

Metal-insulator transition in a boundary three chain model

Niels John^{1,*}, Yuval Gefen,² and Bernd Rosenow^{1,†}

¹*Institut für Theoretische Physik, Universität Leipzig, D-04103 Leipzig, Germany*

²*Department of Condensed Matter Physics, Weizmann Institute, 7610001 Rehovot, Israel*

 (Received 15 February 2022; revised 21 July 2022; accepted 18 November 2025; published 29 January 2026)

We study the boundary physics of topologically trivial Mott insulators, representing the boundary by three coupled Hubbard chains in a linear confining potential. In the Hartree-Fock approximation, the ground state at and slightly off the particle-hole symmetric point remains insulating even at large slopes of the confining potential. By contrast, accounting for quantum fluctuations and correlations through a combination of bosonization and renormalization group (RG) methods, we find that there is always a gapless dipole mode, but it does not contribute to a finite compressibility. Moreover, when increasing the slope of the confining potential at half filling, we find a quantum phase transition between an insulating and a metallic state, indicating the formation of a soft edge state of non-topological origin. Away from half filling, a similar transition takes place.

DOI: [10.1103/rq6s-z87r](https://doi.org/10.1103/rq6s-z87r)

I. INTRODUCTION

The boundaries of two-dimensional (2D) topological phases of matter play a pivotal role when the bulk spectrum is gapped [1–7]. Having a gapless spectrum at the edge may render the system conducting underpinned by bulk topological invariants [8,9]. Examples range from quantum Hall phases (integer, fractional, non-Abelian) featuring chiral modes [1,2], helical modes [4,10], and other 2D topological phases [11–13]. On the other hand, interaction-driven Mott insulators [14,15] are usually not characterized by conducting edge modes.

Here, we focus on the physics of a 2D nontopological phase of matter, specifically the bulk-gapped Mott insulator. Addressing the low-energy physics of its boundary, we discover an unexpectedly rich and intriguing edge physics. Accounting for the interplay of the interaction strength at the edge and the confining potential, we find a mechanism of a metal-insulator transition. This transition occurs as function of the steepness of the confining potential and the band filling (see Fig. 1), notwithstanding the fact that the excitation spectrum remains gapless. Our results are depicted in Fig. 2. Note that despite the presence of a gapless mode, the system may be an insulator since that particular mode (associated with dipole excitations) contributes neither to the compressibility nor to the dc conductance. These main features crucially rely on the inclusion of quantum fluctuations, and cannot be captured by a mean-field Hartree-Fock analysis. We stress that quantum

fluctuations are responsible for the existence of a gapless dipole mode, on top of which the closing and opening of a charge gap is controlled by the slope of a confining potential. Our analysis opens the gate to study of other quantum states of matter (and quantum phase transitions) on nontopological edges.

Specifically, we model the edge with three coupled Hubbard chains under the influence of a smooth external potential. These coupled chains are *not* supposed to model the bulk two-dimensional system, which is incompressible (see Fig. 3). They do represent the potentially compressible narrow strip at the edge, which is subject to a confining potential stronger than the Mott gap. The effect of the bulk could be at most a renormalization of the bare chain parameters. This effect is ignored here. As far as modeling the coupled chains, in addition to a standard nearest-neighbor Hamiltonian with Hubbard interactions, we assume a chain-dependent potential (of slope V) which mimics a confining potential in the perpendicular y direction (cf. Fig. 4). Employing bosonization technique tools, we study the collective phases of this many-fermion system. The overall filling of the system is determined by the chemical potential μ . Of the three transverse modes (in the y direction), one (a gapless mode) corresponds to interchain particle-hole excitations (a “dipole”). The other two, featuring charge excitations, do contribute to the conductance and the compressibility. As function of V and μ a gap opens in the charge sector, leading to a metal-insulator quantum phase transition.

Our microscopic analysis relies on a coupled Hubbard-chain model. In the absence of a confining potential, bosonization was employed for both a single and multiple chains [15–21]. For a single Hubbard chain there exists an insulating phase without magnetic order. Considering several coupled Hubbard chains one finds a variety of instabilities as a function of interchain hopping, as well as Hubbard and next-nearest neighbor interaction strength [15–20]. For two coupled chains the metal-insulator transition was observed

*Deceased.

†Contact author: rosenow@physik.uni-leipzig.de

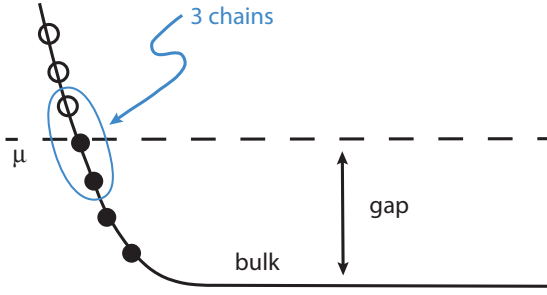


FIG. 1. Setup: We consider a Mott insulator under the influence of a confining potential at the edge. While the bulk is incompressible due to the Mott gap, at the edge compressible regions can appear as a function of the chemical potential and of the confining potential's slope.

as function of the chemical potential [21], the filling [16], the interchain hopping [16,17,21], or the strength of on-site and nearest-neighbor interactions [17,21]. Superconducting *d*-wave or *s*-wave states or charge-modulated states have been observed within the framework of two coupled chains as well [18]. Finally, within a Hartree-Fock (HF) study of the edge (neglecting quantum fluctuations), an alternation of compressible and incompressible boundary regions has been found [22].

II. MODEL

As a first step towards a realistic model for the edge region of a two-dimensional Mott insulator, we allow for a confining potential $V_C(p)$. The confining potential is different for chains with different index p and is constant along each individual chain. In the following, we will linearize the confining potential in the region of the boundary chains, which is a

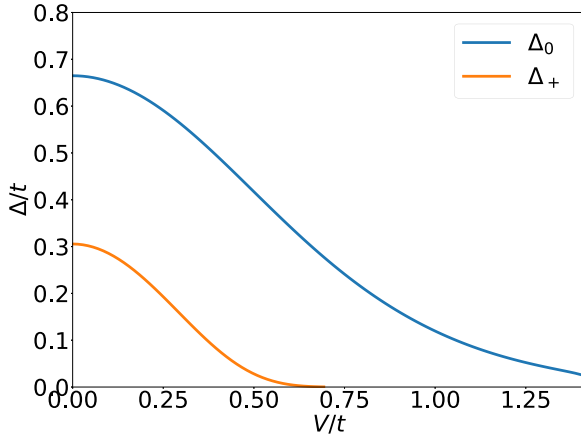


FIG. 2. Gaps of the two modes contributing to charge density and current as a function of the the confining potential strength V at half filling. Δ_+ and Δ_0 are, respectively, the charge gaps of the fields $\phi_{\rho,+}$ and $\phi_{\rho,0}$. The latter remains finite up until $V/t = \sqrt{2}$, whereas the former drops to zero at $V/t \simeq 0.7$. This marks the transition from an incompressible state into a compressible state, which is expected to have a finite conductance. The transition happens at the particle-hole symmetric point and persists for a finite range of values of the chemical potential.

