

Berry curvature of low-energy excitons in rhombohedral graphene

Henry Davenport¹,² Frank Schindler¹,² and Johannes Knolle^{1,2,3}

¹Blackett Laboratory, Imperial College London, London SW7 2AZ, United Kingdom

²Technical University of Munich, TUM School of Natural Sciences, Physics Department, 85748 Garching, Germany

³Munich Center for Quantum Science and Technology (MCQST), Schellingstr. 4, 80799 München, Germany



(Received 19 August 2025; accepted 7 January 2026; published 2 March 2026)

We investigate low-energy excitons in rhombohedral pentalayer graphene encapsulated by hexagonal boron nitride (hBN/R5G/hBN), focusing on the regime at the experimental twist angle $\theta = 0.77^\circ$ and with an applied electric field. We introduce a new low-energy two-band model of rhombohedral graphene that captures the band structure more accurately than previous models while keeping the number of parameters low. Using this model, we show that the centres of the *exciton* Wannier functions are displaced from the moiré unit cell origin by a *quantized amount*—they are instead localized at C_3 -symmetric points on the boundary. We also find that the exciton shift is electrically tunable: by varying the electric field strength, the exciton Wannier center can be exchanged between inequivalent corners of the moiré unit cell. Our results suggest the possibility of detecting excitonic corner or edge modes, as well as novel excitonic crystal defect responses in hBN/R5G/hBN. Lastly, we find that the excitons in hBN/R5G/hBN inherit excitonic Berry curvature from the underlying electronic bands, enriching their semiclassical transport properties. Our results position rhombohedral graphene as a compelling tunable platform for probing exciton topology in moiré materials.

DOI: [10.1103/shrm-4swg](https://doi.org/10.1103/shrm-4swg)

I. INTRODUCTION

The study of Berry curvature effects has played a central role in advancing our understanding of noninteracting electrons in condensed matter systems [1]. More recently, there has been growing interest in extending this framework to interacting systems. Efforts in this direction include the topological classification of ground states in the presence of interactions [2–5], as well as the investigation of Berry curvature effects and the topological classification of interaction-induced excitations, such as excitons [6–11].

Rhombohedral graphene has emerged as a compelling platform for studying nontrivial ground-state topology induced by interactions. Experiments have found that rhombohedral graphene exhibits both the integer and fractional quantum Hall effects [12], and a comprehensive theoretical understanding of these phenomena remains an area of active research [13–20]. In this work, we take an alternative approach and focus on the *excitations* rather than the ground state, specifically the excitons at charge neutrality. We study rhombohedral pentalayer graphene (R5G) encapsulated by hexagonal boron nitride (hBN), shown in Fig. 1(a). When an electric field is applied, we can obtain two well-gapped electronic bands around the Fermi level at charge neutrality [13]. Since the bands immediately above and below the Fermi energy are layer polarized, the twist angle of the hBN on either side

of the R5G strongly influences the topology and dispersion of these electronic bands. The electronic bands are therefore highly tunable making hBN/R5G/hBN an ideal testbed for studying the topology of low-energy excitons emerging from topological electron bands.

In this paper, we show that, at the experimentally relevant twist angle $\theta = 0.77^\circ$ [12], the exciton Wannier states are shifted to the edge of the moiré unit cell, and their Wannier center depends on the electronic potential applied across the rhombohedral graphene layers. They are therefore analogous to obstructed atomic insulators in noninteracting electron systems [21]. We note, however, that this shift is inherited from the electronic bands and so is not like the shift excitons described in Ref. [6] where the excitons have maximally localized exciton Wannier centres [22,23] shifted with respect to the electronic Wannier centres. Despite this, we expect that rhombohedral graphene still exhibits novel “excitonic” defect and edge responses. We also find that the Berry curvature of the excitons can be tuned using the applied electric field, allowing for an experimental exploration of the role of Berry curvature on excitonic properties. Given these attributes, rhombohedral graphene presents a unique system to explore the effects of exciton Berry curvature on exciton transport properties, both experimentally and theoretically.

This paper is structured as follows. We begin by outlining the noninteracting physics underlying the band dispersion and Berry curvature distribution. In Sec. II, we introduce a new simple low-energy two-band model of rhombohedral graphene that better captures the dispersion shape around the Dirac point compared to the well-studied k^5 model [18,24]. In addition, we show how the moiré potential modifies the dispersion and band topology. In Sec. III, we detail our

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

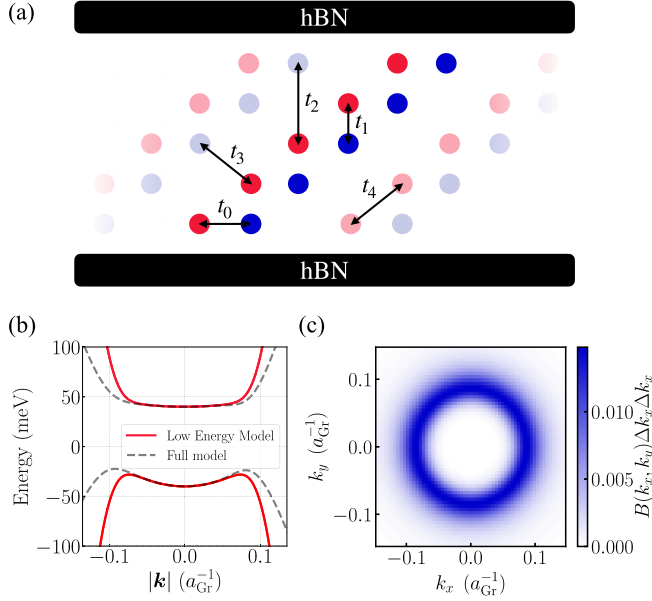


FIG. 1. Rhombohedral graphene encapsulated between two hBN layers. The hopping parameters of our tight-binding model are labeled $t_{0,\dots,4}$ in (a). The A sublattice of graphene is shown in red and the B sublattice in blue. (b) Dispersions of the low-energy model and the full model of the free-standing rhombohedral graphene (with displacement field $u_D = 20$ meV). (c) The Berry curvature $[B(k_x, k_y)]$ distribution of the low-energy model.

methodology for calculating the exciton dispersion and exciton Berry curvature. Our findings indicate that the applied electric field can be used to tune the lowest-energy excitons to regimes where the exciton Berry curvature is enhanced. We predict that this could lead to a measurable thermal Hall effect [25]. Additionally, we identify regimes where the exciton Wannier centres are obstructed from the center of the moiré unit cell, with the Wyckoff position they are centered at tunable using the applied electric field. In Sec. IV, we discuss the experimental consequences of this exciton shift in terms of defect responses detectable using electron energy loss spectroscopy (EELS) [26,27].

II. BAND STRUCTURE

We first derive an economical model that captures the band structure of R5G as well as that of hBN/R5G/hBN with a moiré twist.

A. Low-energy model

We begin by formulating a new low-energy effective model for isolated pentalayer rhombohedral graphene, which serves as the basis for all subsequent analysis. Our starting point is the full ten-band tight-binding model previously studied in Ref. [14], with interlayer and intralayer hopping parameters as illustrated in Fig. 1 (see Appendix A for a full description). This model includes an external perpendicular electric field, known as the displacement field. To model this, we add a layer-dependent onsite potential, where the

potential difference between adjacent layers is uniform and denoted by u_D [14].

When u_D is nonzero, and for momenta near the Dirac point, the electronic wave functions associated with the conduction and valence bands closest to the Fermi level become strongly localized on sublattice A of layer 1 and sublattice B of layer 5 [14]. This layer polarization enables a projection onto the corresponding subspace, yielding a reduced two-band model that captures the low-energy physics at charge neutrality (see Appendix A for detailed derivations). With our method, we can systematically include higher-order corrections beyond the simpler two-band models studied previously [18]. In particular we incorporate a key correction absent from previous work: namely, the intrinsic quadratic dispersion that dominates at small momenta $\mathbf{k}$ near the Dirac point [see Fig. 1(b)]. We show that this curvature can be systematically retained within the effective theory by including a correction quadratic in the momentum. This results in the following two-band Hamiltonian that more accurately reflects the low-energy behavior of rhombohedral graphene,

$$h_{\text{eff}}(\mathbf{k}) = \begin{pmatrix} 2u_D + \gamma_-(k_x^2 + k_y^2) & \frac{v_F^5}{t_1^5}(k_x - ik_y)^5 \\ \frac{v_F^5}{t_1^5}(k_x + ik_y)^5 & -2u_D + \gamma_+(k_x^2 + k_y^2) \end{pmatrix}, \quad (1)$$

where $v_F = \frac{\sqrt{3}}{2}t_0a$ and a is the graphene lattice constant. The numerical values of the hoppings are given in Appendix A [14,28]. The $\gamma_{\pm}$ take a simple form if we assume small u_D (specifically $u_D \ll t_3 \ll t_0$) and expand to first order in u_D ,

$$\gamma_{\pm} \approx \frac{2v_F v_4}{t_1} \pm \frac{u_D v_F^2}{t_1^2}, \quad (2)$$

where $v_4 = \frac{\sqrt{3}}{2}t_4a$. This simplified model predicts that, for $u_D > 0$, the curvature of the higher-energy band's dispersion (at the Dirac point) changes sign when the displacement field reaches $u_D = \frac{2v_4 t_1}{v_F} \sim 35$ meV. In contrast, the curvature of the lower-energy band at the Dirac point remains unchanged for all $u_D \geq 0$. This sign change is absent in the model presented in Ref. [18], which neglects this diagonal quadratic correction.

Figure 1(b) compares the band dispersions of the full and simplified models for a representative set of parameters. As expected, both models closely agree near the Dirac point, with increasing deviations at larger momenta. In addition to capturing the band dispersion, the simplified model also reproduces the Berry curvature distribution of the full model [18]. As shown in Fig. 1(c), the Berry curvature in the simplified model is concentrated in a ring around the Dirac point. This ring occurs at the momenta where the band dispersion transitions from being nearly flat to rapidly increasing or decreasing.

B. Moiré Hamiltonian

We investigate pentalayer rhombohedral graphene (R5G) encapsulated between two layers of hexagonal boron nitride (hBN), as illustrated in Fig. 1. In contrast to most previous work, which has considered coupling of the R5G to hBN on only one side [13,14,17], we assume that both the top and bottom hBN layers couple equally to the R5G (as also studied in Ref. [13]). Due to the lattice parameter mismatch

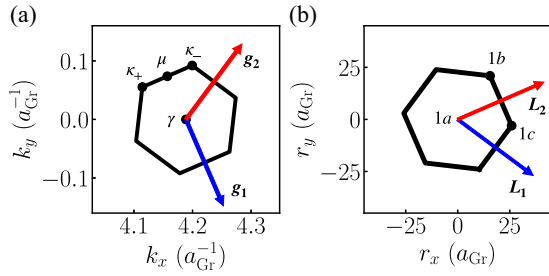


FIG. 2. The mini Brillouin zone (a) and unit cell (b) for hBN/R5G/hBN with twist angle $\theta = 0.77^\circ$. The moiré reciprocal lattice vectors are labeled $\mathbf{g}_1$, $\mathbf{g}_2$, and the high symmetry points are γ , κ_+ , κ_- and μ . The real space unit cell (b) shows the lattice vectors $\mathbf{L}_1$, $\mathbf{L}_2$ and maximal Wyckoff positions $1a$, $1b$, $1c$.

