

Non-Hermitian Landau levels

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We formulate non-Hermitian Landau levels in two-dimensional systems under a complex perpendicular magnetic field. In the symmetric gauge, we derive their discretely spaced, highly degenerate complex spectra and biorthogonal eigenstates. In addition we derive analogous results in the Landau gauge and clarify the role of nonunitary gauge transformations. We find that while normalizable eigenstates exist for both gauges for $\text{Re}(B) > 0$, they are not always related by such a gauge transformation. A non-Hermitian Harper-Hofstadter lattice model confirms the continuum theory and reveals Gaussian wave packet dynamics governed by semiclassical equations with a complex Lorentz force, pointing to possible experimental realizations of complex magnetic fields in mechanical metamaterials or ultracold atom gases.

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I. INTRODUCTION

Understanding the properties of a charged particle in a magnetic field has been pivotal for condensed matter physics. In the presence of a magnetic field in two dimensions a charged particle exhibits equally spaced eigenenergies, called Landau levels [1]. The presence of these highly degenerate Landau levels is responsible for various physical phenomena such as the de Haas-van Alphen effect [2], and the integer and fractional quantum Hall effects [3,4]. In addition, the discretized lattice description of magnetic fields in two dimensions, the so-called Harper-Hofstadter model, is the paradigmatic model for Chern insulators [5].

The recent advances in non-Hermitian topology and the experimental development in non-Hermitian quantum mechanics have increased the interest in higher-dimensional non-Hermitian models [6–13]. The behavior and especially the topology of non-Hermitian systems have thereby been a focus of research, and the non-Hermitian analog of Chern insulators has been studied in detail [14–17]. While there has been interest in the response of non-Hermitian models to real-valued magnetic fields and in the effect of non-Hermiticity on conventional Landau levels [18–28], significantly less research has covered the effects of complex-valued magnetic fields [29–31]. The notable exception is purely imaginary vector potentials, which were introduced in the foundational Hatano-Nelson model and have been pivotal in describing

the non-Hermitian skin effect [32–46]. The imaginary vector potential describes the point-gap topology of one-dimensional systems [9,16,47], and the effects of imaginary vector potentials have been studied in two dimensions as well [29,30]. There the complex energy eigenvalues and the gauge-independent Aharonov-Bohm damping have been discussed.

In spite of the importance of topology in the description of non-Hermitian systems and the examination of the change of conventional Landau levels upon the introduction of non-Hermiticity, the presence of non-Hermitian Landau levels in genuine complex-valued magnetic fields has largely been unexplored. In this paper we show that equally spaced complex-valued eigenenergies arise if a two-dimensional system is subject to a complex-valued magnetic field. We derive both eigenenergies and the normalizable wave functions of the biorthogonal eigenstates of the continuum Hamiltonian under the symmetric gauge, labeled by level index and angular momentum. We show that, upon nonunitary gauge transformations to the Landau gauge, the image of the eigenstates may become non-normalizable depending on the values of the complex magnetic field. Surprisingly this does not imply that there are no normalizable eigenstates in the Landau gauge in this case. For any complex magnetic field with finite real part we find normalizable Landau-level eigenstates in the Landau gauge, but they can only be related to the symmetric gauge if the real part of the magnetic field is larger than the imaginary part. We verify our results numerically and further show that the evolution of a wave packet of the non-Hermitian Harper-Hofstadter model in the symmetric gauge is governed by the semiclassical equations of motion with complex-valued Lorentz force.

II. NON-HERMITIAN LANDAU LEVELS

A charged particle confined to a plane with a perpendicular complex-valued magnetic field evolves under the Schrödinger

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equation

$$i\hbar\partial_t|\psi\rangle = \hat{H}|\psi\rangle. \quad (1)$$

The dynamics are governed by the non-Hermitian Hamiltonian

$$\hat{H} = \frac{1}{2m_q} \left(\hat{\mathbf{p}} - \frac{q}{c} \mathbf{A}(\hat{x}, \hat{y}) \right)^2, \quad (2)$$

with the canonical position and momentum operator denoted by $\hat{x}, \hat{y}$ and $\hat{p}_{x,y}$, respectively, acting on all square integrable functions. Here and throughout the paper vectors are indicated by bold symbols. In the symmetric gauge the non-Hermitian vector potential is given by $\mathbf{A}(\hat{x}, \hat{y}) = \frac{1}{2}B(\hat{x}\mathbf{e}_y - \hat{y}\mathbf{e}_x)$, where $\mathbf{e}_{x,y}$ are unit vectors in the x and y directions. Without loss of generality, we choose the coordinates such that $\text{Re}(B) > 0$. As we point out below, our argument does not hold for the critical case of purely imaginary magnetic fields [$\text{Re}(B) = 0$]. Introducing the complex-valued cyclotron frequency $\omega_c = \frac{qB}{m_q c} \in \mathbb{C}$ we define two distinct creation operators

$$\hat{a}^\S = \frac{1}{\sqrt{2m_q\hbar\omega_c}} \left[\frac{m_q\omega_c}{2}(\hat{x} - i\hat{y}) - i(\hat{p}_x - i\hat{p}_y) \right], \quad (3a)$$

$$\hat{b}^\S = \frac{1}{\sqrt{2m_q\hbar\omega_c}} \left[\frac{m_q\omega_c}{2}(\hat{x} + i\hat{y}) - i(\hat{p}_x + i\hat{p}_y) \right] \quad (3b)$$

and the associated annihilation operators

$$\hat{a} = \frac{1}{\sqrt{2m_q\hbar\omega_c}} \left[\frac{m\omega_c}{2}(\hat{x} + i\hat{y}) + i(\hat{p}_x + i\hat{p}_y) \right], \quad (4a)$$

$$\hat{b} = \frac{1}{\sqrt{2m_q\hbar\omega_c}} \left[\frac{m\omega_c}{2}(\hat{x} - i\hat{y}) + i(\hat{p}_x - i\hat{p}_y) \right]. \quad (4b)$$

