

Broken Lorentz symmetry and violation of the Wiedemann-Franz law in topological insulatorsFeng Liu^{✉,*}, A. Daria Dumitriu-I.^{✉,†} and Alessandro Principi^{✉,‡}*Department of Physics and Astronomy, University of Manchester, Manchester M13 9PL, United Kingdom*

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We study a two-dimensional topological insulator in the presence of a static nonuniform gravitational field, which mimics the variations in the temperature distribution. We derive an effective boundary free energy functional for the gravitational field and show that, in contrast to the case of massive Dirac fermions, the addition of a Newtonian mass term significantly modifies the quantum anomalous behavior of the system. A nonzero bulk thermal current appears, which violates the Wiedemann-Franz law. The systematic approach we develop to calculate the contribution of edge states to thermal Hall conductivity and energy magnetization can easily be extended to other models.

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The thermal Hall effect has the potential to reveal topological properties that are inaccessible through the conventional (electrical) Hall effect. In fact, it enables probing invariants in systems with broken $U(1)$ symmetry [1], with neutral excitations [2], and uncovering complex boundary state structures in strongly correlated systems [3]. The phenomenon is analogous to the electrical Hall effect, with temperature gradients and heat flow in place of electric fields and charge currents. Experimental evidence for the thermal Hall effect spans various systems, including multiferroics [4], topological insulators [5,6], quantum magnets [7], and various other materials [2,8–12]. Significant theoretical effort has been put into the study of this phenomenon [13–21].

The approaches to the thermal Hall effect fall into two categories. On the one hand, from a microscopic perspective [13–15], thermal responses can be calculated using Luttinger’s “trick” [15]. This exploits the equivalence in the linear-response regime between a nonflat metric tensor (also termed “gravitational potential”) and temperature gradients that drive heat currents. On the other hand, from an effective theory perspective [18–20,22] the thermal Hall conductivity κ_{xy} of a topological insulator at low temperatures is determined by the central charge of the corresponding edge theory [3].

Both approaches have limitations that make them not fully satisfactory. Within the microscopic perspective, the fundamental challenge in calculating thermal conductivities, and

especially the thermal Hall one, arises from the intrinsic ambiguity in the definition of the heat current operator. Specifically, the energy continuity equation only fixes the form of the heat current operator up to a curl of some operator, which is crucial for the thermal Hall conductivity. In order to correctly calculate the thermal Hall conductivity and avoid nonphysical divergences, one must then carefully subtract the contribution arising from the so-called energy magnetization [13,14]. Crucially [13,14], to rigorously determine the energy magnetization, one must uniquely specify the exact form of the extra operator appearing in the definition of the heat current. Scaling relations, which provide such a unique determination, have often been assumed without providing a clear microscopic justification [16]. Furthermore, introducing the constraint of a scaling relation inevitably makes the expression of the heat current very complicated, significantly limiting the applicability of such a formalism to more general models of topological insulators. Finally, ambiguities remain about how to extend these methods to interacting systems [14,16], in particular when interactions are not purely local.

A natural way to avoid such ambiguities is to focus explicitly on boundary states, since the ambiguous curl operator in the heat current naturally vanishes both deep into the bulk and at the open boundary. However, existing boundary-state approaches suffer from important limitations: some rely on restrictive assumptions such as Lorentz symmetry [21,23,24], which limits the applicability of the framework to practically a single system, a bulk described by massive Dirac fermions. Other approaches assume universal low-energy effective descriptions based on topological field theories such as the gravitational Chern-Simons theory [19,20]. In the latter case, the edge theory is often modeled as a conformal field theory governed by a fixed central charge. Yet, deriving the gravitational Chern-Simons action directly from a microscopic bulk Hamiltonian poses significant challenges [25], and such theories typically fail to incorporate model-specific details. Recent experimental studies [11,12] have further challenged this framework by revealing finite, nonuniversal bulk thermal Hall currents in topological systems, which cannot be solely explained by topological invariants like the Chern number.

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While these experiments do not probe precisely the regime we consider here, they underscore the broader need for a theoretical framework that clarifies the role of bulk contributions to thermal transport. These findings strongly motivate a reexamination of conventional gravitational effective theories and highlight the need for a more flexible framework that connects microscopic models with measurable thermal transport. Therefore, it would be preferable to derive the conformal edge theory (and its associated thermal Hall conductivity) directly from the bulk microscopic Hamiltonian, without relying on any topological field theory as an intermediate step.

In this paper, we carry out this precise task and thus overcome the limitations of existing approaches. We derive a long-wavelength description of edge states from a bulk microscopic Hamiltonian without assuming Lorentz symmetry, which is generally broken in a topological insulator. As a consequence, the energy current cannot be simply defined as the functional derivative of the Hamiltonian with respect to the vielbein field. To resolve this, we introduce an additional source term in the Hamiltonian. Because this term vanishes when the metric reduces to Luttinger's one, this "extended Hamiltonian" can be used to define the partition function. Despite an apparent initial complication, our formalism rigorously avoids the complexity and ambiguity introduced by the heat magnetization and scaling relation. This significantly expands its applicability range beyond that of existing approaches [23,24].

We demonstrate that the broken Lorentz symmetry significantly modifies the quantum anomalous behavior of edge states. Specifically, unlike the massive Dirac fermion scenario [24], in a generic topological insulator the presence of a Newtonian mass term [26] breaks the energy conservation of the boundary (1+1-dimensional) effective theory when the metric tensor is allowed to vary spatially. Consequently, maintaining energy conservation necessarily requires a compensating energy inflow from the bulk, which physically manifests as a finite bulk thermal Hall current proportional to the gradient of the gravitational potential. Thus, even when the gravitational field gradient is uniform, a bulk thermal Hall current can generically exist due to this anomaly-induced bulk-boundary correspondence. In fact, our result indicates that finite bulk thermal Hall currents may not be an exotic phenomenon occurring in quantum-Hall system alone [11,12], but that they are in fact generic and exist in topological insulators as well. Furthermore, in Appendix F, we find that the thermal Hall conductivity acquires a T^3 correction from the Newtonian mass term, while the charge Hall conductivity is unaffected by it, so that the Wiedemann-Franz law holds only asymptotically at the lowest temperatures. These results cannot be obtained within the gravitational Chern-Simons theory and therefore question its universal applicability to the description of thermal Hall transport.

II. THE MODEL

We consider the system presented in Fig. 1, i.e., a (1+1)-dimensional boundary located at $x^1 = 0$ and oriented along the direction x^2 which separates a (2+1)-dimensional topological insulator system with relativistic mass $m < 0$ from the

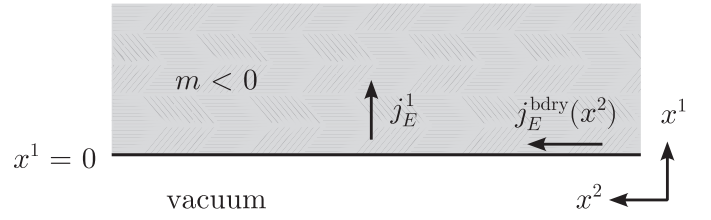


FIG. 1. The boundary located at $x^1 = 0$ separates a (2+1)-dimensional topological insulator system with $m < 0$ from the vacuum. A boundary current j_E^{bdry} flows along the edge, i.e., in the x^2 direction, and bulk currents j_E^1 flow in the direction orthogonal to the boundary (x^1). The textured background represents a nonzero gravitational potential. The equilibrium temperature of the edge mode matches that of the upstream heat bath [27,28]. Throughout the paper, the temperature T is treated as uniform in both space and time. Any spatial variation in temperature is interpreted as a variation in gravitational potential [13,15].

vacuum. The Newtonian mass m_N is assumed to be positive. The action for the (2+1)-dimensional topological insulator coupled to a gravitational field is (hereafter we set both the particles' speed and $\hbar$ to one) [26,29]

$$S = \int_{x,t} \sqrt{g} \left[\frac{i}{2} (\bar{\psi} e^\mu_\alpha \gamma^\alpha \nabla_\mu \psi - (\nabla_\mu \bar{\psi}) \gamma^\alpha e^\mu_\alpha \psi) - m \bar{\psi} \psi + \frac{1}{m_N} \eta^{ab} e^\mu_a e^\nu_b (\nabla_\mu \bar{\psi}) (\nabla_\nu \psi) \right], \quad (1)$$

where we used the short-hand notation $\int_{x,t} = \int d^2x dt$. $\vec{\nabla}_\mu$ ($\overleftarrow{\nabla}_\mu$) is a covariant derivative that acts on the right (left) two-component spinor field ψ , and it is explicitly given by $\vec{\nabla}_\mu \psi = \partial_\mu \psi + [\gamma_\alpha, \gamma_\beta] \omega^\alpha_\mu \psi / 8$. Here, ∂_μ denotes the derivative over the temporal ($\mu = 0$) and spatial ($\mu = x, y$) directions. Finally, the combinations $\gamma^0 \gamma^1$, $\gamma^0 \gamma^2$ and γ^0 correspond to the usual Pauli matrices σ_x , σ_y and σ_z , respectively. In Eq. (1), we have introduced the metric $g_{\mu\nu}$, whose determinant, in modulus, is g . The factor $\sqrt{g}$ ensures invariance of the action under changes of coordinates. Throughout this paper, we use the Greek indices $\mu, \nu = 0, 1, 2$ and $\alpha, \beta, \dots = \hat{0}, \hat{1}, \hat{2}$ to denote the environment and locally flat (or internal) coordinates, respectively. In what follows, when we refer to space-like directions only, we will use the Latin letters $i, j = 1, 2$ for the environment coordinates, and $a, b = \hat{1}, \hat{2}$ for the internal coordinates. The Minkowski metric in the locally-flat space-time is taken to be $\eta_{\alpha\beta} = \text{diag}(+1, -1, -1)$. The environment and flat metrics, $g_{\mu\nu}$ and $\eta_{\alpha\beta}$, respectively, are related by a vielbein field e^α_μ according to the identity [30] $g_{\mu\nu} = e^\alpha_\mu e^\beta_\nu \eta_{\alpha\beta}$. Note that the product $\eta^{ab} e^\mu_a e^\nu_b$ that appears in Eq. (1) does not coincide with $g^{\mu\nu}$, because $a, b \neq \hat{0}$. Thus, such term breaks Lorentz symmetry.

We assume the perturbation to be of the Luttinger's type [15], i.e., $e^\hat{0}_\mu = \delta^\hat{0}_\mu (1 + \phi)$, and $e^\alpha_\mu = \delta^\alpha_\mu$ [19]. In what follows, except when explicitly stated otherwise, we set $m_N = 1$. From Eq. (1), the energy-current is obtained as $j_E^i = -e^\alpha_0 (\delta S / \delta e^\alpha_i)$ [19,29]. Similarly, the Hamiltonian acting in an effectively

locally-flat space-time is

$$H = \int_x \sqrt{g} \psi^\dagger \gamma^0 \left[\frac{i}{2} e_\alpha^0 \gamma^\alpha \omega_0 + \frac{i}{2} \omega_0 \gamma^\alpha e_\alpha^0 - \frac{i}{2} e_\alpha^j \gamma^\alpha \vec{\nabla}_j + \frac{i}{2} \vec{\nabla}_j \gamma^\alpha e_\alpha^j + m - \eta^{ab} e_a^\mu e_b^\nu \vec{\nabla}_\mu \vec{\nabla}_\nu \right] \psi. \quad (2)$$

Because the Hamiltonian breaks Lorentz symmetry, the energy-momentum tensor is no longer symmetric, and thus the energy current cannot be defined as the functional derivative of the Hamiltonian. To address this issue, we introduce the extended Hamiltonian as $H_{\text{ex}} = H + H_{\text{src}}$, where

$$H_{\text{src}} = - \int_x \sqrt{g} e_0^0 e_i^j \psi^\dagger \gamma^0 \left[\vec{\partial}_i \vec{\partial}_0 + \vec{\partial}_0 \vec{\partial}_i \right] \psi. \quad (3)$$

The time derivatives in Eq. (3) should be intended as replaced by the equation of motion generated by the Hamiltonian H . The so-defined Hamiltonian H_{ex} reproduces the energy current obtained from the original action S and the equation of motion obtained from H . Furthermore, the term H_{src} partially restores Lorentz symmetry and allows one to define the energy current as a functional derivative of the Hamiltonian (see Appendix A). For a metric corresponding to Luttinger's gravitational potential and in the absence of Lorentz symmetry, we then find the energy current

$$j_E^2 = -(1 + \phi)^2 \frac{\delta H_{\text{ex}}}{\delta e_0^2} - (1 + \phi) \partial_i \left(\frac{\delta H_{\text{ex}}}{\delta \omega_{i0}^2} \right). \quad (4)$$

This equation is the first fundamental result of our paper. In what follows, we express the energy current as a functional derivative of the extended Hamiltonian H_{ex} . In the case of Luttinger's gravitational potential, Eq. (4) holds at the operator level.

