

Symmetry preservation in commensurate twisted bilayers

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Symmetry plays a key role in materials hosting Dirac electrons and underpins our ability to completely flatten the Dirac cone through the tuning of physical parameters such as twisting in van der Waals heterostructures. The emergent moiré patterns in twisted bilayers at small twist angles appear, at first glance, to be independent of the initial stacking order and hence are only shifted when one layer is translated with respect to the other. However, when the twist angle is large, differences can be seen at the level of both the lattice and electronic structure in the case of twisted bilayer graphene and twisted bilayer kagome. In this work, we show that the rotational and dihedral symmetry of high-symmetry triangular bilayers is preserved for all commensurate twist angles with a sixfold symmetric twist center. Hence, we demonstrate that the exact symmetry of small twist angle systems depends upon the initial stacking of the bilayer. We further apply the principles of our method to twisted bilayer graphene with a threefold symmetric twist center to recover the results of [E. J. Mele, *Phys. Rev. B* **81**, 161405 (2010)]. Finally, we outline how our method can be generalized to commensurate multilayer systems.

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I. INTRODUCTION

Twisted bilayer graphene (TBG) is the forerunner for demonstrating how van der Waals (vdW) heterostructures can be tuned through the simple introduction of a rotational misalignment between layers. Prior to the discovery of superconductivity in magic angle TBG [1], TBG had been at the center of several theoretical works [2–10] that sought to establish its electronic structure, low-energy description, transport characteristics, and optical properties, and predicted the observed magic angle to coincide with the emergence of flat bands [11]. Since the observation of superconductivity, there has been intense effort to unveil the physics underpinning its emergent topological behavior and strongly correlated phases [12–18]. Naturally, Van Hove singularities (VHSs) are of particular interest in our endeavour to model strong correlations due to the vanishing of the Fermi velocity around such points yielding a logarithmic divergence in the density of states (DOS). More recently though, stronger divergences of the DOS have been of interest due to the appearance of higher-order VHSs (HOVHSs) that result in power-law scalings of the DOS rather than logarithmic [19–23] and can lead to an exotic supermetal phase [24,25]. Such HOVHSs have been predicted to appear in TBG [26,27], while evidence for the tunability of VHSs to create a HOVHS has been seen in strontium ruthenates [28,29] and graphene-based vdW heterostructures [30,31]. Twisting in vdW heterostructures can

therefore be viewed as a means by which to engineer band structures and encourage such features.

Inspired by the success of TBG, twisting in other material and lattice combinations has now begun to be explored. Within spintronics, a common layered system of two honeycomb lattices is graphene on transition metal dichalcogenide. Here, twisting can be used to modulate the proximity-induced couplings and alter the Fermi surface spin texture to gain control over spin-charge coupled transport [32–38]. Closely related to the honeycomb lattice is its line graph, the kagome lattice. It has been shown that such lattices will always host at least one flat band [39,40]. Recent studies have looked at the effects of twisting in kagome bilayers and predicted a flattening of the Dirac cone around the moiré Brillouin zone corners with a magic angle around twice that of TBG $\theta_0 \simeq 2.28^\circ$ [41]. Moreover, twisted bilayer kagome (TBK) at large commensurate twist angles has been shown to display a densely packed region of near-flat bands, host a plethora of HOVHSs with C_{6z} , C_{3z} , and C_{2z} symmetry around the various high-symmetry points (HSPs) of the moiré Brillouin zone, and is a potential candidate for strong topological effects with Chern numbers of order 10 [42].

The flattening of a Dirac cone is extremely sensitive to the symmetry of the system [43], determining whether the magic angle corresponds to a drastic reduction in the Fermi velocity or an exact vanishing of the Fermi velocity. Likewise, the types of HOVHSs that a material may host are set by its symmetry. Clearly, the features observed in twisted systems, be it TBG or TBK, are fundamentally underpinned by symmetry. At the heart of the honeycomb and kagome lattices is the triangular lattice: the honeycomb is just the two sublattice variant of the triangular lattice and kagome the three sublattice variant. The symmetry of their twisted bilayers will therefore be the same regardless of the number of sublattices, and so we can seek to understand all three lattices in a single

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unified treatment. Establishing the crystal symmetries of all such systems will enable for a deeper understanding of their topological properties and phase formation [42,44,45].

In this paper, we demonstrate that the rotational aspects of the point group symmetry for all high-symmetry triangular lattice bilayers will be preserved for any commensurate twist centered on a C_{6z} symmetry point of an individual layer. Our approach is underpinned by an overlap function that measures the separation of rotational symmetry points (RSPs) and must vanish when they overlap. We use this general principle to reproduce the two classes of moiré patterns obtained for TBG and extend it for use in commensurate twisted multilayer systems. Additionally, we provide an alternative method for determining the symmetry of twisted bilayers through the construction of a Wigner-Seitz (WS) cell centered on the twist origin. We organize the paper as follows: Section II starts by introducing the various high-symmetry stacking configurations for bilayer graphene, bilayer kagome, and bilayer triangular lattice, provides a labeling scheme for commensurate twist angle, and explains how to identify the appearance of HSPs in a moiré pattern. We then proceed to determine which RSPs are guaranteed to overlap and yield an HSP in Sec. III, and which RSP overlaps are avoided in Sec. IV when the twist is applied about the C_{6z} symmetry point of one layer. In Sec. V, we recover the results of Mele [4] for a twist applied about a C_{3z} RSP of an individual layer and explain how this is connected to the C_{6z} twist center scenario. Section VI determines the symmetry for commensurate twisted multilayers with both alternating and helical-type twisting with one choice commensurate angle and details how to generalize our method for mixtures of commensurate twist angles and stacking orders. Finally, in Sec. VII, we present an alternative approach to discern the symmetry of twisted triangular lattices by considering the twist-centric WS unit cell. We establish the orientation of the WS unit cell and provide compact expressions for the moiré lattice vectors to show what symmetries may arise at the WS corners and edges to reproduce our results from Secs. III and IV. The symmetry analysis we provide here will act as a guide to understand the emergence of HOVHSs in twisted triangular lattice systems. Moreover, it reveals how stacking order plays a central role in determining the allowed HOVHSs upon any commensurate twist and unveils a D_2 scenario that cannot arise upon twisting around a C_{3z} symmetry site.

II. DEFINITIONS AND SETUP

A. High-symmetry stacking orders

We define a high-symmetry stacking as one with a discrete rotational symmetry about the z axis that is not identity and that which possesses a C_2 symmetry about some in-plane axis (i.e., C'_2 symmetry). The resulting point group of the bilayer will thus be D_n , D_{nv} , or D_{nh} , depending on further reflection symmetries that may arise in the aligned bilayer. Figure 1(a) presents all such high-symmetry stackings for kagome, graphene, and triangular bilayers prior to twisting. The stacking orders of bilayer graphene are already well documented in previous literature [46], so we spare a detailed discussion of its stacking order beyond stating that bilayer

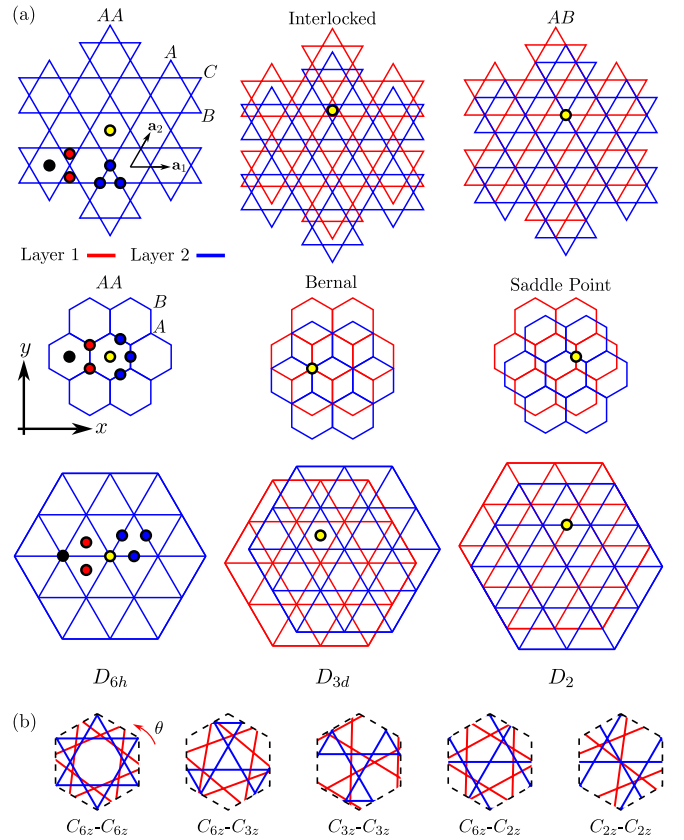


FIG. 1. (a) High-symmetry stackings for kagome (top), graphene (middle), and triangular (bottom) bilayers. The yellow dots indicate high-symmetry centers, while the black, red, and blue dots highlight twist centers with $\theta_p = 60^\circ$, 120° , and 180° , respectively. (b) Types of RSP overlap in TBK that yield a high-symmetry point.

graphene has three high-symmetry stackings: AA (column) with D_{6h} symmetry, AB (Bernal) with D_{3d} symmetry, and saddle point with D_2 symmetry. In comparison, bilayer kagome has only recently started to garner attention [41,47–50], so let us take a moment to define the high-symmetry stackings possible in kagome bilayers.

The most trivial to see is the AA scenario, where one layer is placed directly on top of the other with no relative displacement. This scenario preserves all symmetries of the underlying kagome monolayer and thus has a D_{6h} point group symmetry. The HSP of the AA bilayer kagome is then located at the center of the hexagon. We note that in all cases, the HSP will naturally have a vertical position halfway between the two layers, so we shall only discuss the 2D position of the bilayer HSP.

Turning our attention to the interlocked stacking of kagome bilayers, we see that it is equivalent to Bernal bilayer graphene, with the up and down triangles playing the role of graphene sublattices. To construct an interlocked kagome bilayer, we place the second kagome layer such that the center of a down triangle from one layer lies directly on top of the center of an up triangle from the other. Naturally, we see that the interlocked kagome bilayer possesses the same point group symmetry as Bernal bilayer graphene, D_{3d} , with

the HSP being situated at the center of an up-down triangle overlap.

Lastly, we may choose to arrange the kagome bilayer in an AB configuration where the second layer is shifted relative to the first along one of the bonds by a bond length. This results in two sublattices with direct overlap between the layers and one left floating in the center of the hexagon from the other layer. In this case, while C_{2z} and C'_2 are both present, no mirror planes of the constituent monolayers align, meaning that AB bilayer kagome has no mirror reflection symmetries. Hence, the point group symmetry of this system is simply D_2 . The HSP here is found where the sites of the two layers overlap one another directly.

B. Twisting and commensurate angles

We introduce twisting into a high-symmetry bilayer by rotating one layer by an angle of θ about the z axis and note that there are three natural centers that may be taken as the origin for this rotation; see Fig. 1(a). First, we may retain a 60° twist periodicity, $\theta_p = 60^\circ$, for all choices of stacking by applying this rotation about the isolated HSP of the affected layer. Second, we may increase θ_p to that of the bilayer by choosing to rotate one layer about the HSP of the bilayer rather than of the individual layer. This will result in $\theta_p = 120^\circ$ for Bernal TBG and interlocked TBK, while $\theta_p = 180^\circ$ will arise for AB TBK and saddle point TBG. In this paper, we shall consider the cases where $\theta_p = 60^\circ$ and $\theta_p = 120^\circ$.

