

Electron model on Truchet tiling: Extended-to-localized transitions and asymmetric spectrumJunmo Jeon^{1,2,*} and Shiro Sakai^{2,†}¹*Department of Physics, Korea Advanced Institute of Science and Technology, Daejeon 34141, Korea*²*Physics Division, Sophia University, Chiyoda-ku, Tokyo 102-8554, Japan*

(Received 6 May 2025; revised 14 August 2025; accepted 9 September 2025; published 1 October 2025)

Motivated by recent advances in the realization of Truchet-tiling structures in molecular networks and metal-organic frameworks, we investigate the wave localization issue in this kind of structure. We introduce an electron model based on random Truchet tilings, square lattices with randomly oriented diagonal links, and uncover a rich interplay between spectral and localization phenomena. By varying the strength of diagonal couplings, we explore successive transitions from an extended phase, through a regime with a mobility edge, to a fully localized phase. The energy-resolved fractal dimension analysis captures the emergence and disappearance of mobility edges, while an anomalous shift and asymmetry in the Van Hove singularity are identified as key signatures of the underlying disordered Truchet-tiling structure. Notably, by using the finite-size scaling of level spacing statistics, we clarify that the transition occurs at a finite level of disorder even in the two-dimensional system. Our findings position Truchet-tiling electron systems as a versatile platform for engineering disorder-driven localization and interaction effects in amorphous quantum materials and photonic architectures.

DOI: [10.1103/qg97-s1vt](https://doi.org/10.1103/qg97-s1vt)

Introduction. Truchet tiles are square tiles decorated with a rotationally asymmetric motif [see Fig. 1(a)] [1]. When a plane is covered by these tiles with different orientations, various eye-catching patterns are generated. Although this tiling had long been a purely theoretical construct with no counterpart in nature, it was recently realized in DNA molecular networks [2] and metal-organic frameworks [3]. These experimental realizations have sparked growing interest in engineered forms of disorder that can be precisely controlled and dynamically tuned [4–7]. They can form aperiodic and labyrinthine patterns, characterized by randomly oriented but structurally simple motifs [3,4,8–10]. Because of these characteristics, the Truchet-tiling structure, despite being aperiodic, can exhibit a range of distinctive properties that set it apart from conventional amorphous systems [4,10–12]. In particular, they enable systematic exploration of wave localization and spectral properties in aperiodic yet reproducible geometries [4–7].

Wave localization in the absence of periodicity is a central topic in condensed-matter and photonic physics, underpinning key phenomena such as Anderson localization and metal-insulator transitions [13–18]. In Anderson localization, aperiodicity stems from random on-site energies within a bounded interval [13], while in a random Truchet tiling, it is introduced via the random placement of tiles selected from

a finite set of orientations [3,9]. Thus, unlike conventional Anderson-type models, random Truchet tilings generate disordered networks whose physical properties can be analyzed using graph-theoretic tools [19]. Although structurally disordered systems of this type have been studied in statistical contexts such as percolation and loop models [19–22], their implications for wave localization and spectral properties remain largely unexplored.

In this Letter we introduce a simple electron model which captures the essence of the Truchet-tiling randomness, studying the wave-function localization and anomalous spectral properties in the model. We consider square lattices decorated with randomly oriented diagonal hopping links of uniform strength [Fig. 1(b)]. By tuning the strength of diagonal couplings, we explore the interplay between spectral and localization properties. We identify three distinct regimes: an extended phase, a phase with mobility edges, and a fully localized phase. The energy-resolved fractal dimension captures the emergence and disappearance of mobility edges. Notably, finite-size scaling of level spacing statistics reveals that a localization transition occurs at a finite diagonal hopping strength. This reflects the nature of two-dimensional Anderson localization as a marginal critical dimension, where extremely long localization lengths emerge under weak disorder [14]. Furthermore, we uncover anomalous shift and asymmetry in the Van Hove singularity (VHS), providing insights into the connection between spectral features and localization phenomena. Our results position Truchet tilings as a promising platform for engineering disorder-driven effects in quantum materials.

Random Truchet tiling. Truchet tiling is comprised of unit tiles that are decorated with a motif breaking the rotational symmetry of the lattice [1,3,9,23]. These tiles are placed in such a way that they form visually complex patterns [24,25].

