(k_1 + q + lb), \quad (\text{A2})$$

where we use the shorthand $lb \equiv l_1 b_1 + l_2 b_2$, and

$$\sum_{x'_2 \in \mathcal{L}_2} e^{-i(k_2 + q)x'_2} = |\hat{\mathcal{L}}_2| \sum_{m \in \mathbb{Z}^2} \delta(k_2 + q + mb'). \quad (\text{A3})$$

Noting that $\hat{c}_{2,k_1+lb-mb',\alpha} \equiv \hat{c}_{2,k_1+lb,\alpha}$, $\hat{c}_{1,k'_2+mb'-lb,\alpha} \equiv \hat{c}_{1,k'_2+mb',\alpha}$, we finally obtain (16).

Rewriting (A1) in terms of Grassmann variables, with the added imaginary time component, reads

$$V = \frac{\beta\lambda}{|\hat{\Lambda}_1|^2} \sum_{x_1 \in \mathcal{L}_1} \sum_{x'_2 \in \mathcal{L}_2} \sum_{\alpha \in \{a,b\}} \int_{\mathbb{R}^2} \frac{dq}{4\pi^2} \int_{\hat{\Lambda}_1} d\mathbf{k}_1 \int_{\hat{\Lambda}_2} d\mathbf{k}'_2 \delta(k_{1,0} - k'_{2,0}) \times \left(e^{i(k_1 x_1 - k'_2 x'_2 + q(x_1 - x'_2))} e^{iq(d_\alpha - R d_\alpha)} e^{i\xi(k'_2 - k_1)} \hat{\zeta}(q) \hat{\psi}_{1,\mathbf{k}_1,\alpha}^+ \hat{\psi}_{2,\mathbf{k}'_2,\alpha}^- + e^{-i(k_1 x_1 - k'_2 x'_2 - q(x_1 - x'_2))} e^{iq(d_\alpha - R d_\alpha)} e^{-i\xi(k'_2 - k_1)} \hat{\zeta}(q) \hat{\psi}_{2,\mathbf{k}'_2,\alpha}^+ \hat{\psi}_{1,\mathbf{k}_1,\alpha}^- \right), \quad (\text{A4})$$

where we use the notation $\mathbf{k}_1 = (k_{1,0}, k_1)$ and $\mathbf{k}'_2 = (k'_{2,0}, k_2)$. Again, using the Poisson formula, we find (29).

APPENDIX B: PROOF OF LEMMA 1

To prove Lemma 1, we will first prove a general result on a Diophantine condition for a generic function from $[0, 2\pi)$ to $\mathbb{R}^2$. We will then apply this result to $|K_\omega^i, K_\omega^j + lb + mb'|$, viewed as a function of θ , for the various values of i, j, ω, ω' .

1. Diophantine condition from $\mathbb{R}$ to $\mathbb{R}^2$

Let us consider an interval $[\theta_0, \theta_1] \subset [0, 2\pi]$, and define, given constants $C_1 > 0$, $\tau > 4$ that are fixed once and for all, two twice-differentiable functions $x : [0, 2\pi) \rightarrow \mathbb{R}$, $f : [0, 2\pi) \rightarrow \mathbb{R}^2$, and a subset $\Omega(x, f) \subset [\theta_0, \theta_1]$,

$$\mathcal{D}(x, f) := \{\theta \in \Omega(x, f) : \forall k \in \mathbb{Z}^2 \setminus \{0\}, \forall l \in \mathbb{Z}, \|x(\theta) + l + k \cdot f(\theta)\| \geq C_1 |k|^{-\tau}\}. \quad (\text{B1})$$

We will show that, provided Ω is chosen appropriately, under certain conditions on f and x , $\mathcal{D}$ has a large measure. The novelty of this result is that f takes values in $\mathbb{R}^2$, but is a function of a single variable; if f were a function from $\mathbb{R}^n$ to $\mathbb{R}^n$, then the fact that $\mathcal{D}$ has a large measure would follow from standard arguments. Our result is stated for $\mathbb{R}^2$, but it could easily be adapted to any other dimension, provided f takes a single real-valued argument.

In order to make our argument work, we will assume that $f'(\theta)$ (the derivative of f) remains inside a cone, that is, we assume that $\exists \xi \in \mathbb{R}^2$ with $|\xi| = 1$ and $\alpha \in [0, \frac{\pi}{4})$ such that, $\forall \theta \in [\theta_0, \theta_1]$,

$$f'(\theta) \in \mathcal{C}_\xi(\alpha) := \{y \in \mathbb{R}^2, |y \cdot \xi| > |y| \cos(\alpha)\}. \quad (\text{B2})$$

We take the set $\Omega(x, f)$ in (B1) to be

$$\Omega(x, f) := \{\theta \in [\theta_0, \theta_1] : \forall k \in \zeta, |x'(\theta) + k \cdot f'(\theta)| \geq C_3 |f'(\theta)| |k|^{-\epsilon}\}, \quad (\text{B3})$$

where $C_3 > 0$ is a constant, $\epsilon \in (1, \tau - 3)$, and

$$\zeta := \mathbb{Z}^2 \setminus (\{0\} \cup \mathcal{C}_\xi(\beta)), \quad (\text{B4})$$

where $\beta \in (\alpha, \frac{\pi}{2} - \alpha)$ (the reason why we choose Ω in this way will become apparent in the proof of Lemma B 1 below).

