

Long-range spin-polarized Josephson effect in ballistic S/F/S junctions with precessing magnetization

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We present a theory of ballistic N/F/S and S/F/S junctions with a uniformly precessing magnetization, which generates long-range equal-spin superconducting correlations [Takahashi *et al.*, *Phys. Rev. Lett.* **99**, 057003 (2007), Houzet, *Phys. Rev. Lett.* **101**, 057009 (2008)]. The nonequilibrium distribution of Andreev bound states leads to a strongly nonsinusoidal current-phase relationship for large precession angles. We derive detailed results for ballistic junctions involving partially and fully polarized ferromagnets. In the fully polarized half-metal limit, the magnetization precession switches the junction from an “off” state with vanishing subgap current to an “on” state with finite Andreev conductance and finite Josephson current.

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I. INTRODUCTION

Proximity effects at the boundary between a superconductor (S) and a nonsuperconducting material have been extensively studied since the mid-20th century [1–5]. When a superconductor is brought into contact with a normal metal (N), a region of induced superconducting coherence in the normal metal near the interface is created—a phenomenon known as the proximity effect [6,7]. Manifestations of the superconductor proximity effect are a suppressed density of states in the normal metal [8,9] and the flow of a dissipationless supercurrent through the normal metal in an S/N/S Josephson junction [10,11].

By tailoring the normal metal or the N/S interface, the proximity effect can be used to induce nonconventional (i.e., non-*s*-wave/spin-singlet) forms of superconductivity, thus significantly increasing the materials base beyond that of intrinsically superconducting materials. Via this principle, proximity superconductivity has been proposed as an avenue to engineer one- and two-dimensional topological superconductors [12–19] and as an avenue towards superconductors with odd-frequency spin-polarized Cooper pairs [20–37]. Long-range dissipationless spin-polarized currents are key to applications in superconducting spintronics [38–40].

If a superconductor is brought into contact with a ferromagnetic metal (F), coherent pairs of electrons with opposite spin appear in F [41], with spin quantization axis along the magnetization direction. They oscillate between the spin singlet and unpolarized spin triplet states [42], with period characterized by the difference $2\kappa = k_{\uparrow} - k_{\downarrow}$ of the Fermi

momenta for the two spin projections in F. In two and three dimensions, the destructive interference of many such oscillating trajectories leads to a rapid decay of coherence away from the S/F interface. Such a decay does not occur for spin-polarized Cooper pairs, since these consist of two electrons with the same Fermi momentum in F. Spin-polarized Cooper pairs can emerge when a conventional superconductor is placed in contact with a magnetic material with a noncollinear spin texture [20,27,43]. The inhomogeneous magnetization converts spin-singlet Cooper pairs into spin-polarized pairs aligned with the (local) magnetization, thus inducing spin-polarized proximity superconductivity [20–24,27,41,43–46]. Experimental realizations of this mechanism have been reported for magnetic multilayers with engineered noncollinearity [29,35–37], disordered magnetic interfaces [31,32], helical magnetization profiles [26,33], and noncollinear antiferromagnets [47,48]. Long-range supercurrents have also been observed in a half-metallic ferromagnet (HM) [25,30]. Since a half metal has carriers of one spin direction only, it only admits spin-polarized forms of superconductivity [22].

In pioneering works by Takahashi *et al.* [49] and Houzet [50], it was proposed that a precessing magnetization can also serve as a source of spin-polarized superconducting correlations. The mechanism for the generation of equal-spin Cooper pairs is similar to that in systems with spatially noncollinear magnetization, albeit relying instead on noncollinearity in time [49–57]. In contrast to the static effect with a spatially nonuniform magnetization, where the spin-polarized component is always present, temporal noncollinearity enables dynamic control and switching of the spin-polarized superconducting component. Although supercurrents facilitated by magnetization precession have not been realized experimentally to the best of our knowledge, several experiments have reported signatures of a coupling between magnetization dynamics and superconductivity [58–61].

In this work, we consider spin-polarized proximity superconductivity in a ballistic junction involving a ferromagnetic

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metal with a uniformly precessing magnetization. This limit is particularly relevant for vertical Josephson junctions (as opposed to lateral junctions), where transport occurs along the growth direction of the heterostructure. In such samples, the mean free path can be comparable to the sample thickness, which is typically on the order of several nanometers. As a result, transport may be close to ballistic. We relate the Josephson current I to the dependence of Andreev bound state energies on the phase difference ϕ across the junction. This approach is especially useful in the limit of short junctions (junction length L much shorter than the superconductor coherence length ξ), because such junctions contain only a few Andreev levels, which dominate the supercurrent response. In addition to junctions with two superconducting contacts, we also consider junctions with one superconducting and one normal contact. In such junctions, the dependence of the conductance G on voltage, precession frequency, and temperature gives key information on the equal-spin Andreev reflection facilitating the spin-polarized proximity effect in these junctions. Our work complements previous theories of the Josephson effect with a precessing magnetization, which focused on the diffusive limit and tunneling barriers separating superconducting and normal regions [49,50].

In a conventional Josephson junction, Andreev bound states occupy the full subgap range $0 \leq E \leq |\Delta|$, while the spectrum is continuous only for excitation energies $E > |\Delta|$. In the “short-junction limit”, the Josephson current I is then determined entirely by the bound states [62,63]. For a junction with precessing magnetization, this picture changes qualitatively. Bound states exist only in the reduced energy range $0 \leq E < |\Delta| - \hbar|\Omega|/2$, so that part of the subgap region is replaced by a continuous spectrum. As a result, unlike in the conventional case, the continuous part of the spectrum also contributes to the Josephson current. At the same time, the discrete Andreev bound states still leave a strong imprint on the current, which changes abruptly whenever a level crosses $\hbar\Omega/2$ or merges with the continuum. This interplay between discrete and continuous contributions gives rise to a strongly nonsinusoidal current-phase relation for large tilt angles θ [55] and to abrupt jumps in the Josephson current as the junction length L is increased.

For small amplitudes of the precessing magnetization, which is measured by the tilt angle θ , see Fig. 1, the Josephson current I (in an S/F/S junction) and the conductance G (in an N/F/S junction) are proportional to θ^2 . As pointed out by Takahashi *et al.*, this dependence on the amplitude of the precessing magnetization implies a resonant enhancement of the critical current and the conductance if the precession frequency Ω matches the ferromagnetic resonance frequency Ω_{res} [49]. This effect, which is a unique signature of microwave-controlled switching of the spin-polarized Josephson effect, is illustrated in Fig. 1.

The remainder of this article is organized as follows: We introduce the scattering formalism for a ballistic one-dimensional junction in Sec. II. In Sec. III, we specialize to the case that the magnetic element F is fully spin polarized. Precession-induced effects are particularly striking in such a half-metallic junction, because there is no proximity superconductivity without precession. Junctions containing a partially polarized ferromagnetic element F are considered in

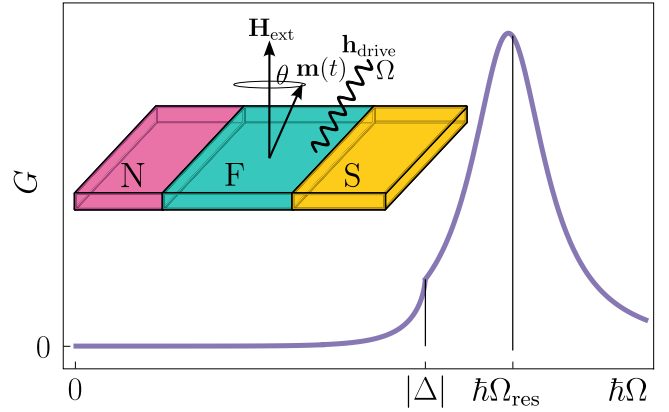


FIG. 1. Zero-bias conductance $G(\Omega)$ of a ballistic N/HM/S junction at $T \rightarrow 0$ as a function of the frequency Ω of the ac magnetic field driving the magnetization precession. The zero-frequency conductance vanishes because Andreev reflection is forbidden in the absence of precession. The conductance exhibits a peak for Ω equal to the ferromagnetic resonance frequency Ω_{res} (which we chose to be $1.25|\Delta|$ for this plot) and has a discontinuous derivative with respect to Ω at $\hbar\Omega = |\Delta|$, where the continuum spectrum of the superconductor begins.

Sec. IV. In this case, spin-polarized and opposite-spin proximity superconductivity coexist. A key difference between the two forms of superconductivity is that the latter oscillates starkly with junction length L with zero average, whereas the former has a component that remains finite after averaging over L . As a result, in the partially polarized case, the opposite-spin proximity effect is short-ranged in two and three dimensions, whereas the equal-spin proximity effect is long-ranged, as we discuss in Sec. V. We conclude with a brief discussion of our results in Sec. VI.

II. ONE-DIMENSIONAL JUNCTIONS: FORMALISM

We first consider a one-dimensional junction consisting of a central ferromagnetic or half-metallic layer F of length L and one superconducting and one normal-metal contact N or two superconducting contacts S with a phase difference ϕ . We choose coordinates such that the magnetic part of the junction is in the region $-L/2 < x < L/2$ and the two leads are at $x < -L/2$ and $x > L/2$, as shown schematically in Figs. 2(a) and 2(b).

The junction is described by the second-quantized Hamiltonian

$$\hat{H}(t) = \int dx \sum_{\sigma, \sigma'} \hat{\psi}_{\sigma}^{\dagger}(x) \mathcal{H}_{\sigma, \sigma'}(x, t) \hat{\psi}_{\sigma'}(x) + \int dx [\hat{\psi}_{\uparrow}^{\dagger}(x) \hat{\psi}_{\downarrow}^{\dagger}(x) \Delta(x) + \text{H.c.}], \quad (1)$$

where the superconducting order parameter $\Delta(x)$ is nonzero in the superconducting regions only,

$$\Delta(x) = |\Delta| e^{i\phi} \Theta(x - L/2) \quad (2)$$

for the case of one superconducting contact [Fig. 2(a)] and

$$\Delta(x) = |\Delta| e^{i \text{sgn}(x)\phi/2} \Theta(|x| - L/2) \quad (3)$$

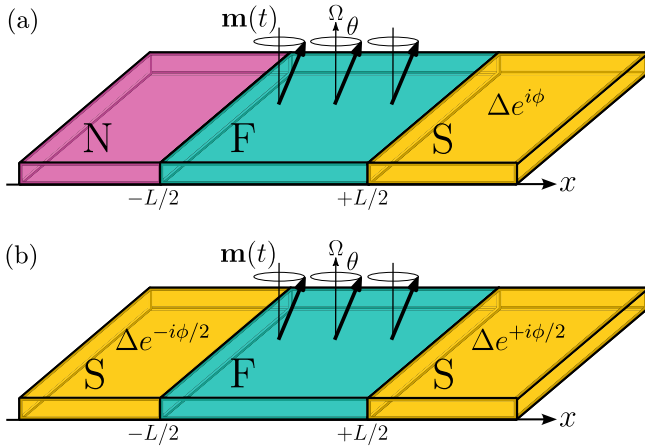


FIG. 2. Geometry of the (a) N/F/S and (b) S/F/S junctions. In both panels, the region $-L/2 < x < L/2$ contains a magnet with precessing magnetization $\vec{m}(t)$ with precession frequency Ω and tilt angle θ . For the S/F/S junction (b), the two superconducting contacts have phase difference ϕ .

for the case of two superconducting contacts [Fig. 2(b)]. In the leads, the normal-state Hamiltonian $\mathcal{H}$ is time-independent and takes the simple form

$$\mathcal{H}(x) = \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - \mu \right) \sigma_0, \quad (4)$$

where σ_0 is the identity matrix for the spin degrees of freedom. In the magnetic part of the junction, $-L/2 < x < L/2$, the Hamiltonian $\mathcal{H}$ is time-dependent because of the precessing magnetization. It reads

$$\mathcal{H}(x, t) = \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V_F - \mu \right) \sigma_0 - h_F \vec{m}(t) \cdot \boldsymbol{\sigma}, \quad (5)$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, V_F is a potential offset, h_F the exchange field, and $\vec{m}(t)$ is the time-dependent magnetization direction, which precesses at angular frequency Ω around the z axis,

$$\vec{m}(t) = (\sin \theta \cos \Omega t, \sin \theta \sin \Omega t, \cos \theta). \quad (6)$$

A. Rotating frame

The time-dependence of the Hamiltonian (1) can be removed by switching to a rotating reference frame for the spin degree of freedom. Such a shift of reference frame is accomplished by the transformation [49,50,54–56,64]

$$\hat{\psi}_\sigma(x) = e^{\frac{1}{2}i\sigma\Omega t} \tilde{\psi}_\sigma(x), \quad (7)$$

where we use a tilde $\tilde{\cdot}$ to indicate operators in the rotating frame. In the rotating frame, the Hamiltonian is time independent,

$$\begin{aligned} \hat{H} = & \int dx \sum_{\sigma, \sigma'} \tilde{\psi}_\sigma^\dagger(x) \tilde{\mathcal{H}}_{\sigma, \sigma'}(x) \tilde{\psi}_{\sigma'}(x) \\ & + \int dx [\tilde{\psi}_\downarrow^\dagger(x) \tilde{\psi}_\uparrow^\dagger(x) \Delta(x) + \text{H.c.}], \end{aligned} \quad (8)$$

where

$$\tilde{\mathcal{H}}(x) = \mathcal{H}(x, 0) - \frac{1}{2} \hbar \Omega \sigma_z. \quad (9)$$

The spin-singlet superconducting order parameter $\Delta(x)$ is not affected by the transformation.

The transformation to the rotating frame exchanges the time dependence of the Hamiltonian for an additional exchange field $\frac{1}{2} \hbar \Omega$ along the z direction. Hence, if the tilt angle $\theta \neq 0$, the Hamiltonian (8) has noncollinear exchange fields: In the leads, the effective exchange field is $\frac{1}{2} \hbar \Omega \vec{e}_z$, whereas in F the effective exchange field

$$\vec{h}_{\text{eff}} = h_F \vec{m}(0) + \frac{1}{2} \hbar \Omega \vec{e}_z \quad (10)$$

also has a component perpendicular to $\vec{e}_z$. As discussed in the introduction, the presence of noncollinear exchange fields is known to be a route to transform spin-singlet superconducting correlations into spin-polarized correlations [20,27,43].

We write the second-quantized Hamiltonian (8) in terms of the 4×4 Bogoliubov–de Gennes Hamiltonian $\mathcal{H}_{\text{BdG}}$,

$$\hat{H} = \frac{1}{2} \int dx \hat{\Psi}(x)^\dagger \tilde{\mathcal{H}}_{\text{BdG}}(x) \hat{\Psi}(x), \quad (11)$$

where $\hat{\Psi}(x) = (\hat{\psi}_\uparrow(x), \hat{\psi}_\downarrow(x), \hat{\psi}_\uparrow^\dagger(x), \hat{\psi}_\downarrow^\dagger(x))^T$,

$$\tilde{\mathcal{H}}_{\text{BdG}}(x) = \begin{pmatrix} \tilde{\mathcal{H}}(x) & -i\sigma_y \Delta(x) \\ i\sigma_y \Delta(x)^* & -\tilde{\mathcal{H}}(x)^* \end{pmatrix}, \quad (12)$$

and we neglect an additive constant that does not enter into the calculation of the current.