These operators fulfill the usual commutation relations [48]

$$[\hat{a}, \hat{a}^\S] = [\hat{b}, \hat{b}^\S] = 1 \quad \text{and} \quad (5a)$$

$$[\hat{a}, \hat{b}^\S] = [\hat{b}, \hat{a}^\S] = [\hat{a}, \hat{b}] = [\hat{a}^\S, \hat{b}^\S] = 0. \quad (5b)$$

We stress that $\hat{a}^\dagger \neq \hat{a}^\S$ and $\hat{b}^\dagger \neq \hat{b}^\S$, if the imaginary part of the magnetic field is nonzero. Rewriting the Hamiltonian in terms of the creation and annihilation operators yields

$$\hat{H} = \hbar\omega_c(\hat{a}^\S\hat{a} + \frac{1}{2}), \quad (6)$$

which is the complex-frequency analog of a quantum harmonic oscillator. From the algebraic structure of Eqs. (5) and (6) that the eigenenergies of the non-Hermitian Landau level are given by

$$\epsilon_n = \hbar\omega_c(n + \frac{1}{2}), \quad (7)$$

and that each Landau level is infinitely degenerate, because the second pair of creation and annihilation operators does not appear in the Hamiltonian. As we discuss below, the eigenvectors corresponding to these eigenvalues are square integrable.

A. Angular momentum

The two-dimensional angular momentum operator $\hat{L} = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$ commutes with the Hamiltonian in Eq. (2). Therefore they can be diagonalized simultaneously and we expect that the non-Hermitian Landau-level eigenstates can be distinguished by considering the angular momentum. Expressing

the angular momentum operator in terms of the creation and annihilation operators results in

$$\hat{L} = \hbar(\hat{a}^\S\hat{a} - \hat{b}^\S\hat{b}). \quad (8)$$

The angular momentum eigenvalues $l_{n,m} = \hbar(n - m)$ are determined by the algebraic structure of the creation and annihilation commutation relations Eq. (5). Each eigenstate of the Hamiltonian is labeled by the Landau-level index $n \in \mathbb{N}_0$, and an additional quantum number $m \in \mathbb{N}_0$.

B. Biorthogonal eigenstates

The eigenstates of the non-Hermitian Landau levels can be directly constructed from the creation and annihilation operators in Eqs. (3) and (4). We find the right eigenstates of the Hamiltonian

$$|n, m\rangle_R = \frac{(\hat{a}^\S)^n (\hat{b}^\S)^m}{\sqrt{n!} \sqrt{m!}} |0, 0\rangle_R, \quad (9)$$

where $\hat{H}|n, m\rangle_R = \omega_n|n, m\rangle_R$ and $\hat{L}|n, m\rangle_R = l_{n,m}|n, m\rangle_R$. The right ground state $|0, 0\rangle_R$ is annihilated by both $\hat{a}|0, 0\rangle_R = 0$ and $\hat{b}|0, 0\rangle_R = 0$. The wave functions of the right eigenstates $\psi_{n,m}^R(x, y) = \langle \mathbf{x} | n, m \rangle_R$ are given by

$$\begin{aligned} \psi_{n,m}^R(x, y) &= \frac{a_c^{n+m-1}}{\sqrt{2^{n+m+1}\pi n!m!}} e^{(1/4a_c^2)(x^2+y^2)} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)^m \\ &\times \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)^n e^{-(1/2a_c^2)(x^2+y^2)}, \end{aligned} \quad (10)$$

where we introduced the complex-valued magnetic length $a_c = \sqrt{\hbar/(qB)}$, with the branch cut of the complex square root taken along the negative real axis. The wave functions are normalizable with respect to the norm $\iint dx dy |\psi_{n,m}^R(x, y)|^2$ because of the assumption $\text{Re}(B) > 0$, and they are linearly independent. However, due to the non-Hermiticity the right eigenstates are not orthogonal to each other. Instead we need to construct the left eigenstates of the Hamiltonian, which are biorthogonal to the right eigenstates [49,50]. Left eigenstates are defined by ${}_L\langle n, m | \hat{H} = \omega_n {}_L\langle n, m |$ and in terms of the creation and annihilation operators they are given by

$$|n, m\rangle_L = \frac{(\hat{a}^\dagger)^n (\hat{b}^\dagger)^m}{\sqrt{n!} \sqrt{m!}} |0, 0\rangle_L. \quad (11)$$

The left eigenstates are derived from the left ground state, which is defined from the annihilation conditions $(\hat{a}^\S)^\dagger|0, 0\rangle_L = 0$ and $(\hat{b}^\S)^\dagger|0, 0\rangle_L = 0$. The left wave functions $\psi_{n,m}^L(x, y) = \langle \mathbf{x} | n, m \rangle_L$ are given by

$$\begin{aligned} \psi_{n,m}^L(x, y) &= \frac{(a_c^*)^{n+m-1}}{\sqrt{2^{n+m+1}\pi n!m!}} e^{[1/4(a_c^*)^2](x^2+y^2)} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)^m \\ &\times \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)^n e^{-[1/2(a_c^*)^2](x^2+y^2)}. \end{aligned} \quad (12)$$

From Eqs. (10) and (12) it is clear that in the Hermitian limit $B \in \mathbb{R}$ the left and right wave functions are identical. We have chosen the normalization of the ground-state wave functions such that $\iint dx dy [\psi_{0,0}^L(x, y)]^* \psi_{0,0}^R(x, y) = 1$. The biorthogonality relation between the left and right eigenstates

$${}_L\langle n', m' | n, m \rangle_R = \delta_{n'n} \delta_{m'm} \quad (13)$$

follows from the creation and annihilation operator commutation relations.

