

Random permutation circuits beyond qubits are quantum chaotic

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Random permutation circuits were recently introduced as minimal models for local many-body dynamics that can be interpreted both as classical and quantum. Standard dynamical complexity indicators such as damage spreading and out-of-time-order correlators (OTOCs), show that these systems exhibit sensitivity to initial conditions in the classical setting and operator scrambling in the quantum setting. Here, we address their quantum chaoticity—a stricter property—by studying the time evolution of local operator entanglement (LOE). We show that the behavior of LOE in random permutation circuits depends on the dimension of the local configuration space q . When $q = 2$, i.e., the circuits act on qubits, random permutations are Clifford and the LOE of any local operator is bounded by a constant, indicating that they are not truly chaotic. On the other hand, when the dimension of the local configuration space exceeds two, the LOE grows linearly in time. We prove this in the limit of large q and present numerical evidence that a three-dimensional local configuration space is sufficient for a linear growth of LOE. Our findings highlight that quantum chaos can be produced by essentially classical dynamics. Moreover, we show that LOE can be defined also in the classical realm and put it forward as a universal indicator chaos, both quantum and classical.

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Introduction. Quantum many-body systems in discrete spacetime, also known as quantum circuits, have emerged in recent years as a fruitful arena to study quantum nonequilibrium dynamics [1–3]. In particular, random unitary circuits (RUC), where the evolution is generated by uncorrelated random unitary matrices coupling nearest neighbors, have proved to be especially instructive [2,4–9]. In essence, this is because (i) RUC have minimal structure (only locality is retained in the most basic setting) and thus can be used to study the universal aspects of quantum information spreading, and (ii) certain key properties of these systems can be derived analytically, employing the so called Weingarten calculus to compute averages over the unitary group [10]. The same basic idea was recently adopted also to study classical many-body dynamics and understand how it differs from its quantum counterpart [11–17]. In this case one considers a subset of random unitary circuits, dubbed random permutation circuits (RPC) [15], that in a special basis act as permutations of the basis elements and, hence, “classically” (a similar class is that of random automaton circuits [11], which, besides permuting the special basis elements, also multiplies them by phases). The key feature of these systems is that they can be thought of both as quantum circuits and as classical cellular automata or Boolean networks [18]. In this way, they allow for a quantitative comparison between quantum and classical dynamics [19,20], avoiding the confusion arising when com-

paring completely different setups (e.g., quantum spin chains and few-particle classical systems with a continuous phase space).

A natural application of these systems is the study of chaos, whose quantum and classical versions are currently treated on very different footings. In essence, the problem is that for generic quantum systems there is no analog to the familiar concept of “sensitivity to initial conditions,” which defines classical chaos. An analog exists in the semiclassical regime (or, more generally, in the presence of a large parameter identifying a small effective Planck constant $\hbar_{\text{eff}} \ll 1$) where it is quantified by out-of-time-ordered correlators (OTOCs) [21–25], but not when $\hbar_{\text{eff}} \sim 1$. In the latter case OTOCs only measure “operator scrambling,” not chaos, meaning that they are unable to distinguish regular and nonregular dynamics [26]. The current belief is that, besides the well established spectral statistics [27,28], another diagnostic of quantum chaos is provided by local operator entanglement (LOE) [29–41]: both these concepts, however, have no obvious classical analog. One can hope that, given their dual “quantum-classical” nature, RPC can help resolving this dissatisfactory situation and establish a unified description of chaos encompassing both classical and quantum systems. So far, however, only their sensitivity to initial conditions in the classical setting (via the so called “damage spreading” [42–44]) and their operator scrambling (via OTOCs) in the quantum setting have been rigorously characterized [15], whereas the current results concerning their quantum chaoticity are not fully conclusive, as they rely on indirect arguments of small-system numerics [11,12,17].

Here we report key developments on this matter. First, we characterize the quantum chaotic properties of RPC.