To diagonalize the rotating-frame Hamiltonian, we introduce the eigenvectors of the Bogoliubov–de Gennes Hamiltonian (12),

$$\tilde{\mathcal{H}}_{\text{BdG}} \Psi_\alpha(x) = \tilde{E}_\alpha \Psi_\alpha(x), \quad \Psi_\alpha(x) = \begin{pmatrix} u_{\alpha;\uparrow}(x) \\ u_{\alpha;\downarrow}(x) \\ v_{\alpha;\uparrow}(x) \\ v_{\alpha;\downarrow}(x) \end{pmatrix}, \quad (13)$$

at positive eigenvalue $\tilde{E}_\alpha > 0$.

In the case of two superconducting contacts [geometry of Fig. 2(b)], the spectrum of $\mathcal{H}_{\text{BdG}}$ is discrete for $0 \leq \tilde{E}_\alpha < |\Delta| - \hbar|\Omega|/2$. In the discrete part of the spectrum, the eigenstates $\Psi_\alpha(x)$ of $\mathcal{H}_{\text{BdG}}$ are Andreev bound states [65], which have a normalized four-component wave function,

$$\sum_{\sigma'} \int dx (|u_{\alpha;\sigma'}(x)|^2 + |v_{\alpha;\sigma'}(x)|^2) = 1. \quad (14)$$

The spectrum of $\mathcal{H}_{\text{BdG}}$ is continuous for eigenvalues $\tilde{E}_\alpha > |\Delta| - \hbar|\Omega|/2$. If there is one normal-metal contact [geometry of Fig. 2(a)], the spectrum of $\mathcal{H}_{\text{BdG}}$ is continuous for all eigenvalues. In the continuous part of the spectrum, we choose the four-component wave function $\Psi_\alpha(x)$ in the form of a scattering state with normalized quasiparticle flux. In this part of the spectrum, the eigenstate index $\alpha = (\tilde{E}, \beta, \tau, \sigma)$ is a composite index, where $\tilde{E}$ is the eigenvalue of $\tilde{\mathcal{H}}_{\text{BdG}}$ and the three labels β , τ , and σ indicate, whether the incoming-wave part of the scattering state $\Psi_\alpha(x)$ is incident from the right ($\beta = 1$) or from the left ($\beta = -1$), whether it is predominantly electron-like ($\tau = 1$) or hole-like ($\tau = -1$), and whether it corresponds to an excitation with spin $\sigma = 1$ or $\sigma = -1$ along the z axis.

Explicitly, the *incoming-wave part* of the wave function $\Psi_\alpha(x)$ with $\alpha = (\tilde{E}, \beta, \tau, \sigma)$ is [62],

$$\Psi_\alpha(x) = \frac{e^{-i\tau q_{\tau\sigma}(\beta x - \frac{1}{2}L)}}{\sqrt{v_{\tau\sigma}}} \Phi_\alpha, \quad \beta x > \frac{1}{2}L, \quad (15)$$

where $q_{\tau\sigma}$ is the positive solution of

$$\frac{(\hbar q_{\tau\sigma})^2}{2m} = \mu + \tau \sqrt{\left(\tilde{E} + \frac{1}{2}\hbar\sigma\Omega\right)^2 - |\Delta|^2}, \quad (16)$$

$v_{\tau\sigma} = |\hbar dq_{\tau\sigma}/d\tilde{E}|^{-1}$ is the velocity, and the four-component spinors Φ_α are

$$\begin{aligned} \Phi_{\tilde{E}, \beta\tau\uparrow} &= \frac{1}{\sqrt{2\cos\eta_+}} \begin{pmatrix} e^{\frac{i}{2}(\beta\phi + \tau\eta_+)} \\ 0 \\ 0 \\ e^{-\frac{i}{2}(\beta\phi + \tau\eta_+)} \end{pmatrix}, \\ \Phi_{\tilde{E}, \beta\tau\downarrow} &= \frac{1}{\sqrt{2\cos\eta_-}} \begin{pmatrix} 0 \\ e^{\frac{i}{2}(\beta\phi + \tau\eta_-)} \\ -e^{-\frac{i}{2}(\beta\phi + \tau\eta_-)} \\ 0 \end{pmatrix}, \end{aligned} \quad (17)$$

with

$$\eta_\sigma = -i \operatorname{arccosh} \frac{\tilde{E} + \frac{1}{2}\hbar\sigma\Omega}{|\Delta|}. \quad (18)$$

The limit $|\Delta| \rightarrow 0$ must be taken for scattering states incident from the normal contact in case of the geometry of Fig. 2(a). In the range $\tilde{E} < |\Delta| - \hbar|\Omega|/2$, only scattering states incident from the normal contact exist for the geometry of Fig. 2(a), so that $\beta = -1$. In the range $|\Delta| - \hbar|\Omega|/2 < \tilde{E} < |\Delta| + \hbar|\Omega|/2$ only scattering states with $\sigma = \operatorname{sgn}\Omega$ exist in the superconducting contact.

Two remarks are in order: (i) If $\theta \neq 0$, it is not possible to assign a unique spin direction to the scattering state Ψ_α as a whole, because of the noncollinear exchange fields in the rotating-frame Hamiltonian $\tilde{H}$ of Eq. (9). However, the incoming part of the Bogoliubov scattering states does have a well-defined spin along z , see Eq. (15), which is what we use to label the scattering state. (ii) If $\theta = 0$, the sectors of the Hilbert space with spin up and spin down decouple. In this case in a junction with two superconducting contacts, Andreev bound states with discrete energies E_α and spin $-\operatorname{sgn}(\Omega)$ coexist with the continuous-spectrum states with spin $\sigma = \operatorname{sgn}(\Omega)$ in the energy range $|\Delta| - \hbar|\Omega|/2 < \tilde{E} < |\Delta| + \hbar|\Omega|/2$.

In the nonrotating reference frame, scattering states incoming from the leads at energy E have the equilibrium occupation $f(E)$, where f is the Fermi–Dirac function. Since the transition to the rotating frame involves a shift of the normal-state Hamiltonian by $-\frac{1}{2}\sigma\hbar\Omega$ for $\sigma = \uparrow (+)$ and $\sigma = \downarrow (-)$, respectively, the rotating-frame energy $\tilde{E}$ of these states is $\tilde{E} = E - \frac{1}{2}\sigma\hbar\Omega$, see Eq. (9). Hence, the scattering state $\alpha = (\tilde{E}, \beta, \tau, \sigma)$ in the rotating frame has the occupation

$$\tilde{n}_\alpha = f_\sigma(\tilde{E}), \quad (19)$$

where we abbreviate

$$f_\sigma(\tilde{E}) = f(\tilde{E} + \frac{1}{2}\hbar\sigma\Omega). \quad (20)$$

To determine the occupation of the Andreev bound states in the discrete part of the spectrum of an S/F/S junction, we imagine that F is (infinitesimally) tunnel-coupled to a normal reservoir, which is in thermal equilibrium and extends homogeneously along x . In the normal reservoir, quasiparticle excitations at spin σ have the occupation $f_\sigma(\tilde{E})$ in the rotating frame. We choose the tunneling rate for spin- σ quasiparticles to and from the fictitious reservoir to be proportional to

$$\rho_{\alpha\sigma} = \int_{-L/2}^{L/2} dx (|u_{\alpha;\sigma}(x)|^2 + |v_{\alpha;-\sigma}(x)|^2), \quad (21)$$

and assume that the steady-state occupation of the Andreev bound state α is

$$\tilde{n}_\alpha = \frac{\rho_{\alpha;\uparrow} f_\uparrow(\tilde{E}_\alpha) + \rho_{\alpha;\downarrow} f_\downarrow(\tilde{E}_\alpha)}{\rho_{\alpha;\uparrow} + \rho_{\alpha;\downarrow}}. \quad (22)$$

Hence, the population of the Andreev bound states/Bogoliubov scattering states α differs from the thermal equilibrium distribution in both the discrete and the continuous parts of the spectrum.

We remark that the choice (21) for weight factors $\rho_{\alpha\sigma}$ is not unique, because it depends on the type of relaxation processes that determine the steady-state population n_α in the rotating frame. These relaxation processes are not included in the simple mean-field Hamiltonian (1). Microscopically, the steady-state population is determined by inelastic scattering processes that couple the occupation of the Andreev bound states to that of highly excited quasiparticles in the superconducting leads. These transitions are mediated by interactions with a bosonic bath, e.g., phonons. The tunnel coupling to a normal reservoir used to construct Eq. (19) serves as a proxy for such inelastic processes in a single-particle framework, similar to Büttiker’s voltage-probe model for dephasing [66]. The choice of the integration limits in Eq. (22) assumes that relaxation takes place in the central magnetic element of the junction only. For comparison, a model in which relaxation takes place equally inside the superconductor and the magnet would use the full support of the Andreev bound states to obtain the weight factors $\rho_{\alpha\sigma}$. We refer to Refs. [54–56,64] for a discussion of the population of the Andreev bound states in the rotating frame in the limit of a tunnel junction connecting the two superconducting contacts.

B. Scattering approach

To find the Andreev bound states (in the discrete part of the spectrum) and the Bogoliubov scattering states (in the continuous part of the spectrum), we follow a scattering approach. We insert ideal leads between F and the leads (see Fig. 3). The length of the ideal leads is sent to zero at the end of the calculation.

In the ideal lead at the interface with the right or left contact ($\beta' = 1, -1$ or R, L, respectively), the four-component wave function $\Psi_\alpha(x)$ in the BdG formalism can be written as a sum of components moving toward F (“in”) and away from

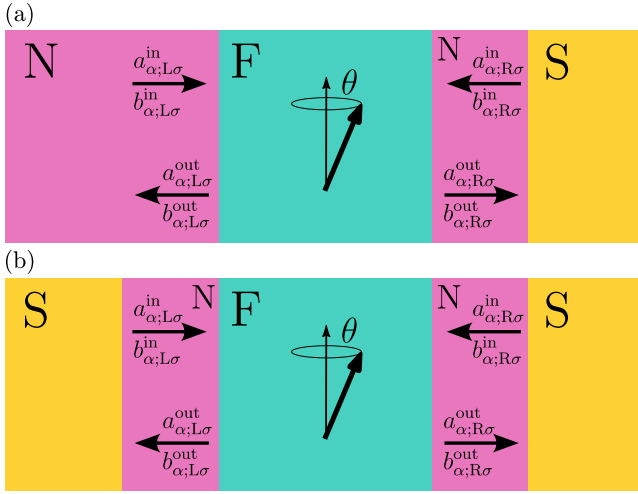


FIG. 3. System geometry with additional ideal leads between the ferromagnet and the left and right contacts for the case of one normal contact and one superconducting contact (a) and two superconducting contacts (b). The amplitudes $a_{\alpha,\beta'\sigma}^{\text{in}}$ and $a_{\alpha,\beta'\sigma}^{\text{out}}$ ($b_{\alpha,\beta'\sigma}^{\text{in}}$ and $b_{\alpha,\beta'\sigma}^{\text{out}}$), with $\beta' = L, R$, refer to electron-like (hole-like) states with spin σ propagating toward and away from the ferromagnet, respectively.

F (“out”),

$$\Psi_{\alpha}(x) = \frac{e^{-ik_0(\beta'x - \frac{L}{2})}}{\sqrt{v}} \begin{pmatrix} a_{\alpha,\beta'\uparrow}^{\text{in}} \\ a_{\alpha,\beta'\downarrow}^{\text{in}} \\ b_{\alpha,\beta'\uparrow}^{\text{out}} \\ b_{\alpha,\beta'\downarrow}^{\text{out}} \end{pmatrix} + \frac{e^{ik_0(\beta'x - \frac{L}{2})}}{\sqrt{v}} \begin{pmatrix} a_{\alpha,\beta'\uparrow}^{\text{out}} \\ a_{\alpha,\beta'\downarrow}^{\text{out}} \\ b_{\alpha,\beta'\uparrow}^{\text{in}} \\ b_{\alpha,\beta'\downarrow}^{\text{in}} \end{pmatrix}, \quad (23)$$

where $\hbar^2 k_0^2 / 2m = \mu$ and $v = \hbar k_0 / m$. (For the wave functions in the ideal leads, we neglect $|\tilde{E}|$ and $\hbar|\Omega|$ with respect to the chemical potential μ .) The amplitudes $a_{\alpha,\beta'\sigma}^{\text{in/out}}$ and $b_{\alpha,\beta'\sigma}^{\text{in/out}}$ are flux normalized, i.e., they have dimension (time) $^{-1/2}$.

The total current through the junction is

$$I = I_R = -I_L, \quad (24)$$

with

$$I_{\beta'} = -\beta' e \sum_{\alpha} \sum_{\sigma'} [(|b_{\alpha,\beta'\sigma'}^{\text{in}}|^2 - |b_{\alpha,\beta'\sigma'}^{\text{out}}|^2)(1 - \tilde{n}_{\alpha}) + (|a_{\alpha,\beta'\sigma'}^{\text{in}}|^2 - |a_{\alpha,\beta'\sigma'}^{\text{out}}|^2)\tilde{n}_{\alpha}], \quad (25)$$

where we recall that in the continuous part of the spectrum $\alpha = (\tilde{E}, \beta, \tau, \sigma)$ is a composite index, so that the symbol $\Sigma_{\alpha} \dots$ should be understood as $\sum_{\beta,\sigma,\tau} \int d\tilde{E} \dots$. For the discrete part of the spectrum, one may make use of the identity [62]

$$\begin{aligned} \frac{\partial \tilde{E}_{\alpha}}{\partial \phi} &= -\beta' \hbar \sum_{\sigma'} [(|b_{\alpha,\beta'\sigma'}^{\text{in}}|^2 - |b_{\alpha,\beta'\sigma'}^{\text{out}}|^2)] \\ &= \beta' \hbar \sum_{\sigma'} [(|a_{\alpha,\beta'\sigma'}^{\text{in}}|^2 - |a_{\alpha,\beta'\sigma'}^{\text{out}}|^2)] \end{aligned} \quad (26)$$

to simplify the calculation of the current.