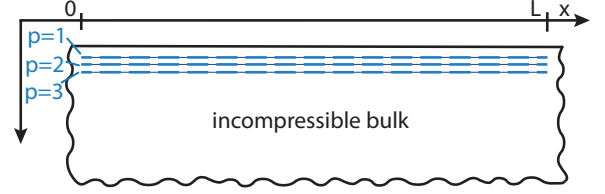


FIG. 3. Boundary region: We model the boundary of the Mott insulating by the three chains closest to the chemical potential, denoted by $p = 1, 2, 3 \dots$. The incompressible bulk can be integrated out, a process which will only renormalize the parameters of the boundary chains. Without loss of generality, we can impose periodic boundary conditions along the x direction.

standard step taken in any analysis of the edge of quantum Hall systems, for example. Without loss of generality, we can then assume that the confining potential is split symmetrically around the chemical. The full Hamiltonian of the system is

$$\begin{aligned}
 H = & -t \sum_{j,\sigma,p} (c_{j,p,\sigma}^\dagger c_{j+1,p,\sigma} + \text{H.c.}) - \mu \sum_{j,\sigma,p} n_{j,\sigma,p} \\
 & - t \sum_{j,\sigma} \sum_{p=1,2} (c_{j,p,\sigma}^\dagger c_{j,p+1,\sigma} + \text{H.c.}) \\
 & + V \sum_{j,\sigma} (c_{j,1,\sigma}^\dagger c_{j,1,\sigma} - c_{j,3,\sigma}^\dagger c_{j,3,\sigma}) \\
 & + U \sum_{j,p} n_{j,p,\uparrow} n_{j,p,\downarrow},
 \end{aligned} \tag{1}$$

where t is the nearest-neighbor hopping strength. Due to the tunneling between the chains, the chain index p is not a good quantum number.

For the noninteracting system with $U = 0$, diagonalization of H yields a Hamiltonian with three bands and pseudospin indices $\tau = -1, 0, +1$. The dispersions are $E_{k,\tau} = -2t \cos(ak) + \tau \sqrt{2t^2 + V^2}$, such that at half filling and weak confining potential $V/t \ll 1$, the Fermi momenta of these bands are $k_{F,\tau} = \pi/2 - \tau(\pi/4 + V^2/t^2)$. Interestingly, one band remains at exactly half filling, independent of the actual value of V . Once the slope of the confining potential reaches the critical value $V/t = \sqrt{2}$, the lower band with $\tau = -1$ is completely filled and the upper band with $\tau = +1$ is empty. Thus, for values $V/t > \sqrt{2}$ and without interaction, the

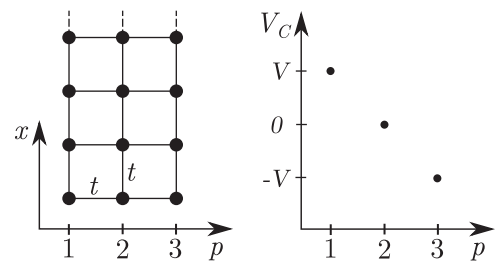


FIG. 4. Geometry of three coupled Hubbard chains subject to a confining potential. In the left panel, the lattice structure with the single-electron hoppings t across and along the chains is depicted. The right panel shows the confining potential as a function of the chain index p .

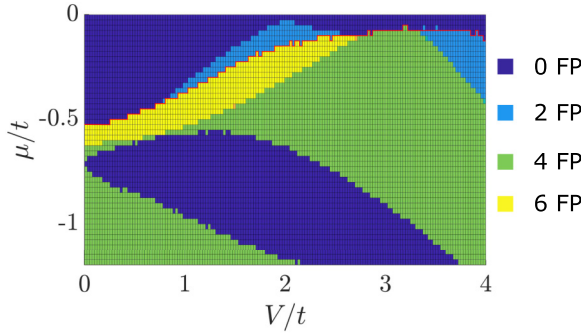


FIG. 5. Mean-field phase diagram for $U/t = 5$. The plot shows the number of Fermi points of the self-consistently determined band structure as a function of the chemical potential μ/t and the confining strength V/t . The red line indicates the transition between antiferromagnetic order and a paramagnet. As an interpretation, we can identify a pair of Fermi points with an ungapped (charge) mode.

system is trivial in the sense that only the middle chain remains at half filling, whereas the one with $p = 3$ is completely filled and thus similar to a bulk state, and the outermost chain with $p = 1$ is empty.

III. HARTREE-FOCK ANALYSIS

In order to gain some intuition into the interacting theory, we use Hartree-Fock (HF) analysis to determine the band structure self-consistently. We reexpress the interaction term $n_{j,p,\uparrow}n_{j,p,\downarrow}$ as $\langle n_{j,p,\uparrow} \rangle_0 \langle n_{j,p,\downarrow} \rangle_0 + n_{j,p,\uparrow} \langle n_{j,p,\downarrow} \rangle_0 - \langle n_{j,p,\uparrow} \rangle_0 \langle n_{j,p,\downarrow} \rangle_0$, where $\langle \dots \rangle_0$ denotes the expectation value with respect to the HF ground state. We perform a restricted HF analysis with

$$\langle n_{j,p,\sigma} \rangle_0 = \frac{n_p}{2} + \sigma S_p (-1)^j, \quad (2)$$

where n_p is the average density in chain p , and the order parameter S_p is the staggered magnetization of chain p . The effective single-particle Hamiltonian in terms of the order parameter is discussed in the Appendix. The densities and staggered magnetizations are determined self-consistently.

For a single half-filled Hubbard chain, the antiferromagnetic order parameter S is finite for any $U > 0$ and the system develops a finite band gap. As the chain is doped away from half filling, the order parameter drops to zero, rendering the system compressible. Then, the band gap closes and the system has two Fermi points at $k = \pm k_F$. We can use a similar picture for the analysis of the three coupled chains: each pair of Fermi points corresponds to one gapless charge mode. Within HF theory and near-half filling, the gapping of Fermi points is only possible due to a finite antiferromagnetic magnetization.

In Fig. 5, the number of Fermi points as a function of the chemical potential and the confining potential is shown. Interestingly, the half-filled system remains gapped and has no Fermi points, independent of the value of V . This constitutes an adiabatic connection from the Mott insulating state at $V = 0$ to an asymptotic state $V/t \gg 1$, where one chain is completely empty, one chain is completely filled, and the middle chain is still a gapped Mott insulator.

We observe transitions into states with a finite number of Fermi points as a function of μ and V . At a finite $\mu/t \simeq -0.2$, increasing V eventually leads to a transition into a state with two Fermi points, which is the equivalent of one gapless charge mode. At the transition point, the band gap closes and the system becomes compressible. If we increase V even further, the system transitions from a state with two Fermi points into one with six Fermi points. In the latter state, not only the band gap vanishes, but the antiferromagnetic magnetization vanishes as well. The regime with six Fermi points, or three ungapped charge modes, is thus equivalent to a system of noninteracting electrons.