The honeycomb and kagome lattices are both examples of a triangular lattice with different sublattice geometries. Hence, their corresponding twisted bilayers possess the same set of commensurate twist angles, θ_c . Each commensurate twist angle, $0 < \theta_c < 60^\circ$, is defined by a pair of coprime integers, m and n , such that $m > n > 0$. The commensurate angle associated to (m, n) is given by [3,51,52]

$$\cos \theta_c = \frac{3m^2 - n^2}{3m^2 + n^2}, \quad \sin \theta_c = \frac{2\sqrt{3}mn}{3m^2 + n^2}. \quad (1)$$

For each θ_c , there is a corresponding commensurate angle $\theta'_c = \pi/3 - \theta_c$. This pair of commensurate angles will always have one associated to $n/3 \in \mathbb{Z}$ (i.e. when n is divisible by 3) while the other is acquired by using $n/3 \notin \mathbb{Z}$ (i.e. an n not divisible by 3) [52]. When n is divisible by 3, we choose to write $n = 3v$ ($v \in \mathbb{Z}$).

In the case where $\theta_p = 60^\circ$, we only need to consider one set of commensurate angles. Specifically, we choose to focus on those where n is divisible by 3. This is a result of the layer being rotated, possessing the mirror reflection symmetry $y \rightarrow -y$ about its HSP prior to rotation. We illustrate the equivalence of these lattices in Appendix A. For the cases where $\theta_p = 120^\circ$, we will need to consider both sets of n as the commensurate angle pair (θ_c, θ'_c) will yield different moiré patterns [4].

C. The role of high-symmetry overlap

For a twisted bilayer to possess a nontrivial rotational symmetry, two points must overlap with compatible nontrivial rotational symmetries; we refer to these points of interest as RSPs. Triangular lattices and their multiple sublattice variants

TABLE I. Resulting point group symmetry of a twisted bilayer due to the overlap of two RSPs (top row and leftmost column). Dihedral symmetry is only preserved when identical RSPs overlap. All other cases yield a point group that is a subgroup of the overlapping RSP point groups, with E being the identity group. Visualization of these overlaps are provided in Fig. 1(b).

	C_{6z}	C_{3z}	C_{2z}
C_{6z}	D_6	C_{3z}	C_{2z}
C_{3z}	C_{3z}	D_3	E
C_{2z}	C_{2z}	E	D_2

possess three RSPs: C_{6z} around the hexagon centers, C_{3z} at the graphene sublattice sites and kagome triangle centers, and C_{2z} about the midpoint between graphene sublattices and at the sublattice sites of kagome; see Fig. 1(a). Let us start by considering AA -stacked bilayers. Naturally, when twisted to a commensurate angle, the C_{6z} RSPs of each are guaranteed to overlap since we may identify the twist center as a moiré lattice site with sixfold symmetry. Therefore, the AA bilayers will have C_{6z} symmetry. In fact, they possess a D_6 point group symmetry since the mirror planes of the individual layers no longer align, while a C'_2 symmetry persists.

If we now move on to consider the Bernal/interlocked cases with a commensurate twist angle, we must study the emergent superlattice more closely as we cannot immediately say whether the twisted system will possess C_{6z} or C_{3z} symmetry. By inspection, we can see that C_{3z} must comprise at least part, if not all, of the rotational symmetry group due to the twist center always possessing this symmetry by construction for both $\theta_p = 60^\circ$ and $\theta_p = 120^\circ$; see Fig. 1(b). Let us first focus on $\theta_p = 60^\circ$. Here, the twist center corresponds to one case of RSP overlap, specifically when a C_{6z} RSP overlaps a C_{3z} RSP. However, two more cases must also be considered: C_{3z} overlapping with C_{3z} , and C_{6z} with C_{6z} . Each of these three cases yields a different point group symmetry which we summarize in Table I. We must therefore check all RSP overlaps that may yield a higher point group symmetry. If we find that C_{6z} - C_{6z} overlap is present, the system will host D_6 symmetry. However, if the system does not host C_{6z} - C_{6z} overlap, we must then check for C_{3z} - C_{3z} overlap, which also preserves the dihedral symmetry of the original untwisted bilayers in contrast to the C_{6z} - C_{3z} overlap naturally present in these systems. We find that, as shall be shown later, C_{3z} - C_{3z} overlap does indeed occur for all commensurate twist angles in Bernal TBG and interlocked TBK, while C_{6z} - C_{6z} overlap will never occur with $\theta_p = 60^\circ$, and so the system will possess a D_3 point group symmetry; see Fig. 2(b) for an example with $\theta_c = 38.21^\circ$ ($m = 5$ and $n = 3$).

For the case where we instead apply our rotation about the bilayer HSP, $\theta_p = 120^\circ$, C_{3z} - C_{3z} overlap occurs by construction, so we only need to check for C_{6z} - C_{6z} overlap. Like the $\theta_p = 60^\circ$ case, every θ_c associated to when 3 divides n avoids C_{6z} - C_{6z} overlap to produce a D_3 point group symmetry. However, for θ_c with n not divisible by 3, C_{6z} - C_{6z} is guaranteed and the system instead exhibits D_6 symmetry [4].

Finally, for AB -stacked kagome, the twist center corresponds to C_{6z} - C_{2z} overlap, meaning the bilayer's minimum

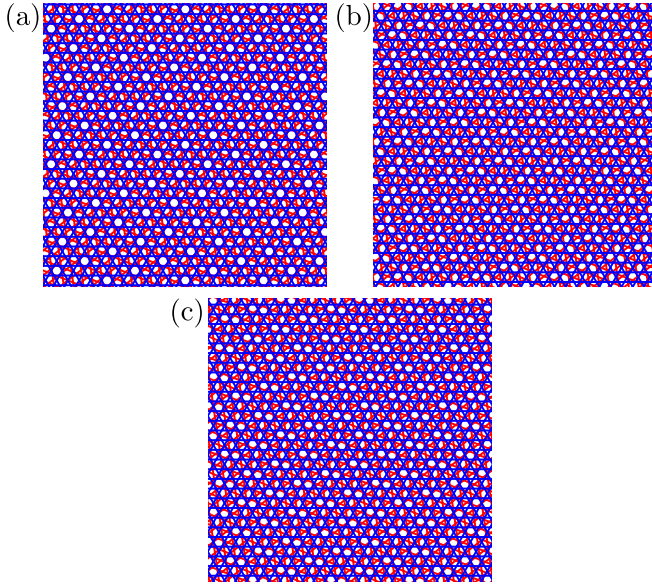


FIG. 2. Moiré patterns for TBK in AA (a), interlocked (b), and AB (c) stacking. The twist is applied about a hexagon center of the red layer to maintain 60° twist periodicity. The commensurate angle used here is $\theta \simeq 38.21^\circ$ corresponding to $(m, n) = (5, 3)$.

point group symmetry is C_{2z} . The overlaps we must check for are thus C_{6z} - C_{6z} , C_{6z} - C_{3z} , C_{3z} - C_{3z} , and C_{2z} - C_{2z} as these will all yield higher point group symmetries than C_{2z} . Given that C_{2z} and C_{3z} are not compatible rotational symmetries, we do not have to check for their overlap. As we shall show later, we find that the only RSP overlap occurring besides C_{6z} - C_{2z} for $\theta_p = 60^\circ$ is C_{2z} - C_{2z} , meaning that commensurate AB TBK will possess a D_2 point group symmetry [Fig. 2(c)] when the twist origin is located at a hexagon center.

D. Identifying RSP overlap

To determine the overlap of RSPs, we construct a function that must vanish when two RSPs lie directly on top of one another. First, let δ_i be the 2D position of the RSP of concern in the bottom layer and δ_j be the 2D position of the RSP of concern in the top layer. The lattice of these RSPs is obtained by adding an arbitrary number of lattice vectors to their respective positions: $\mathbf{v}_i = \delta_i + k\mathbf{a}_1 + l\mathbf{a}_2$ and $\mathbf{v}_j = \delta_j + p\mathbf{a}_1 + q\mathbf{a}_2$, where $k, l, p, q \in \mathbb{Z}$. Next, we introduce the twist via the 2D rotation matrix R_θ such that $R_\theta \mathbf{v}_i = \mathbf{v}_i^\theta$. We now define our overlap function:

$$f(\delta_j, \delta_i; k, l, p, q; \theta) = |\mathbf{v}_j - \mathbf{v}_i^\theta|^2, \quad \mathbf{v}_i = \delta_i + k\mathbf{a}_1 + l\mathbf{a}_2, \quad \mathbf{v}_j = \delta_j + p\mathbf{a}_1 + q\mathbf{a}_2. \quad (2)$$

Clearly, this vanishes when $\mathbf{v}_i^\theta = \mathbf{v}_j$. Specifically, if $f(\delta_j, \delta_i; k, l, p, q; \theta) = 0$ for some $k, l, p, q \in \mathbb{Z}$, then the RSPs at δ_i^θ and δ_j must overlap at some point in the twisted lattice. We provide the full set of RSP positions to be checked in determining RSP overlap for the $\theta_p = 60^\circ$ in Table II, where we denote the nature of the kagome lattice by their subscript using the language of the kagome lattice: Δ/∇ indicate the RSPs of the up/down triangles in the kagome lattice, while $\{A, B, C\}$ are the kagome sublattice

TABLE II. Summary of the RSP locations within each layer for the various stacking choices for $\theta_p = 60^\circ$. We omit AA stacking due to its trivial nature. The shape of the subscript indicates the HSP being considered with a prime denoting a HSP in the second layer, e.g., $\delta_{\nabla'}$ is the C_{3z} HSP location associated to a down triangle in the first layer. When a letter is used in the subscript, it refers to the kagome sublattice. $\delta_{\square} = \mathbf{0}$ indicates that the twist is centered on the C_{6z} RSP of the bottom layer.

HSP	Interlocked	AB kagome
C_{6z}	$\delta_{\square}^{(I)} = \mathbf{0},$ $\delta_{\square'}^{(I)} = \frac{\mathbf{a}_1 + \mathbf{a}_2}{3}$	$\delta_{\square}^{(II)} = \mathbf{0},$ $\delta_{\square'}^{(II)} = \frac{\mathbf{a}_2 - \mathbf{a}_1}{2}$
C_{3z}	$\delta_{\Delta}^{(I)} = \frac{2\mathbf{a}_2 - \mathbf{a}_1}{3},$ $\delta_{\nabla'}^{(I)} = \frac{2\mathbf{a}_2 - \mathbf{a}_1}{3}$	$\delta_{\Delta}^{(II)} = \frac{2\mathbf{a}_2 - \mathbf{a}_1}{3}, \delta_{\nabla}^{(II)} = \frac{\mathbf{a}_1 + \mathbf{a}_2}{3},$ $\delta_{\Delta'}^{(II)} = \frac{\mathbf{a}_1 + \mathbf{a}_2}{6}, \delta_{\nabla'}^{(II)} = -\frac{\mathbf{a}_1 + \mathbf{a}_2}{6},$ $\delta_A^{(II)} = \frac{\mathbf{a}_1}{2}, \delta_B^{(II)} = \frac{\mathbf{a}_2}{2},$
C_{2z}	—	$\delta_C^{(II)} = \frac{\mathbf{a}_2 - \mathbf{a}_1}{2}, \delta_{A'}^{(II)} = \frac{2\mathbf{a}_1 - \mathbf{a}_2}{2},$ $\delta_{B'}^{(II)} = \frac{\mathbf{a}_1}{2}, \delta_{C'}^{(II)} = \mathbf{0}$

sites with C_{2z} symmetry. A primed subscript denotes an RSP in the top layer. For $\theta_p = 120^\circ$, we need only check for C_{6z} - C_{6z} overlap, and so only two RSP positions are required, $\delta_{\square} = (\mathbf{a}_1 - 2\mathbf{a}_2)/3 = -\delta_{\square'}$ (taking the HSP as the origin), corresponding to the C_{6z} RSP in the bottom and top layers, respectively.