In the scattering approach, the solution of the Bogoliubov–Gennes equation (13) is written in terms of matrix equations relating the coefficients in Eq. (23). For the central

F part, the electron (u) components are related as

$$\begin{pmatrix} a_{\alpha;L\uparrow}^{\text{out}} \\ a_{\alpha;L\downarrow}^{\text{out}} \\ a_{\alpha;R\uparrow}^{\text{out}} \\ a_{\alpha;R\downarrow}^{\text{out}} \end{pmatrix} = S_F(\tilde{E}) \begin{pmatrix} a_{\alpha;L\uparrow}^{\text{in}} \\ a_{\alpha;L\downarrow}^{\text{in}} \\ a_{\alpha;R\uparrow}^{\text{in}} \\ a_{\alpha;R\downarrow}^{\text{in}} \end{pmatrix}, \quad (27)$$

where the scattering matrix $S_F(\tilde{E})$ of the central F region is of the form

$$S_F(\tilde{E}) = \sum_{\pm} \begin{pmatrix} r_{\pm}(\tilde{E}) & t'_{\pm}(\tilde{E}) \\ t_{\pm}(\tilde{E}) & r'_{\pm}(\tilde{E}) \end{pmatrix} \otimes P_{\pm}, \quad (28)$$

with

$$P_{\pm} = \frac{1}{2}(\sigma_0 \pm \tilde{m}_{\text{eff}} \cdot \sigma) \quad (29)$$

the projector on spin parallel (+) or antiparallel (−) to $\tilde{m}_{\text{eff}} = \vec{h}_{\text{eff}}/|\vec{h}_{\text{eff}}|$, which is the direction of the effective exchange field in F, see Eq. (10). For the Hamiltonian (5), the reflection and transmission amplitudes read

$$r_{\pm}(\tilde{E}) = r'_{\pm}(\tilde{E}) = \frac{(k^2 - k_{\pm}^2) \sin k_{\pm} L}{(k^2 + k_{\pm}^2) \sin k_{\pm} L + 2ikk_{\pm} \cos k_{\pm} L}, \quad (30)$$

$$t_{\pm}(\tilde{E}) = t'_{\pm}(\tilde{E}) = \frac{2ikk_{\pm}}{(k^2 + k_{\pm}^2) \sin k_{\pm} L + 2ikk_{\pm} \cos k_{\pm} L}, \quad (31)$$

where $\hbar^2 k^2 / 2m = \mu + \tilde{E}$ and $\hbar^2 k_{\pm}^2 / 2m = \mu + \tilde{E} - V_F \pm |\hbar_{\text{eff}}|$. For the hole (v) components, the wave function amplitudes are related by the complex conjugate of the scattering matrix,

$$\begin{pmatrix} b_{\alpha;L\uparrow}^{\text{out}} \\ b_{\alpha;L\downarrow}^{\text{out}} \\ b_{\alpha;R\uparrow}^{\text{out}} \\ b_{\alpha;R\downarrow}^{\text{out}} \end{pmatrix} = S_F(-\tilde{E})^* \begin{pmatrix} b_{\alpha;L\uparrow}^{\text{in}} \\ b_{\alpha;L\downarrow}^{\text{in}} \\ b_{\alpha;R\uparrow}^{\text{in}} \\ b_{\alpha;R\downarrow}^{\text{in}} \end{pmatrix}. \quad (32)$$

For the scattering relations at the interfaces with the superconductors, we discuss the discrete and continuous parts of the spectrum separately. For the discrete part of the spectrum, $0 \leq \tilde{E} < |\Delta| - (1/2)\hbar|\Omega|$, one has

$$\begin{aligned} a_{\alpha,\beta'\sigma'}^{\text{in}} &= \sigma' e^{-i\eta_{\sigma'} + i\beta\phi/2} b_{\alpha,\beta'\sigma'}^{\text{out}}, \\ b_{\alpha,\beta'\sigma'}^{\text{in}} &= \sigma' e^{-i\eta_{\sigma'} - i\beta\phi/2} a_{\alpha,\beta'\sigma'}^{\text{out}}, \end{aligned} \quad (33)$$

where

$$\eta_{\sigma} = \arccos \frac{\tilde{E} + \frac{1}{2}\hbar\sigma\Omega}{|\Delta|} \quad (34)$$

generalizes Eq. (18) to arguments $|\tilde{E} + (1/2)\sigma\hbar\Omega| < |\Delta|$. The combination of Eqs. (27) and (33) allows nontrivial solutions for specific values of the Andreev bound state energy $\tilde{E}_{\alpha}$ only. To find the Andreev bound state wave function, we also impose the normalization condition (14). In the continuous part of the spectrum, Eqs. (33) are replaced by

$$\begin{aligned} a_{\alpha,\beta'\sigma'}^{\text{in}} &= \sigma' e^{-i\eta_{\sigma'} + i\beta\phi/2} b_{\alpha,\beta'\sigma'}^{\text{out}} + s_{\beta\tau\sigma} \delta_{\beta\beta'} \delta_{\tau\sigma} \delta_{\sigma\sigma'}, \\ b_{\alpha,\beta'\sigma'}^{\text{in}} &= \sigma' e^{-i\eta_{\sigma'} - i\beta\phi/2} a_{\alpha,\beta'\sigma'}^{\text{out}} + s_{\beta\tau\sigma} \delta_{\beta\beta'} \delta_{\tau\sigma} \delta_{\sigma\sigma'}, \end{aligned} \quad (35)$$

where

$$s_{\beta\tau\sigma} = e^{\frac{i}{2}\beta\tau\phi} \sqrt{1 - |e^{-i\eta\sigma}|^2}. \quad (36)$$

Equations (27) and (35) have a unique nontrivial solution for all $\tilde{E}$ in the continuous part of the spectrum. The choice of the source terms in Eqs. (35) ensures that this solution is normalized to unit incoming quasiparticle flux in the left or right superconducting lead.

III. N/HM/S AND S/HM/S JUNCTIONS

We begin by considering one-dimensional junctions with a fully polarized half-metallic magnet (HM) with precessing magnetization. In a half metal there are charge carriers of only one spin direction. This is the most illustrative scenario for the mechanism responsible for the generation of spin-polarized superconductivity, since no subgap current can flow through the junction without precessing magnetization [42] (except for a residual contribution associated with the emission of magnons upon Andreev reflection at the HM/S interface [67]).

To simplify the expressions for the scattering matrix S_F of Eq. (28), we set $|h_{\text{eff}}| = V_F \gg \mu$ and we make the Andreev approximation $|\Delta| \ll \mu$ so that

$$\begin{aligned} r_+(\tilde{E}) &= 0, & t_+(\tilde{E}) &= e^{ik_0L + i\tilde{E}L/\hbar v_0}, \\ r_-(\tilde{E}) &= -1, & t_-(\tilde{E}) &= 0, \end{aligned} \quad (37)$$

with $k_0 = \sqrt{2m\mu/\hbar^2}$ and $v_0 = \hbar k_0/m$.

A. N/HM/S junction

We first consider a junction with one normal-metal and one superconducting contact [see Fig. 2(a)]. To find the conductance of the junction and to characterize the induced superconductivity, we calculate the amplitudes $r_{\text{hs},\text{es}'}(\tilde{E}) = r_{\text{es},\text{hs}'}(-\tilde{E})^*$ and $r_{\text{es},\text{es}'}(\tilde{E}) = r_{\text{hs},\text{hs}'}(-\tilde{E})^*$ for Andreev reflection and normal reflection in the rotating frame, respectively. A simple calculation using Eqs. (27)–(35) gives the equal-spin amplitudes

$$\begin{aligned} r_{\text{h}+, \text{e}+}(\tilde{E}) &= -2ie^{2i\tilde{E}L/\hbar v_0 - i\phi} \frac{\sin \eta' \cos \bar{\eta} \sin \theta}{e^{2i\bar{\eta}} + \sin^2 \eta' \sin^2 \theta}, \\ r_{\text{h}-, \text{e}-}(\tilde{E}) &= 0, \\ r_{\text{e}+, \text{e}+}(\tilde{E}) &= e^{2i(k_0 + \frac{\tilde{E}}{\hbar v_0})L} \frac{(\sin \eta' \cos \theta + i \cos \eta')^2}{e^{2i\bar{\eta}} + \sin^2 \eta' \sin^2 \theta}, \\ r_{\text{e}-, \text{e}-}(\tilde{E}) &= -1, \end{aligned} \quad (38)$$

where the labels $\pm$ refer to the spin projection onto the magnetization direction $\tilde{m}_{\text{eff}}$ and we abbreviate

$$\bar{\eta} = \frac{\eta_+ + \eta_-}{2}, \quad \eta' = \frac{\eta_+ - \eta_-}{2}. \quad (39)$$

The opposite-spin amplitudes all vanish, $r_{\tau+, \tau'+}(\tilde{E}) = r_{\tau+, \tau'-}(\tilde{E}) = 0$, $\tau, \tau' \in \{e, h\}$. Scattering amplitudes for spin labels $\sigma, \sigma' = \uparrow, \downarrow$ defined with respect to the z axis are [compare with Eq. (28)]

$$\begin{pmatrix} r_{\tau\uparrow, \tau'\uparrow}(\tilde{E}) & r_{\tau\uparrow, \tau'\downarrow}(\tilde{E}) \\ r_{\tau\downarrow, \tau'\uparrow}(\tilde{E}) & r_{\tau\downarrow, \tau'\downarrow}(\tilde{E}) \end{pmatrix} = \sum_{s, s' = \pm} P_s r_{\tau s, \tau' s'}(\tilde{E}) P_{s'}, \quad (40)$$

with the projector $P_{\pm}$ given by Eq. (29) and $\tau, \tau' \in \{e, h\}$. For $\tilde{E} = 0$ one has $\cos \bar{\eta} = 0$, so that the equal-spin Andreev reflection amplitude $r_{\text{h}+, \text{e}+}(0) = r_{\text{e}+, \text{h}+}(0) = 0$, consistent with the general principle that Andreev reflection must be absent in topologically trivial half-metal–superconductor junctions [68].

The conductance $G(V)$ of the junction at bias voltage V is [69,70],

$$\begin{aligned} G(V) &= \frac{e^2}{h} \sum_{\sigma, \sigma' = \uparrow, \downarrow} \int_{-\infty}^{\infty} d\tilde{E} \left(-\frac{df_{\sigma'}(\tilde{E} - eV)}{d\tilde{E}} \right) \\ &\quad \times (\delta_{\sigma\sigma'} - |r_{\text{es}, \text{es}'}(\tilde{E})|^2 + |r_{\text{hs}, \text{es}'}(\tilde{E})|^2), \end{aligned} \quad (41)$$

where the spin labels σ and σ' refer to the spin projection onto the z axis. Substituting Eq. (38), taking the limit $V \rightarrow 0$, $T \rightarrow 0$, where T is temperature, and expanding to leading order in the precession frequency Ω , we find that the conductance G of the half metal–superconductor junction is

$$G(V) = \frac{e^2}{h} \frac{2(\hbar\Omega)^2}{|\Delta|^2} \sin^2 \theta \left[(eV)^2 + (\hbar\Omega)^2 + \frac{\pi^2}{3} (k_B T)^2 \right] \quad (42)$$

in the limit of small V , T , and Ω . Note that the zero-bias, zero-temperature conductance involves Andreev reflection coefficients at rotating-frame energy $\tilde{E} = -\frac{1}{2}\hbar\Omega$, which is why $G(0)$ is nonzero at $T = 0$ despite the absence of Andreev reflection for $\tilde{E} = 0$.

The Andreev reflection amplitudes also determine the anomalous Green function [71]. In the rotating frame and in the spin-majority/minority basis ($s, s' = \pm$), the retarded anomalous Green function is

$$\tilde{F}_{ss'}^R(x, t; x', t') = -i\Theta(t - t') \langle \{ \hat{\psi}_s(x, t), \hat{\psi}_{s'}(x', t') \} \rangle. \quad (43)$$

Its Fourier transform $\tilde{F}^R(x, x'; \omega)$ is the electron–hole block of the matrix Green function,

$$\tilde{\mathcal{G}}(\omega + i0) = \begin{pmatrix} \tilde{G}^R(x, x'; \omega) & \tilde{F}^R(x, x'; \omega) \\ -\tilde{F}^R(x, x'; -\omega)^* & -\tilde{G}^R(x, x'; -\omega)^* \end{pmatrix}, \quad (44)$$

which satisfies the Gor'kov equation [72,73]

$$(\hbar\omega - \tilde{\mathcal{H}}_{\text{BdG}}) \tilde{\mathcal{G}}(x, x'; \omega) = \mathbb{1} \delta(x - x'), \quad (45)$$

where $\mathbb{1}$ is the identity matrix in spin and particle-hole space. We focus on the induced superconductivity inside the normal region, i.e., on $x, x' < -L/2$. According to the Gor'kov equation (45), the anomalous Green function (43) can then be calculated as the electron component with spin s at position x of the solution of the Bogoliubov–de Gennes equation with a source that injects a hole with spin s' at position x' . This process requires Andreev reflection, so that we find [45,71]

$$\tilde{F}_{ss'}^R(x, x'; \omega) = -\frac{i}{\hbar v_0} e^{-ik_0(x-x')} r_{\text{es}, \text{hs}'}(\hbar\omega). \quad (46)$$

The advanced component of the anomalous Green function can be calculated from particle–hole symmetry

$$\tilde{F}_{ss'}^A(x, x'; \omega) = -\tilde{F}_{s's}^R(x', x; -\omega), \quad (47)$$

and yields

$$\tilde{F}_{ss'}^A(x, x'; \omega) = \frac{i}{\hbar v_0} e^{ik_0(x-x')} r_{\text{es}', \text{hs}}(-\hbar\omega). \quad (48)$$

The expression for the anomalous Green function can be cast in the form

$$\tilde{f}_{ss'}(x, x'; \omega) = -\frac{i}{2} \int dk e^{-ik(x-x')} \tilde{f}_{ss'}(k; \omega) \delta(\varepsilon_k - \omega), \quad (49)$$

where $\varepsilon_k = \frac{\hbar^2}{2m}(k^2 - k_0^2)$ and $\tilde{f}(k; \omega)$ is similar to the anomalous Green function in the quasiclassical theory [71,74],

$$\tilde{f}_{ss'}(k; \omega) = 2 \times \begin{cases} r_{es,hs'}(\omega), & k < 0, \text{Im } \omega > 0, \\ -r_{es,hs}(-\omega), & k > 0, \text{Im } \omega < 0, \\ 0, & \text{else} \end{cases} \quad (50)$$

Note that $f_{ss'}(k, 0) = 0$, because the Andreev reflection amplitudes vanish at $\tilde{E} = 0$. Equation (50) combines the results for both the retarded and advanced Green functions.

For the half metal–superconductor junction, only the anomalous Green function component $\tilde{f}_{++}(k; \omega)$ for $s = s' = +$ is nonzero. This component is even under exchange of the spin labels. To bring out the symmetry of the proximity superconductivity with respect to the momentum k and the frequency ω , we write $\tilde{f}_{ss'}(k; \omega)$ as a sum of a k -odd, ω -even and an ω -odd, k -even component,

$$\tilde{f}_{ss'}(k; \omega) = \tilde{f}_{ss'}^{+-}(k; \omega) + \tilde{f}_{ss'}^{-+}(\omega), \quad (51)$$

with

$$\begin{aligned} \tilde{f}_{ss'}^{+-}(k, \omega) &= \delta_{s,s'} \delta_{s,+} \text{sign}(\text{Im } \omega) r_{e+,h+}(\omega \text{sign}(\text{Im } \omega)), \\ \tilde{f}_{ss'}^{-+}(k, \omega) &= \delta_{s,s'} \delta_{s,+} \text{sign}(k) r_{e+,h+}(\omega \text{sign}(\text{Im } \omega)). \end{aligned} \quad (52)$$

The k -even, ω -odd component is generally robust to disorder scattering and, hence, is responsible for the long-range proximity effect [24].

B. S/HM/S junction

For a junction with two superconducting contacts, we first discuss the bound-state spectrum and its contribution to the Josephson current and include the continuous part of the spectrum afterwards. To further simplify the calculation of the Josephson current, we focus on the short-junction limit $L \ll \xi$, where $\xi = \hbar v_0 / |\Delta|$ is the superconductor coherence length.