Depending on the actual value of V , we observe similar transitions as μ varies. At $V = 0$, lowering μ leads to a transition into a gapless state with three Fermi points around $\mu/t \simeq -0.55$. As before, in this gapless regime the antiferromagnetic order parameter vanishes. If, however, $1 < V/t < 2$, a variation of μ leads to similar transitions as described in the paragraph above: first, the system transits into a state with two Fermi points, which is accompanied by a band-gap closing. For even lower μ , we observe the same transition into a compressible, noninteracting state with six Fermi points and vanishing antiferromagnetic order parameter.

We emphasize that the absence of a gap closing at half filling is an important difference between three coupled and three uncoupled chains placed in a linear confining potential. In the former case, all three chains always have a finite band gap. In the latter case and for sufficiently large V , the chemical potential with respect to the bottom of the band for a given chain will exceed the band gap and the respective order parameters drop to zero. This will render the aforementioned chains compressible and, hence, make the whole system compressible despite the fact that it remains at exactly half filling.

IV. BOSONIZATION

We use the bosonization technique and a first-order renormalization group analysis to investigate the interacting system with $U > 0$. For $V/t < \sqrt{2}$, the three bands are bosonized individually, such that the kinetic energy is diagonal in the τ basis. For each band, there are four bosons fields $\phi_{v,\tau}$ and $\theta_{v,\tau}$ with commutation relations $[\phi_{v,\tau}(x), \partial_{x'} \theta_{v',\tau'}(x')] = i \hbar \pi \delta_{v,v'} \delta_{\tau,\tau'} \delta(x - x')$, where $v = \rho, \sigma$ labels the charge or spin channel. As usual, the density of band τ is related to the fields as $\rho_\tau(x) = \rho_{\tau,0} + (\sqrt{2}/\pi) \partial_x \phi_{\rho,\tau}(x)$, where $\rho_{0,\tau}$ is the average density of the band. The relation between the densities of the bands and the densities of the chains is shown in the Appendix. Keeping only the most relevant contributions of the interaction terms, we can write it as $H_{\text{int}} = H_{\text{int}}^0 + H_{\text{int}}^{2k_F}$, where the first term is a density-density interaction in the new basis and the second term describes umklapp scattering. We note that in H_{int}^0 all three fields are coupled among each other.

To partially decouple H_{int}^0 we introduce the symmetric and antisymmetric combination of the fields $\phi_{v,\tau=+1}$ and $\phi_{v,\tau=-1}$. The fields are transformed as

$$\begin{pmatrix} \phi_{v,\alpha=+} \\ \phi_{v,\alpha=0} \\ \phi_{v,\alpha=-} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \phi_{v,\tau=+1} \\ \phi_{v,\tau=0} \\ \phi_{v,\tau=-1} \end{pmatrix}. \quad (3)$$

The θ fields are transformed accordingly. The field index takes values $\alpha = +, 0, -$. In the α basis, the Hamiltonian $\mathcal{H}_0 = H_0 + H_{\text{int}}^0$ is given by

$$\mathcal{H}_0 = \frac{1}{2\pi} \sum_{\alpha} \sum_{v=\rho,\sigma} \left[u_{v,\alpha} K_{v,\alpha} (\partial_x \theta_{v,\alpha})^2 + \frac{u_{v,\alpha}}{K_{v,\alpha}} (\partial_x \phi_{v,\alpha})^2 \right] + \frac{\gamma}{\pi} \sum_{v=\rho,\sigma} (\partial_x \phi_{v,+}) (\partial_x \phi_{v,0}), \quad (4)$$

with $\gamma = (U/t) \sqrt{2(2(V/t)^2 + 1)} / ((V/t)^2 + 2)^2$ and

$$u_v K_v \hat{=} \begin{pmatrix} \sqrt{2 - (V/t)^2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \sqrt{2 - (V/t)^2} \end{pmatrix}. \quad (5)$$

We can see from Eq. (4) that the transformation from Eq. (3) only decouples the field $\phi_{v,-}$ from the other two in the kinetic Hamiltonian. In the newer basis, the two most relevant umklapp terms are

$$H_{\text{int}}^{2k_F} = -\frac{g_3^{(0,0;0,0)}}{2\pi^2 a^2} \cos(2\sqrt{2}\phi_{\rho,0} - \delta_0 x) - \frac{g_3^{(1,-1;-1,1)}}{2\pi^2 a^2} \cos(2\theta_{\sigma,-}) \cos(2\phi_{\rho,+} - \delta_+ x), \quad (6)$$

where $\delta_+ = \delta_0 \equiv 0$ at half filling. The remaining relevant umklapp terms contain linear combinations of $\phi_{\rho,0}$ and $\phi_{\rho,+}$ as arguments and do not contribute additional physics for the charge modes. The bare values of the coupling constants can be found in the Appendix.

In order to derive the umklapp terms above, we use the standard approach to express fermionic operators as exponentials of bosonic fields [18,23] in the τ basis, multiplied by a plane wave with Fermi momentum $k_{F,\tau}$:

$$\psi_{r,\tau,s}(x) \propto e^{irk_{F,\tau}x} e^{-i/\sqrt{2}(r\phi_{\rho,\tau} - \theta_{\rho,\tau} + s(r\phi_{\sigma,\tau} - \theta_{\sigma,\tau}))}, \quad (7)$$

where $r = \pm 1$ for right- and left movers and $s = \pm 1$ for spin up and spin down. To obtain the umklapp terms given in Eq. (6), we rewrite the interaction term in Eq. (1) in terms of bosonic exponentials, leaving us with terms of the type $\cos(c_\beta \phi_{\rho,\beta} + \delta_\beta x)$, where c_β is a real number and the displacement δ_β is the sum or the difference of all the plane-wave momenta. For the umklapp terms from Eq. (6), the displacement is $\delta_\beta = \pm 2\pi$ at half filling. By construction, the boson fields vary slowly on the scale of inverse Fermi momentum. Indeed, if $\delta_\beta \neq 0$ modulo 2π the cosine operators oscillate on length scales much shorter than $1/k_{F,\tau}$ and thus vanish upon integration over x . This restricts the processes that we should include into the bosonization framework, and, in particular, leaves us with only a few umklapp terms we need to take into consideration.

Remarkably, the charge field $\phi_{\rho,-}$ does not appear in the umklapp terms. This is a direct consequence of the fact that only cosines with vanishing displacements should be kept in the Hamiltonian. Utilizing Eq. (7), we note that cosines with arguments proportional to $\phi_{\rho,-}$ always come with a displacement $\delta_- = \pm 2k_{F,+1} \mp 2k_{F,-1} \equiv \pm\pi$. Keeping in mind that the position variable is understood as $x = ja$ with $j \in \mathbb{N}$, these cosines change sign on adjacent lattice sites and vanish upon integration over x .