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Specifically, by performing a rigorous perturbative expansion in the local Hilbert space dimension q , we prove that RPC with nearest-neighbor interactions generate a linearly growing LOE of diagonal operators in the limit of large q . Next, we extend this result to general operators via a mapping to an effective statistical mechanical model, and present numerical evidence suggesting that the linear growth persists for all $q > 2$. On the other hand, we show that for $q = 2$ LOE is bounded as the circuits belong to the Clifford group [45]. This difference is not captured by damage spreading or OTOCs and showcases the difference between these diagnostics. Finally, noting that LOE of diagonal operators is well defined in the classical realm and is more stringent than damage spreading, we propose it as a universal diagnostic tool for chaos applicability not only for quantum but also for classical systems.

Setting. We consider $2L$ qudits with q internal states each arranged along a one-dimensional lattice with the sites labeled by the elements of $\mathbb{Z}_{2L}/2 = \{0, 1/2, \dots, L - 1/2\}$. The time evolution proceeds in discrete steps and is implemented by the unitary operator $\mathbb{U}(\tau) = \mathbb{U}_e(\tau)\mathbb{U}_o(\tau)$, where

$$\begin{aligned}\mathbb{U}_o(\tau) &= \bigotimes_{x=1}^L U\left(x, \tau - \frac{1}{2}\right), \\ \mathbb{U}_e(\tau) &= \Pi^\dagger \mathbb{U}_o\left(\tau + \frac{1}{2}\right) \Pi;\end{aligned}\quad (1)$$

Π implements a periodic one-site shift, and $U(x, \tau)$ are local two-site time-evolution operators, or gates, acting at time τ on the qudits at sites $x - 1/2$ and x . This defines a “brickwork” quantum circuit implementing local dynamics.

Our focus is on the case where $U(x, \tau)$ are chosen so that there exists a fixed special local basis (computational basis) $\{|n\rangle\}_{n=0}^{q-1}$, in which the gates act as permutations of the q^2 basis elements. Introducing the shorthand notation for a two-qudit state $|n\rangle = |n_1 n_2\rangle$, where $n = qn_1 + n_2$ and $n_{1,2} \in \mathbb{Z}_q$, $U(x, \tau)$ can be expressed as

$$U(x, \tau) |n\rangle = |\pi_{x,\tau}(n)\rangle, \quad \pi_{x,\tau} \in S_{q^2}, \quad (2)$$

where S_{q^2} is the group of permutations of q^2 elements. The resulting circuit is a permutation circuit (a special case of an automaton circuit [11]) and does not generate entanglement in the computational basis. Thus, the dynamics of the system can be thought of as classical: we can associate to the quantum circuit an equivalent classical reversible cellular automaton (or boolean circuit) evolving a classical configuration $(s_0, s_{1/2}, \dots, s_{L-1/2})$ in the same way as the quantum circuit evolves the basis state $|s_0 s_{1/2} \dots s_{L-1/2}\rangle$.

To compute local operator entanglement (LOE), we consider the local operator $\mathcal{O}_x$, acting nontrivially as a $q \times q$ operator $\mathcal{O}$ at site x , and as the identity otherwise. We take $\mathcal{O}$ to be traceless, $\text{tr}[\mathcal{O}] = 0$, and compute its evolution in the Heisenberg picture $\mathcal{O}_x(t) = \mathcal{U}(t)^{-1} \mathcal{O}_x \mathcal{U}(t)$, where

$$\mathcal{U}(t) := \mathbb{U}(t) \dots \mathbb{U}(1). \quad (3)$$

Next, following Refs. [29,30], we compute the “entanglement” of $\mathcal{O}_x(t)$ by viewing it as a quantum state in a doubled Hilbert space, $|\mathcal{O}_x(t)\rangle$, using the mapping

$$|s_1 s_2 \dots s_{2L}\rangle \langle r_1 r_2 \dots r_{2L}| \mapsto |s_1 r_1 s_2 r_2 \dots s_{2L} r_{2L}\rangle, \quad (4)$$

which establishes a correspondence between operators acting on qudits with local dimension q and states of qudits with local dimension q^2 . Then, we estimate its entanglement over a spatial bipartition, $\mathbb{Z}_{2L}/2 = \bar{A} \cup A$. To this end, we compute the second Rényi entropy of $|\mathcal{O}_{L/2}(t)\rangle$, reduced to the subsystem A . Choosing the normalization $\text{tr}[\mathcal{O}\mathcal{O}^\dagger] = q$, we obtain