Lemma B 1. If the following estimates hold:

$$\min_{\theta \in [\theta_0, \theta_1]} |f'(\theta)| > 0, \quad \min_{\theta \in [\theta_0, \theta_1]} \left| \frac{\partial}{\partial \theta} \left(\frac{f'(\theta)}{|f'(\theta)|} \right) \right| > 0 \quad (\text{B5})$$

$\forall \theta \in [\theta_0, \theta_1]$,

$$\min_{0 < |k| < R_1} |x'(\theta) + k \cdot f'(\theta)| - C_3 |f'(\theta)| |k|^{-\epsilon} > 0 \quad (\text{B6})$$

with

$$R_1 := \frac{C_3 + \frac{|x'(\theta)|}{|f'(\theta)|}}{\cos(\alpha + \beta)}, \quad (\text{B7})$$

and, for some $\eta > 0$,

$$\min_{0 < |k| < R_2} \left| \frac{\partial}{\partial \theta} \left(\frac{x'(\theta)}{|f'(\theta)|} \right) + k \cdot \frac{\partial}{\partial \theta} \left(\frac{f'(\theta)}{|f'(\theta)|} \right) \right| - \eta |k| \left| \frac{\partial}{\partial \theta} \left(\frac{f'(\theta)}{|f'(\theta)|} \right) \right| \cos \left(\alpha + \frac{\pi}{2} - \beta \right) > 0 \quad (\text{B8})$$

with

$$R_2 := \eta + \frac{\left| \frac{\partial}{\partial \theta} \left(\frac{x'(\theta)}{|f'(\theta)|} \right) \right|}{\left| \frac{\partial}{\partial \theta} \left(\frac{f'(\theta)}{|f'(\theta)|} \right) \right| \cos \left(\alpha + \frac{\pi}{2} - \beta \right)}, \quad (\text{B9})$$

then the measure of the complement of $\mathcal{D}$ is bounded by

$$|[\theta_0, \theta_1] \setminus \mathcal{D}(x, f)| \leq O(C_3) + O\left(\frac{C_1}{C_3}\right), \quad (\text{B10})$$

where the constants in $O(\cdot)$ depend only on $\theta_0, \theta_1, x, f, \alpha, \beta, \epsilon, \tau$, and η . In particular, if we choose $C_3 \ll \theta_1 - \theta_0$ and $C_1 \ll (\theta_1 - \theta_0)^2$, then $\mathcal{D}(x, f)$ fills most of $[\theta_0, \theta_1]$. If we further choose $C_3 \sim \sqrt{C_1} \ll (\theta_1 - \theta_0)$, then the measure of the complement of $\mathcal{D}(x, f)$ is $O(\sqrt{C_1}) \ll \theta_1 - \theta_0$.

Remark B 1. Conditions (B6) and (B8) concern a finite number of values of k . In the applications of this lemma below, we can make both of these conditions trivial by ensuring that $R_1, R_2 < 1$, which reduces this finite number of values for k to 0.

Proof. Let $|\mathcal{D}_\Omega^c(x, f)|$ denote the Lebesgue measure of the complement $\Omega(x, f) \setminus \mathcal{D}(x, f)$. Let

$$g_{l,k}(\theta) = x(\theta) + l + k \cdot f(\theta) \quad (\text{B11})$$

in terms of which

$$|\mathcal{D}_\Omega^c(x, f)| \leq \sum_{k,l}^* \int_{-C_1|k|^{-\tau}}^{C_1|k|^{-\tau}} \frac{1}{g'_{l,k}} dg_{l,k}, \quad (\text{B12})$$

where $\sum_{k,l}^*$ has the constraint that $\exists \theta \in [\theta_0, \theta_1]$ such that $g_{l,k}(\theta) \in [-C_1|k|^{-\tau}, C_1|k|^{-\tau}]$. Therefore

$$|\mathcal{D}_\Omega^c(x, f)| \leq \sum_{l,k}^* 2C_1 \frac{|k|^{-\tau}}{\min_{\theta \in \Omega(x,f)} |x'(\theta) + k \cdot f'(\theta)|}. \quad (\text{B13})$$

In addition, the number of values of l such that $g_{l,k}(\theta) \in [-C_1|k|^{-\tau}, C_1|k|^{-\tau}]$ is bounded by $C_2|k|$ for some constant C_2 (which depends only on θ_0, θ_1, x, f), and so

$$|\mathcal{D}_\Omega^c(x, f)| \leq 2 \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} C_2 C_1 \frac{|k|^{1-\tau}}{\min_{\theta \in \Omega(x,f)} |x'(\theta) + k \cdot f'(\theta)|}. \quad (\text{B14})$$

In order for this bound to be useful, we must obtain a good lower bound on $|x' + k \cdot f'|$.

To do so, Ω must be chosen appropriately: we wish for $k \cdot f'$ to stay as far away from $-x'$ as possible. Now, it cannot avoid it entirely, as $k \cdot f'$ will cover all possible values as k varies in $\mathbb{Z}^2 \setminus \{0\}$. By choosing Ω as in (B3), we ensure that $k \cdot f'$ may only approach $-x'$ for large values of k . In doing so, we can estimate $\mathcal{D}_\Omega^c$: we split the sum over $\mathbb{Z}^2 \setminus \{0\}$ into a sum over ζ and a sum over its complement $\zeta^c \equiv \mathbb{Z}^2 \cap \mathcal{C}_\xi(\beta)$, and compute a bound for each case.

If $k \in \zeta$, then, by (B3), for $\theta \in \Omega(x, f)$,

$$|x'(\theta) + k \cdot f'(\theta)| \geq C_3 |f'(\theta)| |k|^{-\epsilon}. \quad (\text{B15})$$

If, on the other hand, $k \in \zeta^c \equiv \mathbb{Z}^2 \cap \mathcal{C}_\xi(\beta)$,

$$|k \cdot f'(\theta)| \geq |k| |f'(\theta)| \cos(\alpha + \beta) \quad (\text{B16})$$

so

$$|x'(\theta) + k \cdot f'(\theta)| \geq |k| |f'(\theta)| \cos(\alpha + \beta) - |x'(\theta)|. \quad (\text{B17})$$

We distinguish two cases once more: either

$$|k| \geq \frac{C_3 + \frac{|x'(\theta)|}{|f'(\theta)|}}{\cos(\alpha + \beta)} \equiv R_1 \quad (\text{B18})$$

[see (B7)] in which case (B15) holds true for these k 's as well, or $|k| < R_1$, in which case (B15) holds by virtue of (B6). All in all, whatever the value of k , (B15) holds for all $k \in \mathbb{Z}^2 \setminus \{0\}$.