A further property of the overlap function is that any zeros it exhibits must be minima since $f(\delta_j, \delta_i; k, l, p, q; \theta) \geq 0$. Therefore, a zero in the overlap function will always coincide with $\partial_\mu f(\delta_i, \delta_j; k, l, p, q; \theta) = 0$ for $\mu \in \{k, l, p, q\}$. These partial derivatives will allow us to establish additional relations between the lattice parameters k, l, p , and q to determine whether they can be chosen to be integer such that the overlap function vanishes.

III. GUARANTEED RSP OVERLAPS: $\theta_p = 60^\circ$

A. Bernal TBG and interlocked TBK

We start by considering C_{3z} - C_{3z} overlap in Bernal TBG and interlocked TBK. We find that $\partial_p f(\delta_{\nabla'}, \delta_{\Delta}^{(I)}; k, l, p, q; \theta_c) = 0$ produces $\mathcal{A} = 2p + q$

$$\mathcal{A} = \frac{(2k + l)m^2 - 2(2 + 3l)mv - 3(2k + l)v^2}{m^2 + 3v^3}, \quad (3)$$

and letting $l = -2k$ yields

$$\mathcal{A} = \frac{4(3k - 1)mv}{m^2 + 3v^3}. \quad (4)$$

In demanding that $p, q \in \mathbb{Z}$, we must have $\mathcal{A} \in \mathbb{Z}$. We next notice that $m^2 + 3v^3 - 1$ will always be divisible by 3 courtesy of the coprime nature of m and n such that n is divisible by 3. To see this, we may write $m = 3\mu \pm 1$ ($\mu \in \mathbb{Z}$) to acquire $m^2 + 3v^3 - 1 = 3(3\mu^2 \pm 2\mu + v^2)$. We therefore choose

$$k = \frac{2 + \alpha(m^2 + 3v^2)}{3}(m^2 + 3v^2 - 1) + 1, \quad (\alpha \in \mathbb{Z}) \quad (5)$$

to ensure that both $\mathcal{A}, l \in \mathbb{Z}$,

$$\mathcal{A} = 4mv(2 + \alpha(m^2 + 3v^2 - 1)). \quad (6)$$

Next, let us consider the overlap function with k given by Eq. (5), $l = -2k$, $q = \mathcal{A} - 2p$, and $\mathcal{A}$ given by Eq. (6):

$$f(\delta_{\nabla}^{(I)}, \delta_{\Delta}^{(I)}; k, l, p, q; \theta_c) = \frac{1}{3}[1 - 3p + (m^2 + 6mv - 3v^2)(2 + \alpha(m^2 + 3v^2 - 1))]^2. \quad (7)$$

The overlap function will thus vanish when

$$p = \frac{1}{3}[1 + (m^2 + 6mv - 3v^2)(2 + \alpha(m^2 + 3v^2 - 1))], \quad (8)$$

which we see to be integer upon substituting $m = 3\mu \pm 1$ to obtain a cumbersome but nonetheless integer expression for p :

$$p = 1 + \mu(3\mu \pm 2)(2 + \alpha(1 \pm 3\mu)^2) + 2[6(1 + \alpha)\mu + 27\alpha\mu^2(\mu \pm 1) \pm 2]v + (\alpha - 2)v^2 + 6\alpha(3\mu \pm 1)v^3 - 3\alpha v^4. \quad (9)$$

Therefore, C_{3z} - C_{3z} overlap is guaranteed to occur in Bernal TBG and interlocked TBK at all commensurate twist angles when the twist is applied about a C_{6z} symmetry point. Moreover, because the C_{3z} - C_{3z} overlap we have proven to always occur is between an up and down triangle, the dihedral symmetry of the system will always lie along the x axis of the twist-symmetric frame where the bottom layer is rotated by $\theta_c/2$ and the top layer is rotated by $-\theta_c/2$, see Appendix B.

B. Saddle point TBG and AB TBK

In the commensurate saddle point TBG and AB TBK systems, we shall consider the overlap of C_{2z} RSPs. Naturally, there are several combinations of kagome sublattice overlap we might consider: A - A' , A - B' , A - C' , B - C' , and their layer flipped counterparts. We shall choose to focus on C - A' , A - B' , and A - A' overlap depending on our choice of m and n ,

$$\begin{aligned} C-A' : m + v \text{ is even such that } \frac{m + v + 2}{4} &\in \mathbb{Z}, \\ A-B' : m + v \text{ is even such that } \frac{m + v}{4} &\in \mathbb{Z}, \\ A-A' : m + v \text{ is odd.} \end{aligned} \quad (10)$$

We justify these categorisations in the next section.

1. C-A' overlap

Let us start by moving into the twist-symmetric frame. Clearly, this system is identical to the previous scenario in which only the bottom layer is rotated by θ_c through a global rotation of $-\theta_c/2$. The C and A' sites are then given by

$$\mathbf{v}_C = R_{\frac{\theta_c}{2}}(\delta_C^{(II)} + k\mathbf{a}_1 + l\mathbf{a}_2), \quad (11a)$$

$$\mathbf{v}_{A'} = R_{\frac{\theta_c}{2}}(\delta_{A'}^{(II)} + p\mathbf{a}_1 + q\mathbf{a}_2), \quad (11b)$$

respectively, where $\bar{\theta}_c = -\theta_c$. In this frame, we now search for C - A' overlap along the x axis. Setting the y components $\mathbf{v}_C$ and $\mathbf{v}_{A'}$ to zero yields

$$4vk + 2(m + v)l = -(m - v), \quad (12a)$$

$$4vp - 2(m - v)q = -(m + 3v). \quad (12b)$$

The existence of integer solutions to these diophantine equations can be checked via Bézout's identity [53].

Bézout's Identity: If $a, b \in \mathbb{Z}$ and $d = \alpha \text{GCD}(a, b)$ with $\alpha \in \mathbb{Z}$ (i.e., d is an integer multiple of the greatest common divisor of a and b), then $\exists x, y \in \mathbb{Z}$ such that $ax + by = d$.

Starting with Eq. (12a), we immediately note that when either m or v is even and the other is odd (both cannot be

even due to m and $3v$ being coprime), then $m \pm v$ must be odd and $\text{GCD}(4v, 2(m + v)) = 2$. Hence, Eq. (12a) cannot satisfy Bézout's identity when m and v possess opposite parity. The same can be said for Eq. (12b) since $(m + 3v)$ will also be odd when m and v are of opposite parities.

To progress, we consider the case when m and v are both odd, hence $m \pm v$ is even, and further divide this set of m and v into two equal sets: (i) those where $(m + v)/4 \in \mathbb{Z}$ and (ii) those where $(m + v)/4 \notin \mathbb{Z}$. In identifying these sets, we note that when 4 divides $(m + v)$, $m - v$ is even but not divisible by 4 and vice versa. Focusing on (i), we see that $\text{GCD}(4v, 2(m \pm v)) = 4$. However, 4 does not divide $(m - v)$ and $(m + 3v)$ here: Letting $m + v = 4\alpha$ ($\alpha \in \mathbb{Z}$), we find $m - v = 2(2\alpha - v)$ and $m + 3v = 2(2\alpha + v)$. Therefore, Eqs. (12a) and (12b) again do not satisfy Bézout's identity in this scenario. Moving onto (ii), we again find $\text{GCD}(4v, 2(m \pm v)) = 4$ but now 4 divides $(m - v)$ and $(m + 3v)$: Letting $m + v + 2 = 4\alpha$ yields $m - v = 2(2\alpha - v - 1)$ and $m + 3v = 2(2\alpha + v - 1)$, and we note that $v \pm 1$ is even. Hence, the expressions in Eqs. (12) satisfy Bézout's identity when m and v are both odd such that $(m + v + 2)$ is divisible by 4.

Having identified $\text{GCD}((4, m + v), 2) = 2$, with m and v being odd as the relevant scenario, we use the following ansatz to ensure that $k, p \in \mathbb{Z}$:

$$k = \frac{m + v + 2}{4}, \quad p = \frac{m - v}{4} - 1. \quad (13)$$

Substituting these expressions for k and p into Eqs. (12) yields

$$l = -\frac{v + 1}{2}, \quad q = \frac{1 + v}{2}, \quad (14)$$

which are also integer given that v is odd. Thus, Eqs. (13) and (14) correspond to when C and A' sites lie on the x axis in the twist-symmetric frame. Upon substitution of these expressions into the overlap function, we find that $f(\delta_C^{(II)}, \delta_{A'}^{(II)}; k, l, p, q; \theta_c) = 0$, demonstrating that the C and A' sites possessing C_{2z} symmetry are guaranteed to overlap in scenario (ii) when m and v are both odd, leading to a system with at least D_2 point group symmetry.

2. A-B' overlap

We again consider the twist-symmetric frame but now focus on the appearance of A and B' sites along the x axis in this frame:

$$\mathbf{v}_A = R_{\frac{\theta_c}{2}}(\delta_A^{(II)} + k\mathbf{a}_1 + l\mathbf{a}_2), \quad (15a)$$

$$\mathbf{v}_{B'} = R_{\frac{\theta_c}{2}}(\delta_{B'}^{(II)} + p\mathbf{a}_1 + q\mathbf{a}_2). \quad (15b)$$

Demanding that the y component of these vectors vanishes results in the following diophantine equations for the lattice parameters:

$$2vk + (m + v)l = -v, \quad (16a)$$

$$2vp - (m - v)q = -v. \quad (16b)$$

In this case, we find that only when m and v are of opposite parities can the expressions in Eqs. (16) satisfy Bézout's identity,

$$\text{GCD}(2v, (m \pm v)) = \begin{cases} 1, & \mathcal{P}_m \neq \mathcal{P}_v \\ 2, & \mathcal{P}_m = \mathcal{P}_v = -1, \end{cases} \quad (17)$$

where $\mathcal{P}_m = \pm 1$ when m is even/odd (i.e., $\mathcal{P}_m$ is the parity of m). The latter case requires v to be odd and hence not divisible by $\text{GCD}(2v, (m \pm v))$, whereas the former case is always satisfied for any integer choice of v .

With the above in mind, we employ the ansatz

$$k = \frac{m + v - 1}{2}, \quad p = \frac{m - v - 1}{2}, \quad (18)$$

which are guaranteed to be integer when $\mathcal{P}_m \neq \mathcal{P}_v$, and substitute Eqs. (18) into Eqs. (16) to arrive at

$$l = -v, \quad q = v. \quad (19)$$

Using the integer expressions in Eqs. (18) and (19), we find that the overlap function for the A and B' vanishes, $f(\delta_A^{(\text{II})}, \delta_{B'}^{(\text{II})}; k, l, p, q; \theta_c) = 0$. Therefore, A sites will always overlap with B sites when m and v are opposite in parity, again resulting in a system with at least D_2 symmetry.