between hBN and graphene, a moiré pattern folds the bands into the mini BZ (mBZ) [shown in Fig. 2(a)]. The (real space) unit cell is enlarged compared to the isolated R5G unit cell [Fig. 2(b)]. The moiré unit cell has a periodicity $[a_{\text{moiré}} = |\mathbf{L}_1/2|]$ in Fig. 2(b)], which depends on the relative twist angle θ between the primitive lattice vectors of the hBN and R5G and the lattice constant mismatch between graphene and hBN [28]. We study the regime where both the top and bottom hBN layers are aligned at the same angle relative to the graphene, thereby avoiding the formation of a *supermoiré* pattern. This double alignment is possible experimentally, for example, in Ref. [29], graphene is encapsulated in hBN with the two hBN sheets aligned. This is achieved by aligning the exfoliated straight edges of each flake using images from an optical microscope. We model the hBN layers as an effective potential acting on the outermost layers of the rhombohedral graphene [28]. Here we summarize the method used to model the moiré system, see Appendix B for the full details. Since our effective Hamiltonian retains only the *A* sublattice in the bottom layer and the *B* sublattice in the top layer, we apply the moiré potential exclusively to these sites. For layer l , we use [28]

$$V_{\text{eff}}^l(\mathbf{r}) = V_0 + V_1 \sum_{\substack{\eta \in \{1, -1\} \\ j \in \{1, 2, 3\}}} \exp[i\eta(\mathbf{g}_j \cdot \mathbf{r} + \psi_\xi + \phi_l)]. \quad (3)$$

Here, the sum runs over moiré reciprocal lattice vectors $\mathbf{g}_j$, using the “first harmonic” approximation where we retain only the first shell of reciprocal vectors surrounding the first moiré Brillouin zone [13, 28]. There is a phase shift ψ_ξ which depends on the stacking configuration, for example, the *A* sublattice of graphene may lie above either a boron atom ($\xi = 0$) or a nitrogen atom ($\xi = 1$), and this leads to differing phases for the effective potential [28]. We restrict our study to $\xi = 0$ stacking for both the top and bottom layers (for which $\psi_\xi = 223.5^\circ$) [28]. The layer dependent phase factor is only nonzero on the top layer, $\phi_l = \frac{2\pi N_l}{3} \delta_{l,4}$ [13]. This phase accounts for the relative shift in the moiré potential between the *A* sublattice on the bottom layer and the *B* sublattice on the top layer [13].

We define the fermionic creation operators, $c_{\mathbf{k}, \mathbf{Q}, l}^\dagger = c_{\mathbf{k} + \mathbf{Q}, l}^\dagger$ where the momentum $\mathbf{k}$ is in the mBZ, $\mathbf{Q}$ are the moiré reciprocal lattice vectors, and l is the layer index. The total

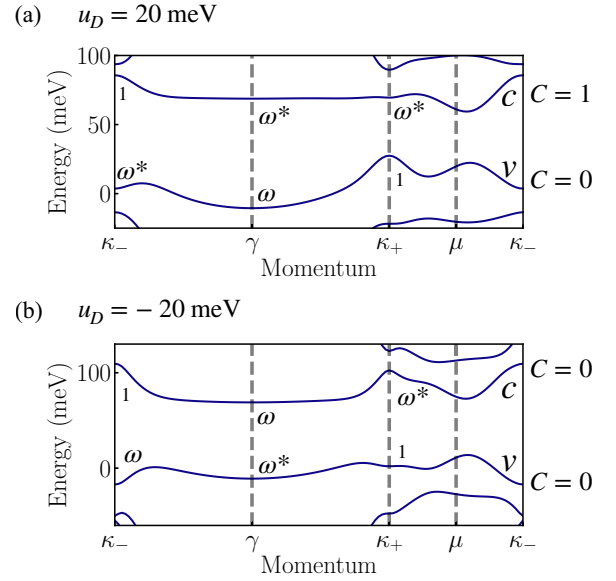


FIG. 3. Electronic (noninteracting) dispersion at $\theta = 0.77^\circ$ for displacement field $u_D = 20$ and -20 meV. The C_3 symmetry eigenvalues are shown at the high symmetry points in terms of $\omega = \exp(i2\pi/3)$.

moiré Hamiltonian is

$$H_M = H_0 + H_V \\ = \sum_{\mathbf{k} \in \text{mBZ}} \sum_{\mathbf{Q}, \mathbf{Q}'} \sum_{l, l'} h_{(\mathbf{Q}, l), (\mathbf{Q}', l')}^{(m)}(\mathbf{k}) c_{\mathbf{k}, \mathbf{Q}, l}^\dagger c_{\mathbf{k}, \mathbf{Q}', l'}, \quad (4)$$

where

$$h_{(\mathbf{Q}, l), (\mathbf{Q}', l')}^{(m)}(\mathbf{k}) = \{[h_{\text{eff}}(\mathbf{k} + \mathbf{Q})]_{l, l'} + V_0 \delta_{l, l'}\} \delta_{\mathbf{Q}, \mathbf{Q}'} \\ + V_1 \sum_{\substack{\eta \in \{1, -1\} \\ j \in \{1, 2, 3\}}} \delta_{\mathbf{Q}, \mathbf{Q}' - \eta \mathbf{g}_j}. \quad (5)$$

Here the first term $h_{\text{eff}}(\mathbf{k})$ represents the original *low-energy* rhombohedral graphene model without the moiré coupling. Although the low-energy two-band model has a continuous rotational symmetry, the hBN moiré potential breaks this but retains C_3 symmetry. As shown in Fig. 2, in the presence of the C_3 symmetry, there are three high symmetry points: γ , κ_+ and κ_- . To obtain a finite matrix size, we truncate the set of $\mathbf{Q}$ vectors in the moiré Hamiltonian to those within a ring around the γ point with $|\mathbf{Q}| \leq 5[2\pi \cdot (a_{\text{moiré}})^{-1}]$.

C. Noninteracting band structure

Using the model outlined above, we calculate the electronic dispersion of the hBN/R5G/hBN stack. Examples of the noninteracting band structures for $\theta = 0.77^\circ$ and $u_D = \pm 20$ meV are shown in Fig. 3. The hBN layers fold the bands of isolated rhombohedral graphene into the moiré Brillouin zone (mBZ) and hybridize the folded bands, resulting in two gapped bands immediately above (labeled *c*) and below the Fermi level (labeled *v*) at charge neutrality. The hBN moiré potential preserves C_3 symmetry, allowing us to use symmetry indicators to compute the Chern number (mod 3) of the electronic bands. The C_3 eigenvalues can take the values 1,

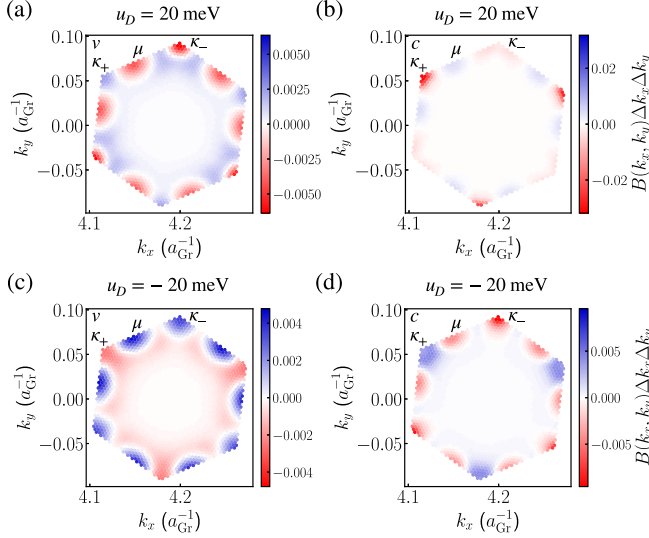


FIG. 4. *Electronic Berry curvature for the occupied (v) and empty band (c) for parameters $\theta = 0.77^\circ$ and $u_D = \pm 20$ meV.*

ω , or ω^* , where $\omega = e^{i\frac{2\pi}{3}}$. The Chern number C is related to the product of the C_3 eigenvalues ($\lambda_k^{C_3}$) at the high-symmetry points as follows [30]:

$$e^{i\frac{2\pi}{3}C} = \prod_{k \in \{\gamma, \kappa_+, \kappa_-\}} \lambda_k^{C_3}. \quad (6)$$

For $u_D = 20$ meV, we find that the empty (conduction) band has a Chern number of 1, while the occupied (valence) band has a Chern number of 0. For $u_D = -20$ meV, both bands have Chern number 0. In Appendix C, we show that a variety of possible Chern numbers for both bands can be achieved in this model by tuning the displacement field u_D and twist angle θ . In addition, we show explicitly how tuning the moiré potential gaps out the bands folded to the mBZ to generate a band with Chern number 1 at $u_D = 20$ meV. Figure 4 shows the Berry curvature of the noninteracting bands for the chosen parameters, revealing that it is concentrated at the edges of the mBZ. At $u_D = 20$ meV, the maxima and minima of the bands are also at the edge of the mBZ, indicating that low-energy charge-neutral excitations are dominated by states near the mBZ edges. We therefore expect this large electronic Berry curvature to also be visible in the exciton Berry curvature.

III. EXCITONS

We next study the exciton bound states of hBN/R5G/hBN, their dispersion, and Berry curvature.

A. Methods

We calculate the exciton dispersion and wave functions in hBN/R5G/hBN stacks that form from the two well separated bands shown Fig. 3. At charge neutrality, the ground state has the lower band (v) occupied and the upper band (c) unoccupied. We will consider excitons formed from these two bands where $c_{k,c}$ ($c_{k,v}$) annihilates an electron in the conduction (valence) band at crystal momentum k . We restrict our calculation to just these two bands since we are interested in the

low-energy spectrum, which will not be affected significantly by the inclusion of higher and lower energy bands. The wave function of an exciton at momentum p is expressed as

$$|\psi^p\rangle = \sum_{k \in \text{mBZ}} \phi_k^p c_{p+k,c}^\dagger c_{k,v} |\text{GS}\rangle, \quad (7)$$

where $|\text{GS}\rangle$ denotes the noninteracting ground state, projected onto the two bands of interest (e.g., $|\text{GS}\rangle = \prod_k c_{k,v}^\dagger |0\rangle$). We study the spin-valley polarized regime, and so we restrict our focus to excitons at just one valley and a single spin species [14].

To model the excitons, we project both the kinetic and interaction Hamiltonians into the basis $c_{p+k,c}^\dagger c_{k,v} |\text{GS}\rangle$ [31,32]. The resulting exciton Hamiltonian, detailed in Appendix D, is written in terms of noninteracting single-particle energies and the products of the noninteracting electronic wave functions (the “form factors”). These products are given by

$$U_{\alpha,\alpha',\beta,\beta'}^{p,k,k'} = \sum_Q V(p+Q) \langle u_{p+k+Q}^\alpha | u_p^{\alpha'} \rangle \langle u_{p'}^\beta | u_{p+k'+Q}^{\beta'} \rangle, \quad (8)$$

where $|u_p^\alpha\rangle$ represents the eigenvector of the moiré Hamiltonian [Eq. (5)] for momentum p and band α . We use the double-gated Coulomb potential, which accounts for the screening effects introduced by the metal electrodes either side of the sample. This potential is given by

$$V(|p|) = \frac{V_0 \tanh(|p|d)}{|p|}, \quad (9)$$

where d is the metal electrode gate distance. We use $d = 250$ Å [14] for the following calculations. The interaction strength $V_0 = \frac{2\pi}{\epsilon_0 \epsilon_r}$ depends on the relative permittivity ϵ_r , we use $\epsilon_r = 6$ [16]. We calculate the exciton Berry curvature using the gauge invariant method of Refs. [23,33] (see Appendix F for details).