Therefore (B14) becomes

$$|\mathcal{D}_\Omega^c(x, f)| \leq \frac{2C_1 C_2}{C_3 \min_{\theta \in [\theta_0, \theta_1]} |f'(\theta)|} \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} |k|^{1-\tau+\epsilon}. \quad (\text{B19})$$

By (B5), this sum is bounded since $\epsilon < \tau - 3$.

We are left with estimating the measure of $\Omega^c(x, f) := [\theta_0, \theta_1] \setminus \Omega(x, f)$. Proceeding in a similar way as for $|\mathcal{D}_\Omega^c(x, f)|$, we find that

$$|\Omega^c(x, f)| \leq 2C_3 \sum_{k \in \zeta} \frac{|k|^{-\epsilon}}{\min_{\theta \in [\theta_0, \theta_1]} \left| \frac{\partial}{\partial \theta} \left(\frac{x'(\theta)}{|f'(\theta)|} \right) + k \cdot \frac{\partial}{\partial \theta} \left(\frac{f'(\theta)}{|f'(\theta)|} \right) \right|}. \quad (\text{B20})$$

Here we see why it was necessary to introduce the set ζ in (B3): if ζ were taken to be $\mathbb{Z}^2 \setminus \{0\}$, then we would run into exactly the same problem as before: the denominator in this bound would not be bounded away from 0. However, the problem that gave rise to the necessity of introducing Ω in the first place actually only occurred for the k 's that are close to being orthogonal to $f'(\theta)$. So we can restrict ζ to only include the k 's that are close to being orthogonal to $f'(\theta)$, which is why we define ζ as in (B4). Because $\frac{\partial}{\partial \theta} \left(\frac{f'}{|f'|} \right)$ is orthogonal to f' ,

$$\frac{\partial}{\partial \theta} \left(\frac{f'}{|f'|} \right) \in \mathcal{C}_{\xi^\perp}(\alpha), \quad (\text{B21})$$

and so, for $k \in \zeta$, because the maximal angle between k and $\frac{\partial}{\partial \theta} \left(\frac{f'}{|f'|} \right)$ is $\alpha + \frac{\pi}{2} - \beta$,

$$\left| k \cdot \frac{\partial}{\partial \theta} \left(\frac{f'(\theta)}{|f'(\theta)|} \right) \right| > |k| \left| \frac{\partial}{\partial \theta} \left(\frac{f'(\theta)}{|f'(\theta)|} \right) \right| \cos \left(\alpha + \frac{\pi}{2} - \beta \right). \quad (\text{B22})$$

Thus

$$\left| \frac{\partial}{\partial \theta} \left(\frac{x'(\theta)}{|f'(\theta)|} \right) + k \cdot \frac{\partial}{\partial \theta} \left(\frac{f'(\theta)}{|f'(\theta)|} \right) \right| > |k| \left| \frac{\partial}{\partial \theta} \left(\frac{f'(\theta)}{|f'(\theta)|} \right) \right| \cos \left(\alpha + \frac{\pi}{2} - \beta \right) - \left| \frac{\partial}{\partial \theta} \left(\frac{x'(\theta)}{|f'(\theta)|} \right) \right|. \quad (\text{B23})$$

Therefore, if

$$|k| \geq \eta + \frac{\left| \frac{\partial}{\partial \theta} \left(\frac{x'(\theta)}{|f'(\theta)|} \right) \right|}{\left| \frac{\partial}{\partial \theta} \left(\frac{f'(\theta)}{|f'(\theta)|} \right) \right| \cos \left(\alpha + \frac{\pi}{2} - \beta \right)} \equiv R_2, \quad (\text{B24})$$

then

$$\left| \frac{\partial}{\partial \theta} \left(\frac{x'(\theta)}{|f'(\theta)|} \right) + k \cdot \frac{\partial}{\partial \theta} \left(\frac{f'(\theta)}{|f'(\theta)|} \right) \right| > \eta |k| \left| \frac{\partial}{\partial \theta} \left(\frac{f'(\theta)}{|f'(\theta)|} \right) \right| \cos \left(\alpha + \frac{\pi}{2} - \beta \right). \quad (\text{B25})$$

If, on the other hand, $|k| < R_2$, then (B25) holds by virtue of (B8). Thus (B25) holds for all $k \in \zeta$. Therefore

$$|\Omega^c(x, f)| \leq \frac{2C_3}{\eta \min_{\theta \in [\theta_0, \theta_1]} \left| \frac{\partial}{\partial \theta} \left(\frac{f'(\theta)}{|f'(\theta)|} \right) \right| \cos \left(\alpha + \frac{\pi}{2} - \beta \right)} \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} |k|^{-1-\epsilon}. \quad (\text{B26})$$

By (B5), this sum is bounded since $\epsilon > 1$. We conclude the proof by combining (B19) with (B26). ■

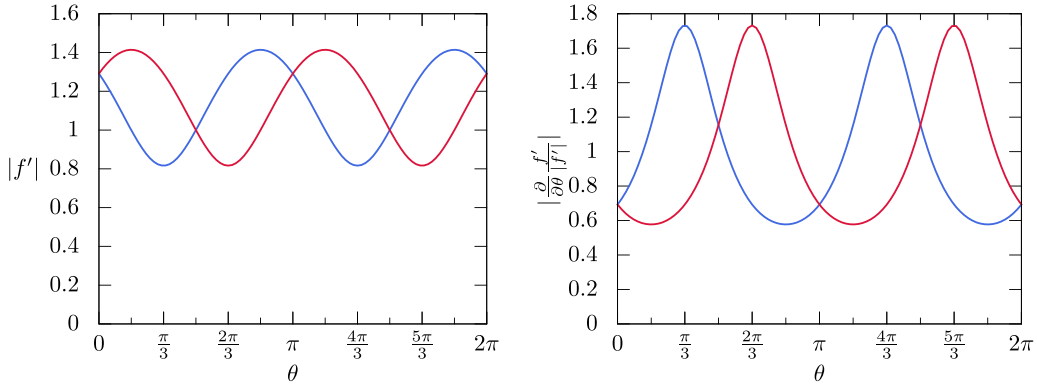


FIG. 4. $|f'_\omega(\theta)|$ and $|\frac{\partial}{\partial\theta} \frac{f'_\omega(\theta)}{|f'_\omega(\theta)|}|$ (blue: $\omega = +$, red: $\omega = -$, color online). These show (B5) holds: neither of these quantities reaches 0.