Bosonic fields appearing as the argument of an RG relevant cosine term are locked in one of the extrema of the cosine, giving the respective boson field a finite expectation value. Expanding the cosine about this minimum, a mass term quadratic in the boson field is obtained, implying that the boson field develops a gap. According to the discussion above, $\phi_{\rho,-}$ remains gapless such that one might conclude that the coupled chains remain compressible. However, $\phi_{\rho,-}$ does not appear in the charge density

$$\rho(x) = \rho_0 + \frac{\sqrt{2}}{\pi} (\partial_x \phi_{\rho,0}) + \frac{2}{\pi} (\partial_x \phi_{\rho,+}), \quad (8)$$

where $\rho_0 = 1$ at half filling. The absence of $\phi_{\rho,-}$ in the charge density implies that the conjugate field will not show up in the associated current density. Invoking this argument, the system can still be an insulator as long as the fields $\phi_{\rho,+}$ and $\phi_{\rho,0}$ are gapped.

To decouple the remaining two fields, we first rescale the fields $\phi_{v,0} \mapsto (2/\sqrt{2-V^2})^{-1/2} \phi_{v,0}$ and $\theta_{v,0} \mapsto (2/\sqrt{2-V^2})^{1/2} \theta_{v,0}$ and diagonalize the remaining Hamiltonian. The rescaling is necessary in order to obey the bosonic commutation relations for the linear combinations $\tilde{\phi}_{v,\alpha}$ of the fields. In matrix notation, we express the transformation as

$$\begin{pmatrix} \phi_{v,+} \\ \phi_{v,0} \\ \phi_{v,-} \end{pmatrix} = \begin{pmatrix} \eta_{v;++} & \eta_{v;0+} & 0 \\ \eta_{v;+0} & \eta_{v;00} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{\phi}_{v,+} \\ \tilde{\phi}_{v,0} \\ \tilde{\phi}_{v,-} \end{pmatrix}. \quad (9)$$

After diagonalization, the full Hamiltonian including the relevant sine-Gordon terms is $H = \tilde{\mathcal{H}}_0 + \tilde{\mathcal{H}}_{\text{int}}^{2k_F}$, with

$$\tilde{\mathcal{H}}_0 = \frac{1}{2\pi} \sum_{\alpha,v=\rho,\sigma} \left[\tilde{u}_{v,\alpha} \tilde{K}_{v,\alpha} (\partial_x \tilde{\theta}_{v,\alpha})^2 + \frac{\tilde{u}_{v,\alpha}}{\tilde{K}_{v,\alpha}} (\partial_x \tilde{\phi}_{v,\alpha})^2 \right], \quad (10)$$

and $\tilde{\mathcal{H}}_{\text{int}}^{2k_F}$ is the same as Eq. (6) except the fields are transformed according to Eq. (9). Further, $\tilde{u}_{v,\alpha} \tilde{K}_{v,\alpha} = \sqrt{2 - (V/t)^2}$. For $\tilde{u}_{v,\alpha} / \tilde{K}_{v,\alpha}$ see the Appendix.

Without confining potential $V/t = 0$, both of the sine-Gordon terms in Eq. (6) are relevant, meaning that the fields $\phi_{\rho,+}$ and $\phi_{\rho,0}$, and $\theta_{\sigma,-}$ are gapped. In the following, we study the charge gaps for the fields $\phi_{\rho,+}$, $\phi_{\rho,0}$ as a function of the confining potential strength V/t . We calculate the gaps utilizing a variational approach outlined in Ref. [18]. The gap for each charge mode is given by

$$\Delta^{(\gamma)} = u_\rho^{(\gamma)} \Lambda \left(\frac{4K_\rho^{(\gamma)} y^{(\gamma)}}{(\alpha\Lambda)^2} \right)^{1/(2-2K_\rho^{(\gamma)})}, \quad (11)$$

where $K_\rho^{(\gamma)}$ is obtained from the scaling dimension of the respective sine-Gordon term, $u_\rho^{(\gamma)}$ is the excitation velocity, and γ is a multi-index that takes values $\gamma = 0, 0; 0, 0$ and $\gamma = 1, -1; -1, 1$. In the following discussion, we use the abbreviations $\Delta_0 \equiv \Delta^{(0,0;0,0)}$ and $\Delta_+ \equiv \Delta^{(1,-1;-1,1)}$, such that Δ_0 is the charge gap for the field $\phi_{\rho,0}$, and Δ_+ for the field $\phi_{\rho,+}$. $\Lambda = \pi/(2a)$ is a cutoff in momentum space, and

$y^{(\gamma)} = g_3^{(\gamma)}/(\pi \hbar v_F)$ is the dimensionless coupling constant. We note that the renormalization of the bare gaps inside the brackets in Eq. (11) happens due to the exponent $1/(2 - 2K_\rho^{(\gamma)})$, which is the inverse of the scaling dimension of the corresponding term [24]. Note that in principle one should consider the renormalization of the Luttinger parameters $K_\rho^{(\gamma)}$ due to the interaction couplings appearing in $\mathcal{H}_{\text{int}}^{2k_F}$ [18]. However, from knowledge of the full Kosterlitz-Thouless (KT) RG flow, one can deduce that the intersection of the KT separatrix with the $K_\rho^{(\gamma)}$ -axis corresponds to the point where the relevancy of the cosine operator changes. This change in relevancy however can be obtained correctly without taking into account the renormalization of Luttinger parameters.

The magnitudes of the gaps as a function of V/t are shown in Fig. 2. One sees that both charge gaps are finite but different in magnitude at $V/t = 0$. Further, we find that the magnitudes of the gaps calculated in Hartree-Fock theory agree quite well with the result from bosonization. As V/t increases, both gaps decrease in magnitude. Again, this is in agreement with the results obtained from HF. At $V/t \simeq 0.7$, the charge gap Δ_+ drops to zero. The closing of this gap is accompanied by the corresponding sine-Gordon term from Eq. (6) becoming an irrelevant operator. More importantly, we observe a metal-insulator transition as a function of the confining potential strength. This is an important difference compared to HF. For the latter, at the symmetry point $\mu = 0$ the system remains gapped independent of the actual value of V/t . The charge gap Δ_0 remains finite all the way up to $V/t \simeq \sqrt{2}$.

V. DISCUSSION

We have considered a simplified picture of an edge of a strongly correlated 2D electron gas, modeled by three coupled chains subject to a “confining potential.” Our bosonization approach is compared with a HF analysis. For the latter, only when being away from the particle-hole symmetry point, we find that as we increase the steepness of the confining potential, we transition from a gapped phase to an ungapped one, from a vanishing to a finite conductance phase, and from a finite to a zero magnetization phase. The magnetization transition does not coincide with the other two. By contrast, our bosonization analysis, both at and away from the symmetry point, reveals distinctly different features. As we increase the steepness of the confining potential, we observe a transition from a nonconducting to a conducting phase (the spectrum remains gapless throughout, and on both sides of the transition the phases are nonmagnetic). The challenge of pushing be-

yond the three-chain limit with our analysis that goes beyond mean field is left for future work. Interestingly, this effort may address the issue of chain-number parity effects, which have been discussed within the framework of mean field [22].