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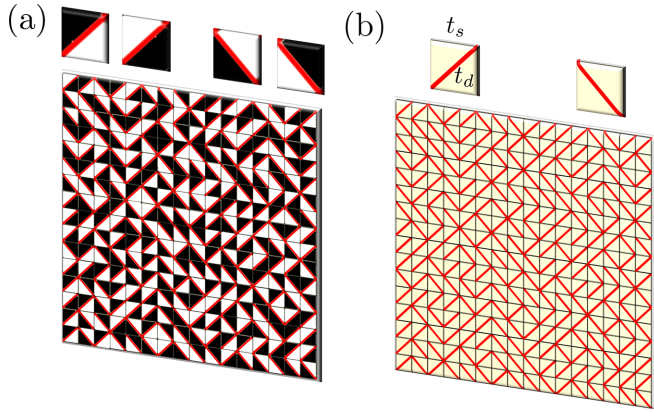


FIG. 1. (a) Random Truchet tiling composed of square tiles split diagonally into black and white triangles, each allowing four distinct orientations. (b) Corresponding tight-binding model on the square lattice, where two types of red diagonal links are introduced depending on the direction of diagonal split of each tile in (a). The black and red links represent two different hopping amplitudes t_s and t_d , respectively.

Figure 1(a) exhibits a random Truchet tiling composed of square tiles diagonally divided into black and white triangles, each randomly oriented among four possible orientations.

Let us introduce the diagonal coupling along the boundary of black and white triangles in each plaquette [see Fig. 1(b)]. The tiling shown in Fig. 1(b) is comprised of only two kinds of mutually exclusive decorated plaquettes: type 1 ($\square$) and type 2 ($\square$).

Such exotic structure of Truchet tiling could be realized experimentally, for example, via a metal-organic framework [3], DNA molecular networks [2], ultracold atoms [26], patterned photonic crystals [27,28], or engineered electronic circuits in which resonant elements are connected along randomized diagonals [29–31]. To probe the resulting electronic (or, more generally, wave) dynamics on this unconventional geometry, we consider a tight-binding model in which, in addition to conventional nearest-neighbor hoppings in a square lattice, a diagonal hopping term is introduced between the two sites of each plaquette. For instance, in ultracold-atom experiments, one can use site-selective Raman-assisted tunneling to imprint a two-photon resonance along one diagonal of each plaquette of a square optical lattice, thereby giving each plaquette an independent next-nearest-neighbor hopping amplitude and realizing the Truchet-tiling tight-binding model [32–34].

The tight-binding Hamiltonian on the Truchet tiling is given by

$$H = -t_s \sum_{\langle i,j \rangle \sigma} c_{i\sigma}^\dagger c_{j\sigma} - t_d \sum_{\langle\langle i,j \rangle\rangle_{\text{diag}} \sigma} c_{i\sigma}^\dagger c_{j\sigma} + \text{H.c.}, \quad (1)$$

where t_s and t_d are the amplitudes for horizontal or vertical and diagonal hoppings, respectively; $c_{i\sigma}$ and $c_{i\sigma}^\dagger$ are electronic annihilation and creation operators at the site i and spin σ , respectively; and $\langle\langle i,j \rangle\rangle_{\text{diag}}$ runs over the randomly chosen diagonal bonds. We adopt open boundary conditions; however, our findings can also be observed under periodic boundary conditions [35]. The strength of disorder inherent in the diagonal pattern is given by the ratio t_d/t_s . To

systematically explore the influence of the unique geometry on the wave localization, we reparametrize the model by setting $t_d = \alpha t$ and $t_s = (1 - \alpha)t$, with $0 \leq \alpha < 1$. We set $t = 1$. By varying α , the electron localization properties of the full range of disorder strengths inherent in the Truchet-tiling pattern can be explored. We perform our calculations over 1000 independent realizations of the random Truchet tiling and average them.

Anomalous Van Hove shift. Before delving into the localization characteristics of the electron states on the Truchet tiling, we explore the notable features of the density of states (DOS), $\rho(E) = \sum_n \delta(E - E_n)$, where E_n is the n th eigenenergy. We focus on the DOS of a random Truchet tiling and compare it with that of an anisotropic triangular (AT) lattice, a long-range-ordered counterpart [36,37]. Note that the AT lattice is comprised of only one of two tile orientations in Fig. 1(b).