2. Applying the Diophantine condition to $|K_\omega^i - K_{\omega'}^j + lb + mb'|$

We now apply Lemma B 1 repeatedly to prove Lemma 1. Throughout this section, we set $\beta = \frac{\pi}{4}$ and take α very small. Our proof is computer assisted.

Let us first consider the case $i = j$, $\omega = \omega'$, and $y = m$, that is, we wish to find a condition on θ such that

$$|K_\omega^i - K_{\omega'}^j + lb + mb'| \equiv |lb + mb'| \equiv |M| \geq \frac{C_0}{|m|^\tau} \quad (\text{B27})$$

[recall (4) and (35)]. We recall (36):

$$|M| \geq \frac{2\pi}{\sqrt{3}}(l_\omega + m \cdot f_\omega) \quad (\text{B28})$$

with $l_+ \equiv l_1$ and $l_- \equiv l_2$, and

$$f_\omega(\theta) := \left(\frac{\varphi_1 - \omega\varphi_3}{2}, \frac{\varphi_2 + \omega\varphi_4}{2} \right). \quad (\text{B29})$$

Now, recalling the definition (B1), we have that if $\theta \in \mathcal{D}(0, f_\omega)$, then the inequality (4) with $i = i'$, $\omega = \omega'$, and $y = m$ holds with $C_0 := \frac{2\pi}{\sqrt{3}}C_1$. We therefore just need to use Lemma B 1 to ensure that the measure of this set is large. Taking θ_0 and θ_1 to be sufficiently close to each other (so that the value of α is arbitrarily small and the cone $\mathcal{C}_\xi(\alpha)$ is arbitrarily narrow), it suffices to verify the conditions at $\alpha = 0$, and conclude by continuity.

We first verify (B5). In Fig. 4, we see that neither f' nor $\frac{\partial}{\partial\theta} \frac{f'_\omega(\theta)}{|f'_\omega(\theta)|}$ vanish for any value of $\theta \in [0, 2\pi)$, so (B5) holds.

Since $x = 0$, the other assumptions trivially hold: we choose $C_3 < 1/\sqrt{2}$ and $\eta < 1$ such that $R_1, R_2 < 1$, in which case the minima in (B6) and (B8) are taken over empty sets, so (B6) and (B8) hold trivially. Thus, by Lemma B 1, choosing $C_1 \ll (\theta_1 - \theta_0)^2$, the set $\mathcal{D}(0, f_\omega)$ has a large measure.

We now repeat the argument for the other values of i, j, y , and ω, ω' . First, note that $K_+ - K_- = \frac{1}{3}(b_1 - b_2)$ so the condition (4) holds for $\omega \neq \omega'$ whenever it holds for $\omega = \omega'$. Next, note that

$$|K_\omega^i - K_{\omega'}^j + lb + mb'| = |R^T(K_\omega^i - K_{\omega'}^j + lb + mb')| \quad (\text{B30})$$

which corresponds to exchanging m and l , and flipping the sign of θ . The arguments made for θ may be adapted in a straightforward way to the case $-\theta$ so our derivation for $y = m$ also applies to $y = l$.

We are thus left with the case $i \neq j$, $\omega = \omega'$, and $y = m$. Without loss of generality, we choose $i = 1$ and 2, and we wish to bound

$$|K_\omega^1 - K_\omega^2 + lb + mb'| \geq \frac{C_0}{|m|^\tau}. \quad (\text{B31})$$

Proceeding as we did above, we bound

$$|K_\omega^1 - K_\omega^2 + lb + mb'| \geq \frac{2\pi}{\sqrt{3}}(x_\omega(\theta) + l_1 + m \cdot f_\omega(\theta)) \quad (\text{B32})$$

with

$$x_\omega(\theta) := \frac{1}{3}(1 - c_\theta) + \omega \frac{1}{\sqrt{3}}s_\theta. \quad (\text{B33})$$

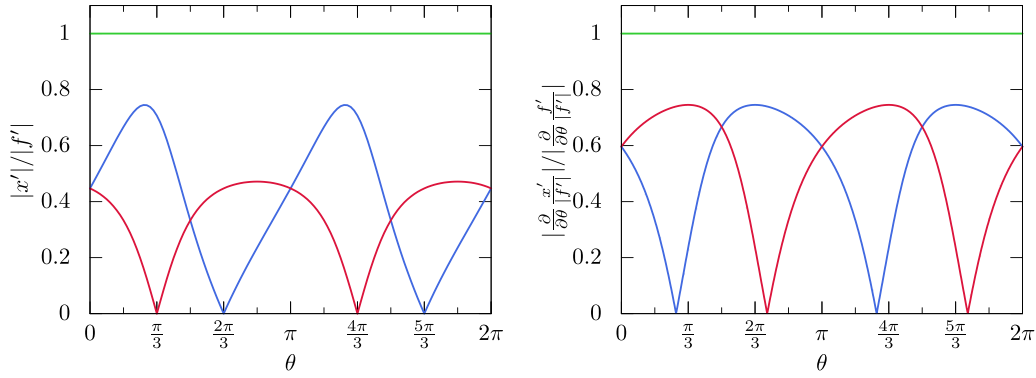


FIG. 5. $\frac{|x'_\omega(\theta)|}{|f'_\omega(\theta)|}$ and $\left| \frac{\partial}{\partial\theta} \frac{x'_\omega(\theta)}{f'_\omega(\theta)} \right| / \left| \frac{\partial}{\partial\theta} \frac{f'_\omega(\theta)}{f'_\omega(\theta)} \right|$ (blue: $\omega = +$, red: $\omega = -$, color online). The green lines represent the value 1. The plot on the left is used to verify (B6): as we see, $\frac{|x'_\omega(\theta)|}{|f'_\omega(\theta)|} < 1$, so, to verify (B6), it suffices to check the cases for which $|k| = 1$. The plot on the right is used to verify (B8): as we see, $\left| \frac{\partial}{\partial\theta} \frac{x'_\omega(\theta)}{f'_\omega(\theta)} \right| / \left| \frac{\partial}{\partial\theta} \frac{f'_\omega(\theta)}{f'_\omega(\theta)} \right| < 1$, so to verify (B8), it suffices to check the cases for which $|k| = 1$.