B. Exciton dispersion and Berry curvature

Figure 5 shows the exciton dispersions at a twist angle of $\theta = 0.77^\circ$ under a displacement field of $u_D = 20$ and -20 meV. For both parameter choices, the exciton binding energy is largest around the γ point since the interaction potential $V(p)$ peaks at $p = 0$. The excitons in both cases are fully gapped across the mBZ but the maximum binding energy for $u_D = -20$ meV is substantially larger at ~ 17 meV compared to ~ 11 meV for $u_D = 20$ meV. We can use symmetry indicators to study the exciton band topology. We note that in Ref. [23] we found that there were an infinite family of exciton Berry connections which give different (but related) exciton Berry curvatures. However, we showed that, despite this, the exciton Chern number is still uniquely defined. We compute the total C_3 eigenvalues of the many-body exciton wave function at the high-symmetry points in the Brillouin zone (see Fig. 5). In both parameter regimes, the exciton Chern number of the band is 0. However, despite the trivial Chern number, there is interesting *crystalline* topology. The symmetry indicators of the excitons can be used to calculate the center of the maximally localized exciton Wannier states for the exciton band [6,22,23]. In Table I, we show the shift of the maximally localized exciton Wannier function from the center of the moiré unit cell s_{exc} (see Appendix E for details

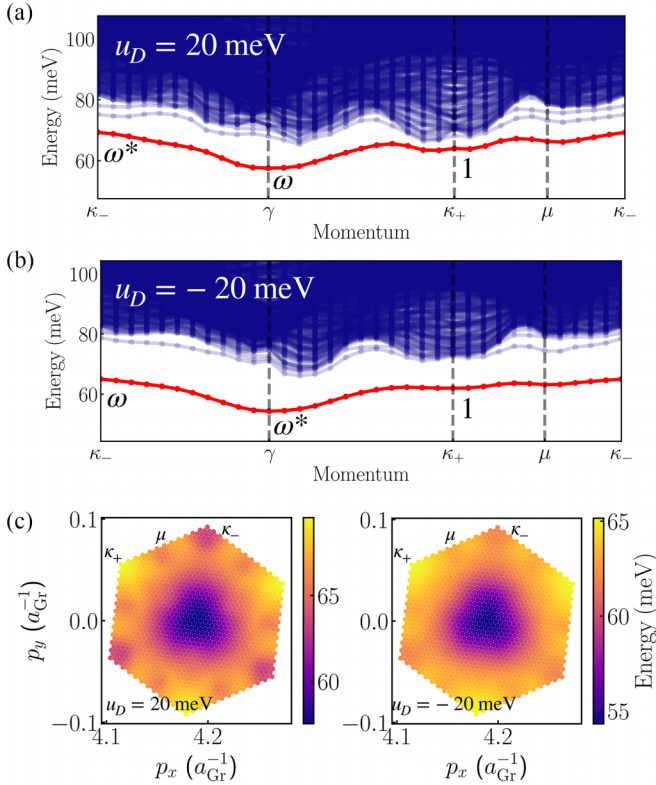


FIG. 5. Exciton dispersion (in red) with particle-hole continuum above (in blue) for $\theta = 0.77^\circ$ (a) $u_D = 20$ and (b) -20 meV. In both cases the C_3 symmetry eigenvalues of the excitons are marked. The dispersion of the excitons throughout the whole BZ for both parameter choices is shown in (c).

of this calculation). In both cases, the exciton Wannier center is not at the center of the moiré unit cell. In addition, the exciton Wannier center varies between the two configurations: at $u_D = 20$ meV, it sits at the $1c$ Wyckoff position but for $u_D = -20$ meV, it is at the $1b$ Wyckoff positions. Beyond the global topology of the exciton band, the exciton Berry curvature (Fig. 6) reveals marked differences between the two displacement field values. For $u_D = 20$ meV, the curvature exhibits pronounced minima and maxima, despite the total Chern number of the exciton band being zero. These features originate from the underlying *electronic* Berry curvature. Specifically, the enhanced *exciton* Berry curvature near

TABLE I. C_3 eigenvalues of the exciton wave function at high-symmetry points in the Brillouin zone for two different parameter choices. The exciton Wannier center s_{exc} is stated in terms of the Wyckoff positions [see Fig. 2(b)].

u_D	$\lambda_{\gamma}^{\text{exc}}$	$\lambda_{\kappa_+}^{\text{exc}}$	$\lambda_{\kappa_-}^{\text{exc}}$	s_{exc}
20 meV	ω	1	ω^*	$1c$
-20 meV	ω^*	1	ω	$1b$

the γ point can be seen to be inherited from the *electronic* Berry curvature. In Fig. 3(a), we can see that the lowest energy particle-hole excitations take an electron from the κ_+ point in the lower band to μ in the upper band. Since the electronic Berry curvature is strongly concentrated at κ_+ [Fig. 4(a)], excitons with momentum $\mathbf{p} \approx \boldsymbol{\mu} - \boldsymbol{\kappa}_+$ (or C_3 rotations of this momentum) inherit this curvature. In contrast, for $u_D = -20$ meV, the exciton Berry curvature is nearly uniform and vanishing [Fig. 6(b)]. This occurs because the electronic bands do not have as pronounced maxima at the edge of the mBZ compared to the $u_D = 20$ meV case. Therefore the exciton bands are mainly composed of electronic states from the center of the mBZ. The Berry curvature of the electronic bands at the center can be seen to be very close to 0 [see Figs. 4(c) and 4(d)] and so the exciton Berry curvature is similarly small in magnitude.

Excitons will thermally occupy the exciton band leading to an enhanced population at the band minimum. Therefore, whilst the overall exciton band is topologically trivial, the locally enhanced Berry curvature in the vicinity of the band minimum (at $u_D = 20$ meV) will influence exciton transport behavior. The integrated exciton Berry curvature, as a function of the energy cutoff, is shown in Fig. 6(c). Due to the presence of both minima and maxima in the Berry curvature, the integrated value oscillates for $u_D = 20$ meV. This will lead to a temperature dependence of response functions sensitive to the exciton Berry curvature, as thermal excitation allows excitons to occupy states beyond the band minimum, altering the Boltzmann-weighted Berry curvature. In contrast, the flat exciton Berry of the $u_D = -20$ meV will not have such oscillatory exciton Berry curvature effects. Hence, rhombohedral graphene provides a platform in which it is possible to electrically tune between regimes with and without *excitonic* Berry curvature effects.

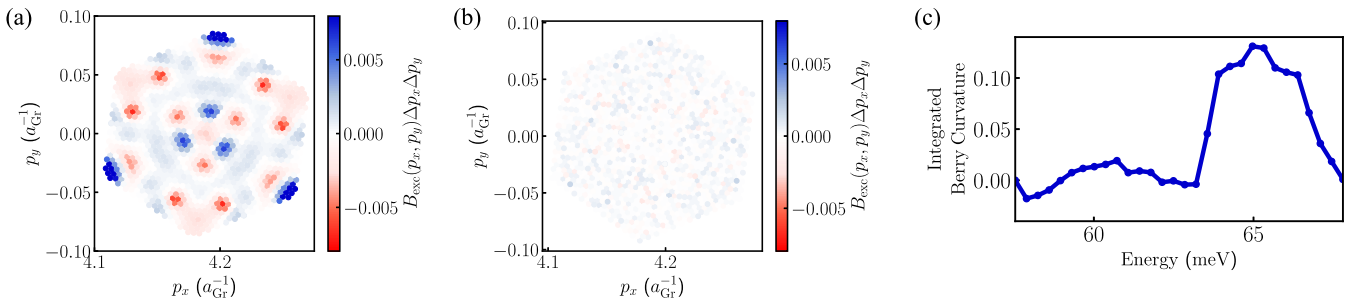


FIG. 6. Exciton Berry curvature for the lowest energy exciton band for $\theta = 0.77^\circ$, (a) $u_D = 20$ and (b) -20 meV. (c) is the exciton Berry curvature integrated up to the given energy cutoff for $u_D = 20$ meV (not shown for full mBZ).

IV. DISCUSSION

The response and transport properties of electrons in non-interacting bands are well known to be influenced by the electronic Berry curvature [1]. Fully elucidating the transport effects of exciton Berry curvature represents a new frontier in the study of quantum transport phenomena. Our work implies that rhombohedral graphene is a possibly important material for exploring the interplay of excitons and topology. For $u_D = -20$ meV, the Berry curvature of the single electron dispersion of rhombohedral graphene is concentrated at the edge of the mBZ, exactly where the low-energy excitons are formed, meaning that the lowest energy excitons in rhombohedral graphene inherit the Berry curvature of the electronic bands. The transport properties of excitons are likely to be influenced by this exciton Berry curvature. One possible measurable consequence of the enhanced Berry curvature at a displacement field of $u_D = 20$ meV is a thermal Hall effect [25]. Applying an in-plane temperature gradient to the sample would generate a transverse exciton current, i.e., most clearly visible as a chiral exciton current at the edge of the system. Typically such exciton currents are hard to measure because excitons are neutral, however, as we have seen, the conduction band and valence band electronic wave functions are layer polarized. Hence, the exciton flow could be detected by measuring the electron or hole current on either the top or bottom layer of the sample similar to a drag current [34,35]. We note that in general the thermal hall effect is suppressed by the gap from the ground state to the excitation band. Despite this the thermal Hall effect can still be measurable experimentally, e.g., for topological magnons [36]. However, observing the thermal Hall effect with excitons may in fact in general be easier than for magnons because excitons are easy to pump unlike magnons.

We have also predicted that the maximally localized exciton Wannier states are shifted away from the center of the unit cell, directly analogous to the concept of an obstructed atomic insulator in noninteracting electron systems [21]. Obstructed atomic insulators are known to possess crystal defect responses that depend on the Wannier state shift, as well as corner or edge polarization [37]. Hence we expect that the exciton Wannier state shift in rhombohedral graphene could analogously lead to new excitonic crystal defect responses and exciton corner or edge modes [6]. The current spatial resolution of exciton probes would not allow it to be possible to detect these effects in bare graphene with a unit cell of order 2.46 Å, but the enhanced moiré unit cell size increases the shift to within the resolution range of techniques like EELs, which can achieve spatial resolutions on the order of $\sim$ nm [26,27]. Moiré systems such as hBN/R5G/hBN therefore offer a promising platform for experimentally probing exciton crystalline topology. However, understanding the full range of experimental signatures, beyond edge states [6], remains a compelling and open direction for future research.

ACKNOWLEDGMENTS

We would like to thank Johannes Lischner for the useful discussions. We acknowledge support from the Imperial-TUM flagship partnership. This work was supported by

a UKRI Future Leaders Fellowship MR/Y017331/1. H.D. acknowledges support from the Engineering and Physical Sciences Research Council (Grant No. EP/W524323/1). J.K. acknowledges support from the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy-EXC- 2111-390814868, DFG Grants No. KN1254/1-2, No. KN1254/2- 1, and No. TRR 360 - 492547816; as well as the Munich Quantum Valley, which is supported by the Bavarian state government with funds from the Hightech Agenda Bayern Plus.

DATA AVAILABILITY

The data that support the findings of this article are openly available [38].

APPENDIX A: PENTALAYER RHOMBOHEDRAL GRAPHENE

We derive the low-energy model for pentalayer rhombohedral graphene (R5G) introduced in the main text. The model is derived from the full ten-band model used by Ref. [14]. We begin by summarising this full ten-band model before introducing the low-energy two-band model we use for the remainder of the paper.