This more reliable bosonization analysis, combined with renormalization group insights, which account for quantum fluctuations, reveals that the phase transition reported here takes place notwithstanding the fact that the spectrum is always gapless. The insulating-to-conducting phase transition is accompanied by a gap closing of a collective charge mode as the confinement strength increases. We thus find that as a function of the confining potential strength, there can be a quantum phase transition at the boundary of interaction-driven Mott insulators, with experimentally observable consequences. In the presence of very shallow confinement potentials, edge reconstruction can occur. Then, additional neutral (i.e., dipole or even higher-order multipole) modes can exist, which couple to the charge mode and modify its dynamic properties [25]. Such edge reconstruction can be described by adding fluctuation corrections to a Hartree-Fock ground state [22].

Our prediction of a gap closing in an interacting system as a function of the slope of a confining potential can be tested with present-day experimental techniques. In bilayer graphene, applying a perpendicular electric field can open up a bulk gap at the neutrality point [26]. A combination of top gates and a back gate allows to independently tune the gap and the position of the Fermi level in the regions underneath the gates [27], such that a confining potential near the sample boundary with electrostatic control over its slope is experimentally feasible.

ACKNOWLEDGMENTS

Y.G. and B.R. acknowledge support by Grant No. DFG RO 2247/11-1. Y.G. was supported by the Minerva Foundation, Grant No. MI 658/10-2, and by the German Israeli Foundation (Grant No. I-118-303.1-2018). Y.G. is an incumbent of an Infosys chair at IISc, Bangalore.

APPENDIX A: NONINTERACTING SYSTEM WITH $U = 0$

To diagonalize the Hamiltonian Eq. (1) with $U = 0$, we apply the Fourier transformation

$$c_{j,p,\sigma} = \frac{1}{\sqrt{N}} \sum_k e^{ijk} c_{k,p,\sigma}, \quad (\text{A1})$$

and diagonalize the remaining Hamiltonian in p space afterwards. The operators are transformed as

$$\begin{pmatrix} c_{k,+1,\sigma} \\ c_{k,0,\sigma} \\ c_{k,-1,\sigma} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\tilde{V}^2 + \sqrt{\tilde{V}^2 + 2\tilde{V} + 1}}{2\tilde{V}^2 + 4}} & -\frac{\sqrt{\tilde{V}^2 + 2\tilde{V}}}{\sqrt{2}\sqrt{(\tilde{V}^2 + 2)(\tilde{V}(\sqrt{\tilde{V}^2 + 2\tilde{V} + 1}) + 1)}} & \frac{1}{\sqrt{2}\sqrt{(\tilde{V}^2 + 2)(\tilde{V}(\sqrt{\tilde{V}^2 + 2\tilde{V} + 1}) + 1)}} \\ -\frac{1}{\sqrt{\tilde{V}^2 + 2}} & -\frac{\tilde{V}}{\sqrt{\tilde{V}^2 + 2}} & \frac{1}{\sqrt{\tilde{V}^2 + 2}} \\ \sqrt{\frac{\tilde{V}^2 - \sqrt{\tilde{V}^2 + 2\tilde{V} + 1}}{2\tilde{V}^2 + 4}} & \frac{\sqrt{\tilde{V}^2 - 2\tilde{V}}}{\sqrt{2}\sqrt{(\tilde{V}^2 + 2)(\tilde{V}^2 - \sqrt{\tilde{V}^2 + 2\tilde{V} + 1})}} & \frac{1}{\sqrt{2}\sqrt{(\tilde{V}^2 + 2)(\tilde{V}^2 - \sqrt{\tilde{V}^2 + 2\tilde{V} + 1})}} \end{pmatrix} \begin{pmatrix} c_{k,1,\sigma} \\ c_{k,2,\sigma} \\ c_{k,3,\sigma} \end{pmatrix}, \quad (\text{A2})$$

where $\tilde{V} = V/t$ is the scaled strength of the confining potential. These bands have an index $\tau = -1, 0, +1$ and the respective Fermi momenta are $k_{F,\tau} = \arccos(-\mu/2t + \tau\sqrt{2t^2 + V^2}/2t)$. At half filling, $V = \sqrt{2}$ is an upper bound for the strength of the confining potential. Otherwise, the expression given before is ill-defined. In physical terms, $V/t = \sqrt{2}$ marks the point where the upper band with $\tau = +1$ is empty and the lower band with $\tau = -1$ is completely filled. Consequently, only the middle band with $\tau = 0$ has Fermi points at $k_{F,0} = \pm\pi/2$. This is crucial for bosonization, since the starting point of the whole procedure is band dispersions, which we can linearize around their respective Fermi points [18].

APPENDIX B: MEAN-FIELD ANALYSIS

We are interested in antiferromagnetic solutions and rewrite the local spin density as

$$n_{j,p,\sigma} = \frac{n_p}{2} - \sigma S_p(-1)^j, \quad (\text{B1})$$

where n_p is the average electron density and S_p the staggered magnetization of chain p . Substituting Eq. (B1) into Eq. (1) and applying the standard mean field decoupling $n_{j,\uparrow}n_{j,\downarrow} \mapsto n_{j,\downarrow}\langle n_{j,\uparrow} \rangle_0 + n_{j,\uparrow}\langle n_{j,\downarrow} \rangle_0 - \langle n_{j,\uparrow} \rangle_0\langle n_{j,\downarrow} \rangle_0$, where the expectation value is respect to the Hartree-Fock ground state, we obtain the mean field Hamiltonian $H_{\text{MF}} = H_1 + H_2 + H_3 + H_{\perp} + H_C$ with

$$\begin{aligned} H_p = & \sum_{\sigma} \sum_{k \in I} \epsilon_k (c_{k,p,\sigma}^{\dagger} c_{k,p,\sigma} - c_{k-\pi,p,\sigma}^{\dagger} c_{k-\pi,p,\sigma}) \\ & + U \frac{n_p}{2} \sum_{\sigma} \sum_{k \in I} (c_{k,p,\sigma}^{\dagger} c_{k,p,\sigma} + c_{k-\pi,p,\sigma}^{\dagger} c_{k-\pi,p,\sigma}) \\ & + U S_p \sum_{\sigma} \sum_{k \in I} \sigma (c_{k,\sigma}^{\dagger} c_{k-\pi,p,\sigma} + c_{k-\pi,p,\sigma}^{\dagger} c_{k,p,\sigma}) \\ & - UN \left(\frac{n_p^2}{4} - S_p^2 \right) \end{aligned} \quad (\text{B2})$$

$$\begin{aligned} H_{\perp} = & t \sum_{p=1,2} \sum_{\sigma} \sum_{k \in I} (c_{k,p,\sigma}^{\dagger} c_{k,p+1,\sigma} \\ & + c_{k-\pi,p,\sigma}^{\dagger} c_{k-\pi,p+1,\sigma} + \text{H.c.}), \end{aligned} \quad (\text{B3})$$

$$\begin{aligned} H_C = & V \sum_{\sigma} \sum_{k \in I} (c_{k,1,\sigma}^{\dagger} c_{k,1,\sigma} + c_{k-\pi,1,\sigma}^{\dagger} c_{k-\pi,1,\sigma} \\ & - c_{k,3,\sigma}^{\dagger} c_{k,3,\sigma} - c_{k-\pi,3,\sigma}^{\dagger} c_{k-\pi,3,\sigma}), \end{aligned} \quad (\text{B4})$$