Figure 2 compares the DOS of the random Truchet tiling and AT lattice for different α values. Recall that the square lattice ($\alpha = 0$) has a symmetric logarithmically divergent VHS at the zero energy. First, we observe that the number of VHSs of the random Truchet tiling remains one for nonzero α , unlike a general case of an AT lattice with two VHSs [37,38]. Second, interestingly, the VHS of the random Truchet tiling shows an anomalous nonmonotonic shift as a function of α . In detail, the VHS moves toward higher energy until $\alpha = 0.4$ and shifts back toward zero energy as α increases further. Notably, the Van Hove shift exhibits a sudden change around $\alpha = 0.4$, marking a turning point that drastically alters the behavior of the shift [see the green curve in Fig. 3(c)]. This contrasts with the Van Hove shifts in the AT lattice, where two VHSs always shift monotonically in opposite energy directions with increasing α and merge at $\alpha = 0.5$ [35]. Furthermore, while the VHSs of the AT lattice are symmetric in their vicinity for general α , in the random Truchet tilings, the VHS for $0.4 \leq \alpha < 0.7$ exhibits an asymmetric peak shape, in contrast to the cases of $\alpha < 0.4$ and $\alpha \geq 0.7$ (compare the blue, orange, and brown curves in Fig. 2, for instance). Third, although both systems recover the bipartite nature as α approaches unity, in contrast to the AT lattice, where two VHSs appear at the spectral edges, the most prominent VHS in the random Truchet tiling moves to zero energy.

Interestingly, unlike the VHS at the band edge of the AT lattice, the zero-energy VHS in the random Truchet tiling is protected by the chiral symmetry restored as $\alpha \rightarrow 1$. Note that this chiral symmetry stems from the bipartite nature of the graph formed solely by diagonal links and appears in both the AT lattice and the random Truchet tiling. However, while the AT lattice reduces to multiple decoupled one-dimensional chains, the random Truchet tiling develops many disconnected diagonal-link graphs that individually possess features such as loop-induced symmetry within a sublattice or sublattice imbalance, thereby hosting a macroscopically large number of symmetry-protected zero-energy states [37]. (See the Supplemental Material [35] for detailed information, which also includes Refs. [39–42].)

We emphasize that, beyond a certain threshold of disorder in random Truchet tilings, the resulting asymmetry in both the VHS and DOS, reminiscent of that observed in face-centered-cubic lattices [38,43], can lead to notable consequences in the presence of electron correlations. This kind of asymmetry