1. Full pentalayer model

We describe the full model of R5G, see Ref. [14] for more details. Graphene has a lattice parameter $a = 2.46$ Å and lattice vectors

$$\mathbf{R}_1 = (a, 0), \quad (\text{A1})$$

$$\mathbf{R}_2 = a \left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right). \quad (\text{A2})$$

The corresponding reciprocal lattice vectors are

$$\mathbf{G}_1 = \frac{2\pi}{a} \left(1, -\frac{1}{\sqrt{3}} \right), \quad (\text{A3})$$

$$\mathbf{G}_2 = \frac{2\pi}{a} \left(0, \frac{2}{\sqrt{3}} \right). \quad (\text{A4})$$

The high symmetry points of graphene are

$$\mathbf{K}_{Gr} = \frac{2}{3}\mathbf{G}_1 + \frac{1}{3}\mathbf{G}_2, \quad (\text{A5})$$

$$\mathbf{K}'_{Gr} = \frac{1}{3}\mathbf{G}_1 + \frac{2}{3}\mathbf{G}_2. \quad (\text{A6})$$

We can write the Hamiltonian for R5G in layer space as

$$h_{R5G}(\mathbf{k}) = \begin{bmatrix} h_0^{(0)} & h^{(1)} & h^{(2)} & 0 & 0 \\ [h^{(1)}]^\dagger & h_1^{(0)} & h^{(1)} & h^{(2)} & 0 \\ [h^{(2)}]^\dagger & [h^{(1)}]^\dagger & h_2^{(0)} & h^{(1)} & h^{(2)} \\ 0 & [h^{(2)}]^\dagger & [h^{(1)}]^\dagger & h_3^{(0)} & h^{(1)} \\ 0 & 0 & [h^{(2)}]^\dagger & [h^{(1)}]^\dagger & h_4^{(0)} \end{bmatrix}, \quad (\text{A7})$$

where the rows and columns correspond to different layers of graphene. The $h_l^{(0)}$, $h^{(1)}$, and $h^{(2)}$ matrices are written in the

basis of the graphene sublattice orbitals for each layer (l) of graphene in the pentalayer stack. These matrices are

$$h_l^{(0)}(\mathbf{k}) = \begin{pmatrix} u_l & -t_0 \tilde{f}_k \\ -t_0 \tilde{f}_k & u_l \end{pmatrix}, \quad (\text{A8})$$

$$h^{(1)}(\mathbf{k}) = \begin{pmatrix} t_4 & t_3 \tilde{f}_k \\ t_1 \tilde{f}_k & t_4 f_k \end{pmatrix}, \quad (\text{A9})$$

$$h^{(2)}(\mathbf{k}) = \begin{pmatrix} 0 & \frac{t_2}{2} \\ 0 & 0 \end{pmatrix}. \quad (\text{A10})$$

We define

$$f_k = \sum_{i=0}^2 e^{ik \cdot \delta_i}, \quad (\text{A11})$$

with δ_n defined as the vectors between nearest neighbor carbon atoms in each layer [14]. We use the hopping values $t_0 = 3.1$ eV, $t_1 = 380$ meV, $t_2 = -21$ meV, $t_3 = 290$ meV, and $t_4 = 141$ meV [14,28]. The applied electric field gives a layer dependent electric potential

$$u_l = u_D l \quad (\text{A12})$$

with the layer index l and the tunable displacement field u_D .

2. Low-energy model

We now derive the low-energy model of pentalayer rhombohedral graphene. For finite u_D there are two bands immediately above and below the Fermi energy at charge neutrality. These two bands are layer polarized around the $\mathbf{K}_{Gr}$ point. The band which is layer polarized to layer 1 has most of the wave function weight on the A sublattice whereas the layer 5 polarized band is on the B sublattice. The low-energy physics, around $\mathbf{K}_{Gr}$, can therefore be well described by a model solely in the subspace $1A, 5B$ (where we label the positions in the unit cell by the layer 1, 5 and the sublattice A, B). We rewrite the equation for the energy dispersion $\epsilon(\mathbf{k})$ as

$$\begin{pmatrix} h(\mathbf{k})_{11} & h(\mathbf{k})_{12} \\ h(\mathbf{k})_{21} & h(\mathbf{k})_{22} \end{pmatrix} \begin{pmatrix} c_1(\mathbf{k}) \\ c_2(\mathbf{k}) \end{pmatrix} = \epsilon(\mathbf{k}) \begin{pmatrix} c_1(\mathbf{k}) \\ c_2(\mathbf{k}) \end{pmatrix}, \quad (\text{A13})$$

where the matrix $h(\mathbf{k})_{11}$ represents the block of the full matrix $h(\mathbf{k})$ in the subspace $1A, 5B$, $h(\mathbf{k})_{22}$ is the high energy subspace (*i.e.* $2B, \dots, 5A$), and $h(\mathbf{k})_{12}$ and $h(\mathbf{k})_{21}$ represents the coupling between the subspaces. The components of the Bloch function in each subspace are given by the functions $c_1(\mathbf{k})$ and $c_2(\mathbf{k})$. We wish to find a 2×2 effective Hamiltonian in the low-energy subspace [39]. To do this we expand Eq. (A13) to give two relations,

$$c_1(\mathbf{k}) = (\epsilon - h(\mathbf{k})_{11})^{-1} h(\mathbf{k})_{12} c_2(\mathbf{k}), \quad (\text{A14})$$

$$c_2(\mathbf{k}) = (\epsilon - h(\mathbf{k})_{22})^{-1} h(\mathbf{k})_{21} c_1(\mathbf{k}). \quad (\text{A15})$$

Combining these, we obtain an equation for the wave function components in the low-energy subspace $c_1(\mathbf{k})$,

$$[h_{11}(\mathbf{k}) + h_{12}(\mathbf{k})(h_{22}(\mathbf{k}) - \epsilon)^{-1} h_{21}(\mathbf{k}) - \epsilon] c_1(\mathbf{k}) = 0. \quad (\text{A16})$$

We expand to first order in energy ϵ and obtain the effective Hamiltonian [39]

$$h_{\text{eff}}(\mathbf{k}) = \{1 - h(\mathbf{k})_{12}[h(\mathbf{k})_{22}]^{-2} h(\mathbf{k})_{21}\}^{-1} \times \{h(\mathbf{k})_{11} + h(\mathbf{k})_{12}[h(\mathbf{k})_{22}]^{-1} h(\mathbf{k})_{21}\}. \quad (\text{A17})$$

We consider small momenta around the $\mathbf{K}_{Gr}$ valley of the R5G, and so we Taylor expand $h_{\text{eff}}(\mathbf{k})$ in powers of $\mathbf{k}$ about $\mathbf{K}_{Gr}$. For pentalayer rhombohedral graphene, the dominant off-diagonal terms in $h_{\text{eff}}(\mathbf{k})$ go as $\sim |\mathbf{k}|^5$ (as described in Ref. [18]). When we expand the diagonal elements to zeroth order in $\mathbf{k}$ we also obtain a contribution from the applied displacement field u_D . We introduce a further correction beyond the model described in Ref. [18]; the full model has a characteristic quadratic curvature which dominates at small momenta $\mathbf{k}$ from the Dirac point $\mathbf{K}_{Gr}$. We find that this effect can be captured in the low-energy model by expanding the diagonal components to second order in momentum $\mathbf{k}$. This gives an effective Hamiltonian in the low-energy subspace which is

$$h_{\text{eff}}(k_x, k_y) = \begin{pmatrix} 2u_D + \gamma_- (k_x^2 + k_y^2) & \frac{v_F^5}{t_4^5} (k_x - ik_y)^5 \\ \frac{v_F^5}{t_4^5} (k_x + ik_y)^5 & -2u_D + \gamma_+ (k_x^2 + k_y^2) \end{pmatrix}. \quad (\text{A18})$$

The $\gamma_{0/1}$ take a simple form if we assume small u_D (specifically $u_D < v_3/a \ll v_F/a$) and expand to first order in u_D ,

$$\gamma_{\pm} \approx \frac{2v_F v_4}{t_1} \pm \frac{u_D v_F^2}{t_1^2}. \quad (\text{A19})$$

This simplified model predicts a sign change in the curvature of the higher-energy band at the $\mathbf{K}_{Gr}$ point at a displacement field of $u_D = \frac{2v_4 t_1}{v_F} \sim 35$ meV, while the band dispersion curvature at the $\mathbf{K}_{Gr}^{VF}$ point of the lower-energy band remains unchanged for all $u_D \geq 0$. This behavior in good agreement with results from the full model. Note that this curvature sign change is not captured in the simple model described in Ref. [18] so this model more accurately describes the low-energy physics of R5G. A comparison between the dispersions of the full and simplified models for a representative parameter set is shown in the main text. As expected, the two models agree well at momenta close to $\mathbf{K}_{Gr}$, with increasing deviation at larger momenta. Notably, the full model lacks inversion symmetry about the $\mathbf{K}_{Gr}$ point—*i.e.*, it is not symmetric under $(k_x, k_y) \rightarrow (-k_x, -k_y)$ —unlike the low-energy approximation. This asymmetry arises from the inclusion of intra-layer hopping t_3 . When t_3 is neglected, the system possesses a unitary symmetry $U = \bigoplus_{l=0}^{N-1} (-1)^l \sigma_z$, obeying $U h(\mathbf{k}) U^\dagger = h(-\mathbf{k})$. This makes the spectrum symmetric about about the $\mathbf{K}_{Gr}$. Since the low-energy 2×2 model does not include t_3 , it is symmetric about the $\mathbf{K}_{Gr}$ point. In addition to capturing the low-energy dispersion, the low-energy model derived above also captures the Berry curvature distribution of the full model (as the model without the quadratic correction does too [18]).

APPENDIX B: MOIRE HAMILTONIAN

We study R5G between hBN layers, i.e., hBN/R5G/hBN stacks. We use the same model as Ref. [14]. The lattice mismatch between the hBN and the R5G leads to a moiré pattern. There are two sources of the lattice mismatch: firstly there is a lattice constant mismatch between hBN and graphene, $\epsilon = a_{\text{hBN}}/a_{\text{Gr}} = 0.018$. Secondly we can rotate the hBN by some angle θ to change the moiré periodicity. We define a rotation matrix R_θ to describe rotation of the hBN graphene by angle θ with respect to the R5G. Similarly, the matrix $M = (1 + \epsilon)I$ can be used to calculate the effects of the lattice constant mismatch. The moiré reciprocal lattice vectors are

$$\mathbf{g}_j = [I - M^{-1}R_\theta]\mathbf{G}_j, \quad (\text{B1})$$

for $j \in \{1, 2\}$ [14,28]. In addition, we define

$$\mathbf{g}_3 = \mathbf{g}_1 + \mathbf{g}_2. \quad (\text{B2})$$

The high symmetry points in the new moiré Brillouin zone are

$$\boldsymbol{\gamma} = \mathbf{K}_{\text{Gr}}, \quad (\text{B3})$$

$$\boldsymbol{\kappa}^+ = \boldsymbol{\gamma} - \frac{1}{3}\mathbf{g}_1 + \frac{1}{3}\mathbf{g}_2, \quad (\text{B4})$$

$$\boldsymbol{\kappa}^- = \boldsymbol{\gamma} - \frac{2}{3}\mathbf{g}_1 - \frac{1}{3}\mathbf{g}_2, \quad (\text{B5})$$

$$\boldsymbol{\mu} = \boldsymbol{\gamma} - \frac{1}{2}\mathbf{g}_1. \quad (\text{B6})$$

We model the effect of the hBN layers as an effective potential acting on the outermost layers ($l = 1, 5$) of the R5G. This moiré potential can be written as

$$H_V = \sum_{\mathbf{r}} \sum_{l \in \{\text{top}, \text{btm}\}} V_{\text{eff}}^l(\mathbf{r}) c_{\mathbf{r},l}^\dagger c_{\mathbf{r},l}, \quad (\text{B7})$$

$\mathbf{r}$ goes over all unit cells and $c_{\mathbf{r},l}^\dagger$ creates an electron at position $\mathbf{r}$ and layer l . Since our effective Hamiltonian retains only the A sublattice in the bottom layer and the B sublattice in the top layer, we apply the moiré potential exclusively to these sites. The moiré potential has the moiré periodicity so we can write the potential as

$$V_{\text{eff}}^l(\mathbf{r}) = \sum_{\substack{\eta \in \{1, -1\} \\ j \in \{1, 2, 3\}}} V_{l, \eta \mathbf{g}_j} \exp[i\eta \mathbf{g}_j \cdot \mathbf{r}]. \quad (\text{B8})$$

Here, the sum runs over the first shell of moiré reciprocal lattice vectors around the first moiré BZ. This is known as the “first harmonic” approximation. We use values for the prefactors $V_{l, \eta \mathbf{g}_j}$ introduced in Ref. [28] such that the potential can be expressed as

$$V_{\text{eff}}^l(\mathbf{r}) = V_0 + V_1 \sum_{\substack{\eta \in \{1, -1\} \\ j \in \{1, 2, 3\}}} \exp[i\eta(\mathbf{g}_j \cdot \mathbf{r} + \psi_\xi + \phi_l)]. \quad (\text{B9})$$

There is a phase shift ψ_ξ which depends on the stacking configuration, for example, the A sublattice of graphene may lie above either a boron atom ($\xi = 0$) or a nitrogen atom ($\xi = 1$), as described in Ref. [28], and this leads to differing phases for the effective potential. We use the values in Refs. [28] and restrict our study to the $\xi = 0$ stacking for both the top and bottom layers, for which $\psi_\xi = 223.5^\circ$. The additional phase factor ϕ_l is 0 for the bottom layer but for N_l layers of graphene,

the top layer gains a phase $\phi_l = \frac{2\pi N_l}{3}$. This phase accounts for the relative shift in the moiré pattern that arises because, in a given unit cell, the B sublattice site in the top layer is not directly aligned above the A sublattice sites in the bottom layer [13]. Note that the moiré coupling does not couple the top and bottom layers. We use $V_0 = 28.9$ meV and $V_1 = 21.0$ meV following Refs. [14,28].