For $\theta = 0$, one finds electron-like and hole-like bound states at the energies given by the solution(s) of the conditions $e^{2ik_0L - 4i\eta_+} = 1$ and $e^{-2ik_0L - 4i\eta_-} = 1$, respectively. For more detail, we refer to Appendix A. Defining $0 \leq \zeta < \pi/2$ such that $e^{4i\zeta} = e^{2ik_0L}$, i.e.,

$$\zeta = \frac{1}{2}(k_0L \bmod \pi), \quad (53)$$

in the short-junction limit $L \ll \xi$ we then find two electron-like bound states and the two hole-like bound states at energies

$$\begin{aligned} \tilde{E}_{e1} &= -\tilde{E}_{h1} = |\Delta| \cos \zeta - \frac{1}{2} \hbar \Omega, \\ \tilde{E}_{e2} &= -\tilde{E}_{h2} = -|\Delta| \sin \zeta - \frac{1}{2} \hbar \Omega. \end{aligned} \quad (54)$$

Figure 4 shows the four bound-state energies $\tilde{E}_{e1}$, $\tilde{E}_{e2}$, $\tilde{E}_{h1}$, and $\tilde{E}_{h2}$ as a function of k_0L for three representative values of the precession frequency Ω . The majority wave function amplitudes of these bound states are

$$a_{R\uparrow j}^{\text{in}} = -e^{-2i\eta_+} a_{R\uparrow j}^{\text{out}} = \frac{1}{\sqrt{\mathcal{N}_+}}, \quad b_{R\uparrow j}^{\text{out}} = b_{R\uparrow j}^{\text{in}} = 0, \quad (55)$$

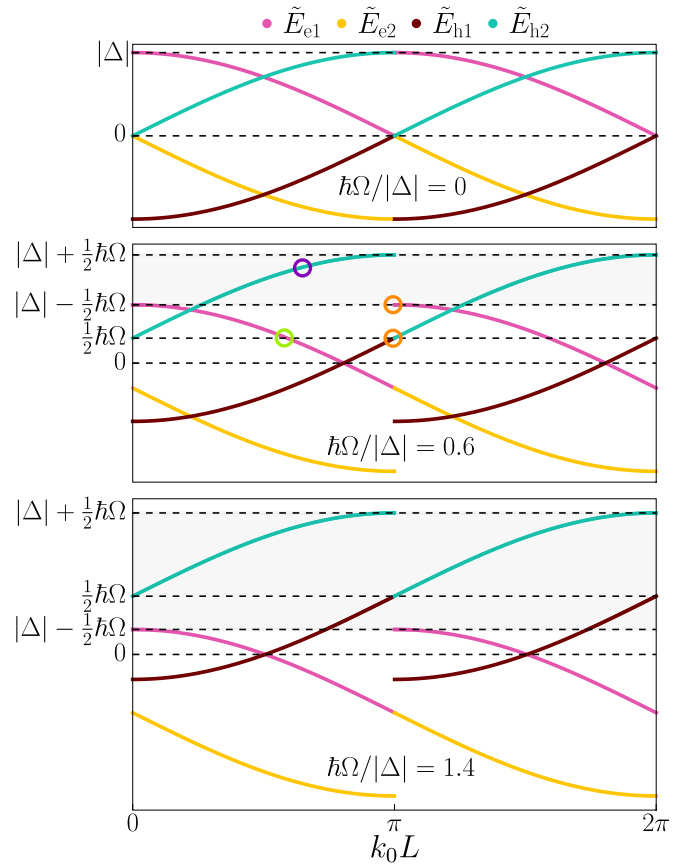


FIG. 4. Bound-state energies $\tilde{E}_{ej}$ and $\tilde{E}_{hj}$, $j = 1, 2$, of a static, i.e., for $\theta = 0$, S/HM/S junction in the short-junction limit $L \ll \xi$ as a function of k_0L for $\hbar\Omega/|\Delta| = 0$ (top), 0.6 (middle), and 1.4 (bottom) in the rotating frame. Bound states with energy $\tilde{E}_\alpha > |\Delta| - \hbar|\Omega|/2$ hybridize with the continuum of scattering states (shaded region). The circles in the middle panel, corresponding to $\hbar\Omega = 0.6|\Delta|$, indicate (i) the example for $k_0L = 2.0$ of a typical point where one of the bound states hybridizes with the continuum spectrum (purple circle), (ii) an abrupt change in the occupation of a bound state (green and bottom orange circles) upon varying k_0L , and (iii) merging of a bound state with the continuum spectrum (top orange circle). For the energy $\tilde{E}$ corresponding to (i) the energy-resolved current $j_\uparrow(\tilde{E})$ exhibits a sharp peak, see Fig. 5. Case (ii) leads to a jump in the current as a function of system parameters, as illustrated in the right panel of Fig. 6.

$j = 1, 2$, for the electron-like bound states, where the phases η_σ should be evaluated at the respective energies $\tilde{E}_{ej}$ and the normalization constant

$$\mathcal{N}_\sigma = \frac{2\hbar}{|\Delta| \sin \eta_\sigma}. \quad (56)$$

The minority wave function amplitudes follow from Eq. (33). The majority and minority wave function amplitudes of the hole-like bound states follow by particle-hole conjugation. The $\theta = 0$ bound states exist for arbitrarily short junction length L and have their support entirely in the superconducting regions in the short-junction limit $L \ll \xi$. In this sense they are like standard Andreev bound states, such as they occur in a superconductor–normal metal–superconductor (S/NM/S) junction. However, their energies do not depend on the phase

difference ϕ and, hence, the bound states at $\theta = 0$ do not carry a supercurrent.

Electron-like bound states exist for $-\lvert\Delta\rvert - \hbar\Omega/2 < \lvert\Delta\rvert - \hbar\Omega/2$; hole-like bound states exist for $-\lvert\Delta\rvert + \hbar\Omega/2 < \lvert\Delta\rvert + \hbar\Omega/2$. For $\lvert\tilde{E}\rvert < \lvert\Delta\rvert - \hbar\Omega/2$ bound states of both types exist. In this energy range, the electron-like bound states acquire a small hole-like component and vice versa for small but nonzero tilt angle θ , whereas the bound state energy $\tilde{E}$ now depends on the phase difference ϕ . The hole-like component of the electron-like bound state already appears to linear order in θ ,

$$\begin{aligned} b_{R\uparrow j}^{\text{out}} &= -\theta B_j (e^{-i\eta_-} - e^{i(2\zeta + \eta_- - \phi + \pi j)}), \\ b_{R\uparrow j}^{\text{in}} &= \theta B_j (e^{i\eta_-} - e^{-i(2\zeta + \eta_- + \phi + \pi j)}), \end{aligned} \quad (57)$$

where

$$B_j = -\frac{i}{\sqrt{N_+}} \frac{\sin \eta' \cos \bar{\eta}}{\sin(2(\zeta + \eta_-))} e^{\frac{i}{2}(2\zeta - \pi j)}, \quad j = 1, 2 \quad (58)$$

and we omitted corrections beyond linear order in θ . [The electron-like wave functions are still given by Eq. (55).] We refer to Appendix A for a more detailed derivation of the wave functions presented above.

The electron-like components of the hole-like bound states again follow by particle-hole conjugation. The ϕ -dependent contribution to the energies $\delta\tilde{E}_{ej} = -\delta\tilde{E}_{hj}$, $j = 1, 2$, is quadratic in θ ,

$$\frac{\partial\delta\tilde{E}_{ej}}{\partial\phi} = 2\theta^2(-1)^{j-1}\lvert\Delta\rvert \frac{\sin \eta_+(\sin \eta' \cos \bar{\eta})^2}{\sin(2(\zeta + \eta_-))} \sin \phi. \quad (59)$$

In Eqs. (57)–(59), the phases η_σ should be evaluated at $\tilde{E}_{e1}$ and $\tilde{E}_{e2}$ for $j = 1$ and $j = 2$, respectively. Note that the factor $\cos^2 \bar{\eta}$ implies that $\partial\delta\tilde{E}_{ej}/\partial\phi \propto \Omega^2$ for small precession frequency Ω .

Since the Andreev bound states are fully spin-polarized inside the central part of the junction and have electron-like and hole-like parts of equal weight, one has

$$\rho_{\alpha\uparrow} = \rho_{\alpha\downarrow}, \quad (60)$$

see Eq. (21). (The label α takes the values e1, e2, h1, and h2.) The occupation of the bound state α then follows from Eq. (22),

$$n_\alpha = \frac{1}{2}[f_\uparrow(\tilde{E}_\alpha) + f_\downarrow(\tilde{E}_\alpha)]. \quad (61)$$

At zero temperature $n_\alpha = 1/2$ for $\lvert E_\alpha \rvert < \hbar\Omega/2$ and the contribution of the bound state α to the current through the junction vanishes, see Eqs. (25) and (26). Since bound states with $\tilde{E}_\alpha > \lvert\Delta\rvert - \hbar\Omega/2$ hybridize with the continuum spectrum, as we discuss below, only bound states in the energy range $\hbar\Omega/2 < \tilde{E}_\alpha < \lvert\Delta\rvert - \hbar\Omega/2$ contribute to the ground-state current with the occupation (61).

For $\lvert\Delta\rvert - \hbar\Omega/2 < \tilde{E} < \lvert\Delta\rvert + \hbar\Omega/2$, bound states coexist with scattering states if the tilt angle $\theta = 0$. At finite tilt angle θ the bound states and scattering states hybridize into scattering resonances, so that for nonzero θ it is sufficient to consider the scattering states only. For $\tilde{E} > \lvert\Delta\rvert + \hbar\Omega/2$ there are scattering states of both spin directions σ , but no bound states and, hence, no scattering resonances.

Combining contributions from the discrete and continuous parts of the spectrum, the Josephson current I through a

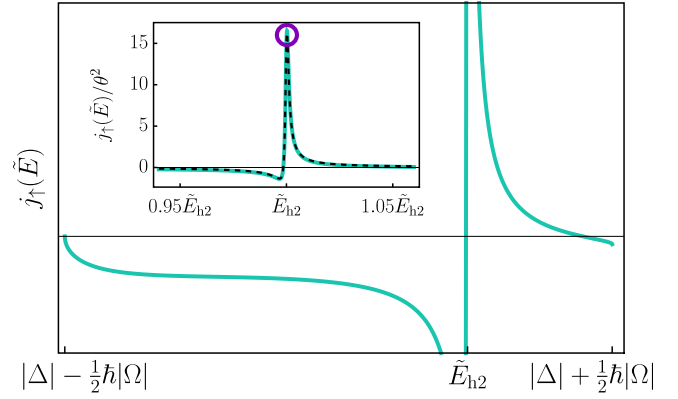


FIG. 5. Probability current $j_\uparrow(\tilde{E})$ carried by the continuum part of the spectrum in the S/HM/S junction ($L \ll \xi$) as a function of energy $\tilde{E} > \lvert\Delta\rvert - \frac{1}{2}\hbar\lvert\Omega\rvert$, for $\hbar\Omega = 0.6\lvert\Delta\rvert$, $\theta = 0.1$, $k_0L = 2.0$, and phase bias $\phi = \pi/2$. The current $j_\uparrow(\tilde{E})$ exhibits a sharp peak, with height $\propto \theta^{-2}$ and width $\propto \theta^2$, near the bound-state energy $\tilde{E}_{h2}$ (purple circle; see Fig. 4). (Inset) A magnified view near the resonance at $\tilde{E}_{h2}$, together with the line shape (63) (dashed).

S/HM/S junction is expressed as

$$\begin{aligned} I &= -\frac{e}{\hbar} \sum_{0 \leq \tilde{E}_\alpha < \lvert\Delta\rvert - \frac{1}{2}\hbar\lvert\Omega\rvert} \frac{\partial\tilde{E}_\alpha}{\partial\phi} (1 - 2n_\alpha) \\ &+ \frac{e}{\hbar} \sum_{\sigma} \int_{\lvert\Delta\rvert - \frac{1}{2}\hbar\Omega}^{\infty} d\tilde{E} j_\sigma(\tilde{E}) (1 - 2f_\sigma(\tilde{E})), \end{aligned} \quad (62)$$

with n_α given by Eq. (61).

For small tilt angle θ , $j_\sigma(\tilde{E})$ has sharp resonances if $\sigma = \text{sign } \Omega$ and the energy $\lvert\Delta\rvert - \frac{1}{2}\hbar\lvert\Omega\rvert < \tilde{E} < \lvert\Delta\rvert + \frac{1}{2}\hbar\lvert\Omega\rvert$ is close to one of the bound state energies $\tilde{E}_\alpha$ at $\theta = 0$. In that case, for small θ one has

$$j_\sigma(\tilde{E}) = \theta^2 J_\alpha \frac{\tilde{E} - \tilde{E}_\alpha + \theta^2 \delta_\alpha}{\lvert\tilde{E} - \tilde{E}_\alpha + \theta^2 \Gamma_\alpha\rvert^2}, \quad (63)$$

where J_α , δ_α , and Γ_α are resonance-specific constants with the dimension of energy. (J_α and δ_α are real; Γ_α is complex.) We refer to Appendix B for explicit expressions. As an illustration, in Fig. 5 below we show $j_\uparrow(\tilde{E})$ in the short-junction limit for $\hbar\Omega/\lvert\Delta\rvert = 0.6$ and $k_0L = 2.0$, in which case it has a resonance near the bound state energy $\tilde{E}_{h2}$. (See Fig. 4 for the bound-state spectrum at $\theta = 0$.) We note that Eq. (63) is valid only in the immediate vicinity of the resonance, i.e., when $\lvert\tilde{E} - \tilde{E}_\alpha\rvert \sim \theta^2 \lvert\text{Im } \Gamma_\alpha\rvert$, and requires the resonance width $\theta^2 \lvert\text{Im } \Gamma_\alpha\rvert$ to be much smaller than the other relevant energy scales, such as $\hbar\lvert\Omega\rvert$ or the distance between the resonance position $\tilde{E}_\alpha$ and the continuum edge at $\lvert\Delta\rvert \pm \hbar\lvert\Omega\rvert/2$. When the opposite is true, the full scattering expression should be used instead of the isolated-resonance approximation.

For a small tilt angle θ , the Josephson current we obtain from Eq. (62) is proportional to $\theta^2 \sin \phi$. The ϕ dependence of the Josephson current is no longer sinusoidal for larger θ , as we illustrate in Fig. 6 (left) for a junction in the short-junction limit. A nonsinusoidal current-phase relationship was also observed in Ref. [55] for a Josephson junction coupled to a precessing spin. The jumps in the current-phase relation for large θ originate from an abrupt change in the steady-state

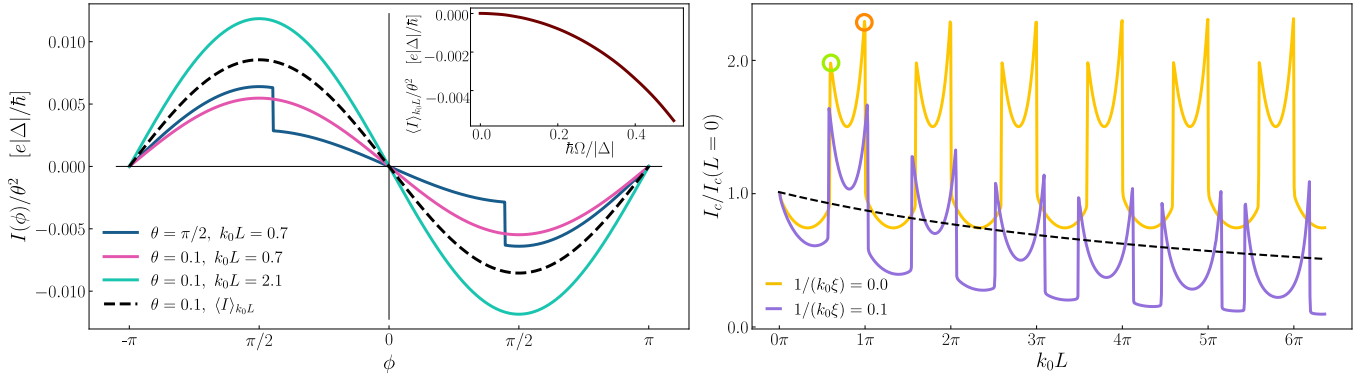


FIG. 6. Supercurrent of a one-dimensional S/HM/S Josephson junction at zero temperature. (Left) Current-phase relation in the short-junction limit for different tilt angles and junction lengths, for $\hbar\Omega = 0.6|\Delta|$. The turquoise and pink curves illustrate the current-phase relation in the limit of small tilt angles for two representative values of k_0L ; the dashed curve shows that the average $\langle I \rangle_{k_0L}$ of $I(\phi)$ over k_0L at fixed ϕ is well defined and nonzero. The dark blue curve illustrates the large tilt-angle case $\theta = \pi/2$, where a strong deviation from the sinusoidal behavior is observed. Inset: dependence of $\langle I \rangle_{k_0L}$ on the precession frequency Ω at phase bias $\phi = \pi/2$ and $\theta = 0.1$ in the short-junction limit. For small values of $\hbar\Omega/|\Delta|$, the Josephson current has a quadratic dependence on the tilt angle θ and it vanishes at $\Omega = 0$. (Right) critical current $I_c = \max_\phi I(\phi)$ as a function of k_0L for $\hbar\Omega = 0.6|\Delta|$ and $\theta = 0.1$, shown for the short-junction case $1/(k_0\xi) = 0$ (yellow curve) and in the crossover toward the long-junction limit, $1/(k_0\xi) = 0.1$ (purple curve). The two circles on the yellow curve mark the same points as in Fig. 4. They indicate the points where the bound states $\tilde{E}_{e1}$ and $\tilde{E}_{h2}$, respectively, undergo an abrupt change in their occupation; see Eq. (61). Similar behavior is also seen for $1/(k_0\xi) = 0.1$, although the positions of the jumps are now shifted. In addition, the current is no longer periodic in k_0L and instead shows an overall decaying trend, which is similar to the decay of the supercurrent with L in a superconductor–normal-metal–superconductor junction (dashed line).

population of an Andreev bound state once its energy crosses the energy splitting $\frac{1}{2}\hbar|\Omega|$ [see Eqs. (22) and (61)]. In our analysis, the steady state is established independently for each fixed set of system parameters, such as the superconducting phase difference ϕ or the junction length L . Therefore, at zero temperature the occupation, and consequently the Josephson current, may change discontinuously as a function of these parameters [75].