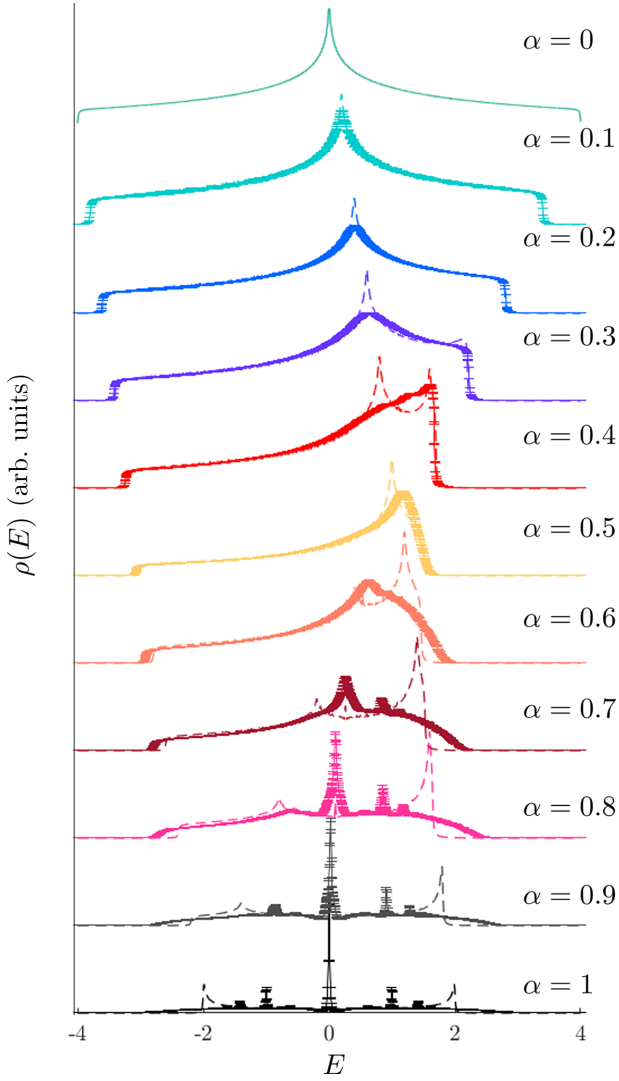


FIG. 2. Density of states in arbitrary units of random Truchet tilings (solid curves with error bar) and anisotropic triangular lattice (dashed curves) for different α values. For $\alpha = 0$, the system is a simple square lattice (random Truchet tilings). As α increases, the energy at which the VHS appears initially starts at zero and gradually shifts upward, reaching the top of the energy band near $\alpha \approx 0.4$. Beyond this point, it decreases and approaches zero again. For $0.4 \leq \alpha < 0.7$, the DOS in the vicinity of the VHS becomes markedly asymmetric (anisotropic triangular lattice). There are two VHSs in general. For every α , the vicinity of the VHSs is symmetric. As α increases, the VHSs monotonically shift in opposite energy directions to the spectral edges. The system size is 130×130 .

can stabilize ferromagnetism by suppressing Fermi surface nesting and inhibiting antiferromagnetic ordering, both of which generically rely on a symmetric band structure and well-nested Fermi surfaces [44–46]. Moreover, the asymmetric VHS near the Fermi level significantly enhances the DOS at the Fermi energy $\rho(E_F)$, thereby reducing the critical interaction strength required for ferromagnetic instability according to the Stoner criterion $U\rho(E_F) > 1$, where U is the interaction strength [44]. As a result, the system becomes more susceptible to ferromagnetic ordering.

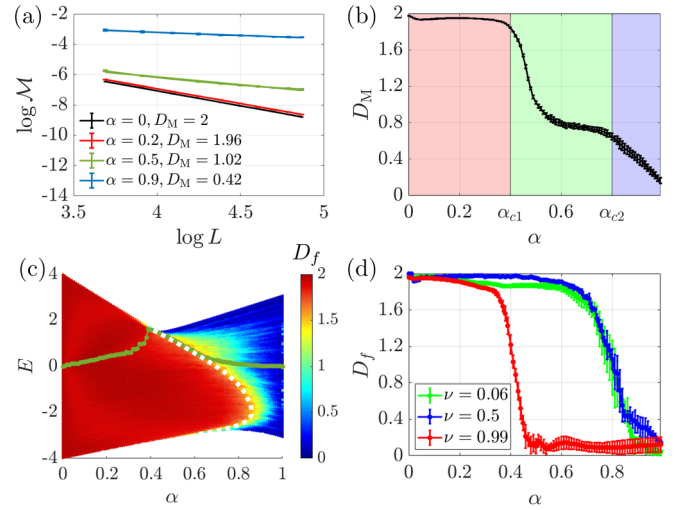


FIG. 3. Transition of the electron localization property in the Truchet tiling. (a) Power-law scaling behavior of the MIPR for different α values as a function of L ($40 \leq L \leq 130$). (b) Three different electron phases in terms of the mean fractal dimension D_M as a function of α . In the red shaded region ($0 \leq \alpha \leq \alpha_{c1}$), D_M stays close to 2 even for nonzero α . In the green shaded region ($\alpha_{c1} < \alpha \leq \alpha_{c2}$), D_M drops to 0.8 and remains nearly constant; D_M decreases again in the blue shaded region ($\alpha > \alpha_{c2}$). Here $\alpha_{c1} \approx 0.4$ and $\alpha_{c2} \approx 0.8$. (c) Landscape of the generalized fractal dimension D_f as a function of energy and α . The white dashed and green solid curves are drawn to emphasize the mobility edge and VHS (E_{VHS}), respectively. (d) Generalized fractal dimension at the Fermi energy for given filling fraction ν as a function of α . The critical value of α , where the generalized fractal dimension rapidly drops, becomes small for $\nu \simeq 1$.

Wave localization and mobility edge. Now let us explore the localization characteristics of electron wave functions. To identify the degree of localization of the wave function, we adopt the inverse participation ratio (IPR) $\mathcal{I}$. The IPR of the eigenstate $|\psi_n\rangle$ is given by $\mathcal{I}_n = \sum_i |\psi_n(i)|^4$, where $\psi_n(i)$ is the normalized wave function of the n th eigenstate at site i [47]. For uniformly extended states, $\mathcal{I} \sim N^{-1}$ for large system size $N = L^2$ in two-dimensional systems, while the IPR of the localized states is almost independent of the entire system size [47]. Generally, $\mathcal{I}_n \sim L^{-D_n}$, where D_n is the fractal dimension [48,49]. In two-dimensional system, $D_n = 0$ (2) for localized (extended) states. When $0 < D_n < 2$, the state is called critical, neither fully extended nor exponentially localized [49–51].