Therefore, if $\theta \in \mathcal{D}(x_\omega, f_\omega)$, then (B31) holds with $C_0 = \frac{2\pi}{\sqrt{3}}C_1$. To show that this set has a large measure, we check the assumptions of Lemma B 1, as we did above. Again, we check the assumptions at $\alpha = 0$, and argue by continuity.

We check (B6). We choose C_3 in such a way that $R_1 < \sqrt{2}$. In Fig. 5 (left), we see that $\frac{|x'_\omega(\theta)|}{|f'_\omega(\theta)|} < 1$ for all $\theta \in [0, 2\pi)$, so we choose C_3 such that $C_3 < 1 - \frac{|x'_\omega(\theta)|}{|f'_\omega(\theta)|}$ (recall that, since $\beta = \frac{\pi}{4}$, $\sqrt{2}\cos(\beta) = 1$). Thus, to verify (B6), it suffices to check the cases for which $|k| = 1$, that is, $k = (\pm 1, 0)$ and $k = (0, \pm 1)$. And, again choosing C_3 to be small enough, it suffices to check that $x'_\omega(\theta) + k \cdot f'_\omega(\theta) \neq 0$. Using a symbolic calculator, we find that this holds as long as

$$\theta \notin \mathcal{B}_1 := \left\{ 0, \frac{\pi}{3}, \arctan(3\sqrt{3}), \pi - \arctan(3\sqrt{3}), \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \pi + \arctan(3\sqrt{3}), 2\pi - \arctan(3\sqrt{3}), \frac{5\pi}{3} \right\}. \quad (\text{B34})$$

Figure 6 shows the plot of $x'_\omega(\theta) + k \cdot f'_\omega(\theta)$.

Finally, we check (B8). We choose η in such a way that $R_2 < \sqrt{2}$. In Fig. 5 (right), we see that $\left| \frac{\partial}{\partial\theta} \frac{x'_\omega(\theta)}{f'_\omega(\theta)} \right| < \left| \frac{\partial}{\partial\theta} \frac{f'_\omega(\theta)}{f'_\omega(\theta)} \right|$ for all $\theta \in [0, 2\pi)$, so we choose η such that $\eta < \sqrt{2} - \sqrt{2} \left| \frac{\partial}{\partial\theta} \frac{x'_\omega(\theta)}{f'_\omega(\theta)} \right| / \left| \frac{\partial}{\partial\theta} \frac{f'_\omega(\theta)}{f'_\omega(\theta)} \right|$ (recall that, since $\beta = \frac{\pi}{4}$, $\cos(\frac{\pi}{2} - \beta) = \frac{1}{\sqrt{2}}$). Thus, to verify (B8), it suffices to check the cases for which $|k| = 1$, that is, $k = (\pm 1, 0)$ and $k = (0, \pm 1)$. And, again choosing η to be small enough, it suffices to check that $\frac{\partial}{\partial\theta} \frac{x'_\omega(\theta)}{f'_\omega(\theta)} + k \cdot \frac{\partial}{\partial\theta} \frac{f'_\omega(\theta)}{f'_\omega(\theta)} \neq 0$. Using a symbolic calculator, we find that this holds as long as

$$\theta \notin \mathcal{B}_2 := \left\{ \frac{\pi}{6}, \arccos\left(\frac{5\sqrt{3}}{2\sqrt{31}}\right), \arccos\left(\frac{\sqrt{3}}{2\sqrt{7}}\right), \pi - \arccos\left(\frac{\sqrt{3}}{2\sqrt{7}}\right), \pi - \arccos\left(\frac{5\sqrt{3}}{2\sqrt{31}}\right), \frac{5\pi}{6}, \frac{7\pi}{6}, \right. \\ \left. \pi + \arccos\left(\frac{5\sqrt{3}}{2\sqrt{31}}\right), \pi + \arccos\left(\frac{\sqrt{3}}{2\sqrt{7}}\right), 2\pi - \arccos\left(\frac{\sqrt{3}}{2\sqrt{7}}\right), 2\pi - \arccos\left(\frac{5\sqrt{3}}{2\sqrt{31}}\right), \frac{11\pi}{6} \right\}. \quad (\text{B35})$$

Figure 7 shows the plot of $\frac{\partial}{\partial\theta} \frac{x'_\omega(\theta)}{f'_\omega(\theta)} + k \cdot \frac{\partial}{\partial\theta} \frac{f'_\omega(\theta)}{f'_\omega(\theta)}$.