Since the source term H_{src} in Eq. (3) is zero when the metric is of the Luttinger's type, the extended Hamiltonian H_{ex} can be used to define the partition function. Thus, the expectation value of the energy current at finite temperature is given by

$$j_E^2 = -(1 + \phi)^2 \frac{\delta F_{\text{ex}}}{\delta e_0^2} - (1 + \phi) \partial_i \left(\frac{\delta F_{\text{ex}}}{\delta \omega_{i0}^2} \right). \quad (5)$$

As usual, we define the free energy $F_{\text{ex}} = -\beta^{-1} \ln Z_{\text{ex}}$, where $Z_{\text{ex}} = \text{Tr}(e^{-\beta H_{\text{ex}}})$ is the partition function and $\text{Tr}(\dots)$ represents the trace in the Fock space [31].

It is important to emphasize that the functional derivative in Eq. (5) is taken assuming the spin connection to be completely general, with all its components different from zero. However, after differentiation, we consider the special case of Luttinger gravity, where Lorentz symmetry is absent. This leads to a vanishing spin connection [19,29].

III. BOUNDARY THEORY

We derive the Hamiltonian for boundary fermions using the standard procedure from [23]. Further details can be found in Appendix B. We define $e_\mu^\alpha = \delta_\mu^\alpha + h_\mu^\alpha/2$, $e_\alpha^\mu = \delta_\alpha^\mu - h_\alpha^\mu/2$ and $\sqrt{g} = 1 + h/2$, where h , h_μ^α and h_α^μ are generic deviations of the vielbein field and its determinant from their value in a flat spacetime. We assume the deviations h , h_μ^α , and h_α^μ to be

small. Therefore, the unperturbed Hamiltonian is

$$H_0 = \int d^2x \psi^\dagger \left[-\frac{i}{2} \gamma^0 \gamma^j \vec{\partial}_j + \frac{i}{2} \gamma^0 \vec{\partial}_j \gamma^j + m \gamma^0 - \gamma^0 \eta^{ab} \vec{\partial}_a \vec{\partial}_b \right] \psi, \quad (6)$$

while other parts in H_{ex} are treated as perturbations.

We assume that close to the boundary, the metric only depends on x^2 , such that the two directions x^1 and x^2 completely decouple in the boundary Hamiltonian. Thus, the wave function of the boundary mode obtained from the Hamiltonian H_0 factorizes as $\psi(x^1, x^2) = \psi_2(x^2) \psi_1(x^1)$, where $\psi_2(x^2)$ is a plane wave in the x^2 direction and $\psi_1(x^1)$ is a two-component evanescent spinor wave function in the x^1 direction. In Appendix E, we have shown the formal expression of the latter is given by [26]

$$\psi_1(x^1) = (e^{-\lambda_1 x^1} - e^{-\lambda_2 x^1}) |s\rangle \quad (7)$$

with $\lambda_{1,2} = (1 \pm \sqrt{1 + 4m})/2$, and the two-component spinor $|s\rangle$, corresponding to the edge bound states, satisfies $i\gamma^1 |s\rangle = |s\rangle$. Therefore, the one-body boundary Hamiltonian obtained from the unperturbed bulk Hamiltonian H_0 is $\mathcal{H}_0 = (i/2)(\vec{\partial}_2 - \vec{\partial}_2)$.

The derivation of the extended *boundary* Hamiltonian $\mathcal{H}_{\text{ex}}$ is more involved. We therefore relegate it to the Appendix B and quote only the final result

$$\mathcal{H}_{\text{ex}} = \frac{i}{2}(\vec{\partial}_2 - \vec{\partial}_2) + \frac{i}{2} \zeta(x^2)(\vec{\partial}_2 - \vec{\partial}_2) + i\chi(x^2)(-\vec{\partial}_2 \vec{\partial}_2^2 + \vec{\partial}_2^2 \vec{\partial}_2), \quad (8)$$

where

$$\begin{aligned} \zeta(x^2) &= \frac{h}{2} + \left(1 + \frac{h}{2}\right) \left((h_0^2/2) - (h_2^2/2)\right), \\ \chi(x^2) &= \left(1 + \frac{h}{2}\right) (-h_0^2/2). \end{aligned} \quad (9)$$

We write the partition function as $Z_{\text{ex}} = \int \mathcal{D}\psi^* \mathcal{D}\psi \exp(-S)$ with imaginary-time boundary action [31]

$$S = \int_0^\beta d\tau \int dx \psi^*(x^2, \tau) (\partial_\tau + \mathcal{H}_{\text{ex}}) \psi(x^2, \tau). \quad (10)$$

Integrating over fermionic fields, the effective free energy functional of the gravitational field is obtained as $\mathcal{F}[\zeta, \chi] = \beta^{-1} \mathcal{S}[\zeta, \chi]$.

Up to a constant which is independent of ζ and χ , the effective action can be expressed as

$$\mathcal{S}[\zeta, \chi] = \sum_{l=1}^{\infty} \text{tr}[(G_0 \Sigma)^l] / l, \quad (11)$$

where the trace is to be taken over the real space direction x^2 and over imaginary time τ . Moreover, since $\chi(x)$ equals zero for a gravitational potential of Luttinger's type, in the free energy we only need to consider the zeroth- and first-order term in powers of the variable χ [32], as higher-order terms will not contribute to Eq. (5).

After some lengthy algebra [33] (see Appendix C), we obtain the following expression for the boundary free energy in the long-wavelength limit [34]

$$\mathcal{F}[\zeta, \chi] = \frac{\pi T^2}{12} \int dx \frac{\zeta(x)}{1 + \zeta(x)} - \frac{7\pi^3 T^4}{60} \int dx \frac{\chi(x)}{(1 + \zeta(x))^4}. \quad (12)$$

This equation is the second main finding of our work.

IV. EDGE ENERGY CURRENT

Substituting Eq. (12) into Eq. (5), the energy current flowing along the boundary is found to be (restore m_N from here)

$$j_E^{\text{bdry}}(x^2) = -\frac{\pi T^2}{12} - \frac{7\pi^3 T^4}{60m_N^2} \frac{1}{(1 + \phi(x))^2}. \quad (13)$$

We stress that the long-wavelength limit has been taken throughout this paper. This means that we have neglected terms in the boundary current that depend on derivatives of the metric. As shown below, such terms would correspond to contributions to the bulk thermal Hall current that are proportional to at least the second derivative of the gravitational potential. Thus, these terms do not contribute to the linear response thermal Hall conductivity κ_{xy} .

In fact, conservation of energy currents requires the presence of a bulk thermal Hall current to compensate for anomalies at the boundary. In other words, if a (1+1)-dimensional current j_E^{bdry} flows along the boundary, then the bulk current j_E^1 crossing the boundary must satisfy the continuity equation [23]

$$j_E^1(x^1 = +0) = -\partial_2 j_E^{\text{bdry}}(x^2). \quad (14)$$

Thus, the result of Eq. (13) indicates that, relative to the massive Dirac fermions model [24], the introduction of the Newtonian mass term leads to an additional higher-order term in the boundary current, causing the system to exhibit a finite bulk thermal current in the linear regime. According to the continuity equation (14), the bulk thermal current reads

$$j_E^1(x^1 = +0) = -\frac{7\pi^3 T^4}{30m_N^2} \partial_2 \phi(x^2). \quad (15)$$

Using linear response theory and the Tolman-Ehrenfest relation [14,15], the thermal conductivity contributed by this term is $\kappa_{xy}^{\text{bulk}} = 7\pi^3 T^3 / 30m_N^2$.

Finally, we can use Eq. (13) to calculate the system's thermal Hall conductivity based on the generalized Streda formula [21,23]. According to the definition of energy magnetization M_E^z in terms of energy current [13], the boundary energy current satisfies the relation [23,35]

$$j_E^{\text{bdry}} = -M_E^z(x^1 = +\infty). \quad (16)$$

From this, we get

$$M_E^z(\phi = 0) = \pi T^2 / 12 + 7\pi^3 T^4 / (60m_N^2). \quad (17)$$

Therefore, the thermal Hall conductivity is given by [14,36]

$$\kappa_{xy} = \kappa_{xy}^{\text{bulk}} + \frac{2M_E^z}{T} = \frac{\pi T}{6} + \frac{7\pi^3 T^3}{15m_N^2}. \quad (18)$$

This result is identical to that obtained directly using the generalized Streda formula [21,35]

$$\kappa_{xy} = \frac{\partial M_E^z}{\partial T} = \frac{\pi T}{6} + \frac{7\pi^3 T^3}{15m_N^2}. \quad (19)$$

The first term in Eq. (18) corresponds to a quantized thermal Hall conductivity with Chern number $C = -1$, and the second contribution, which breaks the Wiedemann-Franz law, corresponds to the bulk thermal current induced by the Newtonian mass term in the topological insulator's Hamiltonian. When the Newtonian mass term vanishes (as $m_N \rightarrow \infty$), the second contribution approaches zero.

V. DISCUSSION AND CONCLUSION

In this work, we have developed a systematic theoretical approach to rigorously compute the boundary-state contributions to the thermal Hall conductivity without relying on further assumptions such as Lorentz symmetry [20,23,24] or scaling laws. By systematically resumming all-order contributions in powers of the metric tensor in the long-wavelength limit, we have obtained a rigorous expression for the boundary free energy. From this, we derived the boundary current generated by a Luttinger-type gravitational potential. By explicitly incorporating nonrelativistic bulk terms, our method reveals significant deviations from gravitational Chern-Simons predictions. In particular, the emergence of nonuniversal bulk thermal Hall currents consistent with recent experimental observations [11,12], indicating that finite bulk thermal currents generally occur beyond finely tuned or exceptional scenarios. Their contribution leads to an intrinsic violation of the Wiedemann-Franz law at any finite temperature that is beyond what is predicted by the gravitational Chern-Simons theory. These findings question the universal applicability of such theory and show that, in order to correctly interpret experiments, it is imperative to accurately model the low-energy Hamiltonian beyond the massive Dirac fermion approximation.

It is instructive to clarify the physical origin behind the breakdown of the Wiedemann-Franz law observed here. Firstly, unlike its longitudinal counterpart, the thermal Hall conductivity arises from topological mechanisms, and it is therefore independent of specific scattering processes. Furthermore, our analysis demonstrates that the Wiedemann-Franz law fails not because of changes in the edge-state dispersion. In fact, bulk nonrelativistic terms affect neither the linear energy dispersion relation of edge states nor their wave function. Instead, the breakdown of the Wiedemann-Franz law is due to a subtle yet fundamental correction to the heat current operator arising from the Newtonian mass and associated quantum anomalies. While these corrections vanish at the Hamiltonian level, they still affect the heat current, which is defined as a functional derivative of the action with respect to perturbations of the metric [19,29]. These bulk-originated corrections invalidate the classical quasiparticle description implicitly assumed by the Wiedemann-Franz law, i.e., the heat current can no longer be simply expressed as a product of quasiparticle velocity and energy expectation values, and reflect an intrinsic nonuniversality which is not captured by the commonly-used gravitational Chern-Simons theory.

We stress that our approach overcomes technical limitations encountered by previous microscopic and topological theories, providing a rigorous, broadly applicable, and extendable theoretical toolbox for future modeling and understanding of topological thermal transport phenomena. In principle, our formalism could be extended to describe other systems possessing chiral edge states. For example it could be employed to describe energy transport due to chiral photonic edge states in topological photonic metamaterials [37,38]. Nonetheless, system-specific features such as the non-number-conserving nature of photons, as well as with the typically nonequilibrium conditions in which they exist, imply that additional considerations would be necessary for a precise adaptation of our method.