$$S_{\mathcal{O},2}(t) = -\log \text{tr}[\rho_{\mathcal{O},A}^2(t)] =: -\log \mathcal{P}_{\mathcal{O}}(t), \quad (5)$$

where we set $\rho_{\mathcal{O},A}(t) := q^{-2L} \text{tr}_{\bar{A}}[|\mathcal{O}_{L/2}(t)\rangle \langle \mathcal{O}_{L/2}(t)|]$ and introduced its purity $\mathcal{P}_{\mathcal{O}}(t)$. We focus on the case of a contiguous bipartition, with $\bar{A}$ containing the first $L + 2\ell$ sites, and A the remaining $L - 2\ell$, with $\ell \geq 0$. This quantity, sometimes called LOE Rényi entropy, has been proposed as a means to assess the ergodicity properties of systems in which the usual definition of quantum chaos through the spectral statistics (see, e.g., Refs. [5,46,47]) cannot be applied due to the lack of a fixed propagator. In essence, its linear growth in the thermodynamic limit is associated with chaotic dynamics.

The purity in Eq. (5) can be represented as a matrix element in a replicated space via a standard folding mapping (see, e.g., Ref. [3]). More specifically, we introduce the following states in the eight-replica space of a single qudit:

$$\begin{aligned}|\bigcirc^{(8)}\rangle &= |\bigcirc^{(4)}\rangle \otimes_r |\bigcirc^{(4)}\rangle, \\ |\bullet_{\mathcal{O}}^{(8)}\rangle &= (\mathcal{O} \otimes_r \mathbb{1} \otimes_r \mathcal{O}^\dagger \otimes_r \mathbb{1})^{\otimes 2} |\bigcirc^{(8)}\rangle, \\ |\Delta^{(8)}\rangle &= \frac{1}{q^2} \sum_{a,b,c,d=0}^{q-1} |abcdcba\rangle_r, \\ |\square^{(8)}\rangle &= |\square^{(4)}\rangle \otimes_r |\square^{(4)}\rangle,\end{aligned}\quad (6)$$

where $\otimes_r$ denotes the tensor product among different replicas, we use the shorthand notation $|a_1 a_2 \dots a_8\rangle_r = |a_1\rangle \otimes_r \dots \otimes_r |a_8\rangle$, and

$$\begin{aligned}|\square^{(4)}\rangle &= \frac{1}{q} \sum_{a,b=0}^{q-1} |abba\rangle_r, \\ |\bigcirc^{(4)}\rangle &= \frac{1}{q} \sum_{a,b=0}^{q-1} |aabb\rangle_r\end{aligned}\quad (7)$$

are states in the four-replica space. In terms of these vectors we can write the purity as

$$\mathcal{P}_{\mathcal{O}}(t) = q^{4L} \langle \square_{L+2\ell}^{(8)} \Delta_{L-2\ell}^{(8)} | \mathcal{U}_8(t) | \bigcirc_L^{(8)} \bullet_{\mathcal{O}}^{(8)} \bigcirc_{L-1}^{(8)} \rangle, \quad (8)$$

where we introduced the replicated time evolution operator $\mathcal{U}_8(t) = (\mathcal{U}(t) \otimes_r \mathcal{U}(t)^*)^{\otimes 4} = \mathcal{U}(t)^{\otimes 8}$, used $\square_x^{(8)}$ to denote a sequence of x symbols $\square^{(8)}$ (analogously for the others), and adopted the convention that sequences of symbols now denote tensor products in space, e.g., $|\square_2^{(8)} \Delta_1^{(8)}\rangle = |\square^{(8)} \square^{(8)} \Delta^{(8)}\rangle = |\square^{(8)}\rangle \otimes |\square^{(8)}\rangle \otimes |\Delta^{(8)}\rangle$.