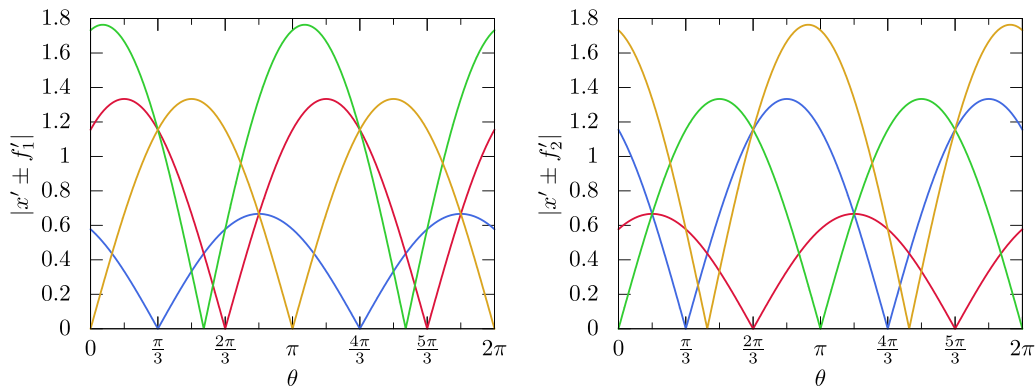


FIG. 6. (Left) $|x' \pm f'_1|$, right: $|x' \pm f'_2|$, in each graph, blue and green are $\omega = +$, red and yellow are $\omega = -$, blue and red are $\pm = +$, green and yellow are $\pm = -$. These plots are used to verify (B6) for $|k| = 1$. As we see, $x' + k\partial f$ only reaches 0 at a finite number of points, enumerated in (B34).

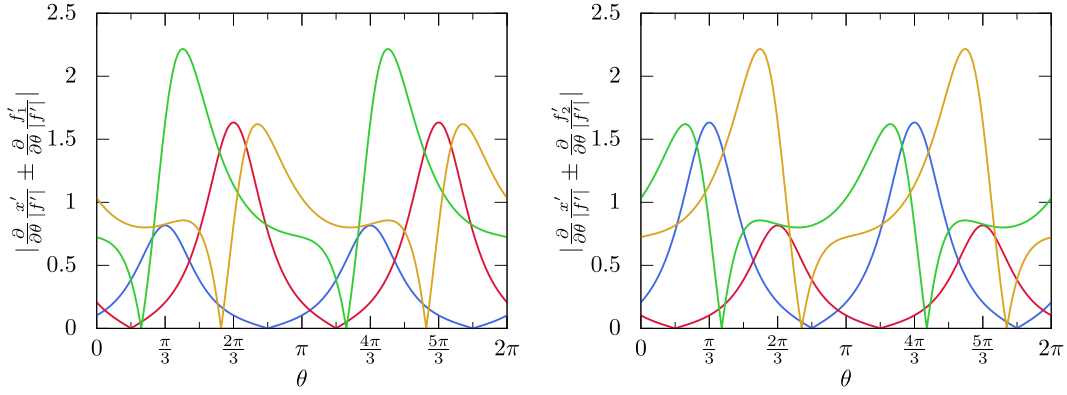


FIG. 7. (Left) $\left| \frac{\partial}{\partial \theta} \frac{x'_\omega}{|f'_\omega|} \pm \frac{\partial}{\partial \theta} \frac{f'_{\omega,1}}{|f'_\omega|} \right|$, right: $\left| \frac{\partial}{\partial \theta} \frac{x'_\omega}{|f'_\omega|} \pm \frac{\partial}{\partial \theta} \frac{f'_{\omega,2}}{|f'_\omega|} \right|$, in each graph, blue and green are $\omega = +$, red and yellow are $\omega = -$, blue and red are $\pm = +$, green and yellow are $\pm = -$. These plots are used to verify (B8) for $|k| = 1$. As we see, $\frac{\partial}{\partial \theta} \frac{x'_\omega}{|f'_\omega|} + k \cdot \frac{\partial}{\partial \theta} \frac{f'_\omega}{|f'_\omega|}$ only reaches 0 at a finite number of points, enumerated in (B35).

All in all, we have found that if we restrict the values of θ to a small interval $[\theta_0, \theta_1]$ that steers clear of any of the angles in $\mathcal{B}_1 \cup \mathcal{B}_2$ (see (B34) and (B35)), and if we further restrict θ to an intersection of Diophantine sets:

$$\theta \in [\theta_0, \theta_1] \cap \bigcap_{\omega=\pm} \bigcap_{\sigma=\pm} \mathcal{D}(0, f_\omega(\sigma\theta)) \cap \bigcap_{\omega=\pm} \bigcap_{\sigma=\pm} \mathcal{D}(x_\omega(\sigma\theta), f_\omega(\sigma\theta)), \quad (\text{B36})$$

then (4) is satisfied for any value of i, i', ω, ω' , and y with a constant $C_0 = O((\theta_1 - \theta_0)^2)$. Because each set has an arbitrarily large measure (relative to $[\theta_0, \theta_1]$), their intersection also does.

APPENDIX C: NAIVE PERTURBATION THEORY

The *two-point function* function is computed using perturbation theory: formally,

$$(S_{1,1}(\mathbf{k}))_{\alpha',\alpha} = \sum_{N=0}^{\infty} \sum_{\alpha_0, \dots, \alpha_{2N+1}} (g_1(\mathbf{k}))_{\alpha, \alpha_0} \left(\prod_{n=0}^N \left(\sum_{l_{2n}} \tau_{l_{2n}, \alpha_{2n}}^{(1)} (k + m_{2n-1}b' + l_{2n}b) (g_2(\mathbf{k} + l_{2n}b))_{\alpha_{2n}, \alpha_{2n+1}} \right. \right. \\ \left. \left. \cdot \sum_{m_{2n+1}} \tau_{m_{2n+1}, \alpha_{2n+1}}^{(2)} (k + l_{2n}b + m_{2n+1}b') ((g_1(\mathbf{k} + m_{2n+1}b'))_{\alpha_{2n+1}, \alpha_{2n+2}})^{\mathbb{1}_{n < N}} \right) \right) (g_1(\mathbf{k}))_{\alpha_{2N+1}, \alpha'} \quad (\text{C1})$$

where $m_{-1} \equiv l_{-1} \equiv 0$ and $\mathbb{1}_{n < N} \in \{0, 1\}$ is equal to 1 if and only if $n < N$,