The left panel of Fig. 6 also shows the average $\langle I(\phi) \rangle_{k_0L}$ of $I(\phi)$ over k_0L for the short-junction limit, where $I(\phi)$ is a periodic function of k_0L . The fact that the average $\langle I(\phi) \rangle_{k_0L}$ is nonzero is an indicator for the long range of the proximity effect in a superconductor–half-metal–superconductor junction with a precessing magnetization.

In the right panel of Fig. 6, we show the critical current $I_c = \max_\phi I(\phi)$ as a function of k_0L , both in the short-junction limit $L \ll \xi$ and in the crossover towards the long-junction limit. The sharp discontinuities as a function of k_0L are the result of the abrupt change of population of the Andreev bound states with increasing k_0L at zero temperature, see Fig. 4. The contribution to the Josephson current from the continuous part of the spectrum, which is the second term in Eq. (62), is a continuous function of k_0L . Abrupt changes of the Josephson current also appear for the opposite-spin contribution to the Josephson current in junctions with a nonprecessing ferromagnet, see Ref. [76].

Our numerical results indicate that the length-averaged current $\langle I \rangle_{k_0L}$ depends quadratically on the precession frequency Ω ,

$$\langle I \rangle_{k_0L} \propto \left(\frac{\hbar\Omega}{|\Delta|} \right)^2 \theta^2 \sin \phi, \quad (64)$$

see Fig. 6. As will be discussed in Sec. V, the quadratic Ω dependence of the average $\langle I \rangle_{k_0L}$ implies a quadratic Ω

dependence of the Josephson current I at fixed k_0L in two- and three-dimensional junctions. For a one-dimensional junction, we find that the (nonaveraged) Josephson current I at a fixed value of k_0L is proportional to Ω^2 for most values of k_0L , but not for all k_0L . An explicit example in which $I(\Omega)$ is not proportional to Ω^2 is shown in Appendix C.

IV. N/F/S AND S/F/S JUNCTIONS

In this section, we consider a Josephson junction with a partially polarized ferromagnetic metal F and a precessing magnetization. Since a partially polarized magnet admits an opposite-spin proximity effect, such a junction already has a Josephson current without precessing magnetization. In a one-dimensional junction, this opposite-spin contribution to the supercurrent strongly oscillates as a function of the junction length L . A precessing magnetization induces spin-polarized proximity superconductivity, which gives a nonoscillating contribution to the Josephson current, which is proportional to θ^2 for small tilt angle θ .

To keep the expressions simple, we take the limit $|\Delta| \ll |h_F| \ll \mu$. In this limit, the reflection and transmission matrices of the ferromagnetic metal are

$$r_{\pm}(\tilde{E}) = 0, \quad t_{\pm}(\tilde{E}) = e^{i(k_0 \pm \kappa)L + i\tilde{E}L/\hbar v_0}, \quad (65)$$

where we abbreviated

$$\kappa = \frac{h_F}{\hbar v_0}. \quad (66)$$

We again focus on the short-junction limit $L \ll \xi$.

A. N/F/S junction

The amplitudes $r_{hs,es'}(\tilde{E}) = r_{es,hs'}(-\tilde{E})^*$ for Andreev reflection are

$$\begin{aligned} r_{h+,e+}(\tilde{E}) &= e^{-i\phi-i\bar{\eta}} \sin \eta' \sin \theta, \\ r_{h-,e+}(\tilde{E}) &= e^{-i\phi-2i\kappa L-i\bar{\eta}} (\cos \eta' - i \sin \eta' \cos \theta), \\ r_{h+,e-}(\tilde{E}) &= -e^{-i\phi+2i\kappa L-i\bar{\eta}} (\cos \eta' - i \sin \eta' \cos \theta), \\ r_{h-,e-}(\tilde{E}) &= -e^{-i\phi-i\bar{\eta}} \sin \eta' \sin \theta, \end{aligned} \quad (67)$$

where we recall that $\bar{\eta} = (\eta_+ + \eta_-)/2$, $\eta' = (\eta_+ - \eta_-)/2$. The normal reflection amplitudes $r_{es,es'}(\tilde{E}) = r_{hs,hs'}(-\tilde{E})^*$ all vanish for the simple limit of Eq. (65). Note that the opposite-spin Andreev reflection amplitudes $r_{h\mp,e\pm}(\tilde{E})$ oscillate with the length L of the ferromagnetic segment, whereas the equal-spin Andreev reflection amplitudes $r_{h\pm,e\pm}(\tilde{E})$ do not.

We now use the Andreev reflection amplitudes to calculate the conductance and the anomalous Green function. Since there is no normal reflection, the zero-bias, zero-temperature conductance,

$$G(0) = 4 \frac{e^2}{h}, \quad (68)$$

independent of θ and Ω . This is the same conductance as in an ideal normal-metal-superconductor junction. For the anomalous Green function, we again make use of Eqs. (49) and (50). We write the quasiclassical Green function in the rotating frame $\tilde{f}(k; \omega)$ in matrix form,

$$\tilde{f}(k; \omega) = i\sigma_y(\chi(k; \omega) + \mathbf{d}(k; \omega) \cdot \boldsymbol{\sigma}). \quad (69)$$

The singlet component $\chi(k; \omega)$ is

$$\begin{aligned} \chi(k; \omega) &= \frac{1}{2} \sum_{s=\pm} (r_{e+,h-}(s\omega) - r_{e-,h+}(s\omega)) \\ &\quad \times \Theta(-sk) \Theta(s\text{Im } \omega). \end{aligned} \quad (70)$$

It contains k -even, ω -even and k -odd, ω -odd components. The triplet components in the rotating frame read

$$\begin{aligned} d_x(k; \omega) &= - \sum_{s=\pm} s r_{e+,h+}(s\omega) \Theta(-sk) \Theta(s\text{Im } \omega), \\ d_y(k; \omega) &= 0, \\ d_z(k; \omega) &= \frac{1}{2} \sum_{s=\pm} s (r_{e+,h-}(s\omega) + r_{e-,h+}(s\omega)) \\ &\quad \times \Theta(-sk) \Theta(s\text{Im } \omega). \end{aligned} \quad (71)$$

They represent nonpolarized spin-triplet pairing (given by d_z), which vanishes if $h_F = 0$ but remains if $\Omega = 0$ or $\theta = 0$, as expected, as well as spin-polarized components:

$$\tilde{f}_{++}(k; \omega) = -d_x + id_y, \quad \tilde{f}_{--}(k; \omega) = d_x + id_y. \quad (72)$$

Both spin-polarized triplet components vanish without precession. They are sums of k -even, ω -odd and ω -even, k -odd contributions. Since the opposite-spin Andreev reflection amplitudes oscillate with the length L of the ferromagnetic segment, the spin-singlet and unpolarized triplet contributions to the proximity effect are short-ranged. The equal-spin Andreev reflections do not oscillate with L , implying that the spin-polarized proximity superconductivity is long-ranged.

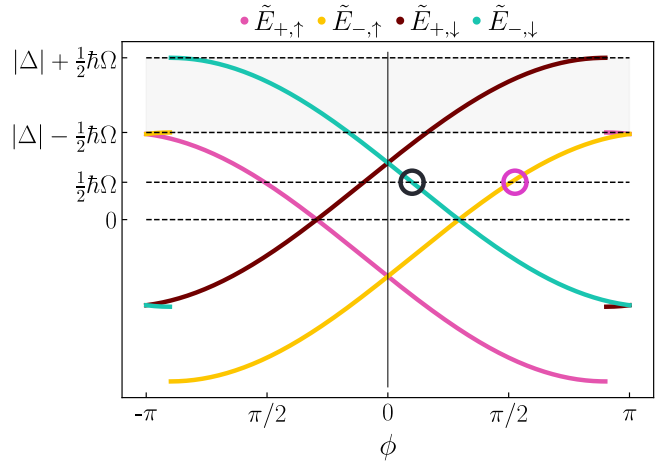


FIG. 7. Dependence of the bound-state energies $\tilde{E}_{\pm, \uparrow}$ and $\tilde{E}_{\pm, \downarrow}$ of a one-dimensional S/F/S Josephson junction on the phase difference ϕ for $\hbar\Omega = 0.6|\Delta|$, $\kappa L = 0.55\pi$, and $\theta = 0$. The black and pink circles mark the points where the occupation $n_{\pm, \sigma}$ of a bound state changes abruptly. Around these points, the current can undergo a sudden change both in the nonprecessing case (around the black circle) and in the regime where features distinguishing nonprecessing and precessing junctions appear (both black and pink circles), see Fig. 8.

B. S/F/S junction

For $\theta = 0$, one finds Andreev bound states at the energy $\tilde{E}_{\pm, \sigma}$ given by the solution of $e^{2i\kappa\sigma L - 2i\eta_{\sigma} \pm i\phi} = 1$, with $\sigma = 1$ or $\sigma = -1$. (The index σ refers to the physical spin of the excitation: The bound state with $\sigma = 1$ has a particle-like part with spin-up and a hole-like part with spin-down; the spins are opposite for the bound state with $\sigma = -1$.) In the short-junction limit, this gives the bound-state energies

$$\begin{aligned} \tilde{E}_{\pm, \uparrow} &= |\Delta| \cos \zeta_{\pm} - \frac{1}{2}\hbar\Omega, \\ \tilde{E}_{\pm, \downarrow} &= -|\Delta| \cos \zeta_{\pm} + \frac{1}{2}\hbar\Omega, \end{aligned} \quad (73)$$

where $0 \leq \zeta_{\pm} < \pi$ is defined such that $e^{2i\zeta_{\pm}} = e^{2i\kappa L \pm i\phi}$, i.e.,

$$\zeta_{\pm} = \kappa L \pm \frac{1}{2}\phi \mod \pi. \quad (74)$$

The Andreev levels $E_{\pm, \sigma}$ are shown in Fig. 7 for a representative value of κL . The occupation of the bound states is

$$n_{\pm, \sigma} = f_{\sigma}(\tilde{E}_{\pm, \sigma}). \quad (75)$$

At zero temperature, the distribution changes sharply at $\tilde{E} = \frac{1}{2}\hbar|\Omega|$ (see Fig. 7).

Since the bound-state energies $\tilde{E}_{\pm, \sigma}$ in a partially polarized ferromagnet depend on the phase difference ϕ already in the limit $\theta = 0$, the Andreev bound states carry a current,

$$\begin{aligned} j_{\pm, \sigma} &= -\frac{e}{\hbar} \frac{\partial \tilde{E}_{\pm, \sigma}}{\partial \phi} (1 - 2n_{\pm, \sigma}) \\ &= \pm \frac{e\sigma}{2\hbar} |\Delta| \sin \zeta_{\pm} (1 - 2f_{\sigma}(\tilde{E}_{\pm, \sigma})). \end{aligned} \quad (76)$$

Only contributions from levels $\tilde{E}_{\pm, \sigma} \geq 0$ are taken into account when calculating the supercurrent. In addition to the contribution from the discrete Andreev bound states, there is a

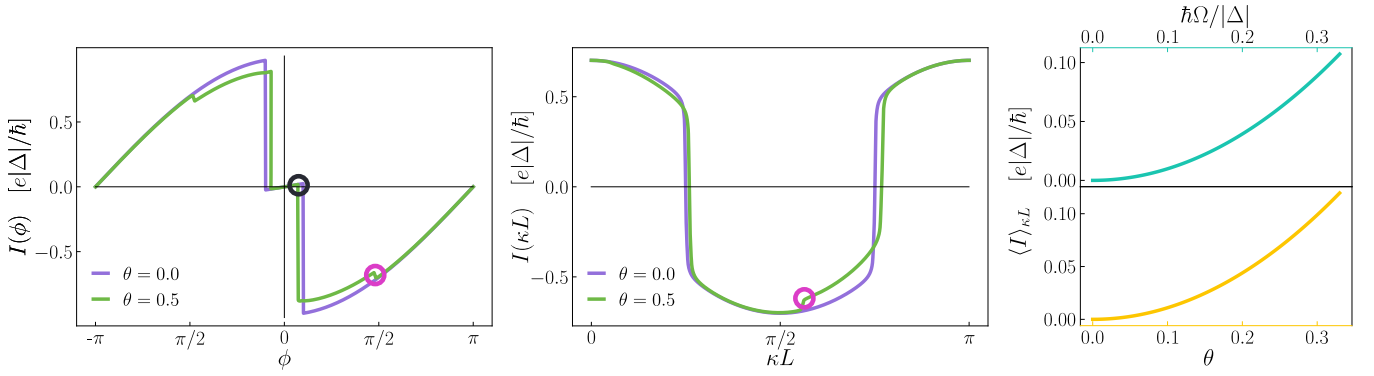


FIG. 8. S/F/S Josephson junction at zero temperature in one dimension. (Left) Current-phase relation in the short-junction limit, $1/(\kappa\xi) = 0$, for $\hbar\Omega = 0.6|\Delta|$, $\kappa L = 0.55\pi$, and $\theta = 0$ (violet, nonprecessing magnetization) compared to $\theta = 0.5$ (green, precessing magnetization). In the latter case, the shape closely matches the $I(\phi)$ curve of the nonprecessing junction, except near the points where the occupation changes abruptly (black and pink circles), cf. Fig. 7, where the small- θ corrections become visible. (Center) Current $I(\kappa L)$ as a function of junction length κL for $\hbar\Omega = 0.6|\Delta|$ at fixed phase bias $\phi = \pi/2$, comparing $\theta = 0$ (violet) and $\theta = 0.5$ (green). The former curve oscillates with κL and averages to zero, implying the short-ranged nature of nonspin-polarized superconductivity in higher dimensions, whereas the precessing junction (green line) is nonzero on average. (Right) Average $\langle I \rangle_{\kappa L}$ at phase bias $\phi = \pi/2$ in the short-junction limit vs θ (bottom, $\hbar\Omega/|\Delta| = 0.6$) and Ω (top, $\theta = 0.5$). For small $\hbar\Omega/|\Delta|$ and θ , the current exhibits a quadratic dependence on Ω and θ , respectively.

contribution to the supercurrent from the continuous spectrum for $\tilde{E} > |\Delta| - \frac{1}{2}\hbar|\Omega|$. For the total supercurrent, one then arrives at an expression similar to Eq. (62), but with

$$j_{\sigma}(\tilde{E}) = \frac{-i\sigma \sin(2\eta_{\sigma}) \sin(2\kappa L) \sin \phi}{\pi [\cos(2\eta_{\sigma}) - \cos(2\zeta_{+})][\cos(2\eta_{\sigma}) - \cos(2\zeta_{-})]}. \quad (77)$$

All contributions to the supercurrent oscillate as a function of κL and the average of the oscillations vanishes. The current-phase relationship is illustrated in Fig. 8 (center).