While Eq. (8) works for any unitary $\mathcal{U}$, the expression can be simplified for automaton circuits when $\mathcal{O}$ is diagonal in the computational basis [11], in which case it reduces to an expectation value in the four-replica space

$$\mathcal{P}_{\mathcal{O}}(t) = q^{2L} \langle \bigcirc_{L+2\ell}^{(4)} \square_{L-2\ell}^{(4)} | \mathcal{U}_4(t) | -_L^{(4)} -_{\mathcal{O}}^{(4)} -_{L-1}^{(4)} \rangle, \quad (9)$$

where we introduced

$$\begin{aligned} |-(4)\rangle &= \frac{1}{q^2} \sum_{a,b,c,d=0}^{q-1} |abcd\rangle_r, \\ |-(4)\rangle &= (\mathcal{O} \otimes_r \mathcal{O}^\dagger)^{\otimes_{r^2}} |-(4)\rangle. \end{aligned} \quad (10)$$

The first represents a flat sum of states in the four-replica space and the second is orthogonal to it because of the traceless property of $\mathcal{O}$. A proof of Eq. (9) is contained in the Supplemental Material (SM) [48].

Equation (9) is well defined also in the classical setting and corresponds to the following experiment. Take four copies of the system and initialize the first two in random initial states. For $x \leq L + \ell$, initialize the third copy to agree with the first and the fourth with the second; and analogously for $x > L + \ell$, but with the roles of third and fourth copy exchanged. Then evolve all copies under the same evolution $\mathcal{U}$. In the end, record the state of all four copies on site $L/2$, and compute the product of values of $\mathcal{O}$ in each of the four local states. The expression in Eq. (9) corresponds to repeating this many times over different initial states and recording the average result. Compared to other indicators of irregular or chaotic dynamics used in the discrete space-time setting (e.g., damage spreading), this quantity involves more replicas of the system and it is natural to expect it to provide a more stringent criterion for chaos.

Let us now provide a quantitative characterization of LOE growth. First we note that permutation gates [cf. Eq. (2)] for $q = 2$ are part of the Clifford group [14,16] and therefore map products of Pauli matrices into products of Pauli matrices [45]. While this property can be directly verified since there are only 24 permutation matrices for $q = 2$, it can be intuitively understood by noting that, for $q = 2$, permutations do not implement universal classical computation. In contrast, they do so for $q > 2$ as they can also implement Toffoli gates. The Clifford property means that at all times $|\mathcal{O}_x(t)\rangle$ is the superposition of at most three product states and thus the LOE is bounded by a constant, which excludes chaotic behavior. Notably, this lack of chaoticity for $q = 2$ is missed by damage spreading and OTOCs, which show the same behavior for all $q \geq 2$ [15]. Indeed, damage spreading is the classical analog of OTOCs. As such, it signals “classical scrambling” rather than chaos.

To make progress for $q > 2$, we consider the case where $U(x, \tau)$ are random permutation gates [i.e., random unitary matrices fulfilling Eq. (2)] independently distributed for each x and τ [15]. In this case we make the customary approximation of considering the annealed average of Eq. (5), i.e., we bring the average inside the logarithm and focus on the average purity given by

$$\bar{\rho}_{\mathcal{O}}(t) = q^{2L} \quad (11)$$

with the averaged local gate

$$\boxed{2k} := \overline{U(x, \tau)^{\otimes_{r^2} 2k}} =: \bar{U}_{2k}. \quad (12)$$

The top and bottom boundary states in Eq. (11) are the same as in Eq. (9), and we assume periodic boundary conditions in space (i.e., the open legs on the left are connected to those on the right). For $q \geq 2k$, the average over random permutations reduces the number of states on each leg from q^{2k} to B_{2k} , where the latter is the Bell number [49,50] fulfilling $B_0 = B_1 = 1$, and $B_n = \sum_{i=0}^{n-1} \binom{n-1}{i} B_i$. These states correspond one-to-one to partitions of a set of $2k$ elements, therefore we will refer to them as partition states and denote them by $\{|\Pi_j\rangle\}_{j=1}^{B_{2k}}$. More precisely, the averaged local gate $\bar{U}_{2k}$ becomes a projector to the subspace $\text{Span}\{|\Pi_j\rangle \otimes |\Pi_j\rangle\}_{j=1}^{B_{2k}}$ [48]. For instance, in the case of $2k = 8$, the number of partition states is $B_8 = 4140$, and include $|\bigcirc^{(8)}\rangle$, $|\square^{(8)}\rangle$, $|\Delta^{(8)}\rangle$, while for $2k = 4$ we have $B_4 = 15$ partition states, including $|-(4)\rangle$, $|\bigcirc^{(4)}\rangle$, and $|\square^{(4)}\rangle$.