$$(S_{2,2}(\mathbf{k}))_{\alpha',\alpha} = \sum_{N=0}^{\infty} \sum_{\alpha_0, \dots, \alpha_{2N+1}} (g_2(\mathbf{k}))_{\alpha, \alpha_0} \left(\prod_{n=0}^N \left(\sum_{m_{2n}} \tau_{m_{2n}, \alpha_{2n}}^{(2)} (k + l_{2n-1}b + m_{2n}b') (g_1(\mathbf{k} + m_{2n}b'))_{\alpha_{2n}, \alpha_{2n+1}} \right. \right. \\ \left. \left. \cdot \sum_{l_{2n+1}} \tau_{l_{2n+1}, \alpha_{2n+1}}^{(1)} (k + m_{2n}b' + l_{2n+1}b) ((g_2(\mathbf{k} + l_{2n+1}b))_{\alpha_{2n+1}, \alpha_{2n+2}})^{\mathbb{1}_{n < N}} \right) \right) (g_2(\mathbf{k}))_{\alpha_{2N+1}, \alpha'}, \quad (\text{C2})$$

$$(S_{2,1}(\mathbf{k}))_{\alpha',\alpha} = \sum_{N=0}^{\infty} \sum_{\alpha_0, \dots, \alpha_{2N}} (g_1(\mathbf{k}))_{\alpha, \alpha_0} \left(\prod_{n=0}^N \left(\sum_{l_{2n}} \tau_{l_{2n}, \alpha_{2n}}^{(1)} (k + m_{2n-1}b' + l_{2n}b) ((g_2(\mathbf{k} + l_{2n}b))_{\alpha_{2n}, \alpha_{2n+1}})^{\mathbb{1}_{n < N}} \right. \right. \\ \left. \left. \cdot \left(\sum_{m_{2n+1}} \tau_{m_{2n+1}, \alpha_{2n+1}}^{(2)} (k + l_{2n}b + m_{2n+1}b') (g_1(\mathbf{k} + m_{2n+1}b'))_{\alpha_{2n+1}, \alpha_{2n+2}} \right)^{\mathbb{1}_{n < N}} \right) \right) (g_2(\mathbf{k}))_{\alpha_{2N+1}, \alpha'}, \quad (\text{C3})$$

$$(S_{1,2}(\mathbf{k}))_{\alpha',\alpha} = \sum_{N=0}^{\infty} \sum_{\alpha_0, \dots, \alpha_{2N}} (g_2(\mathbf{k}))_{\alpha, \alpha_0} \left(\prod_{n=0}^N \left(\sum_{m_{2n}} \tau_{m_{2n}, \alpha_{2n}}^{(2)} (k + l_{2n-1}b + m_{2n}b') ((g_1(\mathbf{k} + m_{2n}b'))_{\alpha_{2n}, \alpha_{2n+1}})^{\mathbb{1}_{n < N}} \right. \right. \\ \left. \left. \cdot \left(\sum_{l_{2n+1}} \tau_{l_{2n+1}, \alpha_{2n+1}}^{(1)} (k + m_{2n}b' + l_{2n+1}b) (g_2(\mathbf{k} + l_{2n+1}b))_{\alpha_{2n+1}, \alpha_{2n+2}} \right)^{\mathbb{1}_{n < N}} \right) \right). \quad (\text{C4})$$

APPENDIX D: SYMMETRY CONSTRAINTS ON THE RESONANT TERMS

1. Symmetries of the system

Let us state the symmetries of the model, which will play an important role in our discussion. Note that the C_2T symmetry in (c) only holds if $\xi = (0, 1/2)$. (a) *Complex conjugation*: every complex constant is conjugated and

$$\begin{aligned}\hat{\psi}_{1,k_0,k,\alpha}^\pm &\mapsto e^{\mp i\xi(b_1+b_2)} \hat{\psi}_{1,-k_0,-k,\alpha}^\pm, \\ \hat{\psi}_{2,k_0,k,\alpha}^\pm &\mapsto e^{\mp i\xi(b'_1+b'_2)} \hat{\psi}_{2,-k_0,-k,\alpha}^\pm.\end{aligned}\quad (\text{D1})$$

Indeed, it is straightforward to check that H_1 and H_2 [see (8) and (12)] are left invariant by (D1) [following from the fact that $\Omega(-k) = \Omega(k)^*$]. To check the invariance of the interlayer hopping term V [see (A4)], there is one subtlety: because $\hat{\mathcal{L}}_1$ and $\hat{\mathcal{L}}_2$ have different periodicity, we cannot simply change k_1 to $-k_1$ and k'_2 to $-k'_2$, which would not leave $\hat{\mathcal{L}}_i$ invariant. Instead, we map k_1 to $-k_1 + b_1 + b_2$ and k'_2 to $-k'_2 + b'_1 + b'_2$. It is then straightforward to check [using (A4)] that V remains invariant under (D1), using $e^{i(1,1)b x_1} = e^{i(1,1)b' x'_2} = 1$ and periodicity. (b) *Particle-hole*

$$\begin{aligned}\hat{\psi}_{1,k_0,k,\alpha}^\pm &\mapsto i e^{\pm i\xi(b_1+b_2)} \hat{\psi}_{1,k_0,-k,\alpha}^\mp, \\ \hat{\psi}_{2,k_0,k,\alpha}^\pm &\mapsto i e^{\pm i\xi(b'_1+b'_2)} \hat{\psi}_{2,k_0,-k,\alpha}^\mp.\end{aligned}\quad (\text{D2})$$

The argument is substantially the same as for the conjugation symmetry (a). (c) C_2T symmetry

$$\hat{\psi}_{j,k_0,k,\alpha}^\pm \mapsto e^{\pm i\chi_j(k)} \hat{\psi}_{j,k_0,-k,\bar{\alpha}}^\pm, \quad (\text{D3})$$

where if $\alpha = a$ then $\bar{\alpha} = b$ and if $\alpha = b$ then $\bar{\alpha} = a$, and

$$\begin{aligned}\chi_1(k) &:= \frac{d_b}{2}(b_1 + b_2) - kd_b - \sigma_{k,2}bd_b, \\ \chi_2(k) &:= \frac{d_b}{2}(b'_1 + b'_2) - kRd_b - \sigma_{k,1}b'd_b.\end{aligned}\quad (\text{D4})$$