Finally, we note that our formulation is developed within a continuum field-theory framework. Establishing a fully microscopic correspondence to lattice Hamiltonians is a subtle and important problem. While a complete lattice derivation lies beyond the scope of this paper, our approach naturally admits a projection onto lattice degrees of freedom. More broadly, our results highlight that thermal Hall responses can acquire model-dependent, intrinsic bulk contributions even in idealized topological insulators. This provides a transparent theoretical framework that can be directly tested in well-controlled settings such as cold-atom realizations of topological models [39–41], and lays the groundwork for future studies on lattice-continuum correspondence.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: ENERGY CURRENT OPERATOR WITHOUT LORENTZ SYMMETRY

The model action for a (2+1)-dimensional topological insulator in flat spacetime is given by

$$\begin{aligned}
 S &= \int dt d^2x \psi^\dagger \gamma^0 \left(\frac{i}{2} (\gamma^\mu \vec{\partial}_\mu - \overleftarrow{\partial}_\mu \gamma^\mu) \right. \\
 &\quad \left. - (m - \eta^{ij} \overleftarrow{\partial}_i \overrightarrow{\partial}_j) \right) \psi \\
 &= \int dt d^2x \psi^\dagger \frac{i}{2} (\overrightarrow{\partial}_0 - \overleftarrow{\partial}_0) \psi - \int dt H \\
 H &= \int d^2x \psi^\dagger \left(-\frac{i}{2} \gamma^0 \gamma^j \overrightarrow{\partial}_j + \frac{i}{2} \gamma^0 \overleftarrow{\partial}_j \gamma^j \right. \\
 &\quad \left. + \gamma^0 (m - \eta^{ij} \overleftarrow{\partial}_i \overrightarrow{\partial}_j) \right) \psi
 \end{aligned} \tag{A1}$$

with the Minkowski metric $\eta_{\alpha\beta} = \text{diag}(1, -1, -1)$ and $\eta_{ab} = \text{diag}(-1, -1)$ for the spatial components. To incorporate gravitational effects, this action can be generalized to curved spacetime by coupling the system to a background metric. According to Refs. [19,30], the resulting action takes the form

$$\begin{aligned}
 S &= \int dt d^2x \sqrt{g} \psi^\dagger \gamma^0 \left(\frac{i}{2} (e^\mu_\alpha \gamma^\alpha \overrightarrow{\nabla}_\mu - \overleftarrow{\nabla}_\mu \gamma^\alpha e^\mu_\alpha) \right. \\
 &\quad \left. - m + \eta^{ab} e^\mu_a e^\nu_b \overleftarrow{\nabla}_\mu \overrightarrow{\nabla}_\nu \right) \psi \\
 &= \int dt d^2x \sqrt{g} \psi^\dagger \gamma^0 \frac{i}{2} (e^0_\alpha \gamma^\alpha \overrightarrow{\partial}_0 - \overleftarrow{\partial}_0 \gamma^\alpha e^0_\alpha) \psi \\
 &\quad - \int dt H, \\
 H &= \int d^2x \sqrt{g} \psi^\dagger \gamma^0 \left(\frac{i}{2} e^0_\alpha \gamma^\alpha \omega_0 + \frac{i}{2} \omega_0 \gamma^\alpha e^0_\alpha \right. \\
 &\quad \left. - \frac{i}{2} e^j_\alpha \gamma^\alpha \overrightarrow{\nabla}_j + \frac{i}{2} \overleftarrow{\nabla}_j \gamma^\alpha e^j_\alpha + m \right. \\
 &\quad \left. - \eta^{ab} e^\mu_a e^\nu_b \overleftarrow{\nabla}_\mu \overrightarrow{\nabla}_\nu \right) \psi.
 \end{aligned} \tag{A2}$$

The covariant derivative $\overrightarrow{\nabla}_\mu$ ($\overleftarrow{\nabla}_\mu$) acts on the right (left) two-component spinor field ψ ,

$$\begin{aligned}
 \overrightarrow{\nabla}_\mu \psi &= \left(\overrightarrow{\partial}_\mu + \frac{1}{8} [\gamma^\alpha, \gamma^\beta] \omega_{\mu\alpha\beta} \right) \psi \equiv (\overrightarrow{\partial}_\mu - \omega_\mu) \psi \\
 \overleftarrow{\nabla}_\mu \psi &= \overleftarrow{\psi} \left(\overleftarrow{\partial}_\mu - \frac{1}{8} [\gamma^\alpha, \gamma^\beta] \omega_{\mu\alpha\beta} \right) \equiv \overleftarrow{\psi} (\overleftarrow{\partial}_\mu + \omega_\mu).
 \end{aligned} \tag{A3}$$

According to the formulation provided in Refs. [19,29], the energy current in the Luttinger gravitational potential is given by

$$\begin{aligned}
 j_E^i &= \sqrt{g} \tau_{v=0}^i = -e_0^\alpha \frac{\delta S}{\delta e_i^\alpha} = -e_0^0 \frac{\delta S}{\delta e_i^0} \\
 &= \sqrt{g} \psi^\dagger \gamma^0 \left(\frac{i}{2} \gamma^i \overrightarrow{\partial}_0 - \frac{i}{2} \overleftarrow{\partial}_0 \gamma^i + \eta^{ib} e_b^\nu \overleftarrow{\nabla}_0 \overrightarrow{\nabla}_\nu \right. \\
 &\quad \left. + \eta^{ai} e_a^\mu \overleftarrow{\nabla}_\mu \overrightarrow{\partial}_0 \right) \psi.
 \end{aligned} \tag{A4}$$

Since we have completed all the functional derivatives, we can represent the energy current in the Luttinger case (without Lorentz symmetry) as follows:

$$\begin{aligned}
 j_E^i &= (1 + \phi) \psi^\dagger \gamma^0 \left(\frac{i}{2} \gamma^i \overrightarrow{\partial}_0 - \frac{i}{2} \overleftarrow{\partial}_0 \gamma^i \right) \psi \\
 &\quad - (1 + \phi) \psi^\dagger \gamma^0 (\overleftarrow{\partial}_0 \overrightarrow{\partial}_i + \overleftarrow{\partial}_i \overrightarrow{\partial}_0) \psi.
 \end{aligned} \tag{A5}$$

Following an approach similar to that in Ref. [29], we will now use the equations of motion to convert the time derivatives in the energy current expression of Eq. (A5) into spatial derivatives. The form of action and the Hamiltonian in the

Luttinger case is (with definition $\phi_g = 1 + \phi$)

$$\begin{aligned}
 S &= \int dt \bar{\psi} \frac{i}{2} (\gamma^0 \vec{\partial}_0 - \overleftarrow{\partial}_0 \gamma^0) \psi \\
 &\quad - \int dt d^2x \phi_g \bar{\psi} \left(-\frac{i}{2} \gamma^j \vec{\partial}_j + \frac{i}{2} \overleftarrow{\partial}_j \gamma^j \right. \\
 &\quad \left. + m + \overleftarrow{\partial}_1 \vec{\partial}_1 + \overleftarrow{\partial}_2 \vec{\partial}_2 \right) \psi \\
 H &= \int dt d^2x \phi_g \psi^\dagger \gamma^0 \left(-\frac{i}{2} \gamma^j \vec{\partial}_j + \frac{i}{2} \overleftarrow{\partial}_j \gamma^j \right. \\
 &\quad \left. + m + \overleftarrow{\partial}_1 \vec{\partial}_1 + \overleftarrow{\partial}_2 \vec{\partial}_2 \right) \psi. \quad (A6)
 \end{aligned}$$

The Heisenberg equation for ψ (and $\psi^\dagger$) immediately yields the equation of motion, which is given by

$$\begin{aligned}
 i\partial_0 \psi &= \phi_g \gamma^0 (-i\gamma^j \vec{\partial}_j + m - \vec{\partial}^2) \psi \\
 &\quad - \frac{i}{2} \gamma^0 (\partial_j \phi_g) \gamma^j \psi - \gamma^0 ((\partial_1 \phi_g) \vec{\partial}_1 + (\partial_2 \phi_g) \vec{\partial}_2) \psi \\
 i\partial_0 \psi^\dagger &= -\phi_g \psi^\dagger \gamma^0 (i\overleftarrow{\partial}_j \gamma^j + m - \overleftarrow{\partial}^2) - \frac{i}{2} \psi^\dagger \gamma^0 (\partial_j \phi_g) \gamma^j \\
 &\quad + \psi^\dagger \gamma^0 (\overleftarrow{\partial}_1 (\partial_1 \phi_g) + \overleftarrow{\partial}_2 (\partial_2 \phi_g)). \quad (A7)
 \end{aligned}$$

Here we use the identity $[A, BC] = \{A, B\}C - B\{A, C\}$. Substituting these results into the first part of Eq. (A5), we can express the first part of the energy current as follows:

$$\begin{aligned}
 I &= \frac{i}{2} \phi_g \psi^\dagger \gamma^0 \gamma^i \vec{\partial}_0 \psi \\
 &= -\frac{1}{2} \phi_g \psi^\dagger \gamma^i \left(-i\phi_g \gamma^j \vec{\partial}_j + \phi_g m - \phi_g \vec{\partial}^2 \right. \\
 &\quad \left. - \frac{i}{2} (\partial_j \phi_g) \gamma^j - (\partial_j \phi_g) \vec{\partial}_j \right) \psi \\
 II &= -\frac{i}{2} \phi_g \psi^\dagger \gamma^0 \overleftarrow{\partial}_0 \gamma^i \psi \\
 &= \frac{1}{2} \phi_g \psi^\dagger \left(-\phi_g i \overleftarrow{\partial}_j \gamma^j + \phi_g m - \phi_g \overleftarrow{\partial}^2 \right. \\
 &\quad \left. - \frac{i}{2} (\partial_j \phi_g) \gamma^j - \overleftarrow{\partial}_j (\partial_j \phi_g) \right) \gamma^i \psi. \quad (A8)
 \end{aligned}$$

Adding them together, we find

$$\begin{aligned}
 I + II &= -\frac{i}{2} \phi_g^2 \psi^\dagger (\vec{\partial}_i - \overleftarrow{\partial}_i) \psi + \frac{i}{2} \phi_g \partial_i (\phi_g \psi^\dagger \gamma^i \gamma^j \psi) \\
 &\quad + \frac{1}{2} \phi_g \partial_j (\phi_g \psi^\dagger \gamma^i (\vec{\partial}_j - \overleftarrow{\partial}_j) \psi) \quad (A9)
 \end{aligned}$$

with $\bar{i} \neq i \neq 0$. For the second part in Eq. (A5), we will not explicitly write the equations of motion to avoid excessive complexity. However, the time derivatives in Eq. (A10) should be understood as corresponding to the equations of motion Eq. (A7).

$$\begin{aligned}
 j_i^E &= -\frac{i}{2} \phi_g^2 \psi^\dagger (\vec{\partial}_i - \overleftarrow{\partial}_i) \psi + \frac{i}{2} \phi_g \partial_i (\phi_g \psi^\dagger \gamma^i \gamma^j \psi) \\
 &\quad + \frac{1}{2} \phi_g \partial_j (\phi_g \psi^\dagger \gamma^i (\vec{\partial}_j - \overleftarrow{\partial}_j) \psi)
 \end{aligned}$$

$$\begin{aligned}
 &- \phi_g \psi^\dagger \gamma^0 (\overleftarrow{\partial}_0 \vec{\partial}_i + \overleftarrow{\partial}_i \vec{\partial}_0) \psi \\
 &= -\frac{i}{2} \phi_g^2 \psi^\dagger (\vec{\partial}_i - \overleftarrow{\partial}_i) \psi \\
 &\quad + \frac{1}{2} \phi_g \partial_i (\phi_g \psi^\dagger (i\gamma^i \gamma^j + \gamma^i (\vec{\partial}_j - \overleftarrow{\partial}_j)) \psi) \\
 &\quad + \frac{1}{2} \phi_g \partial_i (\phi_g \psi^\dagger \gamma^i (\vec{\partial}_i - \overleftarrow{\partial}_i) \psi) \\
 &\quad - \phi_g \psi^\dagger \gamma^0 (\overleftarrow{\partial}_0 \vec{\partial}_i + \overleftarrow{\partial}_i \vec{\partial}_0) \psi. \quad (A10)
 \end{aligned}$$

Next, we turn to the question of how to express the energy current given in Eq. (A10) as a functional derivative of the Hamiltonian. As we will see, the first term can be directly written as the functional derivative of the Hamiltonian H with respect to e_0^i .