A small but nonzero tilt angle θ leads to a correction $\delta\tilde{E}_{\pm,\sigma}$ to the bound-state energy that is proportional to θ^2 ,

$$\delta\tilde{E}_{\pm,\sigma} = -\theta^2 \sin \eta_{\sigma} \frac{\sin(\eta_{\sigma} \pm \frac{1}{2}\sigma\phi) \sin \eta'}{2 \sin(\bar{\eta} \mp \frac{1}{2}\sigma\phi)}, \quad (78)$$

where η_{σ} , η' , and $\bar{\eta}$ should be evaluated at $\tilde{E}_{\pm,\sigma}$. The additional current carried by the Andreev bound states α is

$$\delta j_{\pm,\sigma} = -\frac{e}{\hbar} \frac{\partial \delta\tilde{E}_{\pm,\sigma}}{\partial \phi} (1 - 2f_{\sigma}(\tilde{E}_{\pm,\sigma})) - 2\frac{e}{\hbar} \frac{\partial \tilde{E}_{\pm,\sigma}}{\partial \phi} \left(-\frac{\partial f_{\sigma}(\tilde{E}_{\pm,\sigma})}{\partial \tilde{E}_{\pm,\sigma}} \right) \delta\tilde{E}_{\pm,\sigma}. \quad (79)$$

The first term is independent of κL [except for the indirect dependence via the distribution function $f_{\sigma}(\tilde{E}_{\pm,\sigma})$], so that its average over the length L of the ferromagnetic segment is nonzero. The second term oscillates as a function of κL . For small tilt θ , the contribution from the continuous part of the spectrum $\tilde{E} > |\Delta| - \frac{1}{2}\hbar|\Omega|$, also acquires a change $\delta j_{\sigma}(\tilde{E})$, which is proportional to θ^2 and has a nonzero average with respect to κL . We refer to the Appendix D for explicit expressions. Figure 8 (center) and (left) compare the current-phase relationship and the κL dependence for zero tilt angle and at a finite tilt angle. For small tilt angle θ , the difference is small, except near values of ϕ for which the occupation of the Andreev bound states $E_{\pm,\sigma}$ changes. For the average over κL , this difference is significant: Whereas the average $\langle I \rangle_{\kappa L}$

vanishes for $\theta = 0$, it is nonzero and proportional to θ^2 for finite θ . For the κL -averaged Josephson current, we find the scaling

$$\langle I_c \rangle_{\kappa L} \propto \left(\frac{\hbar\Omega}{|\Delta|} \right)^2 \theta^2 \quad (80)$$

for small precession frequencies Ω and small precession angles θ , see Fig. 8 (right). The nonvanishing average with respect to κL is key in higher-dimensional junctions, as we discuss in the next section.

V. JUNCTIONS IN TWO AND THREE DIMENSIONS

So far, the discussion has been for one-dimensional junctions. Our results can easily be generalized to two- and three-dimensional junctions.

For two- and three-dimensional junctions, the momentum components k_y and k_z parallel to the interfaces are conserved and each value of k_y and k_z is separately described by a one-dimensional Hamiltonian of the form of Eq. (1), but with the substitution

$$\mu \rightarrow \mu - \frac{\hbar^2}{2m} k_{\parallel}^2, \quad k_{\parallel}^2 = k_y^2 + k_z^2, \quad (81)$$

in the normal-state first-quantized Hamiltonians of Eqs. (4) and (5). (For a two-dimensional junction one sets $k_z = 0$.) This substitution has no consequences for the boundary conditions of Eqs. (33) and (35) at the superconductor interfaces, as long as the Andreev approximation $|\Delta| \ll \mu$ holds. In the expressions (30) and (31) for the reflection and transmission amplitudes of the central magnetic segment, one now has to obtain k and $k_{\pm}$ from

$$\frac{\hbar^2 k^2}{2m} = \mu + \tilde{E} - \frac{\hbar^2 k_{\parallel}^2}{2m}, \quad (82)$$

$$\frac{\hbar^2 k_{\pm}^2}{2m} = \mu + \tilde{E} - V_F \pm |h_{\text{eff}}| - \frac{\hbar^2 k_{\parallel}^2}{2m} \quad (83)$$

for each value of k_y and k_z separately. To obtain the total supercurrent in a junction with two superconducting contacts, the sum over all momenta $k_{y,z}$ parallel to the interface must be performed. For a two-dimensional junction this gives

$$I^{(2D)} = \int_0^{k_0} \frac{dk_{\parallel}}{\pi} I^{(1D)}(k_{\parallel}), \quad (84)$$

where

$$\frac{\hbar^2 k_0^2}{2m} = \mu - V_F + |h_{\text{eff}}| \quad (85)$$

and $I^{(1D)}(k_{\parallel})$ is the current in a one-dimensional junction with the substitution (81). Similarly, the supercurrent in a three-dimensional junction is

$$I^{(3D)} = \int_0^{k_0} \frac{k_{\parallel} dk_{\parallel}}{2\pi} I^{(1D)}(k_{\parallel}). \quad (86)$$

For the calculation of the conductance $G(V)$ and the anomalous Green function of a junction with one normal contact and one superconducting contact, we note that the Andreev reflection amplitudes r_{eh} and r_{he} are now functions of $k_{\parallel}$. The conductance $G(V)$ is found by integration over $k_{\parallel}$ similar to Eqs. (84) and (86). The anomalous Green function is now a function of k_x and $k_{\parallel}$. It can be found by inserting the $k_{\parallel}$ -dependent Andreev reflection amplitudes into Eq. (50).

Results for the conductance and the anomalous Green functions in two- and three-dimensional junctions do not differ qualitatively from those for a one-dimensional junction discussed in detail in Secs. III A and IV A. We therefore focus the discussion of the remainder of the section on the Josephson effect.

The supercurrent in one-dimensional superconductor-magnet-superconductor junctions shows oscillations as a function of the junction length L . For a junction involving a half-metal, the oscillations involve the dimensionless product $k_0 L$; for a junction involving a metallic ferromagnet, the oscillations involve the product κL , where $\kappa = \hbar_F / \hbar v_0 = \hbar_F m / \hbar^2 k_0$. In the short-junction limit, the supercurrent is strictly periodic in $k_0 L$ or κL for a junction involving a half metal or a ferromagnetic metal, respectively. Beyond the short junction limit, the supercurrent has a slower dependence on L/ξ in addition to the fast dependence on $k_0 L$ or κL . In both cases, since k_0 and κ depend on $k_{\parallel}$, the integration over the transverse momentum $k_{\parallel}$ in Eqs. (84) and (86) effectively amounts to an average over $k_0 L$ or κL . This is especially true if $k_0 L$ or $\kappa L \gg 1$, indicating that contributions to the supercurrent that vanish upon averaging over $k_0 L$ or κL are short-ranged in a two-dimensional or three-dimensional junction, whereas contributions to the supercurrent that have a nonzero average over $k_0 L$ or κL are long ranged. Our calculations for the one-dimensional junction have shown that the average of the supercurrent I over $k_0 L$ and κL is nonzero if the magnetization precesses. This implies that the Josephson effect is long-ranged in two- and three-dimensional junctions. Without precessing magnetization, the supercurrent is zero in a junction featuring a half-metallic magnet. For a junction with a ferromagnetic metal, it vanishes once an average over κL is performed, consistent with the well-known fast decay of the Josephson effect with junction length L in this case.

To illustrate our findings, in Fig. 9 we show the Josephson current $I(\pi/2)$ at phase difference $\phi = \pi/2$ as a function of junction length L . We focus on the short-junction limit $L/\xi \ll 1$. Unlike the one-dimensional case, in two and three dimensions, the Josephson current is a nonperiodic function of L even in the short-junction limit because of the $k_{\parallel}$ -integrations in Eqs. (84) and (86). The long-range nature of the Josephson current carried by equal-spin Cooper pairs is reflected in the nonzero large- L asymptote within the short-junction limit $L/\xi \ll 1$. Upon going beyond the short-junction limit, a further decay with L sets in, which is similar in nature to that of a nonmagnetic Josephson junction [24,38]. For the numerical evaluation, we use the full model as described in Sec. II and not the simplifications used for Eqs. (37) and (65), which are used for the numerical results for one-dimensional junctions shown in Secs. III and IV. (The reason is that these simplifications cannot be made simultaneously for all values of $k_{\parallel}$.)

VI. DISCUSSION AND CONCLUSIONS

In this article, we considered the Josephson effect in a ballistic Josephson junction containing a ferromagnetic metal with a precessing uniform magnetization. Without precession, the supercurrent in such a junction is carried by opposite-spin Cooper pairs. It shows decaying oscillations with the junction length L , even for a ballistic junction (although for a ballistic junction the suppression is much less than in the presence of disorder). In the short-junction limit $L/\xi \ll 1$ one has $I(L) \propto L^{-1/2}$ for a two-dimensional junction and $I(L) \propto L^{-1}$ in three dimensions [41,42]. The magnetization precession induces an additional contribution to the supercurrent that is carried by equal-spin Cooper pairs. Although this precession-induced contribution is small for small precession amplitudes—it is proportional to the square θ^2 of the tilt angle—it does not suffer from the same suppression with junction length L as the opposite-spin contribution. In the short-junction limit $L \ll \xi$, the precession-induced contribution is independent of L . For larger L , it decays with L in the same manner as the supercurrent in a superconductor-normal-metal-superconductor junction.

We could obtain detailed results in the limit of a junction with a half-metallic, i.e., fully spin-polarized magnet. In this limit, there is no contribution to the supercurrent from opposite-spin Cooper pairs, so that the magnetization precession switches the junction from an “off”-state without current to an “on”-state with finite Andreev conductance and finite Josephson current. For a half metal, the current-phase relationship is sinusoidal for small tilt angle θ , but not for larger tilt angles $\theta \sim 1$. In a partially polarized ferromagnet, the precession-induced term exists next to the supercurrent carried by opposite-spin Cooper pairs. Both have a highly nonsinusoidal current-phase relationship for a short, ballistic Josephson junction [55].

Our work can be compared to the original studies of the precession-induced Josephson effect [49,50], which focused on disordered Josephson junctions, and to similar studies for tunneling junctions coupled to a precessing spin [54–56]. In particular, our result $I_c \propto \theta^2 \Omega^2$ is reminiscent of Eq. (14) in Ref. [50]. A key difference, however, is that Refs. [49,50]

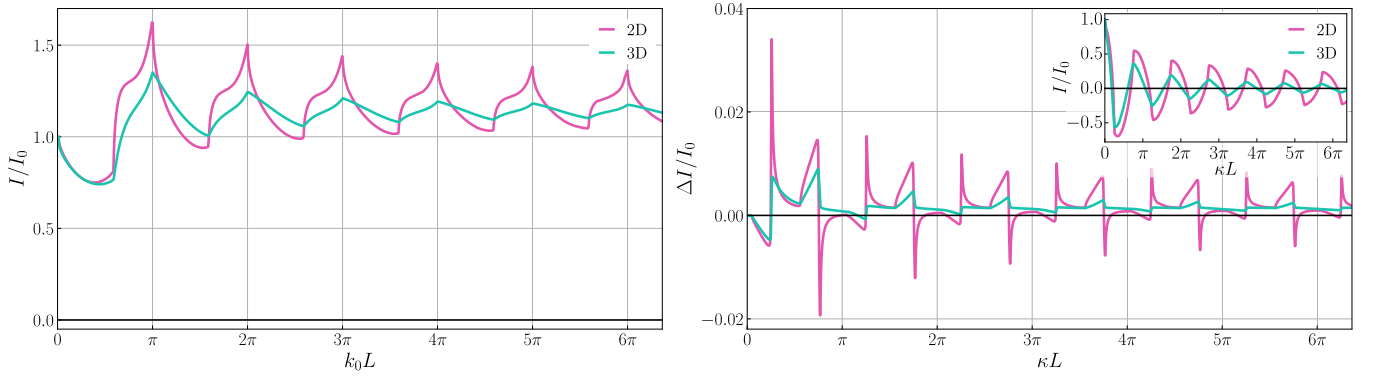


FIG. 9. Josephson current $I(\phi = \pi/2)$ in two-dimensional (blue) and three-dimensional (purple) junctions in the short-junction limit $L/\xi \ll 1$ for a fully polarized S/HM/S junction (left) and a partially polarized S/F/S junction (right). The main right panel shows the difference $\Delta I = I(\Omega) - I(\Omega = 0)$ between the Josephson current with and without precession in F; the inset shows the full current I . The parameters used in the calculations are $\hbar\Omega = 0.6|\Delta|$ and $\theta = 0.1$ (left) and $\theta = 0.5$ (right). In the large- L limit, the Josephson current saturates to a nonzero value. For the S/F/S junction, this can be seen from the difference ΔI in the main panel. In both panels, the current is normalized by its value I_0 at $L = 0$.

obtain closed-form expressions for the supercurrent in the vicinity of T_c , whereas our analytical results apply to zero temperature. The current-phase relationship is sinusoidal for the diffusive junctions considered in Refs. [49,50]. For a ballistic junction, we find $I \propto \sin \phi$ only in the limit of small tilt angles $\theta \ll 1$, whereas the current-phase relationship is strongly nonsinusoidal otherwise. This latter finding is consistent with current-phase relationships obtained in Ref. [55] for a tunneling junction coupled to a precessing spin.

The observation of a long-range proximity effect in the ballistic regime requires that the junction length L be smaller than the mean free path ℓ , but larger than κ^{-1} , where $\kappa = (k_{F\uparrow} - k_{F\downarrow})/2$ is the difference of Fermi wavenumbers for majority and minority electrons, see Eq. (66). For Fe, κ^{-1} is approximately 0.3 nm [77,78], whereas ℓ can be up to 10 nm in a high-quality thin film [79]. This indeed leaves a parameter window within which ballistic Josephson current can be observed. These lengths are significantly shorter than the superconducting coherence length ξ in an Fe/Nb junction, which is ≈ 40 nm (using the velocity of minority electrons in Fe $v_0 \approx 3 \times 10^5$ m/s [78] and Nb gap $|\Delta| \approx 2.3$ meV [80]), so that a ballistic Nb/Fe/Nb junction is typically in the short-junction limit.