3. A-A' overlap

Moving once more into the twist-symmetric frame, we apply an additional global rotation of $-\pi/6$ and then search for overlap of A and A' sites along the x axis. In this frame of reference, the positions of the A and A' sites are given by

$$\tilde{\mathbf{v}}_A = R_{\frac{\theta_c}{2} - \frac{\pi}{6}}(\delta_A^{(\text{II})} + k\mathbf{a}_1 + l\mathbf{a}_2), \quad (20a)$$

$$\tilde{\mathbf{v}}_{A'} = R_{\frac{\theta_c}{2} - \frac{\pi}{6}}(\delta_{A'}^{(\text{II})} + p\mathbf{a}_1 + q\mathbf{a}_2). \quad (20b)$$

Setting the y components of these vectors to zero, we obtain

$$2(m - 3v)k - 2(m + 3v)l = -(m - 3v), \quad (21a)$$

$$2(m + 3v)p - 2(m - 3v)q = -3(m + v). \quad (21b)$$

In checking Bézout's identity, we note that

$$\text{GCD}(2(m - 3v), 2(m + 3v)) = \begin{cases} 2, & \mathcal{P}_m \neq \mathcal{P}_v \\ 4, & \mathcal{P}_m = \mathcal{P}_v = -1, \end{cases} \quad (22)$$

and hence the diophantine equations in Eqs. (21) can only satisfy Bézout's identity if 4 divides $(m + v)$, which in turn gives 4 divides $(m - 3v)$. We thus see that the current setup corresponds to (i) when m and v are both odd.

We next use the ansatz

$$k = \frac{m + 3v - 2}{4}, \quad p = \frac{m - 3v}{4} - 1, \quad (23)$$

noting that when $(m + v)$ is divisible by 4 while m and v are both odd, 4 does not divide $(m + 3v)$ despite $m + 3v$ being

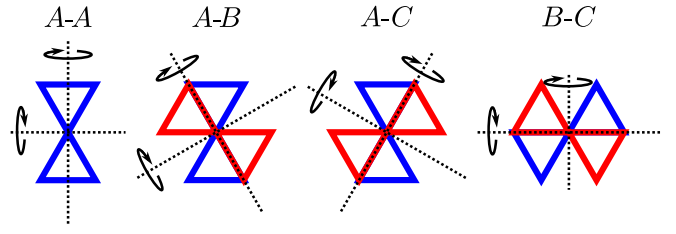


FIG. 3. Examples of C_{2z} overlaps for the aligned bilayers. The dashed lines show the dihedral symmetry axes for the different stacking variations of the C_{2z} RSP overlaps which are preserved within the twist-symmetric frame.

even, implying that 4 divides $(m + 3v \pm 2)$. By substituting Eqs. (23) into Eqs. (21), we find

$$l = \frac{m - 3v}{4}, \quad q = \frac{m + 3v + 2}{4}. \quad (24)$$

Finally, by considering the overlap function for sites A and A' with the lattice parameters identified in Eqs. (23) and (24), we find a vanishing result, $f(\delta_A^{(\text{II})}, \delta_{A'}^{(\text{II})}; k, l, p, q; \theta_c) = 0$. Therefore, in the final case where m and v are both odd such that their sum is divisible by 4, A and A' sites are guaranteed to overlap and hence the system possesses at least a D_2 point group symmetry.

4. Axis of dihedral symmetry

The direction of the dihedral symmetry in the D_2 point groups shown above can be determined by which kagome sublattices overlap in the twist-symmetric frame. In fact, the untwisted D_2 bilayers possess two dihedral symmetries that lie orthogonally to one another, and so the same must be true for the twisted case. With three sublattices, we consider four possible combinations of overlap, A - A , A - B , A - C , and B - C , the remainder can be obtained by an appropriate global rotation to transform between the sublattice labels (e.g., a 60° anticlockwise rotation shifts the sublattice labels as $A \rightarrow C \rightarrow B \rightarrow A$). Perhaps the simplest cases are A - A and B - C overlap, with both possessing dihedral symmetries along the x and y axes in the untwisted scenario and hence the same in the twist-symmetric frame; see Fig. 3. Quickly moving through the remainder of the cases involving A , we find that A - B (A - C) overlap yields dihedral symmetry along the -60° (60°) and 30° (-30°) lines defined by the lines $y = -\sqrt{3}x$ ($y = \sqrt{3}x$) and $y = x/\sqrt{3}$ ($y = -x/\sqrt{3}$), respectively. We therefore establish the dihedral symmetries of the D_2 point group to lie along the following lines for the three cases outlined in Eqs. (25) within the twist-symmetric frame:

$$\begin{aligned} \frac{m + v + 2}{4} \in \mathbb{Z} : y = \sqrt{3}x \quad \text{and} \quad y = -\frac{x}{\sqrt{3}}, \\ \frac{m + v}{4} \in \mathbb{Z} : y = -\sqrt{3}x \quad \text{and} \quad y = \frac{x}{\sqrt{3}}, \\ m + v \text{ is odd} : x \text{ axis and } y \text{ axis.} \end{aligned} \quad (25)$$

IV. AVOIDED RSP OVERLAPS: $\theta_p = 60^\circ$

Having demonstrated that C_{jz} - C_{jz} ($j = 2, 3$) is guaranteed in commensurate twisted triangular, graphene, and Kagome bilayers that started in a high-symmetry stacking configuration, with the rotational symmetry matching that of the untwisted system, we now move on to proving that overlap of RSPs yielding a higher rotational symmetry cannot occur. This section shall draw solely on searching for the minimum of the overlap function as explained in Sec. II D.

A. Bernal TBG and interlocked TBK

Starting with the Bernal/interlocked systems, we find that $\partial_p f(\delta_{\square}^{(I)}, \delta_{\square}^{(I)}; k, l, p, q; \theta_c) = 0$ imposes

$$\mathcal{A} = 2p + q = \frac{(2k + l - 1)m^2 - 6lmv - 3(2k + l + 1)v^2}{m^2 + 3v^2}. \quad (26)$$

Rewriting this, we arrive at an expression for k in terms of $\mathcal{A}$ and l :

$$k = \frac{(1 + \mathcal{A} - l)m^2 + 6lmv + 3(1 + \mathcal{A} + l)v^2}{2(m^2 - 3v^2)}. \quad (27)$$

Let us now assume that $\exists l, \mathcal{A} \in \mathbb{Z}$ such that $k \in \mathbb{Z}$. Substituting Eq. (27) and $q = \mathcal{A} - 2p$ into the overlap function $f(\delta_{\square}^{(I)}, \delta_{\square}^{(I)}; k, l, p, q; \theta_c) = 0$ yields

$$l = \frac{1}{m^2 + 3v^2} \left[m^2(\mathcal{A} - 2p) - 2(1 + \mathcal{A})mv - (1 + 3\mathcal{A} - 6p)v^2 + \frac{m^2}{3} \right]. \quad (28)$$

Since m cannot be divided by 3, we immediately see that l cannot be integer when the overlap function vanishes. Therefore, by contradiction, the C_6 RSPs in the Bernal/interlocked cases never overlap. The point group of Bernal TBG and interlocked TBK is thus D_3 .

B. Saddle point TBG and AB TBK

The case of saddle point TBG and AB TBK requires more steps to establish that the highest symmetry RSP overlap is C_{2z} . In addition to checking the C_{6z} - C_{6z} overlap, we must check that all possible combinations of C_{6z} - C_{3z} and C_{3z} - C_{3z} overlap.

1. C_{6z} - C_{6z} overlap

Starting with the sixfold overlap, we find that $\partial_p f(\delta_{\square}^{(II)}, \delta_{\square}^{(II)}; k, l, p, q; \theta_c) = 0$ yields

$$\mathcal{A}_{\square}^{\square} = \frac{1}{2} - 2k - l - \frac{2m[2mk + (m - 3v)l]}{m^2 + 3v^2}, \quad (29)$$

where we use the superscript and subscript on $\mathcal{A}$ to denote the RSP symmetry of the top/bottom layer in the overlap function, respectively. Let us consider when m and v are of opposite parities. In this case, $m^2 + 3v^2$ will be odd and the final term of Eq. (29) will never yield a result to cancel the factor of $1/2$. Turning our attention to when m and v are both odd, we note that $m^2 + 3v^2$ will now be even at the same time

as $2m$ and $m - 3v$ becoming even. Then, for the final term of Eq. (29) to be a half-integer to counteract the leading factor of $1/2$, $m^2 + 3v^2$ will need to be divisible by 2^γ with $\gamma \in \mathbb{Z}$ and $\gamma \geq 3$. However, this cannot happen when m and v are both odd: letting $m = 2\alpha + 1$ and $v = 2\beta + 1$ ($\alpha, \beta \in \mathbb{Z}$) gives $m^2 + 3v^2 = 4[1 + \alpha(1 + \alpha) + 3\beta(1 + \beta)]$. Consequently, it is not possible for a minimum of the overlap function to exist in p that corresponds to an integer value of $\mathcal{A}$, which in turn means that there are no integer choices for p and q corresponding to a vanishing of the overlap function. Therefore, C_{6z} - C_{6z} overlap does not occur in commensurate twisted triangular lattice with a D_2 stacking.

2. C_{6z} - C_{3z} overlap

There are four cases that lead to C_{6z} - C_{3z} overlap: (Ia) an up triangle of the top layer with a hexagon center of the bottom layer, (IIa) a down triangle of the top layer with a hexagon center of the bottom layer, (IIIa) an up triangle of the bottom layer with a hexagon center of the top layer, and (IVa) a down triangle of the bottom layer with a hexagon center of the top layer. By considering $\partial_p f(\delta_j^{(II)}, \delta_i^{(II)}; k, l, p, q; \theta_c) = 0$ for cases (Ia) and (IIa), we obtain expressions for $\mathcal{A}_{\square}^{\square}$ and $\mathcal{A}_{\square}^{\nabla}$ that are related to $\mathcal{A}_{\square}^{\square}$ by an integer shift:

$$(Ia): (i, j) = (\square, \triangle') \Rightarrow \mathcal{A}_{\square}^{\triangle} = \mathcal{A}_{\square}^{\square} - 1, \quad (30a)$$

$$(IIa): (i, j) = (\square, \nabla') \Rightarrow \mathcal{A}_{\square}^{\nabla} = \mathcal{A}_{\square}^{\square}. \quad (30b)$$

Hence, the same reasoning applies here as in Sec. IV B 1. Next, we consider the layer-reversed situation with the C_{6z} RSP located in the top layer and C_{3z} RSP in the bottom layer, wherein

$$\mathcal{A}_{\triangle}^{\square} = \frac{1}{2} - 2k - l + \frac{2m[2mk + (m - 3v)l - 2v]}{m^2 + 3v^2}, \quad (31a)$$

$$\mathcal{A}_{\nabla}^{\square} = \frac{1}{2} - 2k - \tilde{l} + \frac{2m[2mk + (m - 3v)\tilde{l} + 2v]}{m^2 + 3v^2}, \quad (31b)$$

for cases (IIIa) and (IVa), respectively, where $\tilde{l} = l - 1$. The fourth term of $\mathcal{A}_{\triangle, \nabla}^{\square}$ again fails to compensate for the leading term of $1/2$ by the same reasoning as used for $\mathcal{A}_{\square}^{\square}$. To see this, we note that only when m and v are both odd is the denominator of the fourth term even, so letting $m = 2\alpha + 1$ and $v = 2\beta + 1$ ($\alpha, \beta \in \mathbb{Z}$) yields

$$(1 + 2\alpha) \frac{1 + (1 + 2\alpha)k + (\alpha - 3\beta - 1)l - 2\beta}{1 + \alpha(1 + \alpha) + 3\beta(1 + \beta)} \quad (32)$$

for the final term of Eq. (31a). The denominator of this writing of the final term will always be odd and hence cannot enable the $1/2$ to be canceled in $\mathcal{A}_{\triangle}^{\square}$.