Before addressing the nonrelativistic fourth term in Eq. (A10), let us first consider the second and third terms in Eq. (A10). In fact, these two terms can be reinterpreted as contributions from the spin current. To make this connection explicit, we define a time-independent spin current as follows:

$$\begin{aligned}
 J_s^{\mu ab} &= \frac{1}{\sqrt{g}} \frac{\delta S}{\delta \omega_{\mu ab}} = -\frac{1}{\sqrt{g}} \frac{\delta}{\delta \omega_{\mu ab}} \int dt H = -\frac{1}{\sqrt{g}} \frac{\delta H}{\delta \omega_{\mu ab}} \\
 J_s^{i0a} &= \psi^\dagger \left(-\frac{i}{4} \gamma^i \gamma^a + \frac{i}{4} \gamma^a \gamma^i + \frac{1}{2} \gamma^a \vec{\partial}_i - \frac{1}{2} \overleftarrow{\partial}_i \gamma^a \right) \psi. \quad (A11)
 \end{aligned}$$

With this definition, the second and third terms in Eq. (A10) can be rewritten in terms of the spin current as

$$\begin{aligned}
 -\phi_g \partial_i \left(\frac{\delta H}{\delta \omega_{i0i}} \right) &= \phi_g \partial_i (\phi_g J_s^{i0i}) \\
 &= \frac{1}{2} \phi_g \partial_i (\phi_g \psi^\dagger (i\gamma^i \gamma^i + \gamma^i (\vec{\partial}_i - \overleftarrow{\partial}_i)) \psi) \\
 -\phi_g \partial_i \left(\frac{\delta H}{\delta \omega_{i0i}} \right) &= \phi_g \partial_i (\phi_g J_s^{i0i}) \\
 &= \frac{1}{2} \phi_g \partial_i (\phi_g \psi^\dagger \gamma^i (\vec{\partial}_i - \overleftarrow{\partial}_i) \psi). \quad (A12)
 \end{aligned}$$

Note that in Eq. (A12), identical indices do not imply summation. In this way, we have successfully expressed the second and third terms in Eq. (A10) in terms of the spin current.

We now proceed to analyze the nonrelativistic fourth term in Eq. (A10). As noted in the main text, this term cannot be expressed as a functional derivative of the Hamiltonian with respect to the vielbein field. The reason is that the Newtonian mass term in the action [Eq. (1)] explicitly breaks Lorentz symmetry, thereby leading to an asymmetric energy-momentum tensor.

From this perspective, it is instructive to consider a hypothetical Lorentz-invariant counterpart of the Newtonian mass term in Eq. (1). To achieve this, we replace the original Newtonian mass term (in which the indices a, b are purely spatial, whereas α, β include the temporal direction)

$$\psi^\dagger \gamma^0 (-\eta^{ab} e_a^\mu e_b^\nu \overleftarrow{\nabla}_\mu \overrightarrow{\nabla}_\nu) \psi \quad (A13)$$

by its Lorentz-invariant generalization

$$\psi^\dagger \gamma^0 (-\eta^{\alpha\beta} e_a^\mu e_b^\nu \overleftarrow{\nabla}_\mu \overrightarrow{\nabla}_\nu) \psi. \quad (A14)$$

Performing this replacement would restore Lorentz symmetry and allow the energy current defined in Eq. (A10) to be equivalently expressed as a functional derivative of the Hamiltonian. However, it is clear we cannot perform such a replacement directly, as it would alter the equations of motion, thus describing a physically different system. Nevertheless, this analogy provides valuable insight into the structural differences between the original (Lorentz-breaking) term and its hypothetical Lorentz-symmetric counterpart.

To clearly illustrate this discrepancy, let us explicitly write out their difference as follows:

$$\begin{aligned}
 & \psi^\dagger \gamma^0 (-\eta^{ab} e_a^\mu e_b^\nu \overleftarrow{\nabla}_\mu \overrightarrow{\nabla}_\nu + \eta^{\alpha\beta} e_\alpha^\mu e_\beta^\nu \overleftarrow{\nabla}_\mu \overrightarrow{\nabla}_\nu) \psi \\
 &= \psi^\dagger \gamma^0 (\eta^{\hat{0}\hat{0}} e_{\hat{0}}^\mu e_{\hat{0}}^\nu + \eta^{\hat{0}\hat{b}} e_{\hat{0}}^\mu e_{\hat{b}}^\nu + \eta^{\hat{b}\hat{0}} e_{\hat{0}}^\mu e_{\hat{b}}^\nu) \overleftarrow{\nabla}_\mu \overrightarrow{\nabla}_\nu \psi \\
 &= \psi^\dagger \gamma^0 (\eta^{\hat{0}\hat{0}} e_{\hat{0}}^\mu e_{\hat{0}}^\nu) \overleftarrow{\nabla}_\mu \overrightarrow{\nabla}_\nu \psi \\
 &= \psi^\dagger \gamma^0 (\eta^{\hat{0}\hat{0}} e_{\hat{0}}^0 e_{\hat{0}}^0 \overleftarrow{\nabla}_0 \overrightarrow{\nabla}_0) \psi \\
 &\quad \times \psi^\dagger \gamma^0 (\eta^{\hat{0}\hat{0}} e_{\hat{0}}^i e_{\hat{0}}^j \overleftarrow{\nabla}_i \overrightarrow{\nabla}_j + \eta^{\hat{0}\hat{0}} e_{\hat{0}}^0 e_{\hat{0}}^i \overleftarrow{\nabla}_0 \overrightarrow{\nabla}_i) \psi.
 \end{aligned} \tag{A15}$$

Our main task now, therefore, reduces to analyzing the difference between the original Lorentz-breaking term and its Lorentz-symmetric counterpart, explicitly presented above.

From the definition of the energy-momentum tensor [Eq. (A4)], one observes that in the Luttinger gravity scenario, the first term in Eq. (A15) does not contribute to the functional derivative with respect to the vielbein:

$$\begin{aligned}
 0 &= \frac{\delta}{\delta e_{\hat{0}}^i} (\psi^\dagger \gamma^0 \eta^{\hat{0}\hat{0}} e_{\hat{0}}^0 e_{\hat{0}}^0 \overleftarrow{\nabla}_0 \overrightarrow{\nabla}_0 \psi) \\
 &= \frac{\delta}{\delta e_{\hat{0}}^i} (\psi^\dagger \gamma^0 \eta^{\hat{0}\hat{0}} e_{\hat{0}}^0 e_{\hat{0}}^0 \overleftarrow{\nabla}_0 \overrightarrow{\nabla}_0 \psi).
 \end{aligned} \tag{A16}$$

Thus, this first term can be safely discarded. The second and third terms in Eq. (A15), however, do generate nontrivial contributions to the energy current according to Eq. (A19). This motivates us to include these remaining terms as external source terms in an extended Hamiltonian

$$H_{\text{ex}} = H + H_{\text{src}} \tag{A17}$$

with

$$\begin{aligned}
 H_{\text{src}} &= \int d^2x \sqrt{g} \psi^\dagger \gamma^0 (-\eta^{\hat{0}\hat{0}} e_{\hat{0}}^i e_{\hat{0}}^j \overleftarrow{\partial}_i \overrightarrow{\partial}_j \\
 &\quad - \eta^{\hat{0}\hat{0}} e_{\hat{0}}^0 e_{\hat{0}}^i \overleftarrow{\partial}_0 \overrightarrow{\partial}_i) \psi.
 \end{aligned} \tag{A18}$$

It is worth emphasizing that these additional terms do not alter the equations of motion, since they vanish identically in the context of Luttinger gravity. Hence, they serve purely as external sources in the generating functional. Moreover, the time derivatives appearing in these extra terms do not represent genuine dynamical derivatives; rather, they should be understood as substitutions via the equations of motion, in analogy with the interpretation in Eq. (A10).

Importantly, while our explicit analysis is based on the Newtonian mass term, the logic behind the construction of the extended Hamiltonian is general. Any operator in the Hamiltonian that contributes to the energy current must involve the metric component $e_{\hat{0}}^i$, as the current is defined via the

functional derivative with respect to $e_{\hat{0}}^i$ [see Eq. (A20)]. In the context of Luttinger gravity, $e_{\hat{0}}^i = 0$ by construction. Thus, all terms that explicitly contribute to the energy current operator vanish under this condition and do not affect the original equations of motion. Conversely, any term that modifies the dynamics must be nonzero under the Luttinger condition—but such terms do not contribute to the energy current operator, as shown in Eq. (A20). This guarantees that our extended Hamiltonian H_{ex} introduces no unphysical corrections to either the energy current or the equations of motion.

On the basis of this information, we can express the energy current as the functional derivative of the extended Hamiltonian. To demonstrate this, we directly calculate the energy current corresponding to the dual-energy-momentum tensor τ_i^0 in this case.

$$\begin{aligned}
 \sqrt{g} \tau_i^0 &= \sqrt{g} e_i^\alpha \tau_\alpha^0 = \sqrt{g} \tau_i^0 = -\frac{\delta S_{\text{ex}}}{\delta e_{\hat{0}}^i} \\
 &= \frac{\delta}{\delta e_{\hat{0}}^i} \int dt H_{\text{ex}} = \frac{\delta H_{\text{ex}}}{\delta e_{\hat{0}}^i} \\
 \frac{\delta H_{\text{ex}}}{\delta e_{\hat{0}}^i} &= \psi^\dagger \left(\frac{i}{2} \overrightarrow{\partial}_i - \frac{i}{2} \overleftarrow{\partial}_i \right) \psi \\
 &\quad + \phi_g^{-1} \psi^\dagger \gamma^0 (\overleftarrow{\partial}_i \overrightarrow{\partial}_0 + \overleftarrow{\partial}_0 \overrightarrow{\partial}_i) \psi.
 \end{aligned} \tag{A19}$$

The following variational identities are used in the derivation:

$$\frac{\delta e_{\hat{0}}^\nu}{\delta e_{\hat{0}}^\alpha} = -e_{\hat{0}}^\mu e_{\hat{0}}^\nu{}_\alpha, \quad \frac{\delta e_{\hat{0}}^0}{\delta e_{\hat{0}}^i} = 0, \quad \frac{\delta}{\delta e_{\hat{0}}^i} \sqrt{g} = 0. \tag{A20}$$

By combining Eqs. (A10), (A12), and (A19), we obtain the operator expression for the energy current

$$\begin{aligned}
 j_i^E &= -\phi_g^2 \frac{\delta H_{\text{ex}}}{\delta e_{\hat{0}}^i} - \phi_g \partial_i \left(\frac{\delta H}{\delta \omega_{i\hat{0}i}} \right) - \phi_g \partial_i \left(\frac{\delta H}{\delta \omega_{i\hat{0}i}} \right) \\
 &= -\phi_g^2 \frac{\delta H_{\text{ex}}}{\delta e_{\hat{0}}^i} - \phi_g \partial_i \left(\frac{\delta H_{\text{ex}}}{\delta \omega_{i\hat{0}i}} \right) - \phi_g \partial_i \left(\frac{\delta H_{\text{ex}}}{\delta \omega_{i\hat{0}i}} \right).
 \end{aligned} \tag{A21}$$

This expression is valid at the operator level. Note that repeated indices do not imply summation in this context.