As discussed by Takahashi *et al.* [49], in an experimental realization of this scenario, the magnetization precession is typically driven by an alternating applied magnetic field $\vec{h}_{\text{drive}}$ at frequency Ω . The tilt angle θ has a strong resonance near the ferromagnetic resonance frequency Ω_{res} ,

$$\theta \approx \frac{\gamma h_{\text{drive}}/\hbar}{\pi \sqrt{(\Omega - \Omega_{\text{res}})^2 + (\alpha\Omega)^2}}, \quad (87)$$

where α is the Gilbert damping constant of the magnet and γ the gyromagnetic ratio. The resonance frequency Ω_{res} has an intrinsic contribution from the magnetic anisotropy and an additional term γH_{ext} , where γ is the gyromagnetic ratio and H_{ext} an applied constant magnetic field. Because of the resonant frequency dependence of the tilt angle θ , the supercurrent I (in an S/F/S junction) and the conductance (of an N/F/S junction) have a sharp increase at $\Omega = \Omega_{\text{res}}$ upon increasing

the driving frequency Ω at fixed external magnetic field (such that the ferromagnetic resonance frequency Ω_{res} remains constant). This is illustrated in Fig. 1 for the conductance of an N/HM/S junction.

The long-range supercurrents in the presence of a precessing magnetization are spin polarized. Hence, a finite spin current is associated with the Josephson current. Since the net spin current vanishes in the spin-singlet superconductors, this spin current “appears” and “disappears” at the superconductor–ferromagnet interfaces. This source or sink of spin current corresponds to a torque on the magnetization. For a steady-state Josephson current, the torques at the two interfaces are equal and opposite, so that their net effect on the total magnetization vanishes.

To optimize the strength of the precession-induced spin-polarized proximity effect, the ratio $\hbar|\Omega|/|\Delta|$ should be as large as possible, but not larger than 2, to avoid direct excitation of particle-hole pairs in S by the precessing magnetization. With respect to a possible experimental realization, we note that for conventional superconductors, typically $|\Delta|$ is in the range 0.1 – 3 meV, corresponding to maximum precession frequencies $\Omega_{2\Delta}/2\pi$ in the range 0.05 – 1.5 THz. (For Nb, one has $|\Delta| \approx 2.3$ meV [80], corresponding to $\Omega_{2\Delta}/2\pi \approx 1.2$ THz, whereas for Al, $|\Delta| \approx 0.3$ meV [81], which gives $\Omega_{2\Delta} \approx 0.1$ THz.) For comparison, typical ferromagnetic-resonance frequencies are of the order $\Omega_{\text{res}}/2\pi \sim 1 - 20$ GHz [61,82–84], which is small enough to avoid particle-hole excitations, but large enough that the precession-induced contribution is not suppressed by its proportionality to $(\hbar\Omega/|\Delta|)^2$.

The tilt angle θ is controlled by two main factors: (i) the Gilbert damping parameter α , which can be very small (in the half-metals CrO₂, Co₂MnGe, and Co₂MnSi, $\alpha \sim 10^{-4}$ [85,86], whereas in conventional ferromagnetic metals it is typically $\alpha \sim 10^{-3} - 10^{-2}$, see, e.g., Ref. [60]), and (ii) the ratio $h_{\text{drive}}/H_{\text{ext}}$, which is generally small. As a result, θ is typically of the order of a few degrees, making it the main small parameter controlling the effect. For observation of the precession-induced effects, half-metallic ferromagnets are not only favored because of their low damping, but

also because they have no “conventional” proximity effect with opposite-spin Cooper pairs, which tends to mask the precession-induced contribution for thin junctions.

Finally, one may wonder about the detrimental effect of the heat dissipated by the precessing magnetization. This could be especially relevant in the case of a “vertically stacked” S/F/S junction, because in that case any excess heat in F must be transported away via the superconductors, which are bad heat conductors at low temperatures. Near ferromagnetic resonance, the power dissipated by Gilbert damping (per cross-sectional area of the junction) can be estimated as

$$P_G = \frac{\alpha M_s}{\gamma} \Omega_{\text{res}}^2 \theta^2 L, \quad (88)$$

where M_s is the saturation magnetization (per unit volume). Gilbert damping transfers energy from magnons to electrons, typically through spin-orbit coupling or exchange interactions [87], thereby heating the conduction electrons in the F region. This energy must be removed through heat transfer to the adjacent superconducting leads, mediated primarily by phonons when $T \ll T_c$ and by both phonons and thermally excited quasiparticles when $T \lesssim T_c$. At very low temperatures, the bottleneck in heat removal is the transfer of energy from the electronic degrees of freedom to phonons in the F region, since the corresponding thermal conductivity scales as $\propto T^4$ at low temperatures [88]. On the other hand, when the temperature is close to the critical temperature T_c , heat can also be transferred directly through thermally excited quasiparticles, without first being transferred to phonons. In either regime, since a sizable long-range proximity effect requires a sizable angle θ , sample heating is an important effect. We expect the effective operating temperature in realistic setups to be comparable to the critical temperatures of conventional superconductors.

Several additional effects may significantly influence the driving-induced temperature ΔT of F at low temperatures. These include the thermal (Kapitza) resistance at the F-S interface, which further impedes heat flow into the superconductor and therefore increases ΔT . Moreover, at lower temperatures, the effective Gilbert damping may be modified by the so-called “bottleneck effect” [89,90], which can decrease the effective value of the damping parameter α in Eq. (88). A more detailed analysis of these contributions, which depend strongly on material-specific properties, is an important step toward the eventual experimental realization of the effects predicted here.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The source code for the numerical calculations can be found in the Zenodo repository [91].

APPENDIX A: SMALL- θ SOLUTION FOR THE S/HM/S BOUND STATES

Equation (57) considers the predominantly electron-like state. We use Eq. (33) to eliminate all minority amplitudes from Eqs. (27) and (32). Then, the scattering amplitudes satisfy

$$\begin{pmatrix} a_{L\uparrow}^{\text{out}} \\ -e^{i\eta_- - i\phi/2} b_{L\uparrow}^{\text{in}} \\ a_{R\uparrow}^{\text{out}} \\ -e^{i\eta_- + i\phi/2} b_{R\uparrow}^{\text{in}} \end{pmatrix} = S_F \begin{pmatrix} a_{L\uparrow}^{\text{in}} \\ -e^{-i\eta_- - i\phi/2} b_{L\uparrow}^{\text{out}} \\ a_{R\uparrow}^{\text{in}} \\ -e^{-i\eta_- + i\phi/2} b_{R\uparrow}^{\text{out}} \end{pmatrix},$$

$$\begin{pmatrix} b_{L\uparrow}^{\text{out}} \\ e^{i\eta_+ + i\phi/2} a_{L\uparrow}^{\text{in}} \\ b_{R\uparrow}^{\text{out}} \\ e^{i\eta_+ - i\phi/2} a_{R\uparrow}^{\text{in}} \end{pmatrix} = S_F^* \begin{pmatrix} b_{L\uparrow}^{\text{in}} \\ e^{-i\eta_+ + i\phi/2} a_{L\uparrow}^{\text{out}} \\ b_{R\uparrow}^{\text{in}} \\ e^{-i\eta_+ - i\phi/2} a_{R\uparrow}^{\text{out}} \end{pmatrix}. \quad (A1)$$

The energy dependence of S_F has been dropped, since we work in the short-junction limit [see Eq. (37)]. To first order in θ , the normal-state scattering matrix S_F reads

$$S_F = \begin{pmatrix} 0 & 0 & e^{ik_0 L} & 0 \\ 0 & -1 & 0 & 0 \\ e^{ik_0 L} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} + \frac{\theta}{2} \begin{pmatrix} 0 & 1 & 0 & e^{ik_0 L} \\ 1 & 0 & e^{ik_0 L} & 0 \\ 0 & e^{ik_0 L} & 0 & 1 \\ e^{ik_0 L} & 0 & 1 & 0 \end{pmatrix}. \quad (A2)$$

For $\theta = 0$, Eq. (A1) separates into disjoint equations for the electron-like and hole-like majority amplitudes. For the electron-like amplitudes, one finds

$$\begin{pmatrix} a_{L\uparrow}^{\text{out}} \\ a_{R\uparrow}^{\text{out}} \end{pmatrix} = \begin{pmatrix} 0 & e^{ik_0 L} \\ e^{ik_0 L} & 0 \end{pmatrix} \begin{pmatrix} a_{L\uparrow}^{\text{in}} \\ a_{R\uparrow}^{\text{in}} \end{pmatrix},$$

$$\begin{pmatrix} a_{L\uparrow}^{\text{in}} \\ a_{R\uparrow}^{\text{in}} \end{pmatrix} = -e^{-2i\eta_+} \begin{pmatrix} a_{L\uparrow}^{\text{out}} \\ a_{R\uparrow}^{\text{out}} \end{pmatrix}. \quad (A3)$$

Similarly, for the hole-like amplitudes,

$$\begin{pmatrix} b_{L\uparrow}^{\text{out}} \\ b_{R\uparrow}^{\text{out}} \end{pmatrix} = \begin{pmatrix} 0 & e^{-ik_0 L} \\ e^{-ik_0 L} & 0 \end{pmatrix} \begin{pmatrix} b_{L\uparrow}^{\text{in}} \\ b_{R\uparrow}^{\text{in}} \end{pmatrix},$$

$$\begin{pmatrix} b_{L\uparrow}^{\text{in}} \\ b_{R\uparrow}^{\text{in}} \end{pmatrix} = -e^{-2i\eta_-} \begin{pmatrix} b_{L\uparrow}^{\text{out}} \\ b_{R\uparrow}^{\text{out}} \end{pmatrix}. \quad (A4)$$

From these conditions, it immediately follows that the coupled equations for the electron-like majority coefficients have a solution only if

$$e^{2ik_0 L} = e^{4i\eta_+(\tilde{E})}, \quad (A5)$$

while the equations for the hole-like majority coefficients have a nontrivial solution if

$$e^{2ik_0L} = e^{-4i\eta_- (\tilde{E})}. \quad (\text{A6})$$

Since we consider an Andreev bound state that is electron-like for $\theta = 0$, we choose the energy $\tilde{E}$ such that the condition (A5) is satisfied, while setting the hole-like majority coefficients to zero. In that case, one has

$$\begin{aligned} a_{R\uparrow}^{\text{in}} &= \frac{1}{\sqrt{\mathcal{N}_+}}, & a_{R\uparrow}^{\text{out}} &= -\frac{e^{2i\eta_+}}{\sqrt{\mathcal{N}_+}}, & a_{L\uparrow}^{\text{in}} &= -\frac{e^{2i\eta_+ - ik_0L}}{\sqrt{\mathcal{N}_+}}, \\ a_{L\uparrow}^{\text{out}} &= \frac{e^{4i\eta_+ - ik_0L}}{\sqrt{\mathcal{N}_+}} = \frac{e^{ik_0L}}{\sqrt{\mathcal{N}_+}} \end{aligned} \quad (\text{A7})$$

for the electron-like coefficients, where $\mathcal{N}_+$ is a normalization factor, see Eq. (56). The condition (A5) gives the two solutions labeled by $j = 1, 2$.

If θ is nonzero, the Andreev bound state also acquires nonzero hole-like majority amplitudes. To find the corresponding coefficients, we consider the coupled linear equations for the hole-like coefficients. These are inhomogeneous equations, in which the solution (A7) for the

electron-like coefficients, multiplied by the second term in Eq. (A2), appears as a source term. Equation (57) follows upon solving this inhomogeneous linear set. The denominator $\sin(2(\zeta + \eta_-))$ in Eq. (58) reflects the fact that a bounded solution for the hole-like coefficients exists only if this set of equations is non-singular. This is the case unless Eq. (A6) is satisfied, which is precisely when there is a hole-like Andreev bound state at energy E .

APPENDIX B: S/HM/S JUNCTION: CURRENT CARRIED BY THE CONTINUUM SPECTRUM

In this Appendix, we provide the explicit expression for the contribution $j_\sigma(\tilde{E})$ from the continuous part of the spectrum for small tilt angle θ . We first consider energies $|\Delta| - (1/2)\hbar|\Omega| < \tilde{E} < |\Delta| + (1/2)\hbar|\Omega|$. In this energy range, there is a contribution from scattering states with $\sigma = \text{sign } \Omega$ only. Away from resonances, one has

$$j_\sigma(\tilde{E}) = \theta^2 j'_\sigma(\tilde{E}), \quad (\text{B1})$$

up to corrections subleading in θ , with

$$j'_\sigma(\tilde{E}) = \delta_{\sigma, \text{sign } \Omega} \frac{i \sin(\eta_\sigma) [\sin(\sigma k_0 L) \cos(2\eta_\sigma) \sin \eta_{-\sigma} - \cos(k_0 L) \cos \eta_\sigma (\sin^2 \eta_\sigma + \sin^2 \eta_{-\sigma})]}{\pi \sin(\sigma k_0 L + 2\eta_{-\sigma}) (\cos^2(k_0 L) - \cos^2(2\eta_\sigma))} \quad (\text{B2})$$

Near a resonance, for small θ , $j_\sigma(\tilde{E})$ is given by Eq. (63) with

$$J_\alpha = \lim_{\tilde{E} \rightarrow \tilde{E}_\alpha} (\tilde{E} - \tilde{E}_\alpha) j_\sigma(\tilde{E}), \quad (\text{B3})$$

$$\Gamma_\alpha = - \lim_{\tilde{E} \rightarrow \tilde{E}_\alpha} (\tilde{E} - \tilde{E}_\alpha) \frac{\cos(2\tilde{\eta}) \cos(2\eta') + \cos(2Lk_0 - 2\tilde{\eta}) - \cos(2[Lk_0 - 2\tilde{\eta}]) - \cos(2\eta') - 4 \cos(\phi) \sin^2 \tilde{\eta} \cos^2 \eta'}{2 \sin(\sigma Lk_0 + 2\eta_{-\sigma}) \sin(\sigma Lk_0 - 2\eta_\sigma)}, \quad (\text{B4})$$

and

$$\delta_\alpha = \lim_{\tilde{E} \rightarrow \tilde{E}_\alpha} (\tilde{E} - \tilde{E}_\alpha) \frac{n(\tilde{E})}{d(\tilde{E})}. \quad (\text{B5})$$

In Eq. (B5), the numerator and denominator are given by the following lengthy expressions:

$$\begin{aligned} n(\tilde{E}) &= 3 \sin(2\sigma k_0 L - \eta_{-\sigma}) - (9 + 2 \cos(2\eta_\sigma)) \sin \eta_{-\sigma} - (3 + 4 \cos(2\eta_\sigma)) \sin(3\eta_{-\sigma}) - 2 \cos(3\eta_\sigma) (\sin(2\eta_{-\sigma}) \\ &\quad + 4 \sin(2\sigma k_0 L + 2\eta_{-\sigma})) - 2(6 + 5 \cos(2\eta_\sigma)) \sin(2\sigma k_0 L + \eta_{-\sigma}) + 2 \cos \eta_\sigma (2 \sin(2\sigma k_0 L) + 9 \sin(2\eta_{-\sigma}) \\ &\quad - \sin(4\eta_{-\sigma}) + 6 \sin(2(\sigma k_0 L + \eta_{-\sigma})) - 4 \sin(2\sigma k_0 L + 4\eta_{-\sigma})) + (3 + 16 \cos(2\eta_\sigma)) \sin(2\sigma k_0 L + 3\eta_{-\sigma}), \end{aligned} \quad (\text{B6})$$

$$d(\tilde{E}) = 48 \sin(\sigma Lk_0 + 2\eta_{-\sigma}) [\sin(\sigma Lk_0) \cos(2\eta_\sigma) \sin \eta_{-\sigma} - \cos(Lk_0) \cos \eta_\sigma (\sin^2 \eta_\sigma + \sin^2 \eta_{-\sigma})]. \quad (\text{B7})$$