In the symmetric gauge the wave functions are normalizable, because they decay exponentially for large $|\mathbf{r}|^2$ with a decay length that is nonzero only if $\text{Re}(0) > 0$. The reason for this is that the real part of the magnetic field induces the cyclotron behavior of the eigenstates, which spatially confines the wave function. In the limiting case $\text{Re}(B) = 0$ the states described by Eq. (10) are not decaying, but instead are either constant for $n = m = 0$ or grow polynomially for large $|\mathbf{r}|^2$. Therefore these states are not normalizable and thus are not eigenstates of the Hamiltonian. For the case of purely imaginary fields a solution continuum eigenvalue problem remains unknown.

C. Derivation of eigenstates in the Landau gauge

In the Landau gauge the non-Hermitian vector potential is given by $A(x, y) = Bxe_y$. The resulting non-Hermitian Hamiltonian is given by

$$\hat{H} = \frac{1}{2m_q} \left[\hat{p}_x^2 + \left(\hat{p}_y - \frac{qB}{c} \hat{x} \right)^2 \right]. \quad (14)$$

By introducing the nonunitary shift operators

$$\hat{V} = e^{-(i/\hbar)[-(c/qB)\hat{p}_y]}, \quad (15)$$

$$\hat{V}^{-1} = e^{(i/\hbar)[-(c/qB)\hat{p}_y]}, \quad (16)$$

with

$$\hat{V}\hat{x}\hat{V}^{-1} = \hat{x} + \frac{c}{qB}\hat{p}_y, \quad (17)$$

we find the transformed Hamiltonian

$$\hat{H}_s = \hat{V}\hat{H}\hat{V}^{-1} = \frac{1}{2m_q}\hat{p}_x^2 + \frac{m_q}{2}\omega_c^2\hat{x}^2. \quad (18)$$

Thus the problem is now reduced to a one-dimensional quantum harmonic oscillator with complex frequency. Since the shifted Hamiltonian does not depend on $\hat{p}_y$ anymore, each eigenstate of $\hat{H}_s$ will be highly degenerate, thus reproducing the degeneracy observed for the symmetric gauge. To solve the shifted Hamiltonian $\hat{H}_s$ we introduce

$$\hat{d}_s^\dagger = \frac{1}{\sqrt{2m_q\hbar\omega_c}}[m_q\omega_c\hat{x} - i\hat{p}_x] \quad \text{and} \quad (19)$$

$$\hat{d}_s = \frac{1}{\sqrt{2m_q\hbar\omega_c}}[m_q\omega_c\hat{x} + i\hat{p}_x]. \quad (20)$$

These operators fulfill again the generalized bosonic commutation relations [48]

$$[\hat{d}_s, \hat{d}_s^\dagger] = 1 \quad \text{and} \quad [\hat{d}_s, \hat{d}_s] = [\hat{d}_s^\dagger, \hat{d}_s^\dagger] = 0, \quad (21)$$

and expressed in these creation and annihilation operators the Hamiltonian reads

$$\hat{H}_s = \frac{1}{2}\hbar\omega_c(\hat{d}_s^\dagger\hat{d}_s + \frac{1}{2}). \quad (22)$$

Using the non-Hermitian creation and annihilation operators we can again construct right and left eigenstates from the right and left vacuum states, defined by

$$|\tilde{n}\rangle_R = \frac{(\hat{d}_s^\dagger)^n}{\sqrt{n!}}|\tilde{0}\rangle_R \quad \text{and} \quad |\tilde{n}\rangle_L = \frac{(\hat{d}_s^\dagger)^n}{\sqrt{n!}}|\tilde{0}\rangle_L, \quad (23)$$

with $\hat{d}|\tilde{0}\rangle_R = 0$ and $(\hat{d}_s^\dagger)^\dagger|\tilde{0}\rangle_L = 0$. The right eigenstates of $\hat{H}_s$ are given by

$$\tilde{\phi}_n^R(x) = \frac{a_c^n}{\sqrt{a_c n! \sqrt{2^{n+1}\pi}}} e^{(1/4a_c^2)x^2} \frac{\partial^n}{\partial x^n} e^{-(1/2a_c^2)x^2} \quad (24)$$

and are normalizable for all magnetic field strengths with finite real part. Thus we find solutions of the two-dimensional Hamiltonian in the Landau gauge for all B with $\text{Re}(B) > 0$. Since the system is translationally invariant in the y direction, we find the plane-wave solutions for the two-dimensional problem in the Landau gauge given by

$$\phi_{n,k}^R(x, y) = e^{-iky} \tilde{\phi}_n^R \left(x - \frac{c\hbar k}{qB} \right), \quad (25)$$

where we applied the inverse shift operator.

To obtain normalizable solutions of the Hamiltonian we need to take superposition for different k , creating a normalizable wave packet. It is sufficient to show that we find normalizable wave packets for the lowest Landau level, because the convergence properties are independent of the polynomial part generated by the derivatives in Eq. (25). We construct the $n = 0$ wave packet

$$\Psi(x, y) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} f(k) \phi_{0,k}^R(x, y) \quad (26)$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi} f(k) e^{-iky} \tilde{\phi}_{0,k}^R(x). \quad (27)$$

We find for the norm of the wave packet

$$\begin{aligned} \|\Psi\|^2 &= \iint_{-\infty}^{\infty} dx dy \iint_{-\infty}^{\infty} \frac{dk dk'}{(2\pi)^2} \\ &\times \{ e^{-i(k-k')y} f(k) [f(k')]^* \tilde{\phi}_{0,k}^R(x) [\tilde{\phi}_{0,k'}^R(x)]^* \} \end{aligned} \quad (28)$$

$$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dk}{2\pi} |f(k)|^2 |\tilde{\phi}_{0,k}^R(x)|^2 \quad (29)$$

$$= \frac{\sqrt{q|B|}}{\sqrt{c\hbar}\sqrt{\text{Re}(B)}} \int_{-\infty}^{\infty} \frac{dk}{2\pi} |f(k)|^2 e^{[c\hbar[\text{Im}(B)]^2/q|B|^2 \text{Re}(B)]k^2}. \quad (30)$$

The condition $\|\Psi\|^2 < \infty$ for $\text{Re}(B) > 0$ can always be satisfied by an appropriate choice of $f(k)$ which decays fast enough. Thus we can construct normalizable eigenstates in the Landau gauge for all $\text{Re}(B) > 0$.