At a finite temperature $1/\beta$, the thermal expectation value of the energy current j_2^E is given by

$$\begin{aligned}
 j_E^2(x) &= -\phi_g^2(x) \frac{1}{Z} \text{Tr} \left[e^{-\beta H} \frac{\delta H_{\text{ex}}}{\delta e_{\hat{0}}^2(x)} \right] \\
 &\quad - \sum_{i=1,2} \phi_g(x) \partial_i \left(\frac{1}{Z} \text{Tr} \left[e^{-\beta H} \frac{\delta H_{\text{ex}}}{\delta \omega_{i\hat{0}2}(x)} \right] \right) \\
 &= -\phi_g^2(x) \frac{1}{Z_{\text{ex}}} \text{Tr} \left[e^{-\beta H_{\text{ex}}} \frac{\delta H_{\text{ex}}}{\delta e_{\hat{0}}^2(x)} \right] \\
 &\quad - \sum_{i=1,2} \phi_g(x) \partial_i \left(\frac{1}{Z_{\text{ex}}} \text{Tr} \left[e^{-\beta H_{\text{ex}}} \frac{\delta H_{\text{ex}}}{\delta \omega_{i\hat{0}2}(x)} \right] \right) \\
 &= -\phi_g^2(x) \frac{\delta F_{\text{ex}}}{\delta e_{\hat{0}}^2(x)} - \sum_{i=1,2} \phi_g(x) \partial_i \left(\frac{\delta F_{\text{ex}}}{\delta \omega_{i\hat{0}2}(x)} \right),
 \end{aligned} \tag{A22}$$

where we have introduced the extended partition function

$$\begin{aligned} Z_{\text{ex}} &= \text{Tr}(e^{-\beta H_{\text{ex}}}) \\ &= \int \mathcal{D}[\psi^*, \psi] \exp \left(- \int_0^\beta d\tau \int d^2x \psi^*(x, \tau) \frac{\partial}{\partial \tau} \psi(x, \tau) \right. \\ &\quad \left. + H_{\text{ex}}[\psi^*(\tau), \psi(\tau)] \right) \end{aligned} \quad (\text{A23})$$

and the corresponding free energy $F_{\text{ex}} = -\beta^{-1} \ln Z_{\text{ex}}$. In this derivation, we have also used the identity

$$\begin{aligned} \frac{1}{Z} \text{Tr} \left(e^{-\beta H} \frac{\delta H_{\text{ex}}}{\delta e_0^2(x)} \right) \Big|_{e_\mu^{\hat{0}} = \delta_\mu^{\hat{0}} \phi_g, e_\mu^a = \delta_\mu^a} \\ = \frac{1}{Z_{\text{ex}}} \text{Tr} \left(e^{-\beta H_{\text{ex}}} \frac{\delta H_{\text{ex}}}{\delta e_0^2(x)} \right) \Big|_{e_\mu^{\hat{0}} = \delta_\mu^{\hat{0}} \phi_g, e_\mu^a = \delta_\mu^a}. \end{aligned} \quad (\text{A24})$$

Equations (A21) and (A22) represent one of the key results of our paper. It demonstrates that even in the absence of Lorentz symmetry, the energy current can still be expressed as the functional derivative of an extended Hamiltonian with respect to the vielbein field. As we have shown above, this extended Hamiltonian differs from the original one only by the addition of a source term H_{src} , which explicitly ensures that the energy current operator is correctly derived—without introducing any unphysical modifications to the equations of motion or altering the original physical content of the nonrelativistic system under consideration.

APPENDIX B: DETAILS OF DERIVATION FOR EQS. (8) AND (9), BOUNDARY EFFECTIVE HAMILTONIAN

In this section, we combine the boundary state theory developed in Refs. [23,24] with the extended Hamiltonian constructed in Appendix A. By projecting the extended Hamiltonian onto the boundary states, we obtain the effective boundary Hamiltonian. Since our ultimate goal is to derive the energy current J_i^E using the functional derivative approach introduced in the previous section, we begin by simplifying the Hamiltonian—eliminating any terms that clearly do not contribute to the energy current. With this in mind, the extended Hamiltonian $H_{\text{ex}} = H + H_{\text{src}}$ can be restated in the following form:

$$\begin{aligned} H &= \int d^2x \sqrt{g} \psi^\dagger \gamma^{\hat{0}} \left(-\frac{i}{2} e_\alpha^j \gamma^\alpha (\vec{\partial}_j - \omega_j) \right. \\ &\quad \left. + \frac{i}{2} (\overleftarrow{\partial}_j + \omega_j) \gamma^\alpha e_\alpha^j + m \right. \\ &\quad \left. - \eta^{ab} e_a^i e_b^j (\overleftarrow{\partial}_i + \omega_i) (\vec{\partial}_j - \omega_j) \right) \psi \\ H_{\text{src}} &= \int d^2x \sqrt{g} \psi^\dagger \gamma^{\hat{0}} (-e_\alpha^i e_\alpha^j \overleftarrow{\partial}_i \overrightarrow{\partial}_j - e_\alpha^i e_\alpha^j \overleftarrow{\partial}_i \overrightarrow{\partial}_i) \psi \\ &= i \int d^2x e_\alpha^i \psi^\dagger \overleftarrow{\partial}_i \left[\phi_g (-i \gamma^j \overrightarrow{\partial}_j + m - \overrightarrow{\partial}^2) \right. \end{aligned}$$

$$\begin{aligned} &\left. - \frac{i}{2} (\partial_j \phi_g) \gamma^j - (\partial_i \phi_g) \overrightarrow{\partial}_i \right] \psi \\ &- i \int d^2x e_\alpha^i \psi^\dagger \left[\phi_g (-i \overleftarrow{\partial}_j \gamma^j + m - \overleftarrow{\partial}^2) \right. \\ &\left. - \frac{i}{2} (\partial_j \phi_g) \gamma^j - \overleftarrow{\partial}_i (\partial_i \phi_g) \right] \overrightarrow{\partial}_i \psi. \end{aligned} \quad (\text{B1})$$

In the derivation above, we used the following identities,

$$\frac{\delta e_\beta^\nu}{\delta e_\mu^\alpha} = -e_\beta^\mu e_\alpha^\nu, \quad \frac{\delta e_\alpha^0}{\delta e_0^i} = 0, \quad \frac{\delta}{\delta e_0^i} \sqrt{g} = 0. \quad (\text{B2})$$

To proceed, we first need to identify the explicit form of the boundary state wave function, which will be used to carry out the projection. Now we consider a boundary at $x^1 = 0$ between a topological insulator with mass $m < 0$ (i.e., in the half-plane $x^1 > 0$) and vacuum (i.e., in the half-plane $x^1 < 0$). The boundary extends to the entire x^2 space. This derivation is based on the assumption that the metric depends only on x^2 near the boundary. Then x^1 and x^2 are completely decoupled in the boundary Hamiltonian. The wave function of the boundary mode of the Hamiltonian $\mathcal{H}_0$ is the product of a plane wave of the x^2 coordinate and a two-component spinor wave function of the x^1 coordinate.

$$\psi(x^1, x^2) = \psi_2(x^2) \psi_1(x^1). \quad (\text{B3})$$

Based on the discussion in Sec. E, the $k = 0$ approximation [26] for the wave function is sufficiently accurate for this system. Therefore, the boundary modes satisfy the following equation:

$$(-i \gamma^{\hat{0}} \gamma^{\hat{1}} \partial_1 + m \gamma^{\hat{0}} - \gamma^{\hat{0}} \partial^2) \psi_1(x^1) = 0 \quad (\text{B4})$$

which, combined with the open boundary condition $\psi_1(0) = 0$, gives the solution

$$\psi_1(x^1) = \frac{1}{\sqrt{N}} (e^{-\lambda_1 x} - e^{-\lambda_2 x}) |s\rangle \equiv \psi_0(x^1) |s\rangle \quad (\text{B5})$$

with $\lambda_{1,2} = \frac{1}{2}(1 \pm \sqrt{1 + 4m})$. The two-component spinor $|s\rangle$ corresponding to the edge bound states satisfies $\sigma_y |s\rangle = -|s\rangle$ or $i \gamma^{\hat{1}} |s\rangle = |s\rangle$.

With the boundary wave function determined, we now proceed to project the original bulk Hamiltonian H [Eq. (B1)] onto the edge state. From Eqs. (B5) and (B1), it follows that the corresponding boundary Hamiltonian can be written as

$$\mathcal{H} = \langle \psi_1 | H | \psi_1 \rangle = I + II + III. \quad (\text{B6})$$

Although the three parts do not carry distinct physical meanings on their own, we separate them here for the sake of clarity in the computation.

The first part, I , collects terms in the Hamiltonian that are independent of the spin connection. After some straightforward computation, we find

$$\begin{aligned} I &= \langle \psi_1 | \sqrt{g} \gamma^{\hat{0}} \left(-\frac{i}{2} e_\alpha^j \gamma^\alpha \overrightarrow{\partial}_j + \frac{i}{2} \overleftarrow{\partial}_j \gamma^\alpha e_\alpha^j \right. \\ &\quad \left. + m - \eta^{ab} e_a^i e_b^j \overleftarrow{\partial}_i \overrightarrow{\partial}_j \right) | \psi_1 \rangle \end{aligned}$$

$$\begin{aligned}
&= \langle \psi_1 | \sqrt{g} \left(-\frac{i}{2} e^j_2 \gamma^0 \gamma^2 \vec{\partial}_j + \frac{i}{2} \overleftarrow{\partial}_j \gamma^0 \gamma^2 e^j_2 \right. \\
&\quad \left. - \frac{i}{2} e^j_0 \gamma^0 \gamma^0 \vec{\partial}_j + \frac{i}{2} \overleftarrow{\partial}_j \gamma^0 \gamma^0 e^j_0 \right) | \psi_1 \rangle \\
&= \sqrt{g} (e^2_2 - e^2_0) \frac{i}{2} (\vec{\partial}_2 - \overleftarrow{\partial}_2). \tag{B7}
\end{aligned}$$

In the derivation above, we used the following identity (note that the metric depends only on x^2 near the boundary):

$$\int dx^1 \psi_0^*(x^1) \partial_1 \psi_0(x^1) = 0 \tag{B8}$$

since the wave function $\psi_1(x^1)$ in Eq. (B5) is either a real function or a purely imaginary function, and vanishes both deep in the bulk and at the boundary.

The second part, II , contains terms that involve the spin connection but no derivatives. It evaluates to

$$\begin{aligned}
II &= \langle \psi_1 | \sqrt{g} \gamma^0 \frac{i}{2} e^j_\alpha (\gamma^\alpha \omega_j + \omega_j \gamma^\alpha) | \psi_1 \rangle \\
&= -\frac{i}{8} \sqrt{g} e^j_\alpha \langle s | \gamma^0 \gamma^\alpha \gamma^x \gamma^y + \gamma^0 \gamma^x \gamma^y \gamma^\alpha | s \rangle \omega_{jxy} \\
&= 0. \tag{B9}
\end{aligned}$$

The reasoning here is as follows: for any $x \neq y$, if α is not equal to either x or y , then the structure inside the matrix is $\langle s | \gamma^0 \gamma^0 \gamma^1 \gamma^2 | s \rangle$, which is proportional to $\langle s | \gamma^0 | s \rangle$, and this is clearly zero. If α equals either x or y , then by the properties of the Clifford algebra, it is evident that the two terms inside the matrix will cancel each other out. Note that the condition $x \neq y$ follows from the definition of the spin connection, which vanishes identically when $x = y$.

The third part, III , includes terms that involve both the spin connection and derivatives. This gives

$$\begin{aligned}
III &= -\eta^{ab} e^i_a e^j_b \langle \psi_1 | \sqrt{g} \gamma^0 (-\overleftarrow{\partial}_i \omega_j + \omega_i \overrightarrow{\partial}_j - \omega_i \omega_j) | \psi_1 \rangle \\
&= \eta^{ab} e^i_a e^j_b \langle \psi_1 | \sqrt{g} \gamma^0 \omega_i \omega_j | \psi_1 \rangle \\
&= \frac{1}{2} \eta^{ab} \sqrt{g} e^i_a e^j_b (\langle s | \gamma^0 \omega_i \omega_j | s \rangle + \langle s | \gamma^0 \omega_j \omega_i | s \rangle) = 0. \tag{B10}
\end{aligned}$$

The justification is as follows: In the first equality of the above equation, we recognize that its first two terms can only contribute terms such as $\omega_{i\hat{0}\hat{1}}$ or $\omega_{i\hat{1}\hat{2}}$. However, these terms do not contribute to the energy current j^E_2 (A21). Moreover, the vielbein field part also does not contribute to the energy current [Eqs. (A21) and (B2)]. Therefore, we can omit this term. In the second line of Eq. (B10), we utilize the symmetry of the metric tensor $\eta^{ab} = \eta^{ba}$, which allows us to express the third term in Eq. (B10) in a symmetric form. The key point here is that the components of the spin connection can always be represented by a single gamma matrix, for example

$$\gamma^{\hat{0}} \gamma^{\hat{1}} \omega_{i\hat{0}\hat{1}} \sim \gamma^{\hat{2}} \omega_{i\hat{0}\hat{1}}. \tag{B11}$$

When the gamma matrices corresponding to the two spin connections are the same, the third term in Eq. (B10) is proportional to $\langle s | \gamma^0 | s \rangle$, which is clearly zero. When the gamma matrices of the two spin connections are different, the

symmetric form ensures that the two terms cancel each other out due to the properties of the Clifford algebra.