The unitarity of the local (nonaveraged) gate $U_{2k} = U(x, t)^{\otimes_{r^2} 2k}$ implies $\langle \bigcirc^{(4)} | U_4 = \langle \bigcirc_2^{(4)} |$ and $\langle \square_2^{(4)} | U_4 = \langle \square_2^{(4)} |$, while the permutation property of the gates [cf. Eq. (2)] gives $U_k |-(4)\rangle = |-(4)\rangle$. Since these properties survive the average, for $L \geq t$ we can reduce Eq. (11) to

$$\bar{\rho}_{\mathcal{O}}(t) = q^{2t+2\ell} [\langle \bigcirc_{4\ell+1}^{(4)} | \otimes \langle \phi_{t-1-\ell} | \otimes \langle \square_1^{(4)} |] |\psi_{t+\ell}\rangle, \quad (13)$$

where we assumed for concreteness $L, 2\ell$ to be even, and introduced the states

$$\langle \phi_n | = \dots, \quad |\psi_n\rangle = \dots \quad (14)$$

In this language, the problem of computing LOE is rephrased as that of estimating an overlap between two vectors in a chain of $2t + 2\ell$ qudits with $B_4 = 15$ internal states, which, in line with our physical interpretation of Eq. (9), we expect to be increasingly more suppressed as the chain length increases. This process can be characterized exactly in the limit of large q .

Large q expansion. The states $\langle \phi_n |$ and $|\psi_n\rangle$ from Eq. (14) can be conveniently defined inductively by introducing the rectangular matrices

$$\begin{aligned} M_m &= \dots, \\ N_m &= \dots \end{aligned} \quad (15)$$

In terms of the latter we have [51]

$$\begin{aligned} |\psi_n\rangle &= M_n |\psi_{n-1}\rangle, & |\psi_1\rangle &= \bar{U}_4 |-(4)\rangle, \\ \langle \phi_n | &= \langle \phi_{n-1} | N_n, & \langle \phi_1 | &= \langle \bigcirc_4 \square_4 | \bar{U}_4. \end{aligned} \quad (16)$$

For the initial conditions of interest here, the action of N_n (M_n) reduces the norm of $\langle \phi_n |$ ($|\psi_n\rangle$) at each application and the idea is to find the contributions that are suppressed the least. In the case of $\langle \phi_n |$, this procedure can be straightforwardly

implemented to give [48]

$$\langle \phi_n | = q^{-n} \sum_{\substack{\ell_1, \ell_2, \ell_3 \geq 0 \\ \ell_1 + \ell_2 + \ell_3 = n}} B_{\ell_1, \ell_2, \ell_3}^{(3,1,6)} \langle \ell_1, \ell_2, \ell_3 | + o\left(\frac{1}{q^n}\right), \quad (17)$$

where $f(x) = o(x)$ is little- o notation and means $\lim_{x \rightarrow 0} f(x)/x = 0$, the q -independent coefficients $B_{\ell_1, \ell_2, \ell_3}$ are determined recursively, and we introduced the states

$$|\ell_1, \ell_2, \dots, \ell_n\rangle^c = |\Pi_{c_1, 2\ell_1} \Pi_{c_2, 2\ell_2} \dots \Pi_{c_n, 2\ell_n}\rangle, \quad (18)$$

involving n different domains of partition states determined by the n -tuple $c = (c_1, c_2, \dots, c_n)$.