H_1 and H_2 are invariant under (D3) for the same reason as the conjugation and particle-hole symmetries. For the interlayer hopping, it is easiest to use the expression of V in (16), which involves two integrals: one over $\hat{\mathcal{L}}_1$ and one over $\hat{\mathcal{L}}_2$. Let us discuss the invariance of the integral over $\hat{\mathcal{L}}_1$, as the invariance of the other follows from a similar argument. We change variables in the integral: $k \mapsto -k + b_1 + b_2$, as well as in the sum over l : $(l_1, l_2) \mapsto -(l_1 + 1, l_2 + 1)$, and in the sum over α : $\alpha \mapsto \bar{\alpha}$. This changes $\tau_{l,\alpha}^{(1)}(k + lb)$ to $\tau_{-l-(1,1),\bar{\alpha}}^{(1)}(-k - lb)$. Now, by (17),

$$\begin{aligned}\frac{\tau_{-l-(1,1),\bar{\alpha}}^{(1)}(-k - lb)}{\tau_{l,\alpha}^{(1)}(k + lb)} &= e^{-i\xi(2l+(1,1)b} e^{i(k+lb)(d_{\bar{\alpha}}+d_{\alpha}-Rd_{\alpha}-Rd_{\bar{\alpha}})} \\ &\times e^{-i\xi(\sigma_{-k-lb,1}-\sigma_{k+lb,1})b'} \frac{\hat{\zeta}(k + lb)}{\hat{\zeta}^*(k + lb)}.\end{aligned}\quad (\text{D5})$$

In addition, if $k + lb - \sigma_{k+lb,1}b' \in \hat{\mathcal{L}}_2$, then $-k - lb + (\sigma_{k+lb,1} + (1, 1))b' \in \hat{\mathcal{L}}_2$, so

$$\sigma_{-k-lb,1} = -\sigma_{k+lb,1} - (1, 1) \quad (\text{D6})$$

and, since $d_a = 0$ and $d_b = (1, 0)$,

$$d_{\bar{\alpha}} + d_{\alpha} - Rd_{\alpha} - Rd_{\bar{\alpha}} = d_b - Rd_b \quad (\text{D7})$$

and, since $\zeta(x) = \zeta(-x)$, $\hat{\zeta} \in \mathbb{R}$. Therefore

$$\begin{aligned}\frac{\tau_{-l-(1,1),\bar{\alpha}}^{(1)}(-k - lb)}{\tau_{l,\alpha}^{(1)}(k + lb)} \\ = e^{-i\xi(2l+(1,1)b} e^{i(k+lb)(d_b-Rd_b)} e^{i\xi(2\sigma_{k+lb,1}+(1,1)b')}.\end{aligned}\quad (\text{D8})$$

Since $\xi = d_b/2$,

$$\begin{aligned}\frac{\tau_{-l-(1,1),\bar{\alpha}}^{(1)}(-k - lb)}{\tau_{l,\alpha}^{(1)}(k + lb)} \\ = e^{ikd_b} e^{-i(k+lb)Rd_b} e^{id_b\sigma_{k+lb,1}b'} e^{i\frac{d_b}{2}(b'_1+b'_2-b_1-b_2)}.\end{aligned}\quad (\text{D9})$$

It is then straightforward to check that this extra phase gets canceled out exactly by $e^{\pm i\chi_j}$ in (D3) (to see this, note that if $k \in \hat{\mathcal{L}}_1$, then $\sigma_{k,2} = 0$).

(d) *Inversion*

$$\hat{\psi}_{j,k_0,k,\alpha}^\pm \rightarrow i(-1)^\alpha(-1)^j \hat{\psi}_{j,-k_0,k,\alpha}^\pm. \quad (\text{D10})$$

It is straightforward to check that H_1 , H_2 , and the interlayer hopping [using (A4)] are invariant under (D10).

2. Constraints on the resonant terms

The discrete symmetry properties seen above implies severely constraint the form of the resonant terms. In the following, we use the notation “ $=^a$ ” to mean “by using symmetry (a) from Sec. D 1 (that is, Complex conjugation), it is equal to”, and similarly for “ $=^b$ ”, “ $=^c$ ”, and “ $=^d$ ”.

(1) Using that $W_{aa}(k_0, k) =^d -W_{aa}(-k_0, k)$, we get $W_{aa}(0, K_\omega) = \partial_1 W_{aa}(0, K_\omega) = \partial_2 W_{aa}(0, K_\omega) = 0$. Similarly, $W_{bb}(0, K_\omega) = \partial_1 W_{bb}(0, K_\omega) = \partial_2 W_{bb}(0, K_\omega) = 0$.

(2) From $W_{ab}(k_0, k) =^b W_{ba}(k_0, -k) =^a W_{ba}^*(-k_0, k)$ we get $W_{ab}(0, K_\omega) = W_{ba}^*(0, K_\omega)$, $\partial_1 W_{ab}(0, K_\omega) = \partial_1 W_{ba}^*(0, K_\omega)$, $\partial_2 W_{ab}(0, K_\omega) = \partial_2 W_{ba}^*(0, K_\omega)$. Moreover $W_{ab}(k_0, k) =^d W_{ab}(-k_0, k)$ hence $\partial_0 W_{ab}(0, K_\omega) = 0$.

(3) $\partial_0 W_{aa}(k_0, k) =^a -\partial_0 W_{aa}^*(-k_0, -k) =^b -\partial_0 W_{aa}^*(-k_0, k)$ hence is pure imaginary at $k_0 = 0$; moreover $\partial_0 W_{aa}(k_0, k) =^c \partial_0 W_{bb}(k_0, -k) =^a -\partial_0 W_{bb}^*(-k_0, k)$ so that $\partial_0 W_{aa}(0, K_\omega) = \partial W_{bb}(0, K_\omega) = iz$ with z real.

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