Combining all three parts, the total boundary Hamiltonian becomes

$$\begin{aligned}
\mathcal{H} &= \langle \psi_1 | H | \psi_1 \rangle = I + II + III \\
&= \sqrt{g} (e^2_2 - e^2_0) \frac{i}{2} (\vec{\partial}_2 - \overleftarrow{\partial}_2). \tag{B12}
\end{aligned}$$

Now we consider the boundary Hamiltonian for the source term H_{src} , which can also be decomposed into three parts:

$$\mathcal{H}_{\text{src}} = \langle \psi_1 | H_{\text{src}} | \psi_1 \rangle = I_{\text{src}} + II_{\text{src}} + III_{\text{src}} \tag{B13}$$

with

$$\begin{aligned}
I_{\text{src}} &= i \langle \psi_1 | e^i_0 \overleftarrow{\partial}_i \phi_g (-i \gamma^j \vec{\partial}_j + m - \vec{\partial}^2) | \psi_1 \rangle \\
&\quad - i \langle \psi_1 | e^i_0 \phi_g (-i \overleftarrow{\partial}_j \gamma^j + m - \overleftarrow{\partial}^2) \vec{\partial}_i | \psi_1 \rangle \\
&\stackrel{(B4)}{=} i \langle \psi_1 | e^i_0 \overleftarrow{\partial}_i \phi_g (-i \gamma^2 \vec{\partial}_2 - \vec{\partial}^2) | \psi_1 \rangle \\
&\quad - i \langle \psi_1 | e^i_0 \phi_g (-i \overleftarrow{\partial}_2 \gamma^2 - \overleftarrow{\partial}^2) \vec{\partial}_i | \psi_1 \rangle \\
&= i e^2_0 \phi_g (-\overleftarrow{\partial}_2 \vec{\partial}_2^2 + \overleftarrow{\partial}_2^2 \vec{\partial}_2) \\
II_{\text{src}} &= i \langle \psi_1 | e^i_0 \left(\overleftarrow{\partial}_i \frac{-i}{2} (\partial_j \phi_g) \gamma^j + \frac{i}{2} (\partial_j \phi_g) \gamma^j \vec{\partial}_i \right) | \psi_1 \rangle \\
&= i \langle \psi_1 | e^i_0 \left(\overleftarrow{\partial}_i \frac{-i}{2} (\partial_2 \phi_g) \gamma^2 + \frac{i}{2} (\partial_2 \phi_g) \gamma^2 \vec{\partial}_i \right) | \psi_1 \rangle \\
&= 0 \\
III_{\text{src}} &= i \langle \psi_1 | e^i_0 \left(-\overleftarrow{\partial}_i (\partial_j \phi_g) \vec{\partial}_j + \overleftarrow{\partial}_j (\partial_i \phi_g) \vec{\partial}_i \right) | \psi_1 \rangle \\
&= i (\partial_2 \phi_g) e^i_0 \langle \psi_1 | -\overleftarrow{\partial}_i \vec{\partial}_2 + \overleftarrow{\partial}_2 \vec{\partial}_i | \psi_1 \rangle \\
&= i (\partial_2 \phi_g) e^1_0 \langle \psi_1 | -\overleftarrow{\partial}_1 \vec{\partial}_2 + \overleftarrow{\partial}_2 \vec{\partial}_1 | \psi_1 \rangle \\
&\stackrel{(B8)}{=} 0. \tag{B14}
\end{aligned}$$

In the derivation of the term II_{src} , we relied on the assumption that the metric depends only on x_2 . Summing all components yields

$$\mathcal{H}_{\text{src}} = i e^2_0 \phi_g (-\overleftarrow{\partial}_2 \vec{\partial}_2^2 + \overleftarrow{\partial}_2^2 \vec{\partial}_2). \tag{B15}$$

Finally, we can express the extended boundary Hamiltonian as

$$\begin{aligned}
\mathcal{H}_{\text{ex}} &= \int dx^2 \psi_2^\dagger(x^2) \left(\frac{i}{2} (1 + \zeta(x^2)) (\vec{\partial}_2 - \overleftarrow{\partial}_2) \right. \\
&\quad \left. + i \chi(x^2) (-\overleftarrow{\partial}_2 \vec{\partial}_2^2 + \overleftarrow{\partial}_2^2 \vec{\partial}_2) \right) \psi_2(x^2) \tag{B16}
\end{aligned}$$

with the following identity:

$$\begin{aligned}
\zeta(x^2) &= \sqrt{g} (e^2_2 - e^2_0) - 1 \\
&= \frac{h}{2} + \left(1 + \frac{h}{2} \right) \left(\frac{1}{2} h^2_{\hat{0}} - \frac{1}{2} h^2_{\hat{2}} \right),
\end{aligned}$$

$$\chi(x^2) = \phi_g e^2_0 = \frac{1}{2} (1 + \phi) (-h^2_{\hat{0}})$$

$$\frac{\delta \zeta(x^2)}{\delta e^2_0(x^2)} = 1, \quad \frac{\delta \chi(x^2)}{\delta e^2_0(x^2)} = -1. \tag{B17}$$

and

$$\begin{aligned} e_\mu^\alpha &= \delta_\mu^\alpha + h_\mu^\alpha/2, \\ e_\alpha^\mu &= \delta_\alpha^\mu - h_\alpha^\mu/2, \\ \sqrt{g} &= 1 + h/2. \end{aligned} \quad (\text{B18})$$

By examining the functions $\zeta(x^2)$ and $\chi(x^2)$, we find that they approach zero when the metric tensor is close to that of flat spacetime. This allows us to treat them as perturbative corrections. In the following section, we will use this observation as the basis for a systematic perturbative calculation at finite temperature $T = 1/\beta$.

APPENDIX C: DETAILS OF DERIVATION FOR EQS. (12) AND (13), BOUNDARY FREE ENERGY AND BOUNDARY CURRENT

In this section, we aim to compute the effective free energy of the boundary modes in the presence of the vielbein field. We begin by presenting the partition function corresponding to the boundary Hamiltonian given in Eqs. (8) and (9), formulated within the framework of finite-temperature quantum field theory [31]

$$\begin{aligned} Z &= \int \mathcal{D}[\psi^*, \psi] \exp(-\mathcal{S}[\psi^*, \psi, \zeta, \chi]) \\ &= \int \mathcal{D}[\psi^*, \psi] \exp(-\mathcal{S}_0[\psi^*, \psi] - \mathcal{S}_1[\psi^*, \psi, \zeta, \chi]), \end{aligned} \quad (\text{C1})$$

where $\mathcal{S}_0$ denotes the free part of the action, and $\mathcal{S}_1$ contains the coupling to the vielbein field. The two terms are given by

$$\begin{aligned} \mathcal{S}_0[\psi^*, \psi] &= \sum_{\omega_n, k} \psi_{n,k}^* (-i\omega_n - k) \psi_{n,k} \\ \mathcal{S}_1[\psi^*, \psi, \zeta, \chi] &= \sum_{\omega_n, k, k'} \psi_{n,k}^* \left(-\frac{k+k'}{2} \zeta(k-k') \right. \\ &\quad \left. + (k^2 k' + k k'^2) \chi(k-k') \right) \psi_{n,k'} \end{aligned} \quad (\text{C2})$$

Within this framework, the boundary free energy can be expanded order by order as follows [23,31]:

$$\mathcal{F}[\zeta, \chi] = \frac{1}{\beta} \sum_{l=1}^{\infty} \frac{1}{l} \text{Tr}[(G_0 \Sigma)^l]. \quad (\text{C3})$$

We now begin the step-by-step calculation of the free energy expansion in Eq. (C3). Starting from the action given in Eq. (C2), the corresponding Green's function and self-energy can be expressed as follows:

$$\begin{aligned} G_0(k, \omega_n) &= (i\omega_n + k)^{-1}, \\ \Sigma(k, k') &= -\frac{k+k'}{2} \zeta(k-k') + (k^2 k' + k k'^2) \chi(k-k'). \end{aligned} \quad (\text{C4})$$

With these expressions, we proceed to evaluate each term in the expansion Eq. (C3). The first-order perturbative

contribution is given by

$$\begin{aligned} \frac{1}{\beta} \text{Tr}[G_0 \Sigma] &= \frac{1}{\beta} \sum_{\omega_n} \int \frac{dp}{2\pi} \Sigma(p, p) G_0(p, \omega_n) \\ &= \frac{1}{\beta} \sum_{\omega_n} \int \frac{dp}{2\pi} (-p \zeta(0) + 2p^3 \chi(0)) G_0(p, \omega_n) \\ &= \frac{1}{\beta} \int dx \zeta(x) \sum_{\omega_n} \int \frac{dp}{2\pi} \frac{-p}{i\omega_n + p} \\ &\quad + \frac{1}{\beta} \int dx \chi(x) \sum_{\omega_n} \int \frac{dp}{2\pi} \frac{2p^3}{i\omega_n + p}. \end{aligned} \quad (\text{C5})$$

We now evaluate each contribution separately. For the first term, we compute

$$\begin{aligned} \sum_{\omega_n} \int \frac{dp}{2\pi} \frac{-p}{i\omega_n + p} &= \sum_{\omega_n} \int \frac{dp}{2\pi} \frac{p}{i\omega_n - p} \\ &= \beta \int \frac{dp}{2\pi} p f(p) = \lim_{\Lambda \rightarrow \infty} \beta \int_{-\Lambda}^{\Lambda} \frac{dp}{2\pi} p f(p) \\ &= \lim_{\Lambda \rightarrow \infty} \frac{1}{2\pi\beta} \int_{-\beta\Lambda}^{\beta\Lambda} dx \frac{x}{e^x + 1} \\ &= \frac{1}{2\pi\beta} \frac{-\beta^2 \Lambda^2}{2} - \frac{1}{2\pi\beta} \int dx \frac{x^2}{2} \frac{-e^x}{(1+e^x)^2} \\ &= -\frac{\beta}{4\pi} \Lambda^2 + \beta \frac{\pi T^2}{12}. \end{aligned} \quad (\text{C6})$$

Similarly, for the second term we find

$$\sum_{\omega_n} \int \frac{dp}{2\pi} \frac{2p^3}{i\omega_n + p} = \beta \frac{\Lambda^4}{4\pi} - \beta \frac{7\pi^3 T^4}{60}. \quad (\text{C7})$$

In the course of this computation, we make use of the identity

$$\int dx \frac{x}{e^x + 1} = \int dx \left(\frac{x^2}{2} \frac{1}{e^x + 1} \right)' - \int dx \frac{x^2}{2} \left(\frac{1}{e^x + 1} \right)' \quad (\text{C8})$$

Putting everything together, the first-order contribution becomes

$$\begin{aligned} \frac{1}{\beta} \text{Tr}[G_0 \Sigma] &= \left(-\frac{1}{4\pi} \Lambda^2 + \frac{\pi T^2}{12} \right) \int dx \zeta(x) \\ &\quad + \left(\frac{\Lambda^4}{4\pi} - \frac{7\pi^3 T^4}{60} \right) \int dx \chi(x) \\ &= \frac{\pi T^2}{12} \int dx \zeta(x) - \frac{7\pi^3 T^4}{60} \int dx \chi(x). \end{aligned} \quad (\text{C9})$$

In this expression, we have omitted nonphysical terms that are independent of temperature. Next, using a similar approach, we compute the second-order perturbative contribution

$$\begin{aligned} \frac{1}{2\beta} \text{Tr}[(G_0 \Sigma)^2] &= \frac{1}{2\beta} \int \frac{dq}{2\pi} \int dx dy e^{iq(x-y)} (\Pi^{(1)}(q) \zeta(x) \zeta(y) \\ &\quad + \Pi^{(2)}(q) \zeta(x) \chi(y) + \Pi^{(2)}(q) \chi(x) \zeta(y)), \end{aligned} \quad (\text{C10})$$

where the functions $\Pi^{(1)}(q)$ and $\Pi^{(2)}(q)$ are defined as follows:

$$\begin{aligned}\Pi^{(1)}(q) &= \sum_{\omega_n} \int \frac{dp}{2\pi} \frac{(2p+q)^2}{4} G_0(p+q, \omega_n) G_0(p, \omega_n) \\ \Pi^{(2)}(q) &= \sum_{\omega_n} \int \frac{dp}{2\pi} \frac{-(2p+q)}{2} ((p+q)^2 p + (p+q)p^2) \\ &\quad \times G_0(p+q, \omega_n) G_0(p, \omega_n).\end{aligned}\quad (\text{C11})$$