where $\epsilon_k = -2t \cos(k)$ and $I = [0, \pi]$. We denote the self-consistent dispersions by $E_{k,\alpha,\gamma} \equiv \alpha \sqrt{E_{k,\gamma}^2}$, with $\alpha = \pm 1$ and $\gamma = -1, 0, +1$. The corresponding eigenspinors are $\Psi_{\alpha,\gamma}^{\tau,p}(k, \sigma)$. We note that the dispersion are doubly spin degenerate, and α distinguishes between particle- and holelike states. The observables n_p and S_p in terms of the dispersions and the eigenspinors are found as

$$n_p = \frac{1}{N} \sum_{k,\tau,\sigma} \sum_{\alpha,\gamma} \bar{\Psi}_{\alpha,\gamma}^{\tau,p}(k, \sigma) \Psi_{\alpha,\gamma}^{\tau,p}(k, \sigma) f_0(E_{k,\alpha,\gamma} - \mu), \quad (\text{B5})$$

and

$$S_p = -\frac{\sigma}{2N} \sum_{k,\tau} \sum_{\alpha,\gamma} \bar{\Psi}_{\alpha,\gamma}^{\tau,p}(k, \sigma) \Psi_{\alpha,\gamma}^{-\tau,p}(k, \sigma) f_0(E_{k,\alpha,\gamma} - \mu). \quad (\text{B6})$$

Here, $f_0(E)$ is the Fermi function and $\bar{\Psi}$ is the complex conjugate of Ψ . The chemical potential shows up explicitly in the equations, whereas V appears implicitly in the dispersions and the eigenspinors. The self-consistent values of n_p and S_p are determined numerically for given μ and V . Figure 5 from the main text is obtained with the help of the self-consistently determined dispersions.

APPENDIX C: BOSONIZATION ANALYSIS

As outlined in the main text, we bosonize each band individually, leaving us with the kinetic Hamiltonian

$$H_0 = \frac{1}{2\pi} \sum_{\tau} \sum_{v=\rho,\sigma} v_{F,\tau} \int dx [(\partial_x \theta_{v,\tau})^2 + (\partial_x \phi_{v,\tau})^2], \quad (\text{C1})$$

with $v_{F,\pm 1} = \sqrt{2 - (V/t)^2}$ and $v_{F,0} = 2$. We express the long-wavelength density-density interaction terms as

$$H_{\text{int}}^0 = \frac{U}{\pi} \int dx (\partial_x \phi_v)^T \mathbf{H}_0 (\partial_x \phi_v), \quad (\text{C2})$$

with $(\partial_x \phi_v) = (\partial_x \phi_{v,+1}, \partial_x \phi_{v,0}, \partial_x \phi_{v,-1})^T$ and

$$\mathbf{H}_0 = \begin{pmatrix} \frac{2\tilde{V}^4 + 4\tilde{V}^2 + 3}{2(\tilde{V}^2 + 2)^2} & \frac{2\tilde{V}^2 + 1}{(\tilde{V}^2 + 2)^2} & \frac{3}{2(\tilde{V}^2 + 2)^2} \\ \frac{2\tilde{V}^2 + 1}{(\tilde{V}^2 + 2)^2} & \frac{\tilde{V}^4 + 2}{(\tilde{V}^2 + 2)^2} & \frac{2\tilde{V}^2 + 1}{(\tilde{V}^2 + 2)^2} \\ \frac{3}{2(\tilde{V}^2 + 2)^2} & \frac{2\tilde{V}^2 + 1}{(\tilde{V}^2 + 2)^2} & \frac{2\tilde{V}^4 + 4\tilde{V}^2 + 3}{2(\tilde{V}^2 + 2)^2} \end{pmatrix}. \quad (\text{C3})$$

Clearly, all of the three modes are coupled to each other in the Hamiltonian $\mathcal{H}_0 = H_0 + H_{\text{int}}^0$. Apply the transformation O_2 from Eq. (3) in the main text yields Eq. (4), with

$$\begin{aligned} \frac{u_v}{K_v} \triangleq & \begin{pmatrix} \sqrt{2 - \tilde{V}^2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \sqrt{2 - \tilde{V}^2} \end{pmatrix} \\ & + \frac{U}{\pi} \begin{pmatrix} \frac{\tilde{V}^4 + 2\tilde{V}^2 + 3}{(\tilde{V}^2 + 2)^2} & \frac{\sqrt{2}(2\tilde{V}^2 + 1)}{(\tilde{V}^2 + 2)^2} & 0 \\ \frac{\sqrt{2}(2\tilde{V}^2 + 1)}{(\tilde{V}^2 + 2)^2} & \frac{\tilde{V}^4 + 2}{(\tilde{V}^2 + 2)^2} & 0 \\ 0 & 0 & \frac{\tilde{V}^2}{\tilde{V}^2 + 2} \end{pmatrix}. \end{aligned} \quad (\text{C4})$$

As outlined in the main text, we rescale the fields $\phi_{v,0}$ and $\theta_{v,0}$ such that $u_{v,\alpha} K_{v,\alpha} = \sqrt{2 - (V/t)^2}$. Any orthogonal transformation will leave the matrix $u_v K_v$ invariant. Applying the final transformation O_3 from Eq. (9) yields the diagonal

Hamiltonian $\tilde{\mathcal{H}}_0$ from Eq. (10), with

$$\begin{aligned}\frac{\tilde{u}_{v,+}}{\tilde{K}_{v,+}} &= \frac{U\tilde{V}^4 + 2U\sqrt{2-\tilde{V}^2}\tilde{V}^4 + 4U\sqrt{2-\tilde{V}^2}\tilde{V}^2 + 6U\sqrt{2-\tilde{V}^2} + 2U - 2\pi(\tilde{V}^2 - 3)(\tilde{V}^2 + 2)^2 - \sqrt{f(U, \tilde{V})}}{4\pi\sqrt{2-\tilde{V}^2}(\tilde{V}^2 + 2)^2} \\ \frac{\tilde{u}_{v,0}}{\tilde{K}_{v,0}} &= \frac{U\tilde{V}^4 + 2U\sqrt{2-\tilde{V}^2}\tilde{V}^4 + 4U\sqrt{2-\tilde{V}^2}\tilde{V}^2 + 6U\sqrt{2-\tilde{V}^2} + 2U - 2\pi(\tilde{V}^2 - 3)(\tilde{V}^2 + 2)^2 + \sqrt{f(U, \tilde{V})}}{4\pi\sqrt{2-\tilde{V}^2}(\tilde{V}^2 + 2)^2} \\ \frac{\tilde{u}_{v,-}}{\tilde{K}_{v,-}} &= \sqrt{2-\tilde{V}^2} + \frac{U}{\pi} \frac{\tilde{V}^2}{\tilde{V}^2 + 2},\end{aligned}\quad (C5)$$

with

$$\begin{aligned}f(U, \tilde{V}) &= U^2[-4\tilde{V}^{10} - (4\sqrt{2-\tilde{V}^2} + 7)\tilde{V}^8 - 8(\sqrt{2-\tilde{V}^2} + 1)\tilde{V}^6 + 4(11\sqrt{2-\tilde{V}^2} + 9)\tilde{V}^4 \\ &\quad + 12(4\sqrt{2-\tilde{V}^2} + 5)\tilde{V}^2 - 8\sqrt{2-\tilde{V}^2} + 76] - 4\pi U(\tilde{V}^2 - 1)((2\sqrt{2-\tilde{V}^2} - 1)\tilde{V}^4 \\ &\quad + 4\sqrt{2-\tilde{V}^2}\tilde{V}^2 + 6\sqrt{2-\tilde{V}^2} - 2)(\tilde{V}^2 + 2)^2 + 4\pi^2(\tilde{V}^2 - 1)^2(\tilde{V}^2 + 2)^4.\end{aligned}\quad (C6)$$