The apparent singularity associated with the factor $\sin(\sigma Lk_0 + 2\eta_{-\sigma})$ in the denominator is regularized by the prefactor $(\tilde{E} - \tilde{E}_\alpha)$, so that Γ_α and δ_α remain finite when taking the limit $\tilde{E} \rightarrow \tilde{E}_\alpha$. Since in the final expression for the current, only the integral of $j_\sigma(\tilde{E})$ over $\tilde{E}$ appears, for $|\tilde{E} - \tilde{E}_\alpha| \ll |\Delta|, |\Omega|$ we may approximate

$$j_\sigma(\tilde{E}) = \text{pv } \theta^2 j'_\sigma(\tilde{E}) + \pi \theta^2 \sum_\alpha \delta(\tilde{E} - \tilde{E}_\alpha) \frac{(\delta_\alpha - \text{Re } \Gamma_\alpha)}{|\text{Im } \Gamma_\alpha|}, \quad (\text{B8})$$

where “pv” denotes the principal value and any correction terms are subleading in the limit $\theta \ll 1$. Substituting the explicit expressions given above, we may then write $j_\sigma(\tilde{E})$ as

$$j_\sigma(\tilde{E}) = \delta_{\sigma, \text{sign } \Omega} \theta^2 \sin \phi [\text{pv } C_0^\sigma + C_1^\sigma \delta(\sin(\sigma k_0 L + 2\eta_{-\sigma}))], \quad (\text{B9})$$

where

$$C_0^\sigma = i \sin \eta_\sigma \frac{\sin(\sigma k_0 L) \cos(2\eta_\sigma) \sin \eta_{-\sigma} - \cos(k_0 L) \cos \eta_\sigma (\sin^2 \eta_\sigma + \sin^2 \eta_{-\sigma})}{\pi \sin(\sigma k_0 L + 2\eta_{-\sigma}) \sin^2(\sigma k_0 L - 2\eta_\sigma)},$$

$$C_1^\sigma = \cos(k_0 L + 2\eta_{-\sigma}) \sin \eta_{-\sigma} \frac{1 - \cos(2\eta_\sigma) \sin^2 \eta_{-\sigma} + \cos \eta_\sigma (\cos(2\eta_\sigma) - 2) \cos \eta_{-\sigma}}{\cos(3\eta_\sigma) - 2 \cos \eta_{-\sigma} (\cos(2\eta_\sigma) + \cos(2\eta_{-\sigma})) + \cos \eta_\sigma (1 + 2 \cos(2\eta_{-\sigma}))}. \quad (\text{B10})$$

For $\tilde{E} > |\Delta| + \hbar|\Omega|/2$, both spin projections of quasiparticles can directly enter the continuum spectrum in the adjacent superconductors and one finds

$$j_\sigma(\tilde{E}) = -i \frac{\sigma \theta^2}{\pi} \frac{\sin(2\eta_+) \sin(2k_0 L) \sin(2\tilde{\eta}) \sin(2\eta') \sin \phi}{(\cos(2k_0 L) - \cos(4\eta_+))(\cos(2k_0 L) - \cos(4\eta_-))}. \quad (\text{B11})$$

When $k_B T \ll \hbar|\Delta|$, the difference between $f_\uparrow(\tilde{E})$ and $f_\downarrow(\tilde{E})$ becomes small for $\tilde{E} > |\Delta|$, so that the two contributions from $\sigma = \uparrow, \downarrow$ cancel each other and do not contribute to the current.

APPENDIX C: S/HM/S JUNCTION: NONTRIVIAL $\hbar\Omega \ll |\Delta|$ ASYMPTOTICS

In this Appendix, we give an example of a 1D S/HM/S Josephson junction for which the Ω^2 dependence breaks down when $\hbar\Omega \ll |\Delta|$ and at $T = 0$. We consider a junction satisfying $\zeta \ll \hbar\Omega/|\Delta| \ll 1$, where $\Omega > 0$ and ζ is defined in Eq. (53). The bound-state spectrum is then close to that shown in the top panel of Fig. 4, near the points $k_0 L = 0, \pi, \dots$. In this regime, $\tilde{E}_{e1}$ and $\tilde{E}_{h2}$ contribute to the current, and their respective contributions, see Eq. (59), are given by

$$\frac{\partial \delta \tilde{E}_{e1}}{\partial \phi} = \theta^2 \sin \phi |\Delta| \zeta \frac{\sqrt{\hbar\Omega/|\Delta|}}{4\sqrt{2}}, \quad (\text{C1})$$

$$\frac{\partial \delta \tilde{E}_{h2}}{\partial \phi} = \theta^2 \sin \phi |\Delta| \frac{(\hbar\Omega)^3}{32|\Delta|^3}. \quad (\text{C2})$$

On the other hand, the current carried by the continuum spectrum with energies $|\Delta| - (1/2)\hbar|\Omega| < \tilde{E} < |\Delta| + (1/2)\hbar|\Omega|$ can be found to be

$$I_{\text{cont}} = \theta^2 \sin \phi |\Delta| \zeta \frac{\sqrt{\hbar\Omega/|\Delta|}}{4\sqrt{2}} \frac{\ln(2^{-3} e^{8+\pi} [\zeta/\sqrt{(\hbar\Omega/|\Delta|)}]^2)}{2\pi}. \quad (\text{C3})$$

Taken together, these results provide a clear example in which I_c exhibits a nontrivial, i.e., nonquadratic, dependence on Ω .

APPENDIX D: S/F/S JUNCTION: CURRENT CARRIED BY THE CONTINUUM SPECTRUM

For the continuum part of the spectrum, we find that the current carried by a state at energy $\tilde{E}$, to leading order in θ , is given by

$$j_\sigma(\tilde{E}) = \sigma \sum_{s=\pm} 2is \frac{\sin(2\eta_\sigma) \sin^2(\zeta_{-s\sigma} + \sigma \eta_{-\sigma}) + \theta^2 \sin \eta_\sigma [\sin^2 \kappa L \cos(\eta_{-\sigma} - s\phi) - \sin \kappa L \cos \eta_\sigma \sin(\kappa L - s\phi + 2\sigma \eta_{-\sigma})]}{|2 \sin(\zeta_{-s\sigma} + \sigma \eta_{-\sigma}) \sin(\zeta_{s\sigma} + \sigma \eta_{-\sigma}) + \theta^2 [\sin^2 \kappa L - \sin(\sigma \kappa L) \sin(\sigma \kappa L + \eta_\sigma + \eta_{-\sigma})]}]. \quad (\text{D1})$$

Setting $\theta = 0$, we retrieve Eq. (77) of the main text. Retaining corrections up to quadratic order in θ in the numerator and denominator yields the leading correction to the supercurrent $\delta j_\sigma(\tilde{E})$, which contains the contribution from equal-spin Cooper pairs.

APPENDIX E: DETAILS OF NUMERICAL CALCULATION OF THE CURRENT

In this Appendix, we discuss the implementation of the numerical evaluation of the Josephson current. While Eq. (26) is instructive, because it highlights the contribution of the Andreev bound states to the supercurrent and, hence, explains the sharp jumps of the supercurrent with system parameters in terms of discontinuities related to the population of Andreev levels, the separation into discrete- and continuous-spectrum contributions to the supercurrent poses a complication from a numerical point of view. These complications can be avoided

by weakly coupling the central magnetic part of the junction to a normal reservoir, as illustrated in Fig. 10.

In the presence of the additional normal reservoir, solutions of the Bogoliubov-de Gennes equation (13) are scattering states with a continuous spectrum for all energies $\tilde{E}$, so that a separate treatment of discrete and continuous parts of the spectrum is no longer necessary. With the additional normal lead, the Andreev bound states turn into scattering resonances. The additional lead acts as a source of thermalization for the Andreev bound states, populating them according to Eq. (22). The resonance width is proportional to the transmission probability Γ between the modes in the reservoir and in the junction. In our numerical calculations, we use $\Gamma = (0.05)^2$, which enables us to resolve these resonances while still ensuring that the coupling to the additional lead has an infinitesimal effect on the Josephson current itself.

To implement the coupling to the additional normal reservoir, we insert an additional ideal lead at the center of the magnetic region, i.e., at coordinate $x = 0$. Like the other two

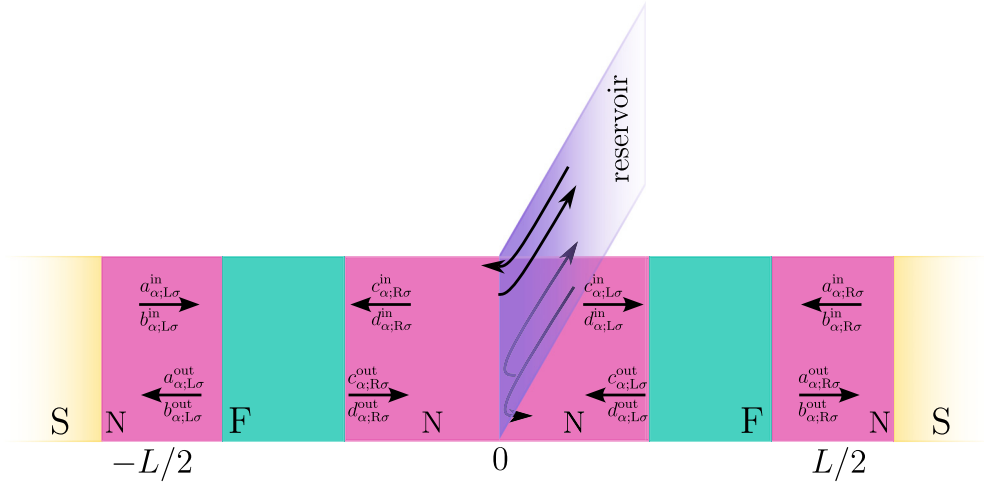


FIG. 10. Geometry used for the numerical calculation of the current in the junction. A normal reservoir in thermal equilibrium is weakly coupled to an ideal lead inserted at $x = 0$, at the center of the magnetic region. The amplitude that electrons scatter from the ideal lead into the normal reservoir or vice versa is $i\sqrt{\Gamma}$; see Eq. (E3). The coupling to the normal lead imposes the population (22) for the Andreev bound states in the junction and turns them into scattering resonances. In our numerical calculations, we set $\Gamma = (0.05)^2$. We verify that the small level broadening due to the coupling to the normal reservoir does not affect our results.

ideal leads, the length of this ideal lead is taken to zero at the end of the calculation. Left- and right-moving states in the ideal lead around $x = 0$ are coupled separately to modes in an additional lead, which is connected to the normal reservoir, see Fig. 10.

The addition of the normal reservoir requires that the composite index $\alpha = (\tilde{E}, \beta, l, \tau, \sigma)$ labeling the scattering states now have five elements, whereby the additional element $l = 0$ for scattering states incident from one of the superconductors and $l = 1$ for scattering states incident from the normal reservoir. For states incident from the normal reservoir, we set $\beta = 1$ (R) if it couples to a left-moving state in the ideal lead (which has amplitudes $c_{\alpha;R\sigma}^{in}$ and $c_{\alpha;L\sigma}^{out}$ for electron-like states and $d_{\alpha;R\sigma}^{in}$ and $d_{\alpha;L\sigma}^{out}$ for hole-like states, see Fig. 10) and $\beta = -1$ (L) if it couples to a right-moving state in the ideal lead (corresponding to the amplitudes $c_{\alpha;L\sigma}^{in}$ and $c_{\alpha;R\sigma}^{out}$, $d_{\alpha;L\sigma}^{in}$ and $d_{\alpha;R\sigma}^{out}$).

Referring to Fig. 10 for the geometry and the notation of the amplitudes of the scattering states in the additional ideal lead around $x = 0$, we replace Eqs. (27) and (32) by the following system of equations:

$$\begin{pmatrix} a_{\alpha;L\uparrow}^{out} \\ a_{\alpha;L\downarrow}^{out} \\ c_{\alpha;R\uparrow}^{out} \\ c_{\alpha;R\downarrow}^{out} \end{pmatrix} = S_F(\tilde{E}) \begin{pmatrix} a_{\alpha;L\uparrow}^{in} \\ a_{\alpha;L\downarrow}^{in} \\ c_{\alpha;R\uparrow}^{in} \\ c_{\alpha;R\downarrow}^{in} \end{pmatrix}, \quad \begin{pmatrix} b_{\alpha;L\uparrow}^{out} \\ b_{\alpha;L\downarrow}^{out} \\ d_{\alpha;R\uparrow}^{out} \\ d_{\alpha;R\downarrow}^{out} \end{pmatrix} = S_F(-\tilde{E})^* \begin{pmatrix} b_{\alpha;L\uparrow}^{in} \\ b_{\alpha;L\downarrow}^{in} \\ d_{\alpha;R\uparrow}^{in} \\ d_{\alpha;R\downarrow}^{in} \end{pmatrix}, \quad (E1)$$

$$\begin{pmatrix} c_{\alpha;L\uparrow}^{out} \\ c_{\alpha;L\downarrow}^{out} \\ a_{\alpha;R\uparrow}^{out} \\ a_{\alpha;R\downarrow}^{out} \end{pmatrix} = S_F(\tilde{E}) \begin{pmatrix} c_{\alpha;L\uparrow}^{in} \\ c_{\alpha;L\downarrow}^{in} \\ a_{\alpha;R\uparrow}^{in} \\ a_{\alpha;R\downarrow}^{in} \end{pmatrix}, \quad \begin{pmatrix} d_{\alpha;L\uparrow}^{out} \\ d_{\alpha;L\downarrow}^{out} \\ b_{\alpha;R\uparrow}^{out} \\ b_{\alpha;R\downarrow}^{out} \end{pmatrix} = S_F(-\tilde{E})^* \begin{pmatrix} d_{\alpha;L\uparrow}^{in} \\ d_{\alpha;L\downarrow}^{in} \\ b_{\alpha;R\uparrow}^{in} \\ b_{\alpha;R\downarrow}^{in} \end{pmatrix}, \quad (E2)$$

Here, Eqs. (E1) and (E2) describe scattering of electron (a and c) and hole (b and d) components, see Fig. 10, around the left and right parts of the central F region ($-L/2 < x < 0$ and $0 < x < L/2$), respectively. The relations between scattering amplitudes at the superconductor interfaces are still given by Eqs. (33) and (35) of the main text, whereby the source terms in Eq. (35) acquire an additional factor δ_{l0} . The set of equations is closed by the relations between the amplitudes on the left and right sides of the normal reservoir (see Fig. 10),

$$\begin{aligned} c_{\alpha;\beta',\sigma'}^{in} &= \sqrt{1-\Gamma} c_{\alpha;-\beta',\sigma'}^{out} + i\sqrt{\Gamma} \delta_{\beta\beta'} \delta_{\sigma\sigma'} \delta_{l1} \delta_{\tau e}, \\ d_{\alpha;\beta',\sigma'}^{in} &= \sqrt{1-\Gamma} d_{\alpha;-\beta',\sigma'}^{out} - i\sqrt{\Gamma} \delta_{\beta\beta'} \delta_{\sigma\sigma'} \delta_{l1} \delta_{\tau h}, \end{aligned} \quad (E3)$$

The current is calculated using Eqs. (24) and (25) from the main text, where the summation over the composite index α should be understood as $\sum_{\beta,l,\sigma,\tau} \int d\tilde{E} \dots$

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