In deriving Eq. (C10), we have neglected quadratic terms in $\chi(x)$, since such contributions vanish identically in the Luttinger case. As a result, if a term in the expression of free energy contains more than one $\chi(x)$ function, its contribution to the energy current will necessarily be zero. Moreover, only the zero-momentum components $\Pi^{(1)}(0)$ and $\Pi^{(2)}(0)$ are needed in our analysis. This is because expanding $\Pi^{(1,2)}(q)$ around $q = 0$ generates terms involving derivatives of $\zeta(x)$ or $\chi(x)$. Since the primary focus is on contributions to the free energy that do not involve such derivatives, we can neglect these terms. Evaluating the expressions at $q = 0$ gives

$$\begin{aligned}\Pi^{(1)}(0) &= \sum_{\omega_n} \int \frac{dp}{2\pi} \frac{p^2}{(i\omega_n + p)^2} = -\beta \frac{\pi T^2}{6} \\ \Pi^{(2)}(0) &= \sum_{\omega_n} \int \frac{dp}{2\pi} \frac{-p(2p^3)}{(i\omega_n + p)^2} \\ &= -\sum_{\omega_n} \int \frac{dp}{2\pi} \frac{2p^4}{(i\omega_n - p)^2} = -\frac{2\beta}{2\pi} \int dp \frac{df(p)}{dp} p^4 \\ &= -\frac{1}{\pi\beta^3} \int dx x^4 \frac{-e^x}{(1+e^x)^2} = \beta \frac{7\pi^3 T^4}{15}.\end{aligned}\quad (\text{C12})$$

Substituting these back, the second-order contribution becomes

$$\begin{aligned}\frac{1}{2\beta} \text{Tr}[(G_0 \Sigma)^2] &= -\frac{\pi T^2}{12} \int dx (\zeta(x))^2 \\ &\quad + \frac{7\pi^3 T^4}{15} \int dx \zeta(x) \chi(x).\end{aligned}\quad (\text{C13})$$

We now turn to the general case of the l -th order contribution, with $l > 2$. In this case, the term in the perturbative expansion takes the form

$$\begin{aligned}\frac{1}{l\beta} \text{Tr}[(G_0 \Sigma)^l] &= \frac{1}{l\beta} U_l^{(1)} \int dx (\zeta(x))^l \\ &\quad + \frac{1}{\beta} U_l^{(2)} \int dx (\zeta(x))^{l-1} \chi(x),\end{aligned}\quad (\text{C14})$$

where the coefficients $U_l^{(1)}$ and $U_l^{(2)}$ arise from performing the Matsubara summation via contour integration, following Hertz's approach [33]. These are given by

$$\begin{aligned}U_l^{(1)} &= \sum_{\omega_n} \int \frac{dp}{2\pi} \frac{(-p)^l}{(i\omega_n + p)^l} = (-1)^{l+1} l\beta \frac{\pi T^2}{12} \\ U_l^{(2)} &= \sum_{\omega_n} \int \frac{dp}{2\pi} \frac{(-p)^{l-1} 2p^3}{(i\omega_n + p)^l} \\ &= \beta (-1)^l \frac{(l+2)(l+1)l\pi^3 T^4}{360}.\end{aligned}\quad (\text{C15})$$

Substituting these results back into the general expression yields

$$\begin{aligned}\frac{1}{l\beta} \text{Tr}[(G_0 \Sigma)^l] &= (-1)^{l+1} \frac{\pi T^2}{12} \int dx (\zeta(x))^l \\ &\quad + (-1)^l \frac{(l+2)(l+1)l\pi^3 T^4}{360} \int dx (\zeta(x))^{l-1} \chi(x)\end{aligned}\quad (\text{C16})$$

Finally, summing over all orders $l \geq 1$, we obtain the complete expression for the boundary effective free energy

$$\mathcal{F}[\zeta, \chi] = \frac{\pi T^2}{12} \int dx \frac{\zeta(x)}{1+\zeta(x)} - \frac{7\pi^3 T^4}{60} \int dx \frac{\chi(x)}{(1+\zeta(x))^4}.\quad (\text{C17})$$

Collecting the results from Eqs. (A22), (B2), (B17), and (C17), we obtain the final expression for the boundary energy current

$$\begin{aligned}j_E^{2,\text{bdry}}(x^2) &= -(1+\phi)^2 \frac{\delta \mathcal{F}}{\delta e_0^2(x^2)} \\ &\quad - \sum_{i=1,2} (1+\phi(x^2)) \partial_i \left(\frac{\delta \mathcal{F}}{\delta \omega_{i02}(x^2)} \right) \\ &= -(1+\phi(x^2))^2 \frac{\delta \mathcal{F}}{\delta e_0^2(x^2)} \\ &= -(1+\phi(x^2))^2 \frac{\pi T^2}{12} \frac{1}{(1+\zeta(x))^2} \frac{\delta \zeta(x^2)}{\delta e_0^2(x^2)} \\ &\quad + (1+\phi(x^2))^2 \frac{7\pi^3 T^4}{60} \frac{1}{(1+\zeta(x^2))^4} \frac{\delta \chi(x^2)}{\delta e_0^2(x^2)} \\ &= -\frac{\pi T^2}{12} - \frac{7\pi^3 T^4}{60} \frac{1}{(1+\phi(x^2))^2}\end{aligned}\quad (\text{C18})$$

APPENDIX D: DETAILS OF DERIVATION FOR Eqs. (14)–(18), ENERGY MAGNETIZATION, AND THERMAL HALL CONDUCTIVITY

In this section, based on the results derived from Eq. (13), we calculate the system's energy magnetization and thermal Hall conductivity. The energy magnetization, denoted as $\mathbf{M}_E$, is defined through the relation

$$\mathbf{j}_E = \nabla \times \mathbf{M}_E.\quad (\text{D1})$$

Under this relation, and within the boundary configuration considered in Fig. 1, the bulk energy magnetization M_E^z can be related to the boundary energy current via

$$\begin{aligned}j_E^{2,\text{bdry}} &\equiv \int_0^{+\infty} dx^1 j_E^2(x^1) \\ &= \int_0^{+\infty} dx^1 (-\partial_1 M_E^z(x^1)) \\ &= -M_E^z(+\infty) + M_E^z(0) = -M_E^z(+\infty),\end{aligned}\quad (\text{D2})$$

where we have used the boundary condition $M_E^z(0) = 0$ as proposed in Ref. [35]. Applying this result to the expression for the boundary current obtained in the previous section, we arrive at the bulk energy magnetization

$$M_E^z \equiv M_z^E(+\infty) = \frac{\pi T^2}{12} + \frac{7\pi^3 T^4}{60}. \quad (\text{D3})$$

It is important to note that, since M_E^z is an equilibrium quantity, we are only required to consider a uniform gravitational potential ϕ . Under such conditions, the approximations made in Appendix C hold rigorously. In addition, since the bulk energy current j_E near the boundary satisfies

$$j_E^1(x^1 = +0) = -\partial_2 j_E^{\text{bdry}}(x^2) \quad (\text{D4})$$

we can relate the boundary energy current derived in Eq. (C18) to the corresponding bulk energy current. This yields

$$\begin{aligned} j_E^1(x^1 = +0) &= -\frac{7\pi^3 T^4}{30} \frac{1}{(1 + \phi(x^2))^3} \partial_2 \phi(x^2) \\ &= -\frac{7\pi^3 T^4}{30} \partial_2 \phi(x^2). \end{aligned} \quad (\text{D5})$$

In the position of the last equality in Eq. (D5), we have applied a linear approximation to ϕ . According to linear response theory and the Tolman-Ehrenfest relation [14,15], the bulk contribution to the thermal Hall conductivity arising from the quantum anomaly is given by

$$\kappa_{xy}^{\text{bulk}} = \frac{7\pi^3 T^3}{30}. \quad (\text{D6})$$

Taking into account the contribution from the energy magnetization [36], the total thermal Hall conductivity becomes

$$\kappa_{xy} = \kappa_{xy}^{\text{bulk}} + \frac{2M_E^z}{T} = \frac{\pi T}{6} + \frac{7\pi^3 T^3}{15}. \quad (\text{D7})$$

It is worth noting that this result is consistent with the one obtained via the generalized Streda formula [21], which expresses the thermal Hall conductivity as the temperature derivative of the energy magnetization

$$\kappa_{xy} = \frac{\partial M_E^z}{\partial T} = \frac{\pi T}{6} + \frac{7\pi^3 T^3}{15}. \quad (\text{D8})$$

APPENDIX E: EDGE MODES OF TOPOLOGICAL INSULATOR

In this section, we construct the wave functions and dispersion relations of the chiral edge states of a topological insulator under open boundary conditions using the method provided in the main text. Unlike the case of the massive Dirac fermion model studied in Refs. [23,24], here we must carefully examine whether the quadratic terms in the bulk Hamiltonian give rise to higher-order contributions in the effective boundary Hamiltonian. As a result, we cannot restrict our analysis to linear terms alone, but must also account for possible higher-order effects.

We start with a standard continuum model describing a topological insulator, considering the model Hamiltonian defined on the half-space $x > 0$ in the $x - y$ plane, as illustrated

in Fig. 1 in the main text.

$$\hat{h} = \hat{k}_x \sigma_x + \hat{k}_y \sigma_y + (m + \hat{k}_x^2 + \hat{k}_y^2) \sigma_z. \quad (\text{E1})$$

According to Ref. [26], we can divide the one-body Hamiltonian $\hat{h}$ into two parts, $\hat{h} = \hat{h}_0 + \hat{h}_1$,

$$\begin{aligned} \hat{h}_0 &= \hat{k}_x \sigma_x + (m + \hat{k}_x^2) \sigma_z, \\ \hat{h}_1 &= \hat{k}_y \sigma_y + \hat{k}_y^2 \sigma_z \end{aligned} \quad (\text{E2})$$

All k_x -dependent terms are included in $\hat{h}_0$. For such a semi-infinite system given by Fig. 1, k_y is a good quantum number, but k_x must be replaced by the operator $-i\partial_x$. Therefore, the Schrödinger equation satisfied by the edge state wave function can be written as

$$\begin{pmatrix} m - \partial_x^2 + k_y^2 & -i\partial_x - ik_y \\ -i\partial_x + ik_y & -m + \partial_x^2 - k_y^2 \end{pmatrix} \psi_{k_y}(x) = E \psi_{k_y}(x) \quad (\text{E3})$$

With the wave function ansatz

$$\psi_{k_y} = e^{\lambda x} |s\rangle = e^{\lambda x} (\psi_u, \psi_d)^T \quad (\text{E4})$$

we can further simplify this Schrödinger equation to

$$\begin{aligned} (m - \lambda^2 + k_y^2) \psi_u - i(\lambda + k_y) \psi_d &= E \psi_u \\ -i(\lambda - k_y) \psi_u - (m - \lambda^2 + k_y^2) \psi_d &= E \psi_d. \end{aligned} \quad (\text{E5})$$

This equation gives four roots $\pm\lambda_1, \pm\lambda_2$

$$\begin{aligned} \lambda_1^2 &= k_y^2 + m + \frac{1}{2} + \frac{1}{2} \sqrt{1 + 4E^2 + 4m}, \\ \psi_{d,1\pm} &= -i \frac{\pm\lambda_1 - k_y}{E + (m - \lambda_1^2 + k_y^2)} \psi_u \\ \lambda_2^2 &= k_y^2 + m + \frac{1}{2} - \frac{1}{2} \sqrt{1 + 4E^2 + 4m}, \\ \psi_{d,2\pm} &= -i \frac{\pm\lambda_2 - k_y}{E + (m - \lambda_2^2 + k_y^2)} \psi_u. \end{aligned} \quad (\text{E6})$$

Therefore, the general solution is given by

$$\psi_{k_y} = a e^{\lambda_1 x} \psi_{1+} + b e^{-\lambda_1 x} \psi_{1-} + c e^{\lambda_2 x} \psi_{2+} + d e^{-\lambda_2 x} \psi_{2-} \quad (\text{E7})$$

Directly solving Eqs. (E5) and (E6) is very challenging. Since we are only interested in the low-energy part of the system, we first consider the case of $k_y = 0$, and then examine how a finite k_y modifies the wave function and the dispersion relation. Owing to the properties of the one-body Hamiltonian h in Eq. (E1), the wave function $\psi_{k_y=0}$ exhibits particle-hole symmetry. This suggests the possible existence of a special edge state $\psi_{k_y=0}$ with zero energy ($E = 0$). Under this condition, Eq. (E6) reduces to

$$\begin{aligned} \lambda_1^2 &= m + \frac{1}{2} + \frac{1}{2} \sqrt{1 + 4m} = \left(\frac{1}{2}(1 + \sqrt{1 + 4m})\right)^2 \\ \lambda_2^2 &= m + \frac{1}{2} - \frac{1}{2} \sqrt{1 + 4m} = \left(\frac{1}{2}(1 - \sqrt{1 + 4m})\right)^2 \end{aligned} \quad (\text{E8})$$

and the two-component spinor in Eq. (E6) becomes

$$\psi_{d,1\pm} = \pm i \psi_u, \quad \psi_{d,2\pm} = \pm i \psi_u. \quad (\text{E9})$$

Now we return to the general case. The coefficients a, b, c, d in Eq. (E7) are determined by two key conditions:

the open boundary condition $\psi_{k_y}(0) = 0$ and the requirement that the wave function be normalizable in the region $x > 0$.