We rewrite the Hamiltonian in momentum space using

$$c_{\mathbf{r},l}^\dagger = \frac{1}{L} \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}} c_{\mathbf{k},l}^\dagger, \quad (\text{B10})$$

where the sum over $\mathbf{k}$ goes over all momenta in the R5G BZ and the system consists of L^2 unit cells. This gives

$$H_V = \sum_{l, \mathbf{r}, \mathbf{k}, \mathbf{k}'} \sum_{\substack{\eta \in \{1, -1\} \\ j \in \{1, 2, 3\}}} e^{i(\mathbf{k} - \mathbf{k}' + \mathbf{g}_j) \cdot \mathbf{r}} V_{l, \eta \mathbf{g}_j} c_{\mathbf{k},l}^\dagger c_{\mathbf{k}',l}. \quad (\text{B11})$$

Performing the summation over $\mathbf{r}$ gives

$$H_V = \sum_{l, \mathbf{k}, \mathbf{k}'} \sum_{\substack{\eta \in \{1, -1\} \\ j \in \{1, 2, 3\}}} \delta_{\mathbf{k}, \mathbf{k}' - \eta \mathbf{g}_j} V_{l, \eta \mathbf{g}_j} c_{\mathbf{k},l}^\dagger c_{\mathbf{k}',l}. \quad (\text{B12})$$

We now express all momenta in terms of the momentum in the mBZ ($\mathbf{k}$) and a moiré reciprocal lattice vector ($\mathbf{Q}$), i.e., $c_{\mathbf{k}, \mathbf{Q}, l}^\dagger \equiv c_{\mathbf{k} + \mathbf{Q}, l}^\dagger$ we obtain

$$H_V = \sum_l \sum_{\mathbf{k}} \sum_{\mathbf{Q}, \mathbf{Q}'} \sum_{\substack{\eta \in \{1, -1\} \\ j \in \{1, 2, 3\}}} (\delta_{\mathbf{Q}, \mathbf{Q}' - \eta \mathbf{g}_j} V_{l, \eta \mathbf{g}_j}) c_{\mathbf{k}, \mathbf{Q}, l}^\dagger c_{\mathbf{k}, \mathbf{Q}', l}. \quad (\text{B13})$$

The sum over $\mathbf{Q}, \mathbf{Q}'$ is over *all* moiré reciprocal lattice vectors, i.e., not restricted to the first ring about the mBZ. We obtain the full moire Hamiltonian

$$\begin{aligned} H_M &= H_0 + H_V \\ &= \sum_{\mathbf{k} \in \text{mBZ}} \sum_{\mathbf{Q}, \mathbf{Q}'} \sum_{l, l'} h_{(\mathbf{Q}, l), (\mathbf{Q}', l')}^{(m)}(\mathbf{k}) c_{\mathbf{k}, \mathbf{Q}, l}^\dagger c_{\mathbf{k}, \mathbf{Q}', l'}, \end{aligned} \quad (\text{B14})$$

where

$$\begin{aligned} h_{(\mathbf{Q}, l), (\mathbf{Q}', l')}^{(m)}(\mathbf{k}) &= \{[h_{\text{eff}}(\mathbf{k} + \mathbf{Q})]_{l, l'} + V_0 \delta_{l, l'}\} \delta_{\mathbf{Q}, \mathbf{Q}'} \\ &+ \sum_{\substack{\eta \in \{1, -1\} \\ j \in \{1, 2, 3\}}} (\delta_{\mathbf{Q}, \mathbf{Q}' - \eta \mathbf{g}_j} V_1). \end{aligned} \quad (\text{B15})$$

Here the first term $h_{\text{eff}}(\mathbf{k})$ represents the *low-energy* R5G without the moiré coupling. The moiré system retains C_3 symmetry and so there are three high symmetry points $\boldsymbol{\gamma}$, $\boldsymbol{\kappa}_+$, and $\boldsymbol{\kappa}_-$. To obtain a finite matrix size, we truncate the set of $\mathbf{Q}$ vectors in the moiré Hamiltonian to those within a ring around the $\boldsymbol{\gamma}$ point with $|\mathbf{Q}| \leq 5[2\pi (a_{\text{moiré}})^{-1}]$, where $a_{\text{moiré}}$ is the moiré lattice parameter.

APPENDIX C: CHERN NUMBERS OF NON-INTERACTING MODEL

We plot the Chern numbers of the conduction and valence bands (e.g. see Fig. 4 in main text) as a function of applied field u_D and twist angle θ . The resulting phase diagrams are shown in Fig. 7. We see that both bands can have Chern numbers 0, 1, and 2 (mod 3). We choose to study excitons

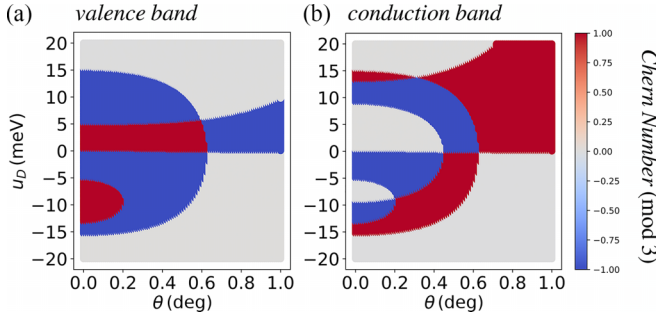


FIG. 7. Chern numbers (mod 3) of the valence and conduction bands as a function of the twist angle of the hBN with respect to the R5G and the applied potential u_D .

at the experimental angle $\theta = 0.77^\circ$ at the displacement fields $u_D = \pm 20$ meV.

We will now show an example of how the moire potential alters the band structure of the free-standing rhombohedral graphene model. We demonstrate this by slowly turning on the moire potential and seeing how the bands gap out to generate the Chern number 1 band at $u_D = 20$ meV. We tune the moire potential by multiplying the moire potential strengths (V_0, V_1) by a parameter α . When $\alpha = 0$, the moire potential is turned off, when $\alpha = 1$ the potentials are at their full values given by Ref. [28]. At $\alpha = 0$, we plot the bands folded into the smaller mBZ (Fig. 8(a)). However, since there is no moire potential applied, the folded bands do not yet hybridize. Note that the states at κ_+ and κ_- (above and below the Fermi level) are three-fold degenerate. Adding the moire potential allows these degenerate states to gap out. We see that in the conduction band, at κ_+ the state with C_3 eigenvalue ω^* becomes lowest in energy. The conduction band states at κ_- remain (close to) degenerate until large α , but by $\alpha = 0.8$, we see that the state with C_3 eigenvalue 1 becomes lowest in energy. The resulting C_3 eigenvalues for the upper band imply that the conduction band has Chern number 1 (mod 3). We see

then that the Chern number 1 band was induced by the moire potential via band folding and then hybridization.

APPENDIX D: EXCITON HAMILTONIAN

We calculate the exciton spectra and wave functions using the following method. We project the fully many-body Hamiltonian into the variational basis,

$$|p, k\rangle = c_{p+k, c}^\dagger c_{k, v} |GS\rangle. \quad (D1)$$

The total momentum of the above state is p , the relative momentum is k .

We project the full Hamiltonian onto the variational basis and calculate its matrix elements,

$$\langle p, k | \hat{H} | p, k' \rangle. \quad (D2)$$

We have a contribution from the kinetic Hamiltonian [Eq. (B14)],

$$\langle p, k | \hat{H}_0 | p, k' \rangle = \delta_{k, k'} (E_{G.S.} - \epsilon_{k, v} + \epsilon_{p+k, c}), \quad (D3)$$

where $\epsilon_{p, c/v}$ are the single particle energies of the kinetic hamiltonian for the conduction/valence bands at momentum p . The ground state energy is $E_{G.S.} = \sum_k \epsilon_{k, v}$.

We also project the interaction term into this basis. We first derive the general form of the matrix elements before showing the results for the moiré Hamiltonian. We begin with the general form of a translationally-symmetric interaction Hamiltonian,

$$\hat{H}_{\text{int}} = \sum_{R, R', i, j} U_{R-R', i, j} \hat{n}_{R, i} \hat{n}_{R', j}, \quad (D4)$$

where $\hat{n}_{R, i}$ is the number operator at unit cell R and i labels the sublattices. Now we write this in the momentum basis. First note that

$$c_{k, i}^\dagger = \frac{1}{\sqrt{V}} \sum_R e^{ik \cdot R} c_{R, i}^\dagger, \quad (D5)$$

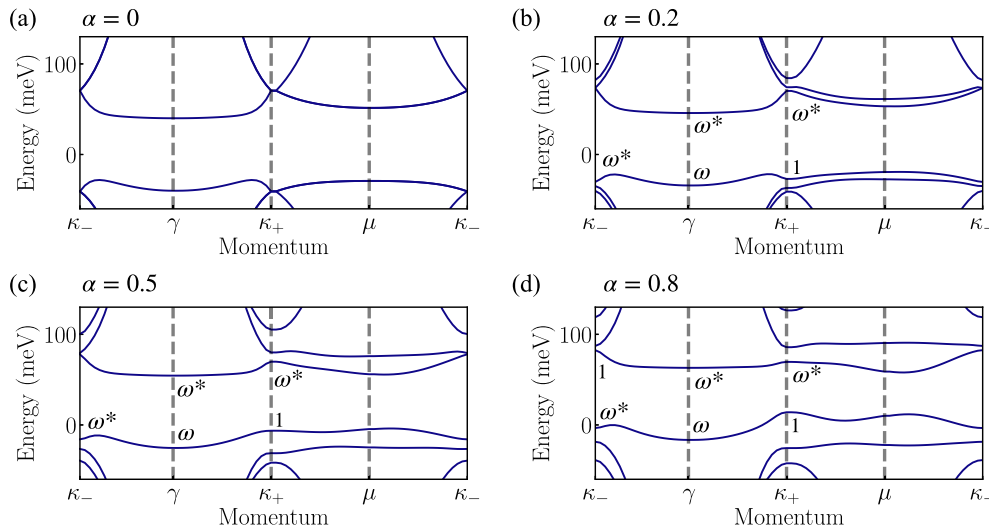


FIG. 8. Tuning the moire potential to show the formation of a conduction band with Chern number 1. The $\alpha = 0$ plot shows the effects of band folding without hybridization. When $\alpha \neq 1$ hybridization can occur to gap out the conduction band. We plot the C_3 eigenvalues at the high symmetry points with $\omega = \exp(i\frac{2\pi}{3})$.