III. NONUNITARY GAUGE TRANSFORMATIONS

We have seen in both the symmetric gauge and the Landau gauge that the non-Hermitian Landau levels exist, and have also explicitly constructed normalizable eigenstates for each gauge. One can also try to obtain eigenstates of one gauge from the other by applying a gauge transformation leaving B invariant. However, as we now discuss, the corresponding gauge transformations are not necessarily unitary for complex-valued magnetic fields. Choosing, for example, the gauge function $\chi(x, y) = -(B/2)xy$ results in $A(x, y) = \frac{1}{2}B(\hat{x}\mathbf{e}_y - \hat{y}\mathbf{e}_x) \rightarrow A(x, y) - \nabla\chi(x, y) = Bxe_y$, which transforms the symmetric gauge to the Landau gauge. The corresponding gauge transformation is given

by $\hat{U} = \exp[-i\frac{q}{c\hbar}\chi(\hat{x}, \hat{y})] = \exp(\frac{i}{2}a_c^{-2}\hat{x}\hat{y})$ [1]. Since $a_c \in \mathbb{C}$ this constitutes a nonunitary gauge transformation $\hat{U}^{-1} \neq \hat{U}^\dagger$, which transforms the Hamiltonian of the system according to $\hat{H} \rightarrow \hat{U}\hat{H}\hat{U}^{-1}$. Due to the unboundedness of $\hat{U}$ the wave functions of the transformed biorthogonal eigenstates $|n, m\rangle_R \rightarrow \hat{U}|n, m\rangle_R$ and $|n, m\rangle_L \rightarrow (\hat{U}^{-1})^\dagger|n, m\rangle_L$ may not be square integrable. For the transformation to the Landau gauge it is straightforward to show that the wave functions $\tilde{\psi}_{n,m}^R = \langle x|\hat{U}|n, m\rangle_R$ scale as

$$|\tilde{\psi}_{n,m}^R| \sim O([x, y]^{n+m})e^{-(q/4c\hbar)[\text{Re}(B)(x^2+y^2)-2\text{Im}(B)xy]}. \quad (31)$$

Thus the wave functions are well-localized and square-integrable functions only for $\text{Re}(B) > |\text{Im}(B)|$. If this condition is violated, the images of the symmetric-gauge eigenstates under $\hat{U}$ are unbounded and thus they are not eigenstates of the transformed Hamiltonian. Similarly, we can perform nonunitary gauge transformation $\hat{U}^{-1}$ to localized eigenstates in the Landau gauge to try to obtain localized eigenstates in the symmetric gauge. One can confirm that this method, again, results in localized eigenstates in the symmetric gauge only when $\text{Re}(B) > |\text{Im}(B)|$. This implies that while normalizable eigenstates exist in both gauges, we cannot always find them by a simple gauge transformation, because of the unbounded nature of the nonunitary gauge for complex-valued magnetic fields.

The nonunitary gauge transformations affect the properties of the eigenstates beyond their normalizability. Let us consider the non-Hermitian Schrödinger equation $i\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle$, and consider the gauge transformation with a nonunitary operator $\hat{U}$ by defining $|\psi'\rangle = \hat{U}|\psi\rangle$. By acting on the Schrödinger equation from the left by $\hat{U}$ we can write $i\frac{\partial}{\partial t}|\psi'\rangle = U\hat{H}U^{-1}|\psi'\rangle$. Thus the Schrödinger equation remains invariant, if we define the transformed Hamiltonian as $\hat{H}' \equiv \hat{U}\hat{H}\hat{U}^{-1}$. It is then natural to define for an arbitrary observable $\hat{O}$ the transformed observable to be $\hat{O}' \equiv \hat{U}\hat{O}\hat{U}^{-1}$. We note that this transformation is a similarity transformation and thus the spectrum of the observable remains unchanged. However, we can see that the expectation value of the observable does change because

$$\langle \hat{O}' \rangle = \frac{\langle \psi'|\hat{O}'|\psi' \rangle}{\langle \psi'|\psi' \rangle} = \frac{\langle \psi|\hat{U}^\dagger\hat{U}\hat{O}|\psi \rangle}{\langle \psi|\hat{U}^\dagger\hat{U}|\psi \rangle} \neq \frac{\langle \psi|\hat{O}|\psi \rangle}{\langle \psi|\psi \rangle} = \langle \hat{O} \rangle. \quad (32)$$

The two quantities are not equal because of the nonunitarity $\hat{U}^\dagger \neq \hat{U}^{-1}$. Consequently, the choice of the gauge for complex-valued magnetic fields has physical consequences and must always be specified when discussing such models. An exception is gauge transformations, which only gauge the real part of the magnetic field and leave the imaginary part of $A(x, y)$ invariant. It is clear that for these gauge transformations we can choose the gauge function to be real valued. Hence, those gauge transformations do not affect physical observables, since the operator $\hat{U}$ is unitary for $\chi(x, y) \in \mathbb{R}$. We note that, while observables such as the spread of the eigenstate change under generic nonunitary gauge transformations, all biorthogonal expectation values are invariant under any gauge transformation of the complex-valued magnetic field.