3. C_{3z} - C_{3z} overlap

We again have four cases that lead to C_{3z} - C_{3z} overlap: (Ib) up triangle overlap, (IIb) down triangle overlap, (IIIb) an up triangle of the top layer with a down triangle of the bottom layer, and (IVb) a down triangle of the top layer with an up triangle of the bottom layer. The avoidance of RSP in each of these cases collapses to the same reasoning, so let us consider (Ib) as the main example for C_{3z} - C_{3z} overlap. Once again, in setting $\partial_p f(\delta_j^{(II)}, \delta_i^{(II)}; k, l, p, q; \theta_c) = 0$ for each of these

cases, we obtain expressions for the various $\mathcal{A}_{\Delta,\nabla}^{\Delta,\nabla}$ that are related to $\mathcal{A}_{\Delta,\nabla}^{\square}$ by simple integer shifts:

$$(Ib): (i, j) = (\Delta, \Delta') \Rightarrow \mathcal{A}_{\Delta}^{\Delta} = \mathcal{A}_{\Delta}^{\square} - 1, \quad (33a)$$

$$(IIb): (i, j) = (\nabla, \nabla') \Rightarrow \mathcal{A}_{\nabla}^{\nabla} = \mathcal{A}_{\Delta}^{\square}|_{l \rightarrow \bar{l}}, \quad (33b)$$

$$(IIIb): (i, j) = (\Delta, \nabla') \Rightarrow \mathcal{A}_{\Delta}^{\nabla} = \mathcal{A}_{\Delta}^{\square}, \quad (33c)$$

$$(IVb): (i, j) = (\nabla, \Delta') \Rightarrow \mathcal{A}_{\nabla}^{\Delta} = \mathcal{A}_{\nabla}^{\square}|_{l \rightarrow \bar{l}} - 1. \quad (33d)$$

Hence, all possible C_{3z} - C_{3z} overlaps are avoided, so we may conclude from the above analysis that only C_{2z} - C_{2z} overlap will occur in this system. Hence, saddle point TBG and AB-stacked TBK will possess a point group symmetry of D_2 .

V. RSP OVERLAPS: $\theta_p = 120^\circ$

Rather than twisting one layer about its isolated HSP with C_{6z} symmetry, we may instead choose to rotate one layer about the bilayer HSP. In the case of Bernal-stacked graphene and interlocked kagome, this corresponds to the AB site and up/down triangle overlap, respectively, which exhibit D_{3d} symmetry. Applying the twist around this center, we immediately guarantee the existence of a lattice of C_{3z} - C_{3z} overlap points given that the twist origin may be labeled as a moiré lattice site. Hence, we need only check for C_{6z} - C_{6z} overlap to determine whether the point group symmetry is D_3 or D_6 . We note that, unlike the $\theta_p = 60^\circ$ scenario, we must consider both when n is divisible by 3 and when n is not divisible by 3.

A. n divisible by 3

Starting with $\partial_p f(\tilde{\delta}_{\square'}, \tilde{\delta}_{\square}; k, l, p, q; \theta_c) = 0$, we acquire

$$\mathcal{A} = \frac{(2k+l)m^2 + 2(2-3l)mv - 3(2k+l)v^2}{m^2 + 3v^2}, \quad (34)$$

which is not necessarily noninteger for $m, n, k, l \in \mathbb{Z}$. Let us therefore rearrange for k and assume $\exists l, \mathcal{A} \in \mathbb{Z}$ such that $k \in \mathbb{Z}$:

$$k = \frac{(\mathcal{A}-l)m^2 + 2(3l-2)mv + 3(\mathcal{A}+l)v^2}{2(m^2 - 3v^2)}. \quad (35)$$

Substituting Eq. (35) into the overlap function with $q = \mathcal{A} - 2p$ and setting the overlap function to zero yields

$$l = -\mathcal{A} + 2p + \frac{2m(m(2+3\mathcal{A}-6p) - 3\mathcal{A}v)}{3(m^2 + 3v^2)}. \quad (36)$$

The final term of this expression cannot be integer as the numerator cannot be divided by 3. Therefore, by contradiction, $\nexists l \in \mathbb{Z}$ such that $k \in \mathbb{Z}$ and $f(\tilde{\delta}_{\square'}, \tilde{\delta}_{\square}; k, l, p, q; \theta_c) = 0$. Hence, when $\theta_p = 120^\circ$ and θ_c is associated to a choice of n that is divisible by 3 Bernal TBG and interlocked TBK will possess a D_3 point group symmetry.

B. n not divisible by 3

As in Sec. III B, we start by moving to the twist-symmetric frame and search for where the C_{6z} RSPs of each layer lie on

the x axis. The RSP vectors in this frame are given by

$$\tilde{\mathbf{v}}_{\square} = R_{\frac{\theta_c}{2}}(\tilde{\delta}_{\square} + k\mathbf{a}_1 + l\mathbf{a}_2), \quad (37a)$$

$$\tilde{\mathbf{v}}_{\square'} = R_{\frac{\theta_c}{2}}(\tilde{\delta}_{\square'} + p\mathbf{a}_1 + q\mathbf{a}_2). \quad (37b)$$

Setting the y components of these vectors to zero yields

$$2nk + (3m+n)l = 2m, \quad (38a)$$

$$2np - (3m-n)q = 2m. \quad (38b)$$

We now note that

$$\text{GCD}(2n, (3m \pm n)) = \begin{cases} 1, & \mathcal{P}_m \neq \mathcal{P}_n \\ 2, & \mathcal{P}_m = \mathcal{P}_n = -1 \\ 3, & n/3 \in \mathbb{Z}, \end{cases} \quad (39)$$

which will always divide $2m$ for the first two cases but never in the third case, so $k, l, p, q \in \mathbb{Z}$ satisfying these diophantine equations are ensured by Bézout's identity when n is not divisible by 3. Given that C_{6z} RSPs from each layer will lie on the x axis, let us now discern if they are able to overlap. Solving Eqs. (38) for l and q , we obtain

$$l = \frac{2(m-nk)}{3m+n}, \quad q = -\frac{2(m-np)}{3m-n}, \quad (40)$$

which we substitute into $\tilde{\mathbf{v}}_{\square} - \tilde{\mathbf{v}}_{\square'}$. Given the y component of the resulting vector is always zero, we search for where its x component vanishes to obtain the following relation between k and p :

$$k = \frac{(3m+n)p - 2m}{3m-n} = p + q. \quad (41)$$

Hence, if $p \in \mathbb{Z}$ such that $q \in \mathbb{Z}$, then $k \in \mathbb{Z}$. Now, let us check if this integer value of k will also give $l \in \mathbb{Z}$. Substituting Eq. (41) into the expression for l in Eqs. (40), we find that $l = -q$, meaning that $l \in \mathbb{Z}$ when $p \in \mathbb{Z}$ such that $q \in \mathbb{Z}$. Therefore, C_{6z} - C_{6z} overlap will always appear in Bernal TBG and interlocked TBK when twisted about their HSP by a commensurate angle attributed to n not being divisible by 3.

C. Connection to $\theta_p = 60^\circ$

When n is divisible by 3 and $\theta_p = 120^\circ$, the rotational misalignment of the layers at a C_{3z} - C_{3z} overlap site will be identical to the twisted bilayers with n divisible by 3 and $\theta_p = 60^\circ$. Hence, the moiré pattern and point group for these two cases will be equivalent. In contrast, when n is not divisible by 3 and $\theta_p = 120^\circ$, the misalignment of overlapping hexagons cannot match that of the AA system with $\theta_p = 60^\circ$ due to the unique labeling of each commensurate angle with (m, n) . However, given that commensurate angles are paired such that $\theta'_c = \pi/3 - \theta_c$ is associated to n not being divisible by 3 when θ_c is given by a choice of n that is divisible by 3 [52], we can obtain the moiré pattern of one from the other provided the twist is applied about a hexagon center. Specifically (see Appendix A for details), by reflecting the AA system in the x axis and applying a global $\pi/3$ rotation, we arrive at the superlattice for a θ'_c . After this transformation, the hexagon overlap for the $\theta_p = 60^\circ$ AA-stacked system will be identical to that of the Bernal/interlocked system twisted about its C_{3z} HSP.

VI. GENERALIZATION TO COMMENSURATE MULTILAYERS

Let us now discuss the emergent symmetry of commensurate twisted multilayers when the twist is applied about a hexagon center of the first layer. There are three cases in which we can think to construct a multilayered system composed of commensurate layers: (i) each layer is twisted at an angle of θ_c relative to the layer beneath it to form a helical stack [54,55], (ii) each layer alternates between $\pm\theta_c$ relative to the previous layer [56–58], and (iii) each layer has its own independent twist relative to the x axis, $\theta_{c,i}$ [59]. The simplest situation is case (ii) for column stacking (i.e., AAA...) or when the stacking alternates between displacements (e.g., ABA graphene), where we note that all odd layers will be aligned with the first and all even layers will be aligned with the second. Immediately, we may say that dihedral symmetry will be preserved in all systems with an even number of layers. In contrast, any system formed with an odd number of layers will not possess dihedral symmetry: in the twist-symmetric frame, dihedral rotation will map layers with $+\theta_c/2$ rotation onto layers with $-\theta_c/2$ rotation, hence having an odd number of layers means that the number of layers with $+\theta_c/2$ differs from those with $-\theta_c/2$. Turning our attention toward case (i) with helical stacking, we choose to work in the twist-symmetric frame. For layered systems which possess a dihedral symmetry point with the same 2D position that aligns with the C_{6z} twist center of the first layer (e.g., ABA and ABCA graphene), which we may choose to be along the x axis, the helical twist will always preserve the dihedral symmetry; see Appendix B. For the cases where the dihedral symmetry is located elsewhere in the unit cell, a more in-depth analysis akin to that above is required to determine whether or not the dihedral symmetry is preserved.

Lastly, given how general scenario (iii) is and the plethora of variations that can be taken in constructing it, let us outline how to approach the question of RSP overlap in such a system. This method can also be applied to check for RSP overlap in case (i) when there is no dihedral symmetry at the twist center. Let us start by considering a trilayer system where θ_1 is the commensurate angle between layers 1 and 2 and θ_2 is the commensurate angle between layers 2 and 3. Layers 1 and 2 will form a moiré lattice of HSPs defined $\delta_0 + k\mathbf{a}_1^M + l\mathbf{a}_2^M$, where δ_0 is the location of an HSP relative to the twist origin and $\mathbf{a}_1^M$ and $\mathbf{a}_2^M$ are the moiré lattice vectors in the frame where only layer 2 is rotated. We then need to check the overlap of the various RSPs of the third layer with the bilayer HSP lattice. The overlap function for this would take the form

$$\tilde{f}(\delta_j, \delta_0; k, p, q, l; \theta_1, \theta_2) = |\delta_j^{\theta_1+\theta_2} + p\mathbf{a}_1^{\theta_1+\theta_2} + q\mathbf{a}_2^{\theta_1+\theta_2} - \delta_0 - k\mathbf{a}_1^M - l\mathbf{a}_2^M|^2. \quad (42)$$

We can then attempt the same analysis as before to determine whether or not integer solutions for k , l , p , and q exist to give a vanishing overlap function. If solutions exist, then we can move on to consider four layers, but now with a new set of moiré lattice vectors for the trilayer HSP lattice. This approach can also be used for cases (i) and (ii) if the layered system is not a simple repetition of stacking orders, such as ABB or rhombohedral graphene.