$$c_{R,i}^\dagger = \frac{1}{\sqrt{V}} \sum_R e^{-ik \cdot R} c_{k,i}^\dagger. \quad (\text{D6})$$

We rewrite the number operators as

$$n_{R,i} = c_{R,i}^\dagger c_{R,i} \quad (\text{D7})$$

$$= \frac{1}{V} \sum_{p,q} e^{-i(p-q) \cdot R} c_{p,i}^\dagger c_{q,i}. \quad (\text{D8})$$

Equation (D4) becomes

$$\hat{H}_{\text{int}} = \sum_{R,R',i,j} \sum_{p,q,p',q'} \frac{1}{V^2} U_{R-R',ij} e^{-i(p-q) \cdot R - i(p'-q') \cdot R'} \times c_{pi}^\dagger c_{qi} c_{p'j}^\dagger c_{q'j} \quad (\text{D9})$$

$$= \sum_{R,R',i,j} \sum_{p,q,p',q'} \frac{1}{V^2} U_{R,ij} e^{-i(p-q) \cdot R - i(p'-q'+p'-q') \cdot R'} \times c_{pi}^\dagger c_{qi} c_{p'j}^\dagger c_{q'j}, \quad (\text{D10})$$

where we set $R - R' \rightarrow R$. This can be further simplified by performing the sum over R' and q' to give

$$\hat{H}_{\text{int}} = \frac{1}{V} \sum_{R,i,j} \sum_{p,q,p'} U_{R,ij} e^{-i(p-q) \cdot R} c_{p,i}^\dagger c_{q,i} c_{p',j}^\dagger c_{p-q+p',j}. \quad (\text{D11})$$

Let $U_{p,ij} = \frac{1}{V} \sum_R U_{R,ij} e^{-ip \cdot R}$,

$$\hat{H}_{\text{int}} = \sum_{pq q' ij} U_{p,ij} c_{p+q,i}^\dagger c_{q,i} c_{q',j}^\dagger c_{p+q',j}. \quad (\text{D12})$$

To calculate the matrix elements

$$\langle p, k | \hat{H}_{\text{int}} | p, k' \rangle, \quad (\text{D13})$$

we express the interaction Hamiltonian in the band basis. The Bloch function for band α , is $u_i^{k,\alpha}$ meaning that

$$c_{k,\alpha}^\dagger = \sum_i u_i^{k,\alpha} c_{k,i}^\dagger, \quad (\text{D14})$$

$$c_{k,\alpha} = \sum_i \bar{u}_i^{k,\alpha} c_{k,i}, \quad (\text{D15})$$

where i sums over the sub-lattices, and $\alpha \in \{c, v\}$. We use the notation $\bar{u}_i^{k,\alpha} = (u_i^{k,\alpha})^*$. Plugging this relation into equation (D12), we obtain

$$\hat{H}_{\text{int}} = \sum_{pq q' ij} U_{p,ij} c_{p+q,i}^\dagger c_{q,i} c_{q',j}^\dagger c_{p+q',j} \quad (\text{D16})$$

$$= \sum_{pq q' ij \alpha \alpha' \beta \beta'} U_{p,ij} \bar{u}_i^{p+q,\alpha} u_i^{q,\alpha'} \bar{u}_j^{q',\beta} u_j^{p+q',\beta'} c_{p+q,\alpha}^\dagger \times c_{q,\alpha'} c_{q',\beta}^\dagger c_{p+q',\beta'}. \quad (\text{D17})$$

Defining $U_{\alpha \alpha' \beta \beta'}^{pq q'} = \sum_{ij} (U_{p,ij} \bar{u}_i^{p+q,\alpha} u_i^{q,\alpha'} \bar{u}_j^{q',\beta} u_j^{p+q',\beta'})$, we can write

$$\hat{H}_{\text{int}} = \sum_{pq q' \alpha \alpha' \beta \beta'} U_{\alpha \alpha' \beta \beta'}^{pq q'} c_{p+q,\alpha}^\dagger c_{q,\alpha'} c_{q',\beta}^\dagger c_{p+q',\beta'}. \quad (\text{D18})$$

The matrix elements are

$$\langle p, k | \hat{H}_{\text{int}} | p, k' \rangle = \sum_{l q q' \alpha \alpha' \beta \beta'} U_{\alpha \alpha' \beta \beta'}^{l q q'} \langle \text{GS} | c_{k,v}^\dagger c_{p+k,c} \times c_{l+q,\alpha}^\dagger c_{q,\alpha'} c_{q',\beta}^\dagger c_{l+q',\beta'} c_{p+k',c}^\dagger c_{k',v} | \text{GS} \rangle. \quad (\text{D19})$$

Wick's theorem can be used to simplify the expectation value,

$$\langle \text{GS} | c_{k,v}^\dagger c_{p+k,c} c_{l+q,\alpha}^\dagger c_{q,\alpha'} c_{q',\beta}^\dagger c_{l+q',\beta'} c_{p+k',c}^\dagger c_{k',v} | \text{GS} \rangle. \quad (\text{D20})$$

For readability, we simplify the notation to

$$\mu^\dagger = c_{k,v}^\dagger, \quad (\text{D21})$$

$$v = c_{p+k,c}, \quad (\text{D22})$$

$$\zeta^\dagger = c_{l+q,\alpha}^\dagger, \quad (\text{D23})$$

$$\sigma = c_{q,\alpha'}, \quad (\text{D24})$$

$$\chi^\dagger = c_{q',\beta}^\dagger, \quad (\text{D25})$$

$$\gamma = c_{l+q',\beta'}, \quad (\text{D26})$$

$$\lambda^\dagger = c_{p+k',c}^\dagger, \quad (\text{D27})$$

$$\xi = c_{k',v}. \quad (\text{D28})$$

The expectation value becomes

$$\langle \mu^\dagger v \zeta^\dagger \sigma \chi^\dagger \gamma \lambda^\dagger \xi \rangle. \quad (\text{D29})$$

This can be expanded using Wick's theorem as

$$\begin{aligned} \langle \mu^\dagger v \zeta^\dagger \sigma \chi^\dagger \gamma \lambda^\dagger \xi \rangle &= \langle \mu^\dagger \sigma \rangle \langle v \zeta^\dagger \rangle \langle \chi^\dagger \xi \rangle \langle \gamma \lambda^\dagger \rangle - \langle v \chi^\dagger \rangle \langle \zeta^\dagger \xi \rangle \langle \gamma \lambda^\dagger \rangle - \langle v \lambda^\dagger \rangle \langle \zeta^\dagger \xi \rangle \langle \chi^\dagger \gamma \rangle - \langle \zeta^\dagger \gamma \rangle \langle \chi^\dagger \xi \rangle \\ &+ \langle \mu^\dagger \gamma \rangle \langle -v \zeta^\dagger \rangle \langle \sigma \lambda^\dagger \rangle \langle \chi^\dagger \xi \rangle + \langle v \chi^\dagger \rangle \langle \zeta^\dagger \xi \rangle \langle \sigma \lambda^\dagger \rangle - \langle v \lambda^\dagger \rangle \langle \zeta^\dagger \sigma \rangle \langle \chi^\dagger \xi \rangle + \langle \zeta^\dagger \xi \rangle \langle \sigma \chi^\dagger \rangle \\ &+ \langle \mu^\dagger \xi \rangle \langle v \zeta^\dagger \rangle \langle \sigma \chi^\dagger \rangle \langle \gamma \lambda^\dagger \rangle + \langle \sigma \lambda^\dagger \rangle \langle \chi^\dagger \gamma \rangle + \langle v \chi^\dagger \rangle \langle \zeta^\dagger \sigma \rangle \langle \gamma \lambda^\dagger \rangle - \langle \zeta^\dagger \gamma \rangle \langle \sigma \lambda^\dagger \rangle \\ &+ \langle v \lambda^\dagger \rangle \langle \zeta^\dagger \sigma \rangle \langle \chi^\dagger \gamma \rangle + \langle \zeta^\dagger \gamma \rangle \langle \sigma \chi^\dagger \rangle. \end{aligned} \quad (\text{D30})$$

We simplify the expression using the values of the two particle correlators. After performing the sum in equation (D19), we obtain

$$\begin{aligned} \langle \mathbf{p}, \mathbf{k} | \hat{H}_{\text{int}} | \mathbf{p}, \mathbf{k}' \rangle = & U_{cvc}^{p,k,k'} + U_{vcc}^{-p,p+k',p+k} - U_{vcc}^{k'-k,k,p+k} - U_{ccv}^{k-k',p+k',k'} + \delta_{k,k'} \sum_{\mathbf{q}} (U_{vvv}^{q,k,k} - U_{vvv}^{0,k,q} - U_{vvv}^{0,q,k} - U_{vcc}^{q,k-q,k-q} \\ & + U_{ccc}^{p+k-q,q,q} + U_{ccv}^{0,p+k,q} + U_{vcc}^{0,q,p+k} - U_{vcc}^{q,p+k,p+k}) + \delta_{k,k'} \sum_{\mathbf{q},\mathbf{q}'} (U_{vvv}^{0,q,q'} + U_{vcc}^{q,q',q'}). \end{aligned} \quad (\text{D31})$$

Finally we note that the final term is equal to the shift of the ground state energy due to the interaction, i.e.,

$$\langle \text{GS} | H_{\text{int}} | \text{GS} \rangle = \sum_{\mathbf{q},\mathbf{q}'} (U_{vvv}^{0,q,q'} + U_{vcc}^{q,q',q'}). \quad (\text{D32})$$

Therefore, to find the exciton spectrum compared to the modified ground state energy, we subtract this term from the exciton Hamiltonian.

For a moiré Hamiltonian [e.g. of the form Eq. (B15)] the U matrices become

$$U_{\alpha,\alpha',\beta,\beta'}^{p,k,k'} = \sum_{\mathbf{Q}} V(\mathbf{p} + \mathbf{Q}) \langle u_{\mathbf{p}+\mathbf{Q}}^{\alpha} | u_{\mathbf{p}}^{\alpha'} \rangle \langle u_{\mathbf{p}'}^{\beta} | u_{\mathbf{p}'+\mathbf{Q}}^{\beta'} \rangle, \quad (\text{D33})$$

where $|u_{\mathbf{p}}^{\alpha}\rangle$ is the eigenstate of the moiré Hamiltonian [Eq. (B15)] with momentum $\mathbf{p}$ and for band $\alpha \in \{c, v\}$. The interactions are modelled as a double-gated Coulomb potential as in Ref. [14],

$$V(\mathbf{p}) = \frac{V_0 \tanh(|\mathbf{p}|d)}{|\mathbf{p}|}, \quad (\text{D34})$$

where d is the gate distance. We set $d = 250 \text{ \AA}$ [14] for the following calculations. The interaction strength $V_0 = \frac{2\pi}{\epsilon_0 \epsilon_r}$ depends on the relative permittivity ϵ_r , for which we use $\epsilon_r = 6$. Note that the expression is in Gaussian units.