The same consideration of gauge transformations holds for lattice models. On such lattice models the difference between gauging the real and the imaginary part of the magnetic field becomes especially clear. While the real part of the gauge function $\chi(x, y)$ affects only the distribution of the phases around a plaquette of the lattice leaving the enclosed flux invariant, the imaginary part of $\chi(x, y)$ changes the directional hopping strengths along the edges of the plaquettes. In the Landau gauge only the hoppings in the y direction experience directional hopping strength, while in the symmetric gauge hopping in all directions exhibits directional hopping strength.

IV. NON-HERMITIAN HARPER-HOFSTADTER MODEL

To confirm our description of the non-Hermitian Landau levels we numerically solve a finite-size non-Hermitian Harper-Hofstadter model with lattice spacing a [51,52]. For $a_c/a \gg 1$ the flux through each plaquette of the lattice is small and we expect that the non-Hermitian Landau levels will be resolved. The tight-binding Hamiltonian for the bare lattice without magnetic fields is given by

$$\hat{H} = - \sum_{\langle i,j \rangle} t_{ij} \hat{c}_i^\dagger \hat{c}_j + \tilde{t}_{ij} \hat{c}_j^\dagger \hat{c}_i, \quad (33)$$

where $t_{ij} = \tilde{t}_{ij} = 1$ and the sum runs over nearest neighbors. For the nonvanishing magnetic field each plaquette of the lattice is pierced by the complex-valued flux $\oint_{\square} A(\mathbf{r}) \cdot d\mathbf{r} = B$, where we set $a = 1$. We introduce the complex-valued flux by Peierls substitution

$$t_{ij} \rightarrow t_{ij} e^{i(q/c\hbar) \int_{r_j}^{r_i} A(\mathbf{r}) \cdot d\mathbf{r}}, \quad (34a)$$

$$\tilde{t}_{ij} \rightarrow \tilde{t}_{ij} e^{-i(q/c\hbar) \int_{r_j}^{r_i} A(\mathbf{r}) \cdot d\mathbf{r}}. \quad (34b)$$

Transforming t_{ij} and $\tilde{t}_{ij}$ individually is important, because they are not related by complex conjugation for complex-valued magnetic fields.

We solve for the eigenvalues of the non-Hermitian Harper-Hofstadter model with $B = 0.1 - 0.001i$ on a 50×50 lattice in the symmetric gauge with centered origin by means of exact diagonalization. We set $q = 1$ and $c = \hbar = 1$ and from the bare hopping $t_{ij} = \tilde{t}_{ij} = 1$ we find $m_q = \frac{1}{2}$ from the Taylor expansion of the band structure around $\mathbf{k} = 0$, resulting in the complex-valued cyclotron frequency $\omega_c = 2B$. The complex-valued eigenenergies are plotted in Fig. 1. Close to the lower boundary of the energy band, where the dispersion approximates the dispersion of the free continuum Hamiltonian from Eq. (2), we find the non-Hermitian Landau levels. The eigenenergies cluster around $\epsilon = -4 + \epsilon_n$. To distinguish the different non-Hermitian Landau-level eigenstates each eigenvalue is colored according to their angular momentum, calculated from the numerical right eigenstates. While the non-Hermitian Landau levels are fully degenerate in energy for the continuum model, the imaginary part of the eigenvalues of the finite-size Harper-Hofstadter Hamiltonian increases for decreasing angular momentum (larger m). For each level this results in pronounced eigenvalue arcs that spiral into the center of the eigenvalue distribution. This is an edge effect of the finite-sized lattice model, which is more pronounced for lower angular momenta due to the increased spread of the wave function. Along the edge the nonreciprocity of the

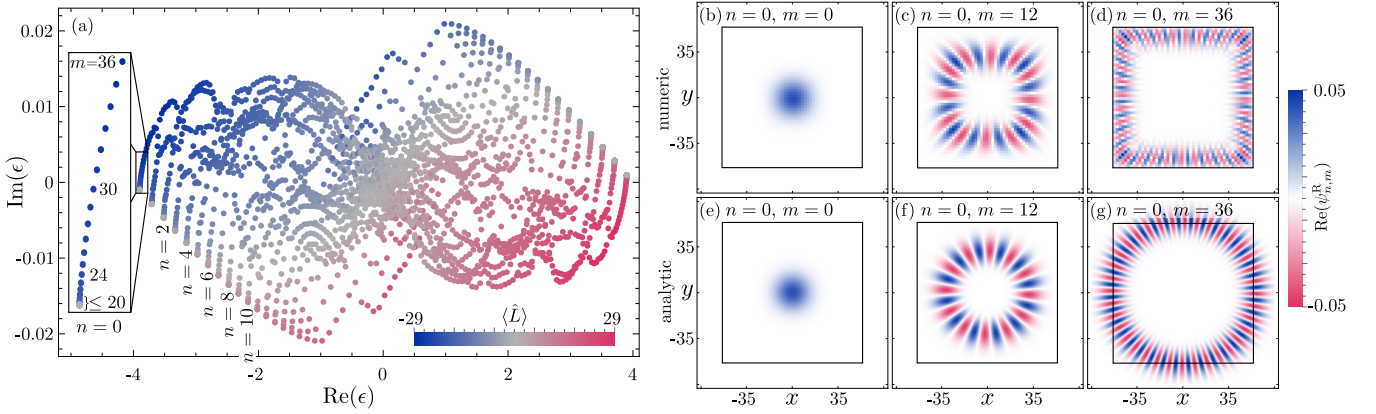


FIG. 1. Eigenvalues and right eigenstates of the non-Hermitian Harper-Hofstadter model on a 50×50 square lattice with lattice spacing $a = 1$ and the complex-valued magnetic field $B = 0.1 - 0.001i$. In panel (a) the complex-valued eigenenergies are plotted, where the color indicates the planar angular momentum calculated from the eigenstates. The first ten non-Hermitian Landau levels can be identified from the eigenvalue distribution. The inset shows that for larger m the numerical eigenenergy deviates from the analytical value as described in the text. To prove that this is indeed a finite-size effect we show the numerical eigenstates from three states from the lowest non-Hermitian Landau level in panels (b)–(d) and compare them to the wave functions given in Eq. (10), which are plotted in panels (e)–(g). The black square is drawn in all plots along the edge of the simulated lattice. The comparison shows that the Landau levels are recovered only as long as the wave functions are not extended beyond the edge of the simulated lattice.