First, normalizability imposes that either $\text{Re}\lambda_{1,2} < 0$, which leads to $b = d = 0$, or $\text{Re}\lambda_{1,2} > 0$, which leads to $a = c = 0$. From the structure of $\lambda_{1,2}$ in the zero-mode limit $\psi_{k_y=0}$, these conditions are only satisfied when $m < 0$, ensuring a localized edge state. The wave function then takes the form

$$\psi_{k_y} = be^{-\lambda_1 x} \psi_{1,-} + de^{-\lambda_2 x} \psi_{2,-} \quad (\text{E10})$$

Next, applying the boundary condition $\psi_{k_y}(0) = 0$ gives two additional constraints: $d = -b$ and $\psi_{d,1-} = \psi_{d,2-}$.

Combining the second constraint with Eq. (E6), we can derive the energy dispersion relation for the edge states.

$$\frac{-\lambda_1 - k_y}{E + (m - \lambda_1^2 + k_y^2)} = \frac{-\lambda_2 - k_y}{E + (m - \lambda_2^2 + k_y^2)} \quad (\text{E11})$$

$$E = -k_y(\lambda_1 + \lambda_2) - k_y^2 - m - \lambda_1 \lambda_2$$

The system of equations formed by Eqs. (E6) and (E11) remains difficult to solve. However, when we restrict our consideration to small k_y , we can compute its derivatives step by step. First, based on the particle-hole symmetry, we know that $E(k_y = 0) = 0$. Using this as a starting point, we proceed to compute its first derivative of $\lambda_{1,2}$:

$$\left. \frac{d\lambda_1}{dk_y} \right|_{k_y=0} = \frac{E(0)}{\sqrt{1+4m+4E(0)^2}} \frac{1}{\sqrt{\frac{1}{2}+m+\frac{1}{2}\sqrt{1+4m+4E(0)^2}}} \left. \frac{dE}{dk_y} \right|_{k_y=0} = 0,$$

$$\left. \frac{d\lambda_2}{dk_y} \right|_{k_y=0} = 0,$$

$$\left. \frac{dE}{dk_y} \right|_{k_y=0} = \left(-(\lambda_1(0) + \lambda_2(0)) - k_y \left(\frac{d\lambda_1}{dk_y} + \frac{d\lambda_2}{dk_y} \right) - 2k_y - \frac{d\lambda_1}{dk_y} \lambda_2 - \lambda_1 \frac{d\lambda_2}{dk_y} \right)_{k_y=0} = -1 \quad (\text{E12})$$

The first-order calculation shows that, to linear order in $-k_y$, the energy satisfies $E(k_y) = -k_y$, which is consistent with our expectation. Next, we use the obtained derivative information to compute the second-order contribution.

$$\left. \frac{d^2\lambda_1}{dk_y^2} \right|_{k_y=0} = \frac{2 + \frac{2}{\sqrt{1+4m}}}{\sqrt{2+4m+2\sqrt{1+4m}}},$$

$$\left. \frac{d^2\lambda_2}{dk_y^2} \right|_{k_y=0} = \frac{2 - \frac{2}{\sqrt{1+4m}}}{\sqrt{2+4m-2\sqrt{1+4m}}}, \quad (\text{E13})$$

and

$$\left. \frac{d^2E}{dk_y^2} \right|_{k_y=0} = -\left(\frac{d\lambda_1}{dk_y} + \frac{d\lambda_2}{dk_y} \right) - 2 - 2 \frac{d\lambda_1}{dk_y} \frac{d\lambda_2}{dk_y} - \lambda_1 \frac{d^2\lambda_2}{dk_y^2} - \frac{d^2\lambda_1}{dk_y^2} \lambda_2$$

$$= -2 - \lambda_1 \frac{d^2\lambda_2}{dk_y^2} - \frac{d^2\lambda_1}{dk_y^2} \lambda_2 = 0. \quad (\text{E14})$$

Continuing this process iteratively, we arrive at the following series of results:

$$\left. \frac{d^3\lambda_1}{dk_y^3} \right|_{k_y=0} = \frac{3\sqrt{2}}{\sqrt{1+4m}\sqrt{1+2m+\sqrt{1+4m}}} \frac{d\lambda_1}{dk_y} \frac{d^2\lambda_1}{dk_y^2} = 0,$$

$$\left. \frac{d^3\lambda_2}{dk_y^3} \right|_{k_y=0} = -\frac{3\sqrt{2}}{\sqrt{1+4m}\sqrt{1+2m-\sqrt{1+4m}}} \frac{d\lambda_2}{dk_y} \frac{d^2\lambda_2}{dk_y^2} = 0.$$

$$\left. \frac{d^3E}{dk_y^3} \right|_{k_y=0} = -\left(\frac{d^2\lambda_1}{dk_y^2} + \frac{d^2\lambda_2}{dk_y^2} \right) - 3 \frac{d\lambda_1}{dk_y} \frac{d^2\lambda_2}{dk_y^2} - 3 \frac{d^2\lambda_1}{dk_y^2} \frac{d\lambda_2}{dk_y} - \lambda_1 \frac{d^3\lambda_2}{dk_y^3} - \lambda_2 \frac{d^3\lambda_1}{dk_y^3} = 0. \quad (\text{E15})$$

Therefore, near $k_y = 0$, we obtain a linear dispersion,

$$E(k_y) = -k_y + O(k_y^4). \quad (\text{E16})$$

Based on this information, we can rewrite the corresponding energy eigenstates in this case in the following form:

$$\psi_d/\psi_u = i \frac{\lambda_1 + k_y}{E + (m - \lambda_1^2 + k_y^2)} = -i + 0 \times \frac{k_y}{m} + 0 \times \frac{k_y^2}{m^2} + 0 \times \frac{k_y^3}{m^3} + O(k_y^4) = -i + O(k_y^4). \quad (\text{E17})$$

From Eqs. (E16) and (E17), we see that up to $O(k_y^4)$, the Newton mass term introduced in the topological insulator model does not alter the linear energy dispersion and wave function of the edge states. Since our focus is on low-energy physics, this level of approximation is more than sufficient. Thus, for an ideal insulator, the $k_y = 0$ approximation accurately captures the edge-state behavior, and the Newton mass term does not introduce any significant quadratic corrections to the linear dispersion. Thus, we can use $\psi_0(x)$ as a substitute for $\psi_{k_y}(x)$, even though $e^{-ik_y y}\psi_0(x)$ is not an exact eigenstate of the Hamiltonian for finite k_y .

APPENDIX F: CORRECTION OF CHARGE HALL CURRENT

In this section, we explicitly construct the boundary effective theory for the charge Hall conductivity to investigate potential corrections arising from the quadratic (Newtonian mass) term in our topological insulator model. We find that, in contrast to the thermal Hall conductivity, the boundary effective theory for the charge Hall conductivity is identical to that of a massive Dirac fermion (which corresponds to taking the Newtonian mass $m_N \rightarrow \infty$). Consequently, the charge Hall conductivity remains unmodified by the Newtonian mass term.

The model action for a (2+1)-dimensional topological insulator coupled to a gauge potential A_μ is given by

$$S = \int dt d^2x \sqrt{g} \psi^\dagger \frac{i}{2} (\vec{\partial}_0 - \overleftarrow{\partial}_0) \psi - \int dt H$$

$$H = \int d^2x \sqrt{g} \psi^\dagger \left(-\frac{i}{2} \gamma^0 \gamma^j \vec{D}_j + \frac{i}{2} \gamma^0 \overleftarrow{D}_j \gamma^j \right. \\ \left. + \gamma^0 \left(m - \frac{1}{m_N} \eta^{ij} \overleftarrow{D}_i \vec{D}_j \right) \right) \psi, \quad (F1)$$

where D_μ is the gauge-covariant derivative, and we set $A_\mu = (0, A_i)$,

$$\vec{D}_i \psi = (\vec{\partial}_i - iA_i) \psi \quad \psi^\dagger \overleftarrow{D}_i = \psi^\dagger (\overleftarrow{\partial}_i + iA_i). \quad (F2)$$

Here, the gauge potential A_μ is introduced purely as an external source term, solely to generate the number current operator. The physical boundary states $|\psi_1\rangle$ remain identical to those discussed previously, independent of the presence of the external gauge potential A_μ . The number current operator, defined by functional differentiation with respect to the gauge

potential A_i , is [19]

$$J^i = \frac{1}{\sqrt{g}} \frac{\delta S}{\delta A_i} \Big|_{A_i=0} = -\frac{1}{\sqrt{g}} \frac{\delta H}{\delta A_i} \Big|_{A_i=0}$$

$$= \psi^\dagger \left(-\frac{i}{2} \gamma^0 \gamma^i (-i) + \frac{i}{2} \gamma^0 (i) \gamma^i + \gamma^0 (i \vec{\partial}_i - i \overleftarrow{\partial}_i) \right) \psi$$

$$= \psi^\dagger (-\gamma^0 \gamma^i + \gamma^0 (i \vec{\partial}_i - i \overleftarrow{\partial}_i)) \psi. \quad (F3)$$

Following the approach detailed in Appendix B, we project the bulk Hamiltonian onto the boundary states to construct the corresponding boundary Hamiltonian

$$\mathcal{H} = \langle \psi_1 | H | \psi_1 \rangle = I + II. \quad (F4)$$

The first term, I , comprises contributions independent of the gauge field A_i , and evaluates to

$$I = \langle \psi_1 | \sqrt{g} \gamma^0 \left(-\frac{i}{2} \gamma^j \vec{\partial}_j + \frac{i}{2} \overleftarrow{\partial}_j \gamma^j + m - \eta^{ij} \overleftarrow{\partial}_i \vec{\partial}_j \right) | \psi_1 \rangle$$

$$= \langle \psi_1 | \sqrt{g} \left(-\frac{i}{2} \gamma^0 \gamma^2 \vec{\partial}_2 + \frac{i}{2} \overleftarrow{\partial}_2 \gamma^0 \gamma^2 \right) | \psi_1 \rangle$$

$$= \frac{i}{2} \sqrt{g} (\vec{\partial}_2 - \overleftarrow{\partial}_2). \quad (F5)$$

The second term, II , includes contributions from the gauge field A_i :

$$II = \langle \psi_1 | \sqrt{g} \left(\gamma^0 \left(-\frac{i}{2} \gamma^j (-iA_j) + \frac{i}{2} (iA_j) \gamma^j \right. \right. \right. \\ \left. \left. - \frac{1}{m_N} \eta^{ij} (iA_i) \overleftarrow{\partial}_j - \frac{1}{m_N} \eta^{ij} \overleftarrow{\partial}_i (-iA_j) \right) \right) | \psi_1 \rangle$$

$$= \frac{i}{2} \langle \psi_1 | \sqrt{g} (\gamma^0 \gamma^2 iA_2 + iA_2 \gamma^0 \gamma^2) | \psi_1 \rangle$$

$$= \frac{i}{2} \sqrt{g} iA_2 (-2) = -\frac{i}{2} \sqrt{g} (iA_2) - \frac{i}{2} \sqrt{g} (iA_2). \quad (F6)$$

In the derivation above, we used the fact that the metric depends only on x^2 near the boundary and $\langle s | \gamma^0 | s \rangle = 0$ [Eq. (B5)]. Combining both contributions, we obtain the effective boundary Hamiltonian for the charge current

$$\mathcal{H} = \sqrt{g} \frac{i}{2} (\vec{\partial}_2 - \overleftarrow{\partial}_2) - \frac{i}{2} \sqrt{g} (iA_2) - \frac{i}{2} \sqrt{g} (iA_2)$$

$$= \frac{i}{2} \sqrt{g} (\vec{D}_2 - \overleftarrow{D}_2). \quad (F7)$$

This result explicitly shows that the boundary effective theory for the charge current matches precisely that of a chiral Dirac fermion. Importantly, it contains no additional terms generated by the Newtonian mass term m_N . Consequently, the charge Hall conductivity remains quantized and uncorrected by the quadratic term, distinguishing it fundamentally from the thermal Hall conductivity scenario analyzed previously.

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