APPENDIX E: EXCITON WANNIER STATES

We now use symmetry indicators to calculate the exciton Wannier centres. We label the exciton eigenstates in the exciton band of interest as $|\psi^p\rangle$. The maximally localized exciton Wannier states then take the form

$$|W^{\mathbf{R}}\rangle = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{R}} e^{i\phi(\mathbf{p})} |\psi^{\mathbf{p}}\rangle, \quad (\text{E1})$$

where $e^{i\phi(\mathbf{p})}$ is a gauge transformation that maximally localizes the exciton Wannier state [22,23]. The C_3 eigenvalues of the exciton bands are shown in Table II. Since the exciton band is trivial, the exciton Wannier states are exponentially localized. The C_3 symmetry then requires that the exciton

TABLE II. C_3 eigenvalues of the exciton wave function at high-symmetry points in the Brillouin zone for two different parameter choices. The exciton Wannier center s_{exc} is stated in terms of the Wyckoff positions [see Fig. 9(b)].

u_D	$\lambda_{\gamma}^{\text{exc}}$	$\lambda_{\kappa_+}^{\text{exc}}$	$\lambda_{\kappa_-}^{\text{exc}}$	s_{exc}
20 meV	ω	1	ω^*	1c
-20 meV	ω^*	1	ω	1b

Wannier centres sit at one of the maximal Wyckoff positions [1a, 1b, 1c in Fig. 9(b)]. We calculate the C_3 eigenvalues we expect for exciton Wannier states localized at each of these locations.

1. Wyckoff position: 1a

The Wyckoff position 1a lies at the center of the moiré unit cell. We wish to determine the relationship between the exciton Wannier center and the exciton C_3 eigenvalues. We first express our exciton wave function in terms of the maximally localized Wannier functions,

$$|\psi^{\mathbf{p}}\rangle = \frac{1}{\sqrt{V}} \sum_{\mathbf{R}} e^{i\mathbf{p}\cdot\mathbf{R}} |W^{\mathbf{R}}\rangle. \quad (\text{E2})$$

Next we consider how the C_3 operator acts on this wave function. The C_3 operator rotates real space positions by $2\pi/3$ about the 1a Wyckoff position within the $\mathbf{R} = 0$ unit cell. Therefore, under C_3 , the exciton Wannier state for unit cell $\mathbf{R}$ is mapped to unit cell $U_{C_3}\mathbf{R}$ (the U_{C_3} operator rotates real space positions by $2\pi/3$). Applying $\hat{C}_3$ to the exciton wave function gives

$$\hat{C}_3 |\psi^{\mathbf{p}}\rangle = \frac{1}{\sqrt{V}} \sum_{\mathbf{R}} e^{i\mathbf{p}\cdot\mathbf{R}} \hat{C}_3 |W^{\mathbf{R}}\rangle \quad (\text{E3})$$

$$= \lambda_W \frac{1}{\sqrt{V}} \sum_{\mathbf{R}} e^{i\mathbf{p}\cdot\mathbf{R}} |W^{U_{C_3}\mathbf{R}}\rangle \quad (\text{E4})$$

$$= \lambda_W \frac{1}{\sqrt{V}} \sum_{\mathbf{R}} e^{i\mathbf{p}\cdot U_{C_3}^{-1}\mathbf{R}} |W^{\mathbf{R}}\rangle \quad (\text{E5})$$

$$= \lambda_W \frac{1}{\sqrt{V}} \sum_{\mathbf{R}} e^{iU_{C_3}\mathbf{p}\cdot\mathbf{R}} |W^{\mathbf{R}}\rangle, \quad (\text{E6})$$

where λ_W is a phase such that $\hat{C}_3 |W^{\mathbf{R}}\rangle = \lambda_W |W^{U_{C_3}\mathbf{R}}\rangle$. Since $[\hat{C}_3]^3$ is the identity, therefore, $\lambda_W \in \{1, \omega, \omega^*\}$. At the high

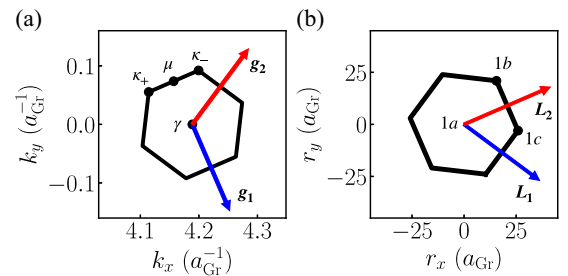


FIG. 9. (a) is the mBZ of hBN/R5G/hBN for $\theta = 0.77^\circ$, moiré reciprocal lattice vectors $\mathbf{g}_1$, $\mathbf{g}_2$, and high symmetry points γ , κ_+ , κ_- and μ . The real space unit cell (b) shows the lattice vectors $\mathbf{L}_1$, $\mathbf{L}_2$ and maximal Wyckoff positions 1a, 1b, 1c.

symmetry points $\tilde{\mathbf{p}} \in \{\boldsymbol{\gamma}, \boldsymbol{\kappa}_+, \boldsymbol{\kappa}_-\}$ (see Fig. 9), the exciton wave function is an eigenfunction of the $\hat{C}_3$ operator,

$$\hat{C}_3 |\psi^{\tilde{\mathbf{p}}}\rangle = \lambda_W \frac{1}{\sqrt{V}} \sum_{\mathbf{R}} e^{i\hat{U}_{C_3} \tilde{\mathbf{p}} \cdot \mathbf{R}} |W^{\mathbf{R}}\rangle \quad (\text{E7})$$

$$= \lambda_W \frac{1}{\sqrt{V}} \sum_{\mathbf{R}} e^{i\mathbf{p} \cdot \mathbf{R}} |W^{\mathbf{R}}\rangle \quad (\text{E8})$$

$$= \lambda_W |\psi^{\tilde{\mathbf{p}}}\rangle. \quad (\text{E9})$$

We see therefore that, when the exciton Wannier centres are at the $1a$ Wyckoff position, the C_3 eigenvalues are the same at all high symmetry points (and equal to λ_W). This is not true for either of the bands in Table II and so their exciton Wannier centres are not at the $1a$ Wyckoff position.

2. Wyckoff position: $1b$

We now find the symmetry eigenvalues of the exciton wave function if the exciton Wannier states sit at Wyckoff position $1b$ (Fig. 9). The $\hat{C}_3$ operator rotates about the $1a$ Wyckoff position. Therefore, when the exciton Wannier state is centered at $1b$, the exciton Wannier center is transformed $\mathbf{R} \rightarrow \hat{U}_{C_3} \mathbf{R} - L_2$ under the action of $\hat{C}_3$ [see Fig. 9(b)].

We consider the action of the C_3 operator on the exciton wave function

$$\hat{C}_3 |\psi^{\mathbf{p}}\rangle = \frac{1}{\sqrt{V}} \sum_{\mathbf{R}} e^{i\mathbf{p} \cdot \mathbf{R}} \hat{C}_3 |W^{\mathbf{R}}\rangle \quad (\text{E10})$$

$$= \lambda_W \frac{1}{\sqrt{V}} \sum_{\mathbf{R}} e^{i\mathbf{p} \cdot \mathbf{R}} |W^{\hat{U}_{C_3} \mathbf{R} - L_2}\rangle \quad (\text{E11})$$

$$= \lambda_W \frac{1}{\sqrt{V}} \sum_{\mathbf{R}} e^{i\mathbf{p} \cdot \hat{U}_{C_3}^{-1}(\mathbf{R} + L_2)} |W^{\mathbf{R}}\rangle \quad (\text{E12})$$

$$= \lambda_W \frac{1}{\sqrt{V}} \sum_{\mathbf{R}} e^{i\hat{U}_{C_3} \mathbf{p} \cdot (\mathbf{R} + L_2)} |W^{\mathbf{R}}\rangle. \quad (\text{E13})$$

Hence at the high symmetry points $\tilde{\mathbf{p}}$,

$$\hat{C}_3 |\psi^{\tilde{\mathbf{p}}}\rangle = \lambda_W \frac{1}{\sqrt{V}} \sum_{\mathbf{R}} e^{i\tilde{\mathbf{p}} \cdot (\mathbf{R} + L_2)} |W^{\mathbf{R}}\rangle \quad (\text{E14})$$

$$= (\lambda_W e^{i\tilde{\mathbf{p}} \cdot L_2}) |\psi^{\tilde{\mathbf{p}}}\rangle. \quad (\text{E15})$$

The C_3 eigenvalues at the high symmetry points are, therefore,

$$\lambda_{C_3}(\tilde{\mathbf{p}}) = \lambda_W e^{i\tilde{\mathbf{p}} \cdot L_2}. \quad (\text{E16})$$

Using $\omega = e^{-i\frac{2\pi}{3}}$, we can write the C_3 eigenvalues of the exciton wave function at the high symmetry points as $\lambda_{C_3}(\boldsymbol{\gamma}) = \lambda_W$, $\lambda_{C_3}(\boldsymbol{\kappa}_+) = \lambda_W \omega$ and $\lambda_{C_3}(\boldsymbol{\kappa}_-) = \lambda_W \omega^*$. For $\lambda_W = \omega^*$ we see that this gives identical symmetry indicators to those shown for the excitons at $u_D = -20$ meV (see Table II). Therefore the exciton Wannier centres at $u_D = -20$ meV are at the $1b$ Wyckoff position.

3. Wyckoff position: $1c$

The same method can be used to calculate the symmetry eigenvalues for exciton Wannier states that sit at the $1c$ Wyckoff position. Here we find that under C_3 the exciton Wannier state at unit cell $\mathbf{R}$ transforms to $\hat{U}_{C_3} \mathbf{R} - L_1$. The symmetry eigenvalues, therefore, are

$$\lambda_{C_3}(\tilde{\mathbf{p}}) = \lambda_W e^{i\tilde{\mathbf{p}} \cdot L_1}. \quad (\text{E17})$$

We find that $\lambda_{C_3}(\boldsymbol{\gamma}) = \lambda_W$, $\lambda_{C_3}(\boldsymbol{\kappa}_+) = \lambda_W \omega^*$ and $\lambda_{C_3}(\boldsymbol{\kappa}_-) = \lambda_W \omega$. For $\lambda_W = \omega$ this gives the symmetry indicators we found for the exciton band at displacement field $u_D = 20$ meV (see Table II). Therefore the exciton Wannier states at this displacement field are centered at the Wyckoff position $1c$.

APPENDIX F: EXCITON BERRY CURVATURE

In Ref. [23], we derived two exciton Wilson loops which relate to the projection of the electron and hole position operator into the exciton band. For 1D systems, we obtain the two exciton Wilson loops for the conduction (c) and valence bands (v) respectively,

$$W_{\text{exc},c} = \prod_p \left[\sum_k \bar{\phi}_{k-p}^{p+\Delta} \phi_{k-p}^p \langle u_{k+\Delta,c} | u_{k,c} \rangle \right], \quad (\text{F1})$$

$$W_{\text{exc},v} = \prod_p \left[\sum_k \bar{\phi}_k^{p+\Delta} \phi_{k+\Delta}^p \langle u_{k+\Delta,v} | u_{k,v} \rangle \right]. \quad (\text{F2})$$

with $\Delta = 2\pi/L$ for system size L . We note that these can both be written as a product of projectors,

$$W_{\text{exc},c/v} = \langle \mathcal{U}_{p_1}^{\text{exc},c/v} | \lim_{R \rightarrow \infty} \prod_{i=R}^2 |\mathcal{U}_{p_i}^{\text{exc},c/v}\rangle \langle \mathcal{U}_{p_i}^{\text{exc},c/v}| | \mathcal{U}_{p_1}^{\text{exc},c/v} \rangle \quad (\text{F3})$$

using

$$|\mathcal{U}_p^{\text{exc},c}\rangle = \sum_k \phi_k^p |k\rangle \otimes |u_{p+k,c}\rangle \otimes |\bar{u}_{k,v}\rangle, \quad (\text{F4})$$

$$|\mathcal{U}_p^{\text{exc},v}\rangle = \sum_k \phi_k^p |p+k\rangle \otimes |u_{p+k,c}\rangle \otimes |\bar{u}_{k,v}\rangle. \quad (\text{F5})$$

See Ref. [23] for details of the derivations.