hopping results in chiral edge amplification, similar to that of a periodic Hatano-Nelson model. The Landau-level eigenstates experience amplification for $\text{Im}(B) < 0$ and damping for $\text{Im}(B) > 0$, due to an interplay of angular momentum and the direction of the edge nonreciprocity. In Fig. 1 three eigenstates are shown for the same non-Hermitian Landau level to confirm the relation between angular momentum and edge-induced gain of the eigenstate. All three eigenstates are compared to the analytic solutions of the continuum model highlighting the differences between the wave functions at smaller angular momentum.

V. WAVE-PACKET DYNAMICS

In addition to the diagonalization of the non-Hermitian Harper-Hofstadter model for weak complex-valued magnetic fields we simulated the evolution of a Gaussian wave packet on the lattice.

We prepared the wave packet initially centered at the origin of the lattice $\psi_0 = \exp[-(x^2 + y^2)/(2\sigma^2)]$ with a width $\sigma = 5$ and imprinted a finite initial momentum $\mathbf{k}_0 = \pi/40\hat{\mathbf{x}}$ onto the wave packet. For a real magnetic field, the center of mass of the wave packet is described by semiclassical cyclotron motion, which forms a circle. In contrast, the motion of the wave packet in the complex-valued magnetic field is described by a spiral, which spirals inwards for $\text{Im}(B) < 0$ and outwards for $\text{Im}(B) > 0$. The motion of the center of mass $\mathbf{r}_c$ is still described by the semiclassical equations of motion with complex-valued Lorentz force. First, recall that, under the Hermitian magnetic field [$\text{Im}(B) = 0$], the equation of motion for the center-of-mass position $\mathbf{r}_c = (x_c, y_c)$ under external electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ is $m_{\text{eff}}\ddot{\mathbf{r}}_c = q(\mathbf{E} + \dot{\mathbf{r}}_c \times \mathbf{B})$. In terms of the complex coordinate $z_c \equiv x_c + iy_c$, this equation can be written as

$$m_{\text{eff}}\ddot{z}_c = q(E_c - iB\dot{z}_c), \quad (35)$$

where $E_c \equiv E_x + iE_y$. We find that the center-of-mass motion of a Gaussian wave packet under a complex magnetic field is described by the same equation where B is replaced by a complex magnetic field. In terms of the original coordinates, this results in

$$m_{\text{eff}}\ddot{\mathbf{r}}_c = q[\mathbf{E} + \dot{\mathbf{r}}_c \times \text{Re}(\mathbf{B}) + \text{Im}(B)\dot{\mathbf{r}}_c]. \quad (36)$$

From this expression, we can see that the imaginary part of the magnetic field results in a drag force. In Appendix A, we show how this equation can be recast as a complex frequency harmonic oscillator for vanishing electric field. The frequency of oscillation is given by $\text{Re}(\omega_c)$ and its imaginary part governs the spiraling behavior. In Fig. 2 the trajectory of the center of mass of the simulated wave packet is compared to the analytic for $m_{\text{eff}} = 0.515$, given the same initial conditions. The deviations between the numerical solution and the analytical spiral can be explained by the width- and position-dependent effective mass m_{eff} of the wave packet dynamics (see Appendix B for the details).

Similar results are obtained for the wave packet initially at rest in a constant electric field $\mathbf{E} = E\hat{\mathbf{x}}$ with $E = 0.1$ in addition to the complex-valued magnetic field. The solution of Eq. (36) is now given by a skipping trajectory, derived in Appendix A. This is confirmed by the numerical simulation of the dynamics, shown in Fig. 2. Deviations from the analytic curve arise again due to the varying effective mass m_{eff} .

VI. EXPERIMENTAL IMPLEMENTATION

Possible platforms for the implementation of the lattice Hamiltonian, Eq. (33), are topoelectric circuits, acoustic, plasmonic, optical metasurfaces, and mechanical metamaterials [35,53–57]. In particular, in mechanical metamaterials [35,57], where macroscopic mechanical oscillators are coupled using springs or magnets, their collective excitations are governed by a lattice model, which describes the transport

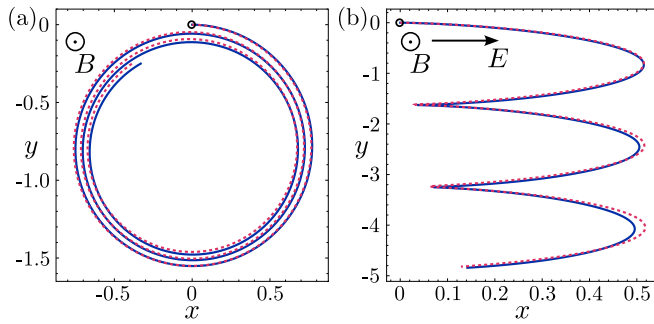


FIG. 2. Trajectories of the center of mass of Gaussian wave packets in the presence of a complex-valued magnetic field. Their evolution is simulated on a 50×50 square lattice governed by the Harper-Hofstadter model with $B = 0.1 - 0.001i$ in the symmetric gauge with centered origin. The initial position is marked by a black circle. In panel (a) the spiral trajectory of a wave packet with finite initial velocity is shown in blue and compared to the solution of the semiclassical equation of motion, plotted in red. In panel (b) the wave packet is initially at rest and an in-plane electric field $\mathbf{E} = 0.005\mathbf{e}_x$ is applied. The center of mass follows a skipping trajectory, shown in blue, and this curve is again compared to the analytic solution, shown in red. The analytic expressions are derived in Appendix A.

of mechanical energy through the system. In these materials complex-valued nonreciprocal hopping amplitudes can be engineered by measuring the position of each oscillator in real time and applying controlled feedback forces to each individual oscillator. Given this site-dependent flexibility, our lattice Hamiltonian can be directly implemented in a two-dimensional mechanical metamaterial and the propagation of wave packets can be directly measured.