VII. SYMMETRY IN THE TWIST-CENTRIC WIGNER-SEITZ UNIT CELL

Above we have shown that the rotational and dihedral symmetries are preserved through a long algebraic expedition that lends itself to a procedural generalization to multiple layers. However, we may also focus entirely upon the WS unit cell constructed with the C_{6z} twist origin at its center and seek to find the HSP within its boundaries. To start this analysis, we will need to know the moiré lattice vectors underpinning the commensurate bilayer. By working in the twist-symmetric frame, the moiré unit cell (MUC) will be one of two orientations courtesy of the $y \rightarrow -y$ symmetry between the 2D positions of lattice sites in differing layers when we consider AA stacking. While other stacking choices do not have this symmetry about the twist center, they will possess the same moiré lattice vectors. We name the two possible orientations of the MUC using graphene nomenclature: zigzag, when we may define a moiré lattice vector parallel to the x axis; armchair, when we may define a moiré lattice vector parallel to the y axis.

Let us denote the position of triangular lattice sites in layer i by $\mathbf{r}_i(k, l) = k\mathbf{a}_1 + l\mathbf{a}_2$. We then search for solutions to $\mathbf{r}_1^{\frac{\theta_c}{2}}(k, l) = \mathbf{r}_2^{\frac{\theta_c}{2}}(p, q)$ that yield $k, l, p, q \in \mathbb{Z}$ such that the y components of $\mathbf{r}_1^{\frac{\theta_c}{2}}(k, l)$ and $\mathbf{r}_2^{\frac{\theta_c}{2}}(p, q)$ vanish (i.e., we search for the zigzag orientation). Under the assumption that $n = 3\nu$, we find these conditions are met when

$$\begin{aligned} k &= \alpha_{mv}(m + \nu), & l &= -2\alpha_{mv}\nu, \\ p &= \alpha_{mv}(m - \nu), & q &= 2\alpha_{mv}\nu, \end{aligned} \quad (43)$$

with

$$\alpha_{mv} = \begin{cases} 1, & m + \nu \text{ is odd} \\ \frac{1}{2}, & m + \nu \text{ is even,} \end{cases} \quad (44)$$

which produce a moiré lattice constant of $a_M = \alpha_{mv}\sqrt{m^2 + 3\nu^2}$. If we instead demanded the x components $\mathbf{r}_1^{\frac{\theta_c}{2}}(k, l)$ and $\mathbf{r}_2^{\frac{\theta_c}{2}}(p, q)$ to vanish, we would obtain a larger moiré lattice constant of $\tilde{a}_M = \alpha_{mv}\sqrt{3}\sqrt{m^2 + 3\nu^2}$. Therefore, when n is divisible by 3, the system will always be in the zigzag orientation. Alternatively, if we assumed n to not be divisible by 3, we would find that the armchair orientation would always be obtained with

$$\begin{aligned} k &= \alpha_{mn}(m - n), & l &= -2\alpha_{mn}m, \\ p &= \alpha_{mn}(m + n), & q &= -2\alpha_{mn}m, \end{aligned} \quad (45)$$

corresponding to $a_M = \alpha_{mn}\sqrt{m^2 + n^2}$. Therefore, given the connection between the moiré patterns with n divisible by 3 and those with n not divisible by 3, we can choose to focus solely on the zigzag orientation with $n = 3\nu$ and moiré lattice vectors

$$\begin{aligned} \mathbf{a}_1^M &= \alpha_{mv}\sqrt{m^2 + 3\nu^2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ \mathbf{a}_2^M &= \frac{\alpha_{mv}}{2}\sqrt{m^2 + 3\nu^2} \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \end{aligned} \quad (46)$$

when working in the twist-symmetric frame.

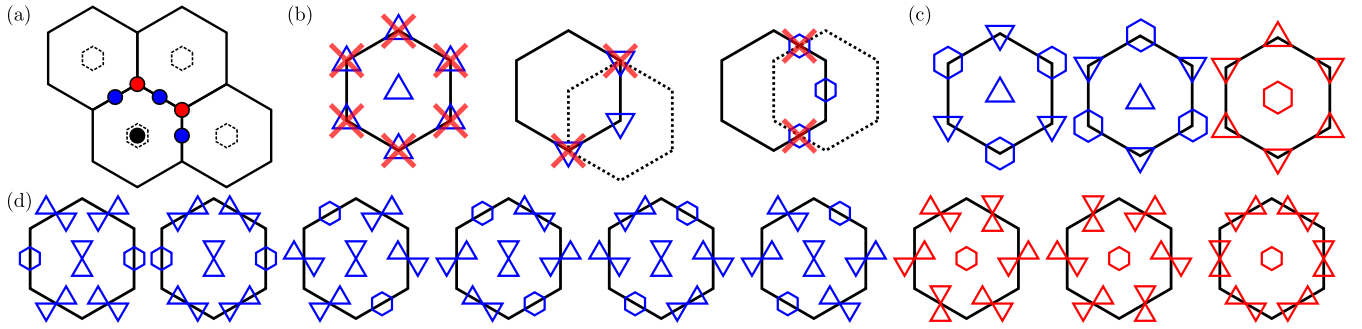


FIG. 4. (a) Generic set of WS unit cells for twisted triangular lattices. The black, red, and blue dots indicate potential C_{6z} , C_{3z} , and C_{2z} RSPs, respectively. The dashed hexagon at each WS unit cell acts as a placeholder for lattice shapes appearing at the twist origin for the various stacking orders. (b) Disallowed combinations of RSP configurations due to neighboring corners/edges not being able to host the same RSP, nor can they host the same RSP as the WS unit cell center. The large dotted hexagon highlights an alternative choice of WS unit cell that places an edge or corner of the original twist-centric WS unit cell at its center. (c) Allowed orientation of RSPs from the first (red) and second (blue) layers for interlocked stacking. Hexagons indicate a C_{6z} center and triangles the two types of C_{3z} centers. (d) Allowed orientations of RSPs from the first (red) and second (blue) layers for AB kagome stacking. The bow ties denote the three types of C_{2z} RSPs.

Equipped with the moiré lattice vectors, we may now consider the geometry of the WS unit cell centered on the twist origin. Figure 4(a) shows a generic set of WS unit cells with the potential RSPs highlighted by the colored dots. It is these positions within the WS unit cell that we need to consider in our search for RSP overlap. Let us now consider which RSPs of the constituent monolayers may appear at the potential WS RSPs. Using the language of the kagome lattice, the bottom layer will always have a hexagon at the center of the WS unit cell, while the top layer will either have a hexagon, triangle, or bow tie (C_{2z} symmetry) at this point. We then ask which of these shapes can appear at the WS corners and edges.

If the WS center belongs to the triangular lattice of a layer, then we find that the corners and edges of the WS cell cannot also belong to the triangular lattice of the same layer. To see this, we simply need to determine whether there exist choices for $p, q \in \mathbb{Z}$ that satisfy

$$p\mathbf{a}_1^{\frac{\theta_c}{2}} + q\mathbf{a}_2^{\frac{\theta_c}{2}} = \begin{cases} \mathbf{a}_1^M + \mathbf{a}_2^M, & \text{corner 1} \\ 2\mathbf{a}_1^M - \mathbf{a}_2^M, & \text{corner 2} \end{cases} \quad (47)$$

for the WS corners and

$$p\mathbf{a}_1^{\frac{\theta_c}{2}} + q\mathbf{a}_2^{\frac{\theta_c}{2}} = \begin{cases} \frac{1}{2}\mathbf{a}_1^M, & \text{edge 1} \\ \frac{1}{2}\mathbf{a}_2^M, & \text{edge 2} \\ \frac{1}{2}(\mathbf{a}_1^M - \mathbf{a}_2^M), & \text{edge 3} \end{cases} \quad (48)$$

for the WS edges. Solving Eq. (47) yields

$$\frac{1}{\alpha_{mv}}(p, q) = \begin{cases} \left(\frac{m-3v}{3}, \frac{m+3v}{3}\right), & \text{corner 1} \\ \left(\frac{2m}{3}, -\frac{m-3v}{3}\right), & \text{corner 2}, \end{cases} \quad (49)$$

which cannot be integer since m is not divisible by 3. Likewise, rearranging Eq. (48) produces

$$\frac{1}{\alpha_{mv}}(p, q) = \begin{cases} \left(\frac{m-v}{2}, v\right), & \text{edge 1} \\ \left(-v, -\frac{m+v}{2}\right), & \text{edge 2} \\ \left(-\frac{m+v}{2}, \frac{m-v}{2}\right), & \text{edge 3}, \end{cases} \quad (50)$$

which cannot give p and q to be integer simultaneously. A similar analysis can be done for the top layer by replacing $p\mathbf{a}_1^{\frac{\theta_c}{2}} + q\mathbf{a}_2^{\frac{\theta_c}{2}}$ in Eqs. (47) and (48) with $k\mathbf{a}_1^{\frac{\theta_c}{2}} + l\mathbf{a}_2^{\frac{\theta_c}{2}}$ to

arrive at a similar conclusion. Therefore, the symmetry shape (hexagon, triangles, bow ties) that appears at the WS origin will not also appear at the WS corners and edges within either layer.

This result allows us to also state that no two neighboring corners or edges may possess the same symmetry shape. This is because we can identify an alternative WS unit cell centered on a corner (edge) of the original twist-centric WS unit cell, where we then see that the corners (edges) of the alternative WS unit cell overlap with a corner (edge) of the original WS unit cell. Figure 4(b) illustrates examples of disallowed symmetry combinations between the WS origin, corners, and edges, with the large dashed hexagon denoting a redefined WS unit cell.

If we consider interlocked stacking, we have a hexagon in the bottom layer and triangle in the top layer at the center of the twistcentric WS unit cell. The only geometry and symmetry-allowed combination is shown in Fig. 4(c), which guarantees that one corner of the WS cell will have a C_{3z} - C_{3z} overlap. Given that there is no point within the WS cell that may exhibit C_{6z} symmetry, we can conclude that AB TBG and interlocked TBK must have a D_3 point group symmetry. Similarly, if we considered AB TBK or saddle point TBG, symmetry and geometry ensure that the configurations shown in Fig. 4(d) are the only possible scenarios. The corners cannot possess C_{3z} symmetry since the WS unit cell has a C_{2z} center, so we need only focus on the WS edges in this case. From Fig. 4(d), we see that C_{2z} - C_{2z} overlap is guaranteed and hence AB TBK and saddle point TBG must possess D_2 symmetry.