In the case of $|\psi_n\rangle$ keeping track of the leading-order contribution is complicated by a large number of cancellations (due to the orthogonality between $|-_4\rangle$ and $|-_4^{\mathcal{O}}\rangle$). To make progress, we note that there exist subspaces of partition states that are invariant under the action of M_n . We start by introducing the following subspaces of partition states on a chain of length n :

$$\begin{aligned} \mathcal{S}_{1,n} &= \text{Span}\{|\Pi_{i_1} \dots \Pi_{i_n}\rangle; i_j \in \{2, 9, 10, 11, 15\}\}, \\ \mathcal{S}_{2,n} &= \text{Span}\{|\Pi_{i_1} \dots \Pi_{i_n}\rangle; i_j \in \{4, 9, 13, 14, 15\}\}, \\ \mathcal{S}_{3,n} &= \text{Span}\{|\Pi_{i_1} \dots \Pi_{i_n}\rangle; i_j \in \{7, 11, 12, 14, 15\}\}, \\ \mathcal{S}_{4,n} &= \text{Span}\{|\Pi_{i_1} \dots \Pi_{i_n}\rangle; i_j \in \{5, 10, 12, 13, 15\}\}. \end{aligned} \quad (19)$$

As shown in the SM [48], the special feature of these states is that they can be factorized in replica space: they can all be written in terms of two factors, one of which is $|-_n^{(1)}\rangle$, e.g., the states in $\mathcal{S}_{1,n}$ can all be written as $|-_n^{(1)}\rangle \otimes_r |\phi_n^{(3)}\rangle$. This implies [48].

Property 1. For every $|\phi\rangle$ in $\mathcal{S}_{2n}$, where $\mathcal{S}_n \equiv \bigoplus_{j=1}^4 \mathcal{S}_{j,n}$, and $\mathcal{S}_n^*$ is its dual space, one has

$$M_n |\phi\rangle \in \mathcal{S}_{2n+2}, \quad \langle \phi | M_{n+1} \in \mathcal{S}_{2n-2}^*. \quad (20)$$

In other words, the evolution defined by M_n is closed for states in $\mathcal{S}_{2n}$. Moreover, recalling that $\langle -_4 | -_4^{\mathcal{O}} \rangle = 0$, property 1 also implies [48].

Property 2. For every $|\phi\rangle \in \mathcal{S}_{2n}$ we have $\langle \phi | \psi_n \rangle = 0$.

These two properties allow us to find the leading order contribution to $|\psi_n\rangle$ by focusing on the components of $|\psi_n\rangle$ that are not in $\mathcal{S}_{2n}$. At each time step we decompose the state as $|\psi_n\rangle = |\psi_n'\rangle + |\psi_n''\rangle$, where $|\psi_n''\rangle \in \mathcal{S}_{2n}$, and $|\psi_n'\rangle \notin \mathcal{S}_{2n}$. Then, property 1 implies $|\psi_n'\rangle = (M_n |\psi_{n-1}'\rangle)'$, as $M_n \psi_{n-1}''$ has no component out of $\mathcal{S}_{2n}$. Using this observation, we are able to consistently determine the leading-order contribution to $|\psi_n'\rangle$, while the orthogonality condition in property 2 implies that $|\psi_n''\rangle$ gives subleading corrections [48]. Finally, we find that the leading order of $|\psi_n\rangle$ takes a form similar to Eq. (17), but the coefficients decay twice as fast—as q^{-2n} —and the relevant states [cf. (18)] are richer. They are given by 18 different configurations of partition states $c = (c_1, c_2, c_3, c_4, c_5)$ with (up to) four domain walls [48].

Results. Plugging the leading-order forms of $\langle \phi_n |$ and $|\psi_n\rangle$ in Eq. (13), we find the following exact result [48]:

$$\bar{\mathcal{P}}_{\mathcal{O}}(t) = \frac{1 + \delta_{\ell,0}}{q^{t-\ell}} + o\left(\frac{1}{q^{t-\ell}}\right), \quad (21)$$

which is the main result of this Letter. This equation shows that for large q , the LOE does indeed grow linearly in time for