Another possibility is to shape the desired Hamiltonian by employing adiabatic potentials [58–61]. In systems with fast internal degrees of freedom, such as multilevel atoms in ultracold atomic gasses, the adiabatic evolution of said atoms is governed in part by adiabatic potentials, derived from the quantum geometry associated with the fast degrees of freedom. If atoms in their excited state are lost from the condensate, these adiabatic potentials become non-Hermitian and thus can result in complex-valued magnetic fields acting on the condensate wave function [62]. A similar idea was utilized recently to create imaginary vector potentials in one dimension in a Bose-Einstein condensate [63]. In this continuum system a spin-orbit coupled quantum gas was prepared in a one-dimensional condensate. By coupling one of the spin states via microwave radiation to a reservoir state, which spontaneously decays into a state other than the condensate, non-Hermiticity is introduced in the dynamics of the condensate. This non-Hermiticity could be tuned to realize the Hatano-Nelson model, described by a one-dimensional imaginary vector potential [32]. Going beyond this one-dimensional example, a generalization to imaginary synthetic magnetic fields for ultracold atoms was proposed [64], in which a method to realize spatially varying imaginary vector potential was discussed. This vector potential results in an imaginary magnetic field acting on the condensate. By combining this novel method with the established techniques for generating real synthetic magnetic fields for ultracold atoms, the

continuum system described by Eq. (2) can be probed experimentally [65].

VII. CONCLUSION

Our work shows that complex-valued magnetic fields can be incorporated into continuum systems. Considering the system first in the symmetric gauge shows that essential properties such as the normalizability and locality of the eigenstate wave functions are intact for complex-valued magnetic fields. It further yields analytic expressions for the biorthogonal wave functions, which are labeled by Landau-level index and angular momentum. While we find analogous results in the Landau gauge, where we constructed normalizable wave packets, we observed that the solutions of symmetric and Landau gauge can only be mapped onto each other by a nonunitary gauge transformation in the case of $\text{Re}(B) > \text{Im}(B)$. This shows the difficulties when dealing with unbounded nonunitary gauge transformations for complex-valued magnetic fields. Further, we highlighted how gauging the system may change physical observables of the system and that these properties are not only an artifact of the continuum model, but are reflected by the lattice model as well. In the lattice model the gauge of the imaginary part of the magnetic field described a distortion of the nonreciprocal hopping strengths, which strongly affects the behavior of the system. We would also like to point out that, although it is known that under the real magnetic fields, the discrete Landau-level eigenvalues exhaust the spectrum [66], we do not have a mathematical argument if the discrete complex Landau-level eigenvalues we found exhaust the spectrum or if there should be additionally a continuous spectrum. The non-Hermitian Landau-level wave functions allows us to construct certain many-body wave functions such as non-Hermitian Laughlin states, which potentially results in fractional topological states in non-Hermitian interacting systems.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: SPECIAL SOLUTIONS OF THE SEMICLASSICAL EQUATIONS OF MOTION FOR THE COMPLEX-VALUED LORENTZ FORCE

Here we present the solutions of the semiclassical equations of motion for the wave-packet center-of-mass dynamics

presented in the main text for both numerically probed configurations. The semiclassical equations of motion in the presence of complex-valued magnetic fields, taken from Eq. (17) in the paper, are given by

$$m_{\text{eff}} \ddot{\mathbf{r}}_c = q[\mathbf{E} + \dot{\mathbf{r}}_c \times \text{Re}(\mathbf{B}) + \text{Im}(\mathbf{B}) \dot{\mathbf{r}}_c]. \quad (\text{A1})$$

The center-of-mass position of the wave packet is given by $\mathbf{r}_c = (x_c, y_c)^T$. In the following we consider first the solution for vanishing electric field and include electric fields later.

1. Spiraling cyclotron orbit

For vanishing electric field and real magnetic field one obtains circular motion with the frequency ω_c . If the magnetic field has a finite imaginary part, we can show that this still holds, now with a complex frequency. The equation of motion for the complex coordinate Eq. (35) yields

$$\ddot{z}_c = -i \frac{qB}{m_{\text{eff}}} \dot{z}_c \Rightarrow \dot{z}_c = v_{z,0} e^{-i\omega_c t}, \quad (\text{A2})$$

where we set $c = 1$ and $v_{z,0} = \dot{x}_c(0) + i\dot{y}_c(0)$ is a constant depending on the initial velocity of the wave packet. We find the center-of-mass position by simply integrating Eq. (A2) resulting in

$$z_c = i \frac{v_{z,0}}{\omega_c} e^{-i\omega_c t} + \left(z_c(0) - i \frac{v_{z,0}}{\omega_c} \right) \quad (\text{A3a})$$

$$= i \frac{v_{z,0}}{\omega_c} e^{-i \text{Re}(\omega_c) t} e^{i \text{Im}(\omega_c) t} + \left(z_c(0) - i \frac{v_{z,0}}{\omega_c} \right). \quad (\text{A3b})$$

Since the frequency of the oscillation $\omega_c \in \mathbb{C}$ the trajectory always spirals inwards or outwards depending on $\text{Im}(B)$, which can be seen by splitting the frequency in the real and imaginary parts. We recover the center-of-mass position by $\text{Re}(z_c) = x_c$ and $\text{Im}(z_c) = y_c$.