The individual Luttinger parameters and excitation velocities are calculated with the help of Eqs. (C5), (C6), and the relation $u_{v,\alpha}K_{v,\alpha} = \sqrt{2 - (V/t)^2}$, as stated earlier in the text.

APPENDIX D: UMKLAPP TERMS

As outlined in the main text, for the umklapp terms the field chiralities of the field operators have to be chosen such that the displacement of the cosines is $\delta = \pm 2\pi$. We find four such terms:

$$\begin{aligned}\mathcal{H}_{\text{int}}^{2k_F} &= -\frac{g_3^{(0,0;0,0)}}{2\pi^2\alpha^2} \cos(2\sqrt{2}\phi_{\rho,0} - \delta_0 x) - \frac{g_3^{(1,-1;-1,1)}}{2\pi^2\alpha^2} \cos(2\theta_{\sigma,-}) \cos(2\phi_{\rho,+} - \delta_+ x) - \frac{g_3^{(1,-1;1,-1)}}{2\pi^2\alpha^2} \cos(2\theta_{\rho,-}) \cos(2\phi_{\rho,+} - \delta_+ x) \\ &\quad + \frac{g_3^{(0,0;1,-1)}}{2\pi^2\alpha^2} [\cos(\theta_{\rho,-} - \theta_{\sigma,-}) \cos(\phi_{\rho,+} - \phi_{\sigma,+} + \sqrt{2}\phi_{\rho,0} + \sqrt{2}\phi_{\sigma,0}) \\ &\quad + \cos(\theta_{\rho,-} + \theta_{\sigma,-}) \cos(\phi_{\rho,+} + \phi_{\sigma,+} + \sqrt{2}\phi_{\rho,0} - \sqrt{2}\phi_{\sigma,0})],\end{aligned}\quad (D1)$$

and the bare values of the interaction couplings are shown in Table I. The scaling dimension in first nonvanishing order is just the dimension of the operator itself. We obtain for the scaling dimensions

$$\begin{aligned}\frac{dg_3^{(0,0;0,0)}(l)}{dl} &= (2 - 2(\eta_{\rho;0+})^2\tilde{K}_{\rho,+} - 2(\eta_{\rho;00})^2\tilde{K}_{\rho,0})g_3^{(0,0;0,0)}(l) \\ \frac{dg_3^{(1,-1;-1,1)}(l)}{dl} &= (2 - 1/\tilde{K}_{\sigma,-} - (\eta_{\rho;++})^2\tilde{K}_{\rho,+} - (\eta_{\rho;+0})^2\tilde{K}_{\rho,0})g_3^{(1,-1;-1,1)}(l) \\ \frac{dg_3^{(1,-1;1,-1)}(l)}{dl} &= (2 - 1/\tilde{K}_{\rho,-} - (\eta_{\rho;++})^2\tilde{K}_{\rho,+} - (\eta_{\rho;+0})^2\tilde{K}_{\rho,0})g_3^{(1,-1;1,-1)}(l) \\ \frac{dg_3^{(0,0;1,-1)}(l)}{dl} &= \left(8 - 1/\tilde{K}_{\rho,-} - 1/\tilde{K}_{\sigma,-} - (\eta_{\rho;++})^2\tilde{K}_{\rho,+} - (\eta_{\rho;+0})^2\tilde{K}_{\rho,0} - (t_{\sigma}^{(+,+)})^2\tilde{K}_{\sigma,+} - (t_{\sigma}^{(+,0)})^2\tilde{K}_{\sigma,0} \right. \\ &\quad \left. - \frac{(\eta_{\rho;0+})^2\tilde{K}_{\rho,+} + (\eta_{\rho;00})^2\tilde{K}_{\rho,0} + (t_{\sigma}^{(0,+)})^2\tilde{K}_{\sigma,+} + (t_{\sigma}^{(0,0)})^2\tilde{K}_{\sigma,0}}{\sqrt{2 - (V/t)^2}}\right)g_3^{(0,0;1,-1)}(l)/4.\end{aligned}\quad (D2)$$

If we compare the respective scaling dimensions, we find that the $g_3^{(0,0;0,0)}$ and $g_3^{(1,-1;-1,1)}$ are more relevant than the other two sine-Gordon terms.

APPENDIX E: COMPRESSIBILITY AND CONDUCTANCE

In the main text, we have used the fact that a gap in the bosonic excitation spectrum of a mode indicates that this mode is incompressible and does not contribute to the conductance. To make our paper self-contained, we describe

this connection here. In Eq. (8), the charge density has been expressed in terms of the fields $\phi_{\rho,0}$ and $\phi_{\rho,+}$. In the case of RG relevant cosine terms Eq. [6], these fields are locked into an extremum of the cosine and acquire gaps according to Eq. [11]. To be specific, we here focus on the field $\phi_{\rho,0}$, which is gapped for $V/t < 0.7$ and ungapped for larger V/t . We consider the following effective Hamiltonian for $\phi_{\rho,0}$:

$$H = \int dx \left[\frac{1}{2\kappa} \rho_0^2(x) - \mu(x)\rho_0(x) + \Delta_0\phi_{\rho,0}^2(x) \right], \quad (E1)$$

TABLE I. Bare values of the interaction constants. The lowest value of U for which $g < 1/2$ is $U/t = 1$.