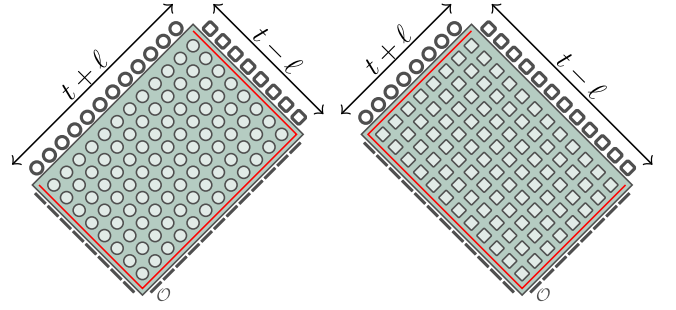


FIG. 1. Schematic illustration of the dominant contribution to local operator purity in cases with $\ell > 0$ and $\ell < 0$. The large rectangle represents (up to normalization) the partition sum from Eq. (11) after taking into account the unitarity and locality of the interaction, and the symbols outside the border represent the boundary conditions given by the partition states. Interpreting purity as a partition sum of a two-dimensional statistical spin model, the leading contribution comes from the bulk configuration that minimizes the lengths of domain walls (red lines) between different partition states [32,52]. Note that for $\ell = 0$, both the configurations above give the same contribution, which explains the prefactor two in Eq. (21).

any finite ℓ , demonstrating a chaotic behavior of random permutation circuits. We note that while it is known that random permutation circuits with $q \geq 3$ form permutation k designs in linear depth [53–55], this does not imply linear growth of LOE at times $t < L$ [56]. Our results apply precisely to the short time regime, $t < L$, showing linear growth of LOE and giving a precise prefactor.

Our result can be interpreted by generalizing the entanglement membrane picture of Refs. [32,52] to the random-permutation setting. As in the standard case, the idea is to interpret circuit-averaged multicopy quantities as the partition function of a classical spin model, whose degrees of freedom are the partition states emerging from the averaging. Notably, there are many more partition states than the "pairing states" occurring in random unitary circuits (B_{2k} vs $k!$) [15]. In this framework, the large- q limit corresponds to the low-temperature limit of the partition function, which is dominated by the lowest energy configuration depicted in Fig. 1. In a given partition state, uniform patches have zero energy and the energy cost only comes from the domain walls between different patches. At leading order, the line tension of different domain walls is given by the logarithm of the overlap between the neighboring partition states, which are $\langle \bigcirc^{(4)} | \square^{(4)} \rangle = \langle \bigcirc^{(4)} | -^{(4)} \rangle = \langle \square^{(4)} | -^{(4)} \rangle = 1/q$. Thus, Fig. 1 indeed captures the leading contribution, Eq. (21).

The membrane analysis can be generalized beyond diagonal operators, and can thus be employed to characterize the general expression in Eq. (8). The dominant configurations are still those depicted in Fig. 1, where, however, $\bigcirc^{(4)}$, $\square^{(4)}$, and $-^{(4)}$ are replaced by $\bigcirc^{(8)}$, $\square^{(8)}$, and $\Delta^{(8)}$, respectively. Computing their overlaps, we now have $\langle \Delta^{(8)} | \bigcirc^{(8)} \rangle = \langle \bigcirc^{(8)} | \square^{(8)} \rangle = \langle \Delta^{(8)} | \square^{(8)} \rangle = 1/q^2$, and therefore we expect that for nondiagonal operators the averaged operator purity decays twice as fast (with the same asymptotic q dependence of random unitary circuits, cf. Ref. [8]).

To test these results in the finite- q regime, we perform a numerical investigation exploiting the automaton nature