VIII. CONCLUSIONS

In this paper, we considered the overlap of points with nontrivial rotational symmetry between in twisted bilayers of triangular lattices, graphene, and kagome. If these rotational symmetry points do not overlap, the resulting moiré superlattice will not possess a nontrivial rotational symmetry. By constructing the untwisted bilayers in a high-symmetry stacking, the system will always possess a nontrivial rotational symmetry when one layer is rotated to a commensurate twist angle

about the bilayer HSP or its own C_{6z} HSP. When the twist is applied around a C_{6z} center for a high-symmetry stacking, the twist center will possess a C_{3z} or C_{2z} point group for the D_3 (Bernal bilayer graphene and interlocked kagome) and D_2 (AB kagome) bilayers. However, we have demonstrated that a higher symmetry point is guaranteed to exist at commensurate twist angles.

In particular, we have shown that RSP overlaps with C_{jz} matching the rotational symmetry of the untwisted bilayer can always be identified in the twisted superlattice. Consequently, in the twist-symmetric frame, the axes of dihedral symmetry of the untwisted bilayer will be preserved. However, the mirror planes of the individual layers will not be aligned, preventing the twisted bilayer from hosting a higher point group symmetry. We therefore conclude that the point group symmetry for twisted triangular lattices, including TBG and TBK, will be D_j with j set by the rotational symmetry of the untwisted bilayer. We further recovered the results of Mele [4] when the twist is applied about the HSP of the interlocked kagome and AB graphene bilayers using the the overlap function defined in Sec. II D. The moiré patterns obtained from this alternative twist center can be mapped onto those generated by the C_{6z} twist center.

Finally, we provided an alternative route to discern the symmetry of twisted triangular bilayers by constructing a real space Wigner-Seitz cell centered on the twist origin. In doing so, we acquired compact expressions for the resulting moiré lattice vectors and showed that the WS cell corners and edges could not belong to the same triangular lattice as the unit cell origin or the same lattice as one another. This restricted the various possible combinations of the RSP overlap at the WS corners and edges to reach the same conclusions obtained by our analysis of RSP separation. The results of this paper will help inform future studies on twisted bilayers of triangular lattices, in particular, TBK as systems hosting the monolayer lattice continue to be discovered and studied in the context of band engineering.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: TWO CLASSES OF COMMENSURATE ANGLES

We start by defining the two lattices constituting a bilayer prior to twisting as L_1 and L_2 for the bottom and top layers, respectively. The top layer will be displaced relative to the first by $\mathbf{d}$ to form a high-symmetry stacking and the origin will be located at the center of a hexagon for L_1 . The lattices are thus given by (ignoring their vertical displacement)

$$\begin{aligned} L_1 &= \{\mathbf{R}_i + \boldsymbol{\tau}_\alpha \mid \mathbf{R}_i \in L, \alpha \in S\}, \\ L_2 &= \{\mathbf{R}_i + \boldsymbol{\tau}_\alpha + \mathbf{d} \mid \mathbf{R}_i \in L, \alpha \in S\}, \end{aligned} \quad (\text{A1})$$

where L is the underlying hexagonal lattice independent of sublattice geometry, S is the sublattice geometry with $S = \{A, B\}$ ($S = \{A, B, C\}$) for graphene (kagome), and $\boldsymbol{\tau}_\alpha$ is the position of sublattice α in the monolayer unit cell. We take $\boldsymbol{\tau}_\alpha$ such that L_1 possesses mirror reflection symmetry in the x axis, M_x (i.e., $y \rightarrow -y$); see Fig. 1. For AA stacking, this reflection also leaves L_2 unchanged, $\tilde{L}_2^\square = M_x L_2^\square = L_2^\square$. In contrast, the top layer for AB graphene and interlocked kagome will be changed in a manner that can be reinterpreted as a rotation, $\tilde{L}_2^{\Delta/\nabla} = M_x L_2^{\Delta/\nabla} = R_{\frac{\pi}{3}} L_2^{\Delta/\nabla}$. Here we use the superscript to indicate the RSP of L_2 located at the twist origin. Regarding the D_2 stacking of lattices, the second layer will transform in one of three ways. In the language of kagome, if the AB stacking were created by overlapping the sublattices as BC' and CB' leaving A' at the twist center, then $L_2^{A'} = M_x L_2^{A'}$. However, if the AB stacking is made by placing B' or C' at the twist center, then reflection yields $\tilde{L}_2^{B'} = M_x L_2^{B'} = R_{\frac{\pi}{3}} L_2^{B'}$ and $\tilde{L}_2^{C'} = M_x L_2^{C'} = R_{-\frac{\pi}{3}} L_2^{C'}$, respectively.

Let us now consider rotating L_1 by an arbitrary angle of $\pm\theta$, $L_1^\theta = R_\theta L_1$ and $L_1^{\bar{\theta}} = R_{-\theta} L_1$. The two twisted bilayers of concern are now $\mathcal{L}^\theta = L_1^\theta \cup L_2$ and $\mathcal{L}^{\bar{\theta}} = L_1^{\bar{\theta}} \cup L_2$. Courtesy of $M_x L_1 = L_1$, we have $M_x L_1^\theta = L_1^\theta$, and so $M_x \mathcal{L}^\theta = L_1^\theta \cup \tilde{L}_2$. If the system has AA stacking, $\tilde{L}_2 = L_2$, and so $\mathcal{L}_\square^\theta = M_x \mathcal{L}_\square^\theta$, where we use the subscript to indicate the RSP of L_2 located at the twist center. If instead the system has interlocked stacking, we apply a global $\pi/3$ rotation to the system to reacquire $L_2^{\Delta/\nabla}$. Moreover, because $R_{\pm\frac{\pi}{3}} L_1^\theta = L_1^\theta$, we see that $R_{\frac{\pi}{3}} M_x \mathcal{L}_{\Delta/\nabla}^\theta = \mathcal{L}_{\Delta/\nabla}^\theta$. Alternatively, if the system is stacked according to AB kagome, then we need no rotation ($L_2^{A'}$), a global $\pi/3$ rotation ($L_2^{B'}$), or a global $-\pi/3$ rotation ($L_2^{C'}$), all of which leave L_1 unchanged. Hence, $M_x \mathcal{L}_{A'}^\theta = \mathcal{L}_{A'}^\theta$, $R_{\frac{\pi}{3}} M_x \mathcal{L}_{B'}^\theta = \mathcal{L}_{B'}^\theta$, and $R_{-\frac{\pi}{3}} M_x \mathcal{L}_{C'}^\theta = \mathcal{L}_{C'}^\theta$. Finally, having applied the twist about the C_{6z} symmetry point of the bottom layer, a twist of $-\theta$ is equivalent to $\pi/3 - \theta$. Therefore, the set of θ_c with 3 not dividing n will have the same point group symmetry as those with a choice of n that is divisible by 3 when $\theta_p = 60^\circ$.

APPENDIX B: DIHEDRAL SYMMETRY IN THE TWIST-SYMMETRIC FRAME

If a system formed of two layers possesses dihedral symmetry along some line, then rotating the constituent layers symmetrically, θ for one layer and $-\theta$ for the other, about their HSP will always preserve their dihedral symmetry axes. To see this, let us consider a system with dihedral symmetry along the x -axis. This implies that the 2D positions of lattice sites within each layer must be related by M_x , $M_x L_1 = L_2$,

$$L_2 = \{M_x \mathbf{R}_i + M_x \boldsymbol{\tau}_\alpha \mid \mathbf{R}_i \in L, \alpha \in S\}, \quad (\text{B1})$$

where we again neglect the vertical displacement of the layers. If we then apply a rotation of θ to L_1 and $-\theta$ to L_2 , we acquire

$$\begin{aligned} L_1^\theta &= \{\mathbf{R}_i^\theta + \boldsymbol{\tau}_\alpha^\theta \mid \mathbf{R}_i \in L, \alpha \in S\}, \\ L_2^{\bar{\theta}} &= \{R_{\bar{\theta}} M_x \mathbf{R}_i + R_{\bar{\theta}} M_x \boldsymbol{\tau}_\alpha \mid \mathbf{R}_i \in L, \alpha \in S\}. \end{aligned} \quad (\text{B2})$$

However, we note that $R_\theta M_x = M_x R_\theta$, and hence $M_x L_1^\theta = L_2^\theta$. Therefore, if a dihedral symmetry exists in a bilayer, applying

a symmetric twist about its HSP will preserve the axis of dihedral symmetry.