$g_3^{(0,0,0)}$	$g_3^{(1,-1;1,-1)}$	$g_3^{(1,-1;-1,1)}$
$\frac{U}{t} \frac{(V/t)^4 + 2}{((V/t)^2 + 2)^2}$	$\frac{U}{t} \frac{3}{2((V/t)^2 + 2)^2}$	$\frac{U}{t} \frac{3}{2((V/t)^2 + 2)^2}$

where $\rho_0 = \frac{\sqrt{2}}{\pi} \partial_x \phi_0$ and $\kappa^{-1} = \pi \hbar u / K$. In order to find the change in charge density due to an external chemical potential $\mu(x)$, we extremize the energy with respect to ϕ_{ρ_0} , differentiate with respect to x , and Fourier transform to momentum space:

$$\rho(q) = \frac{q^2 \kappa}{\kappa \pi^2 \Delta_0 / 2 + q^2} \mu(q). \quad (\text{E2})$$

We see that in the limit $q \rightarrow 0$ there is no change in density when changing the chemical potential; hence, the system is incompressible. In the absence of a gap, however, $\rho(q) = \kappa \mu(q)$ as expected for a compressible system. Following the deriva-

tion described in Ref. [18], we obtain for the conductance of a system with size L

$$G = \frac{e^2}{\hbar \pi} \lim_{\omega \rightarrow 0} \frac{1}{\omega + i\delta} \int_{-\pi/a}^{\pi/a} \frac{dq}{2\pi} K(q, L) \times \left(\frac{\omega_n^2 u K}{\omega_n^2 + u^2 q^2 + \pi^2 \kappa \Delta_0 / 2} \right)_{i\omega_n \rightarrow \omega + i\delta}, \quad (\text{E3})$$

with $K(q, L) = \sin(qL/2)/(qL/2)$. Rescaling momenta according to $q \rightarrow q/\omega_n$, and performing the integral in the limit of small ω_n , we finally obtain

$$G = -\frac{e^2}{\hbar \pi} \frac{K}{2\pi u \pi \sqrt{\kappa \Delta_0 / 2}} \times 2 \arctan \left(\frac{1}{a \sqrt{\kappa \Delta_0 / 2}} \right) \lim_{\omega \rightarrow 0} -i(\omega + i\delta) = 0, \quad (\text{E4})$$

i.e., the conductance vanishes. For $\Delta_0 = 0$, however, one obtains a finite conductance in the usual way.

-
- [1] B. I. Halperin, Quantized Hall conductance, current-carrying edge states, and the existence of extended states in a two-dimensional disordered potential, *Phys. Rev. B* **25**, 2185 (1982).
- [2] R. B. Laughlin, Quantized Hall conductivity in two dimensions, *Phys. Rev. B* **23**, 5632 (1981).
- [3] C. L. Kane and E. J. Mele, Z_2 topological order and the quantum spin Hall effect, *Phys. Rev. Lett.* **95**, 146802 (2005).
- [4] B. A. Bernevig, T. L. Hughes, and S.-C. Zhang, Quantum spin Hall effect and topological phase transition in HgTe quantum wells, *Science* **314**, 1757 (2006).
- [5] M. Z. Hasan and C. L. Kane, *Colloquium: Topological insulators*, *Rev. Mod. Phys.* **82**, 3045 (2010).
- [6] X.-L. Qi and S.-C. Zhang, Topological insulators and superconductors, *Rev. Mod. Phys.* **83**, 1057 (2011).
- [7] M. König, S. Wiedmann, C. Brüne, A. Roth, H. Buhmann, L. W. Molenkamp, X.-L. Qi, and S.-C. Zhang, Quantum spin Hall insulator state in HgTe quantum wells, *Science* **318**, 766 (2007).
- [8] A. Altland and M. R. Zirnbauer, Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures, *Phys. Rev. B* **55**, 1142 (1997).
- [9] X.-G. Wen, *Quantum Field Theory of Many-Body Systems* (Oxford University Press, Oxford, 2007).
- [10] C. L. Kane and E. J. Mele, Quantum spin Hall effect in graphene, *Phys. Rev. Lett.* **95**, 226801 (2005).
- [11] J. M. Kosterlitz and D. J. Thouless, Long range order and metastability in two dimensional solids and superfluids (Application of dislocation theory), *J. Phys. C* **5**, L124 (1972).
- [12] J. M. Kosterlitz and D. J. Thouless, Ordering, metastability and phase transitions in two-dimensional systems, *J. Phys. C* **6**, 1181 (1973).
- [13] F. D. M. Haldane, Model for a quantum Hall effect without Landau levels: Condensed-matter realization of the “parity anomaly”, *Phys. Rev. Lett.* **61**, 2015 (1988).
- [14] N. F. Mott, The basis of the electron theory of metals, with special reference to the transition metals, *Proc. Phys. Soc. Lond. Sect. A* **62**, 416 (1949).
- [15] H. J. Schulz, in *Strongly Correlated Magnetic and Superconducting Systems*, edited by G. Sierra and M. A. Martn-Delgado (Springer-Verlag, Berlin, Heidelberg, 1997), pp. 136–136.
- [16] L. Balents and M. P. A. Fisher, Weak-coupling phase diagram of the two-chain Hubbard model, *Phys. Rev. B* **53**, 12133 (1996).
- [17] K. L. Hur, Critical Ising modes in low-dimensional Kondo insulators, *Phys. Rev. B* **63**, 165110 (2001).
- [18] T. Giamarchi and O. U. Press, *Quantum Physics in One Dimension*, International Series of Monographs (Clarendon Press, Oxford, 2004).
- [19] A. M. Finkel’stein and A. I. Larkin, Conductance fluctuations in one-dimensional systems, *Phys. Rev. B* **47**, 10461 (1993).
- [20] J.-J. Miao, F.-C. Zhang, and Y. Zhou, Instability of three-band Tomonaga-Luttinger liquid: Renormalization group analysis and possible application to $\text{K}_2\text{Cr}_3\text{As}_3$, *Phys. Rev. B* **94**, 205129 (2016).
- [21] M. Tsuchiizu, P. Donohue, Y. Suzumura, and T. Giamarchi, Commensurate-incommensurate transition in two-coupled chains of nearly half-filled electrons, *Eur. Phys. J. B* **19**, 185 (2001).
- [22] U. Khanna, Y. Gefen, O. Entin-Wohlman, and A. Aharony, Edge reconstruction of a time-reversal invariant insulator: Compressible-incompressible stripes, *Phys. Rev. Lett.* **128**, 186801 (2022).
- [23] J. von Delft and H. Schoeller, Bosonization for beginners – refermionization for experts, *Ann. Phys.* **510**, 225 (1998).
- [24] Note that in principle one should consider the renormalization of the $K_\rho^{(\gamma)}$ due to the umklapp couplings appearing in $H_{\text{int}}^{2k_F}$ [18]. However, even when only renormalizing the $g_3^{(\gamma)}$, one still

- obtains the qualitatively correct answer regarding their relevance or irrelevance, which only depends on the signum of the initial value of $2 - 2K_\rho^{(\nu)}$ [18]. Thus, on a qualitative level the answer regarding the relevance of the $g_3^{(\nu)}$ is not changed if we neglect the renormalization of the $K_\rho^{(\nu)}$.
- [25] B. Rosenow and B. I. Halperin, Nonuniversal behavior of scattering between fractional quantum Hall edges, *Phys. Rev. Lett.* **88**, 096404 (2002).
- [26] J. Li, R.-X. Zhang, Z. Yin, J. Zhang, K. Watanabe, T. Taniguchi, C. Liu, and J. Zhu, A valley valve and electron beam splitter, *Science* **362**, 1149 (2018).
- [27] H. Overweg, H. Eggimann, X. Chen, S. Slizovskiy, M. Eich, R. Pisoni, Y. Lee, P. Rickhaus, K. Watanabe, T. Taniguchi, V. Fal'ko, T. Ihn, and K. Ensslin, Electrostatically induced quantum point contacts in bilayer graphene, *Nano Lett.* **18**, 553 (2018).