To compare the analytic solution, Eq. (A3), to the numerical solution we numerically determined the initial velocity and measured the effective mass of the bare lattice, as discussed in Appendix B, given the initial width of the wave packet. The initial position is given from the initiation of the wave packet; however, due to the complex-valued vector potential there is no simple relation between the momentum imprinted on the wave packet and the initial velocity, thus $v_{z,0}$ needed to be determined numerically.

2. Skipping orbit in electric field

We now want to solve Eq. (35) in the presence of a constant electric field. We can solve for $\dot{z}_c$ by variation of parameters finding

$$\dot{z}_c = -i \frac{E_c}{B} + \left(v_{z,0} + i \frac{E_c}{B} \right) e^{-i\omega_c t}. \quad (\text{A4})$$

Integrating this over t and considering the initial conditions results in

$$z_c = -i \frac{E_c}{B} t + i \frac{1}{\omega_c} \left(v_{z,0} + i \frac{E_c}{B} \right) e^{-i\omega_c t} \quad (\text{A5})$$

$$+ \left[z_c(0) - i \frac{1}{\omega_c} \left(v_{z,0} + i \frac{E_c}{B} \right) \right], \quad (\text{A6})$$

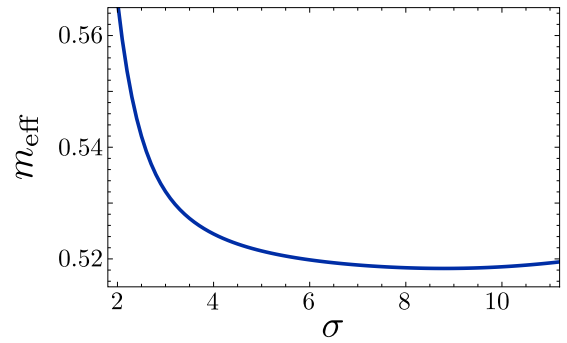


FIG. 3. Effective mass m_{eff} depending on the wave-packet width for a 50×50 square lattice for vanishing magnetic field.

where the first term described the drift perpendicular to the electric field and the second term results in the skipping motion with decreasing amplitude. We again recover the center-of-mass position by $\text{Re}(z_c) = x_c$ and $\text{Im}(z_c) = y_c$.

In the numerical simulation the wave packet is initially at rest, therefore we do not need to fit the initial velocity but simply can set it to $v_{z,0} = 0$. The result is shown in Fig. 2 of the paper.

APPENDIX B: DISCUSSION OF WIDTH-DEPENDENT EFFECTIVE MASS OF WAVE PACKET

As mentioned in the main text, the effective mass of the wave packet m_{eff} depends in general on the width of the wave packet σ and for finite $\text{Im}(B)$ it is also (weakly) position dependent. In the parameter regime used for the simulation, the width of the wave packet affects the effective mass stronger. To highlight this we measured the effective mass of Gaussian wave packets on a 50×50 square lattice without any magnetic fields. We prepared Gaussian wave packets at the center

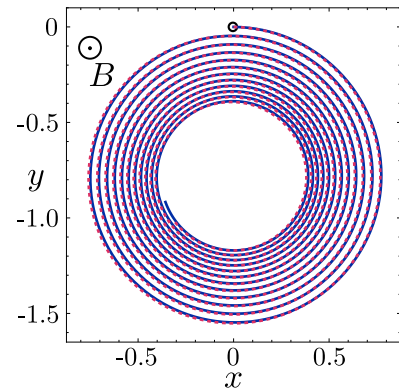


FIG. 4. Trajectory of the center of mass of the $n = m = 0$ eigenstate of the Harper-Hofstadter model. Their evolution is simulated on a 50×50 square lattice with $B = 0.1 - 0.001i$ in the symmetric gauge with centered origin. The initial position is marked by a black circle. The spiral trajectory of a wave packet with finite initial velocity is shown in blue and compared to the solution of the semiclassical equation of motion, plotted in red. It shows that if the wave packet does not breathe, the analytical and numerical curves agree over many periods of the cyclotron motion.

of the lattice with vanishing initial velocity and applied a constant electric field $\mathbf{E} = E\mathbf{e}_x$, accelerating the wave packet. From the numerically determined acceleration rate $a(\sigma)$, we can determine the effective mass as $m_{\text{eff}}(\sigma) = E/a(\sigma)$. We show the results in Fig. 3.

During the evolution in magnetic fields the width of the wave packets changes periodically, which is sometimes called breathing of the wave packet [67,68]. For real magnetic fields this change in width results in a varying velocity of the wave packet along the cyclotron orbit, but the orbit itself is not changed. Due to the damping induced by the imaginary part of the complex-valued cyclotron frequency, this is not the case for complex-valued magnetic fields. If the damping rate is modulated the trajectory changes, resulting in deviations between the numerical and analytical trajectories.

In addition to the width-dependent mass changes there is a small position dependence of the effective mass, given finite $\text{Im}(B)$. However, this position dependence is, in the position range explored by the wave packet in our simulation, about a magnitude smaller than the width dependence. We thus believe that the oscillating width of the wave packet is responsible for the deviations of the numerically determined trajectory from the analytic expressions derived above.

The breathing of the wave packet is expected to be minimal for eigenstates of the Harper-Hofstadter Hamiltonian. Taking the $n = m = 0$ eigenstate of the lattice and imprinting a finite momentum onto it, we observe damped cyclotron motion, which follows the predicted analytic trajectory perfectly for multiple periods. The comparison between the analytic curve for $m_{\text{eff}} = 0.515$ and the numerical trajectory is shown in Fig. 4.

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