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- [1] Y. Cao, V. Fatemi, S. Fang, K. Watanabe, T. Taniguchi, E. Kaxiras, and P. Jarillo-Herrero, Unconventional superconductivity in magic-angle graphene superlattices, *Nature (London)* **556**, 43 (2018).
 - [2] J. M. B. Lopes dos Santos, N. M. R. Peres, and A. H. Castro Neto, Graphene bilayer with a twist: Electronic structure, *Phys. Rev. Lett.* **99**, 256802 (2007).
 - [3] S. Shallcross, S. Sharma, and O. A. Pankratov, Quantum interference at the twist boundary in graphene, *Phys. Rev. Lett.* **101**, 056803 (2008).
 - [4] E. J. Mele, Commensuration and interlayer coherence in twisted bilayer graphene, *Phys. Rev. B* **81**, 161405(R) (2010).
 - [5] R. Bistritzer and A. H. MacDonald, Transport between twisted graphene layers, *Phys. Rev. B* **81**, 245412 (2010).
 - [6] J. M. B. Lopes dos Santos, N. M. R. Peres, and A. H. Castro Neto, Continuum model of the twisted graphene bilayer, *Phys. Rev. B* **86**, 155449 (2012).
 - [7] P. Moon and M. Koshino, Energy spectrum and quantum Hall effect in twisted bilayer graphene, *Phys. Rev. B* **85**, 195458 (2012).
 - [8] P. Moon and M. Koshino, Optical absorption in twisted bilayer graphene, *Phys. Rev. B* **87**, 205404 (2013).
 - [9] P. Moon, Y.-W. Son, and M. Koshino, Optical absorption of twisted bilayer graphene with interlayer potential asymmetry, *Phys. Rev. B* **90**, 155427 (2014).
 - [10] F. Finocchiaro, F. Guinea, and P. San-Jose, Quantum spin Hall effect in twisted bilayer graphene, *2D Mater.* **4**, 025027 (2017).
 - [11] R. Bistritzer and A. H. MacDonald, Moiré bands in twisted double-layer graphene, *Proc. Natl. Acad. Sci. USA* **108**, 12233 (2011).
 - [12] J. Kang and O. Vafek, Strong coupling phases of partially filled twisted bilayer graphene narrow bands, *Phys. Rev. Lett.* **122**, 246401 (2019).
 - [13] M. Koshino, Band structure and topological properties of twisted double bilayer graphene, *Phys. Rev. B* **99**, 235406 (2019).
 - [14] X. Lu, P. Stepanov, W. Yang, M. Xie, M. A. Aamir, I. Das, C. Urgell, K. Watanabe, T. Taniguchi, G. Zhang, A. Bachtold, A. H. MacDonald, and D. K. Efetov, Superconductors, orbital magnets and correlated states in magic-angle bilayer graphene, *Nature (London)* **574**, 653 (2019).
 - [15] B. Lian, Z.-D. Song, N. Regnault, D. K. Efetov, A. Yazdani, and B. A. Bernevig, Twisted bilayer graphene. IV. Exact insulator ground states and phase diagram, *Phys. Rev. B* **103**, 205414 (2021).
 - [16] Z.-D. Song and B. A. Bernevig, Magic-angle twisted bilayer graphene as a topological heavy fermion problem, *Phys. Rev. Lett.* **129**, 047601 (2022).
 - [17] Y. Xie, A. T. Pierce, J. M. Park, D. E. Parker, E. Khalaf, P. Ledwith, Y. Cao, S. H. Lee, S. Chen, P. R. Forrester, K. Watanabe, T. Taniguchi, A. Vishwanath, P. Jarillo-Herrero, and A. Yacoby, Fractional Chern insulators in magic-angle twisted bilayer graphene, *Nature (London)* **600**, 439 (2021).
 - [18] F. Xie, J. Kang, B. A. Bernevig, O. Vafek, and N. Regnault, Phase diagram of twisted bilayer graphene at filling factor $\nu = \pm 3$, *Phys. Rev. B* **107**, 075156 (2023).
 - [19] A. Shtyk, G. Goldstein, and C. Chamon, Electrons at the monkey saddle: A multicritical Lifshitz point, *Phys. Rev. B* **95**, 035137 (2017).
 - [20] D. V. Efremov, A. Shtyk, A. W. Rost, C. Chamon, A. P. Mackenzie, and J. J. Betouras, Multicritical Fermi surface topological transitions, *Phys. Rev. Lett.* **123**, 207202 (2019).
 - [21] A. Chandrasekaran, A. Shtyk, J. J. Betouras, and C. Chamon, Catastrophe theory classification of Fermi surface topological transitions in two dimensions, *Phys. Rev. Res.* **2**, 013355 (2020).
 - [22] A. Chandrasekaran and J. J. Betouras, A practical method to detect, analyze, and engineer higher order Van Hove singularities in multi-band Hamiltonians, *Adv. Phys. Res.* **2**, 2200061 (2023).
 - [23] L. Classen and J. J. Betouras, High-order Van Hove singularities and their connection to flat bands, *Annu. Rev. Condens. Matter Phys.* **16**, 229 (2025).
 - [24] H. Isobe and L. Fu, Supermetal, *Phys. Rev. Res.* **1**, 033206 (2019).
 - [25] P. Castro, D. Shaffer, Y.-M. Wu, and L. H. Santos, Emergence of the Chern supermetal and pair-density wave through higher-order Van Hove singularities in the Haldane-Hubbard model, *Phys. Rev. Lett.* **131**, 026601 (2023).
 - [26] Y. Sherkunov and J. J. Betouras, Electronic phases in twisted bilayer graphene at magic angles as a result of Van Hove singularities and interactions, *Phys. Rev. B* **98**, 205151 (2018).
 - [27] N. F. Q. Yuan, H. Isobe, and L. Fu, Magic of high-order Van Hove singularity, *Nat. Commun.* **10**, 5769 (2019).
 - [28] A. Tamai, M. P. Allan, J. F. Mercure, W. Meevasana, R. Dunkel, D. H. Lu, R. S. Perry, A. P. Mackenzie, D. J. Singh, Z.-X. Shen, and F. Baumberger, Fermi surface and Van Hove singularities in the itinerant metamagnet $\text{Sr}_3\text{Ru}_2\text{O}_7$, *Phys. Rev. Lett.* **101**, 026407 (2008).
 - [29] E. Abarca Morales, G.-R. Siemann, A. Zivanovic, P. A. E. Murgatroyd, I. Marković, B. Edwards, C. A. Hooley, D. A. Sokolov, N. Kikugawa, C. Cacho, M. D. Watson, T. K. Kim, C. W. Hicks, A. P. Mackenzie, and P. D. C. King, Hierarchy of Lifshitz transitions in the surface electronic structure of Sr_2RuO_4 under uniaxial compression, *Phys. Rev. Lett.* **130**, 096401 (2023).
 - [30] P. Rosenzweig, H. Karakachian, D. Marchenko, K. Küster, and U. Starke, Overdoping graphene beyond the Van Hove singularity, *Phys. Rev. Lett.* **125**, 176403 (2020).
 - [31] A. M. Seiler, N. Jacobsen, M. Statz, N. Fernandez, F. Falorsi, K. Watanabe, T. Taniguchi, Z. Dong, L. S. Levitov, and R. T. Weitz, Probing the tunable multi-cone band structure in Bernal bilayer graphene, *Nat. Commun.* **15**, 3133 (2024).

- [32] Y. Li and M. Koshino, Twist-angle dependence of the proximity spin-orbit coupling in graphene on transition-metal dichalcogenides, *Phys. Rev. B* **99**, 075438 (2019).
- [33] C. G. Péterfalvi, A. David, P. Rakyta, G. Burkard, and A. Kormányos, Quantum interference tuning of spin-orbit coupling in twisted van der Waals trilayers, *Phys. Rev. Res.* **4**, L022049 (2022).
- [34] A. Veneri, D. T. S. Perkins, C. G. Péterfalvi, and A. Ferreira, Twist angle controlled collinear Edelstein effect in van der Waals heterostructures, *Phys. Rev. B* **106**, L081406 (2022).
- [35] S. Lee, D. J. P. de Sousa, Y.-K. Kwon, F. de Juan, Z. Chi, F. Casanova, and T. Low, Charge-to-spin conversion in twisted graphene/WS₂ heterostructures, *Phys. Rev. B* **106**, 165420 (2022).
- [36] L. Sun, L. Rademaker, D. Mauro, A. Scarfato, Á. Pásztor, I. Gutiérrez-Lezama, Z. Wang, J. Martínez-Castro, A. F. Morpurgo, and C. Renner, Determining spin-orbit coupling in graphene by quasiparticle interference imaging, *Nat. Commun.* **14**, 3771 (2023).
- [37] D. T. S. Perkins, A. Veneri, and A. Ferreira, Spin Hall effect: Symmetry breaking, twisting, and giant disorder renormalization, *Phys. Rev. B* **109**, L241404 (2024).
- [38] M. Milivojević, M. Gmitra, M. Kurpas, I. Štich, and J. Fabian, Giant asymmetric proximity-induced spin-orbit coupling in twisted graphene/SnTe heterostructure, *2D Mater.* **11**, 035036 (2024).
- [39] A. J. Kollár, M. Fitzpatrick, P. Sarnak, and A. A. Houck, Line-graph lattices: Euclidean and non-Euclidean flat bands, and implementations in circuit quantum electrodynamics, *Commun. Math. Phys.* **376**, 1909 (2020).
- [40] D.-S. Ma, Y. Xu, C. S. Chiu, N. Regnault, A. A. Houck, Z. Song, and B. A. Bernevig, Spin-orbit-induced topological flat bands in line and split graphs of bipartite lattices, *Phys. Rev. Lett.* **125**, 266403 (2020).
- [41] F. Crasto de Lima, R. H. Miwa, and E. Suárez Morell, Double flat bands in kagome twisted bilayers, *Phys. Rev. B* **100**, 155421 (2019).
- [42] D. T. S. Perkins, A. Chandrasekaran, and J. J. Betouras, Designing topological high-order Van Hove singularities: Twisted bilayer Kagomé [arXiv:2504.08707](https://arxiv.org/abs/2504.08707).
- [43] Y. Sheffer, R. Queiroz, and A. Stern, Symmetries as the guiding principle for flattening bands of Dirac fermions, *Phys. Rev. X* **13**, 021012 (2023).
- [44] C. Fang, M. J. Gilbert, and B. A. Bernevig, Bulk topological invariants in noninteracting point group symmetric insulators, *Phys. Rev. B* **86**, 115112 (2012).
- [45] C.-K. Chiu, J. C. Y. Teo, A. P. Schnyder, and S. Ryu, Classification of topological quantum matter with symmetries, *Rev. Mod. Phys.* **88**, 035005 (2016).
- [46] E. McCann and M. Koshino, The electronic properties of bilayer graphene, *Rep. Prog. Phys.* **76**, 056503 (2013).
- [47] L. Ye, M. Kang, J. Liu, F. von Cube, C. R. Wicker, T. Suzuki, C. Jozwiak, A. Bostwick, E. Rotenberg, D. C. Bell, L. Fu, R. Comin, and J. G. Checkelsky, Massive Dirac fermions in a ferromagnetic kagome metal, *Nature (London)* **555**, 638 (2018).
- [48] J.-X. Yin, W. Ma, T. A. Cochran, X. Xu, S. S. Zhang, H.-J. Tien, N. Shumiya, G. Cheng, K. Jiang, B. Lian, Z. Song, G. Chang, I. Belopolski, D. Multer, M. Litskevich, Z.-J. Cheng, X. P. Yang, B. Swidler, H. Zhou, H. Lin *et al.*, Quantum-limit Chern topological magnetism in TbMn₆₆, *Nature (London)* **583**, 533 (2020).
- [49] M. Li, Q. Wang, G. Wang, Z. Yuan, W. Song, R. Lou, Z. Liu, Y. Huang, Z. Liu, H. Lei, Z. Yin, and S. Wang, Dirac cone, flat band and saddle point in kagome magnet YMn₆Sn₆, *Nat. Commun.* **12**, 3129 (2021).
- [50] M. Sinha, H. K. Vivanco, C. Wan, M. A. Siegler, V. J. Stewart, E. A. Pogue, L. A. Pressley, T. Berry, Z. Wang, I. Johnson, M. Chen, T. T. Tran, W. A. Phelan, and T. M. McQueen, Twisting of 2D kagomé sheets in layered intermetallics, *ACS Cent. Sci.* **7**, 1381 (2021).
- [51] S. Shallcross, S. Sharma, E. Kandelaki, and O. A. Pankratov, Electronic structure of turbostratic graphene, *Phys. Rev. B* **81**, 165105 (2010).
- [52] M. G. Scheer, K. Gu, and B. Lian, Magic angles in twisted bilayer graphene near commensuration: Towards a hypermagic regime, *Phys. Rev. B* **106**, 115418 (2022).
- [53] W. A. Coppel, *Number Theory: An Introduction to Mathematics Part A* (Springer, New York, 2006); G. A. Jones and J. M. Jones, *Elementary Number Theory* (Springer, London, 1998).
- [54] T. Devakul, P. J. Ledwith, L.-Q. Xia, A. Uri, S. C. de la Barrera, P. Jarillo-Herrero, and L. Fu, Magic-angle helical trilayer graphene, *Sci. Adv.* **9**, eadi6063 (2023).
- [55] M. Long, A. Jimeno-Pozo, H. Sainz-Cruz, P. A. Pantaleón, and F. Guinea, Evolution of superconductivity in twisted graphene multilayers, *Proc. Natl. Acad. Sci. USA* **121**, e2405259121 (2024).
- [56] E. Khalaf, A. J. Kruchkov, G. Tarnopolsky, and A. Vishwanath, Magic angle hierarchy in twisted graphene multilayers, *Phys. Rev. B* **100**, 085109 (2019).
- [57] J. M. Park, Y. Cao, L.-Q. Xia, S. Sun, K. Watanabe, T. Taniguchi, and P. Jarillo-Herrero, Robust superconductivity in magic-angle multilayer graphene family, *Nat. Mater.* **21**, 877 (2022).
- [58] A. Banerjee, Z. Hao, M. Kreidel, P. Ledwith, I. Phinney, J. M. Park, A. Zimmerman, M. E. Wesson, K. Watanabe, T. Taniguchi, R. M. Westervelt, A. Yacoby, P. Jarillo-Herrero, P. A. Volkov, A. Vishwanath, K. C. Fong, and P. Kim, Superfluid stiffness of twisted trilayer graphene superconductors, *Nature (London)* **638**, 93 (2025).
- [59] F. K. Popov and G. Tarnopolsky, Magic angle butterfly in twisted trilayer graphene, *Phys. Rev. Res.* **5**, 043079 (2023).