Following the method introduced by Ref. [33], we can use these Wilson loop expressions to construct expressions for the flux through loops in the BZ. We consider the exciton wave functions which we have calculated on a grid with spacing $\mathbf{d}_\mu$ in direction μ . We can define the link variables

$$U_{\mu,c/v}(\mathbf{k}) = \frac{\langle \mathcal{U}^{\text{exc},c/v} | \mathcal{U}^{\text{exc},c/v}(\mathbf{k} + \mathbf{d}_\mu) \rangle}{|\langle \mathcal{U}^{\text{exc},c/v}(\mathbf{k}) | \mathcal{U}^{\text{exc},c/v}(\mathbf{k} + \mathbf{d}_\mu) \rangle|}. \quad (\text{F6})$$

We then calculate the lattice field strength,

$$F_{c/v}(\mathbf{k}) = -i \ln[U_{1,c/v}(\mathbf{k} + \mathbf{d}_1) U_{2,c/v}(\mathbf{k} + \mathbf{d}_2) \times U_{1,c/v}(\mathbf{k})^{-1} U_{2,c/v}(\mathbf{k})^{-1}]. \quad (\text{F7})$$

For small $|\mathbf{d}_\mu|$, $F_{c/v}(\mathbf{k})$ is proportional to the Berry curvature [33] and the Chern number of the band can be calculated as

$$C = \frac{1}{2\pi} \sum_{\mathbf{k}} F_{c/v}(\mathbf{k}). \quad (\text{F8})$$

This gives us a method to calculate the exciton Berry curvature without having to first find a smooth gauge. We have two expressions for the Berry curvature but for the system studied in this paper, they both give similar Berry curvature distributions. In addition, as expected, they both integrate to the same value and so give the same Chern number [23]. In the main text, we, therefore, only plot one of the possible Berry curvature plots that calculated using $|\mathcal{U}_p^{\text{exc},c}\rangle$.

- [1] D. Xiao, M.-C. Chang, and Q. Niu, Berry phase effects on electronic properties, *Rev. Mod. Phys.* **82**, 1959 (2010).
- [2] M. O. Soldini, N. Astrakhantsev, M. Iraola, A. Tiwari, M. H. Fischer, R. Valentí, M. G. Vergniory, G. Wagner, and T. Neupert, Interacting topological quantum chemistry of Mott atomic limits, *Phys. Rev. B* **107**, 245145 (2023).
- [3] K. Shiozaki, C. Z. Xiong, and K. Gomi, Generalized homology and Atiyah–Hirzebruch spectral sequence in crystalline symmetry protected topological phenomena, *Prog. Theor. Exp. Phys.* **2023**, 083101 (2023).
- [4] N. Manjunath, V. Calvera, and M. Barkeshli, Characterization and classification of interacting $(2 + 1)$ -dimensional topological crystalline insulators with orientation-preserving wallpaper groups, *Phys. Rev. B* **109**, 035168 (2024).
- [5] J. Herzog-Arbeitman, B. A. Bernevig, and Z.-D. Song, Interacting topological quantum chemistry in 2D with many-body real space invariants, *Nat. Commun.* **15**, 1171 (2024).
- [6] H. Davenport, J. Knolle, and F. Schindler, Interaction-induced crystalline topology of excitons, *Phys. Rev. Lett.* **133**, 176601 (2024).
- [7] Y. H. Kwan, Y. Hu, S. H. Simon, and S. A. Parameswaran, Exciton band topology in spontaneous quantum anomalous Hall insulators: Applications to twisted bilayer graphene, *Phys. Rev. Lett.* **126**, 137601 (2021).
- [8] P. Froese, T. Neupert, and G. Wagner, Topological excitons in moiré $\text{MoTe}_2/\text{WSe}_2$ heterobilayers, *Phys. Rev. Res.* **7**, 023047 (2025).
- [9] W. Yao and Q. Niu, Berry phase effect on the exciton transport and on the exciton Bose-Einstein condensate, *Phys. Rev. Lett.* **101**, 106401 (2008).
- [10] W. J. Jankowski, J. J. P. Thompson, B. Monserrat, and R.-J. Slager, Excitonic topology and quantum geometry in organic semiconductors, *Nat. Commun.* **16**, 4661 (2025).
- [11] H.-Y. Xie, P. Ghaemi, M. Mitran, and B. Uchoa, Theory of topological exciton insulators and condensates in flat Chern bands, *Proc. Natl. Acad. Sci. USA* **121**, e2401644121 (2024),.
- [12] Z. Lu, T. Han, Y. Yao, A. P. Reddy, J. Yang, J. Seo, K. Watanabe, T. Taniguchi, L. Fu, and L. Ju, Fractional quantum anomalous Hall effect in a graphene moiré superlattice, *arXiv:2309.17436*.
- [13] J. Herzog-Arbeitman, Y. Wang, J. Liu, P. M. Tam, Z. Qi, Y. Jia, D. K. Efetov, O. Vafek, N. Regnault, H. Weng, Q. Wu, B. A. Bernevig, and J. Yu, Moiré fractional Chern insulators. II. First-principles calculations and continuum models of rhombohedral graphene superlattices, *Phys. Rev. B* **109**, 205122 (2024).
- [14] J. Dong, T. Wang, T. Wang, T. Soejima, M. P. Zaletel, A. Vishwanath, and D. E. Parker, Anomalous Hall crystals in rhombohedral multilayer graphene. I. Interaction-driven Chern bands and fractional quantum Hall states at zero magnetic field, *Phys. Rev. Lett.* **133**, 206503 (2024).
- [15] T. Soejima, J. Dong, T. Wang, T. Wang, M. P. Zaletel, A. Vishwanath, and D. E. Parker, Anomalous Hall crystals in rhombohedral multilayer graphene. II. General mechanism and a minimal model, *Phys. Rev. B* **110**, 205124 (2024).
- [16] B. A. Bernevig and Y. H. Kwan, “Berry trashcan” model of interacting electrons in rhombohedral graphene, *arXiv:2503.09692*.
- [17] J. Yu, J. Herzog-Arbeitman, Y. H. Kwan, N. Regnault, and B. A. Bernevig, Moiré fractional Chern insulators IV: Fluctuation-driven collapse of FCIs in multi-band exact diagonalization calculations on rhombohedral graphene *Phys. Rev. B* **112**, 075110 (2025).
- [18] Z. Dong, A. S. Patri, and T. Senthil, Stability of anomalous Hall crystals in multilayer rhombohedral graphene, *Phys. Rev. B* **110**, 205130 (2024).
- [19] Z. Dong, A. S. Patri, and T. Senthil, Theory of quantum anomalous Hall phases in pentalayer rhombohedral graphene moiré structures, *Phys. Rev. Lett.* **133**, 206502 (2024).
- [20] B. Zhou, H. Yang, and Y.-H. Zhang, Fractional quantum anomalous Hall effect in rhombohedral multilayer graphene in the moiréless limit, *Phys. Rev. Lett.* **133**, 206504 (2024).
- [21] B. Bradlyn, L. Elcoro, J. Cano, M. G. Vergniory, Z. Wang, C. Felser, M. I. Aroyo, and B. A. Bernevig, Topological quantum chemistry, *Nature (London)* **547**, 298 (2017).
- [22] J. B. Haber, D. Y. Qiu, F. H. da Jornada, and J. B. Neaton, Maximally localized exciton Wannier functions for solids, *Phys. Rev. B* **108**, 125118 (2023).
- [23] H. Davenport, J. Knolle, and F. Schindler, Exciton Berryology *Phys. Rev. B* **113**, 045125 (2026).
- [24] A. Fischer, L. Klebl, D. M. Kennes, and T. O. Wehling, Supercell Wannier functions and a faithful low-energy model for Bernal bilayer graphene, *Phys. Rev. B* **110**, L201113 (2024).
- [25] X.-T. Zhang, Y. H. Gao, and G. Chen, Thermal Hall effects in quantum magnets, *Phys. Rep.* **1070**, 1 (2024).
- [26] H. C. Nerl, K. T. Winther, F. S. Hage, K. S. Thygesen, L. Houben, C. Backes, J. N. Coleman, Q. M. Ramasse, and V. Nicolosi, Probing the local nature of excitons and plasmons in few-layer MoS_2 , *npj 2D Mater. Appl.* **1**, 2 (2017).
- [27] S. Susarla, M. H. Naik, D. D. Blach, J. Zipfel, T. Taniguchi, K. Watanabe, L. Huang, R. Ramesh, F. H. da Jornada, S. G. Louie, P. Ercius, and A. Raja, Hyperspectral imaging of exciton confinement within a moiré unit cell with a subnanometer electron probe, *Science* **378**, 1235 (2022).
- [28] P. Moon and M. Koshino, Electronic properties of graphene/hexagonal-boron-nitride moiré superlattice, *Phys. Rev. B* **90**, 155406 (2014).
- [29] X. Sun, S. Zhang, Z. Liu, H. Zhu, J. Huang, K. Yuan, Z. Wang, K. Watanabe, T. Taniguchi, X. Li, M. Zhu, J. Mao, T. Yang, J. Kang, J. Liu, Y. Ye, Z. V. Han, and Z. Zhang, Correlated states in doubly-aligned hBN/graphene/hBN heterostructures, *Nat. Commun.* **12**, 7196 (2021).
- [30] C. Fang, M. J. Gilbert, and B. A. Bernevig, Bulk topological invariants in noninteracting point group symmetric insulators, *Phys. Rev. B* **86**, 115112 (2012).
- [31] F. Schindler, O. Vafek, and B. A. Bernevig, Trions in twisted bilayer graphene, *Phys. Rev. B* **105**, 155135 (2022).
- [32] E. Khalaf and A. Vishwanath, Baby skyrmions in Chern ferromagnets and topological mechanism for spin-polaron formation in twisted bilayer graphene, *Nat. Commun.* **13**, 6245 (2022).
- [33] T. Fukui, Y. Hatsugai, and H. Suzuki, Chern numbers in discretized Brillouin zone: Efficient method of computing (spin) Hall conductances, *J. Phys. Soc. Jpn.* **74**, 1674 (2005).
- [34] P. Nguyen, L. Ma, R. Chaturvedi, K. Watanabe, T. Taniguchi, J. Shan, and K. Mak, Perfect Coulomb drag in a dipolar excitonic insulator, *Science* **388**, 274 (2025).
- [35] R. Qi, A. Y. Joe, Z. Zhang, J. Xie, Q. Feng, Z. Lu, Z. Wang, T. Taniguchi, K. Watanabe, S. Tongay, *et al.*, Perfect Coulomb

- drag and exciton transport in an excitonic insulator, *Science* **388**, 278 (2025).
- [36] P. Czajka, T. Gao, M. Hirschberger, P. Lampen-Kelley, A. Banerjee, N. Quirk, D. G. Mandrus, S. E. Nagler, and N. P. Ong, Planar thermal Hall effect of topological bosons in the Kitaev magnet α -RuCl₃, *Nat. Mater.* **22**, 36 (2023).
- [37] F. Schindler, S. S. Tsirkin, T. Neupert, B. Andrei Bernevig, and B. J. Wieder, Topological zero-dimensional defect and flux states in three-dimensional insulators, *Nat. Commun.* **13**, 5791 (2022).
- [38] H. Davenport, J. Knolle, and F. Schindler, Berry curvature of low-energy excitons in rhombohedral graphene, (Data) [Data set], Zenodo (2025), <https://doi.org/10.5281/zenodo.16902666>.
- [39] F. Zhang, B. Sahu, H. Min, and A. H. MacDonald, Band structure of *abc*-stacked graphene trilayers, *Phys. Rev. B* **82**, 035409 (2010).