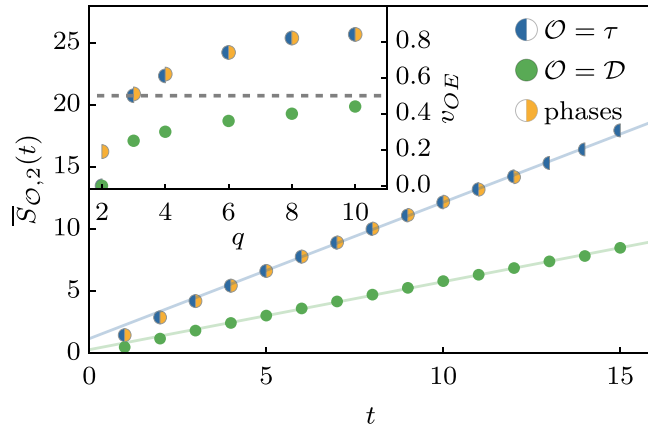


FIG. 2. Averaged Rényi-2 local operator entanglement $\bar{S}_{O,2}(t)$ for $q = 3$ for an off-diagonal operator τ (blue), a diagonal operator $\mathcal{D}$ (green), and τ under the dynamics with phases (yellow), and their linear fits. Note that for the off-diagonal operator $\bar{S}_{O,2}(t)$ grows approximately twice as fast as for the diagonal operator, ($v_{OE,\tau} = 0.50$ vs $v_{OE,\mathcal{D}} = 0.25$), and dynamics with phases is on top of the data without phases. The inset shows the numerically estimated velocities for various q . Note that for $q \gg 1$ and the diagonal operator, the velocity approaches $1/2$, agreeing with the analytical results (dashed line). We also observe that at $q = 2$, only dynamics with phases shows nonzero velocity, as it is not Clifford. The off-diagonal operator τ is the one-site cyclic shift of basis states, $\tau |s\rangle = |s + 1 \pmod{q}\rangle$, while the diagonal operator $\mathcal{D}$ is taken to be $\text{diag}(+1, -1, +1, \dots)$, and for odd q we need to further normalize it as $\text{tr}[\mathcal{D}] = 0$ and $\text{tr}[\mathcal{D}^2] = q$.

of the dynamics [11]. Namely, we randomly sample initial configurations and evolve each one classically. Despite being optimized for automaton circuits, this approach cannot compute LOE for very large times because it requires an exponentially large number of samples since the purity becomes exponentially small [48]. Our results are summarized in Fig. 2, which reports the (annealed) average of $S_{O,2}(t)$, as a function of time t . The finite- t data shows convincing linear scaling allowing us to numerically estimate the speed of entanglement growth $v_{OE} \equiv \lim_{t \rightarrow \infty} \bar{S}_{O,2}(t)/(\log(q^2)t)$ —plotted in the inset. The results in the large q limit for diagonal operators agree with Eq. (21), which predicts $v_{OE} = 1/2$, and also confirms that the entanglement velocity is twice as large

for off-diagonal operators. Interestingly, this property appears to remain true also at finite q .

Finally, we note that one could also consider an immediate generalization of the permutation dynamics by adding two-site random phases to dynamics by multiplying $U(x, t)$ with a randomly chosen diagonal phase matrix, which amounts to changing the r.h.s. in Eq. (2) as $|\pi_{x,\tau}(n)\rangle \mapsto e^{i\phi_{x,\tau}(n)} |\pi_{x,\tau}(n)\rangle$. As we show in Fig. 2 (see also the SM [48]), as long as $q > 2$ the behavior of the averaged purity is similar, and it takes exactly the same form for diagonal observables. The only qualitative difference occurs at $q = 2$, where the two-site permutations with added phases no longer have Clifford property, and therefore can support linear growth of LOE.

Discussion. In this Letter, we characterized quantum chaos in random permutation circuits with nearest-neighbor interactions by means of local operator entanglement [29,30]. We showed that while the latter is bounded for qubits, for qudits it grows linearly in the thermodynamic limit. This suggests that random permutations on qudits are quantum chaotic.

We also argued that local operator entanglement is a better diagnostic of classical chaos than damage spreading—the latter only measures sensitivity to initial conditions—and we proposed it as a unified tool for the detection of chaos in both the quantum and classical realms.

Our work opens many questions for future research. In our view, the two most pressing ones are to (i) develop a comprehensive understanding of local operator entanglement in classical dynamical systems, determining how it is related to the standard concepts of chaos theory such as Lyapunov exponents and Kolmogorov-Sinai entropy [57–60]; and (ii) explore the implications that the quantum chaotic nature of random permutation has on their quantum computational properties.

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