We first consider the mean IPR (MIPR) given by $\mathcal{M} = \frac{1}{N} \sum_{n=1}^N \mathcal{I}_n$ as a function of system size [50]. Figure 3(a) shows that the MIPR also follows similar scaling behavior, $\mathcal{M} \sim L^{-D_M}$. Here D_M is the mean fractal dimension, which represents the average localization property of the system. Figure 3(b) illustrates the mean fractal dimension as a function of α . We determine three different phases in terms of distinct behaviors of D_M . First, D_M stays close to 2 up to $\alpha = \alpha_{c1} \approx 0.4$. This indicates that most states behave like conventional extended state below this critical disorder strength [red shaded region in Fig. 3(b)]. Second, D_M decreases to 0.8 beyond α_{c1} and remains nearly constant until the second

critical value $\alpha_{c2} \approx 0.8$. As we will show, the states in this phase exhibit energy-dependent localization properties with a nontrivial mobility edge. Thus, the localized, extended, and critical states coexist in this phase (green shaded region). Finally, for $\alpha > \alpha_{c2}$, we observe that D_M decreases again as α increases (blue shaded region). It turns out that this second drop in D_M is attributed to the absence of extended states in the spectrum. In the thermodynamic limit, the localized states form an Anderson insulator with an everywhere dense spectrum [35].

To unveil the exotic phase diagram shown in Fig. 3(b), we explore the energy-dependent localization characteristics. We investigate the average IPR of states in the vicinity of a given energy E as a function of system size. In detail, we introduce the density of the IPR $\mathcal{D}$ at the given energy E as

$$\mathcal{D}(E) = \frac{1}{\pi \rho(E)} \sum_n \mathcal{I}_n \frac{\Gamma}{(E - E_n)^2 + \Gamma^2}, \quad (2)$$

where $\rho(E) = \pi^{-1} \sum_n \Gamma / [(E - E_n)^2 + \Gamma^2]$ is the DOS and Γ is half the linewidth. Note that $\mathcal{D}$ is an averaged IPR of the states in the vicinity of a given energy E by using the Lorentzian function. We set $\Gamma = 10^{-2}t$. The $\mathcal{D}(E)$ follows the scaling behavior of $L^{-D_f(E)}$ for large system sizes $N = L^2$. Here $D_f(E)$ is the generalized fractal dimension at the energy E given by

$$D_f(E) = - \lim_{N \rightarrow \infty} \frac{\log \mathcal{D}(E)}{\log L}. \quad (3)$$

When the states around energy E are ideally localized (extended), $D_f(E) = 0$ (2), whereas $0 < D_f(E) < 2$ implies either the coexistence of extended and localized states near E or the presence of sufficiently many critical states. We emphasize that the generalized fractal dimension can be used to capture the presence of the mobility edge, since it generally depends on the energy E , as we will show.

Figure 3(c) exhibits generalized fractal dimensions as a function of energy and the disorder parameter α . Note that until the disorder parameter exceeds the first critical value $\alpha_{c1} \approx 0.4$, D_f remains close to 2 for general energies. As $\alpha > \alpha_{c1}$, the localized states appear in the spectrum, and hence D_M significantly decreases at $\alpha \gtrsim \alpha_{c1}$. Nevertheless, we emphasize that even for $\alpha > \alpha_{c1}$, there are extended states and critical states that mitigate the sharp decline of D_M toward zero. Specifically, Fig. 3(d) plots D_f at the given filling fraction ν as a function of disorder parameter α . Notably, the high-energy states become localized even for smaller α values, while the low-energy states require larger values of α to be localized. Thus, the system possesses a nontrivial mobility edge drawn as the white dashed curve in Fig. 3(c). We note that this mobility edge suddenly disappears at $\alpha \gtrsim \alpha_{c2} \approx 0.8$. This implies that the absence of nearly extended states, previously acting as a buffer, leads to a more rapid decline in D_M for $\alpha > \alpha_{c2}$.

Let us point out a notable relationship between the mobility edge and the VHS. The green solid curve in Fig. 3(c) represents the energy E_{VHS} at which the most prominent VHS appears. The E_{VHS} in random Truchet tiling increases with α until α_{c1} and then decreases, with a divergent slope at α_{c1} . Importantly, the VHS exhibits a pronounced asymmetry in the DOS for $\alpha_{c1} \leq \alpha < 0.7$ (see Fig. 2). This asymmetry in

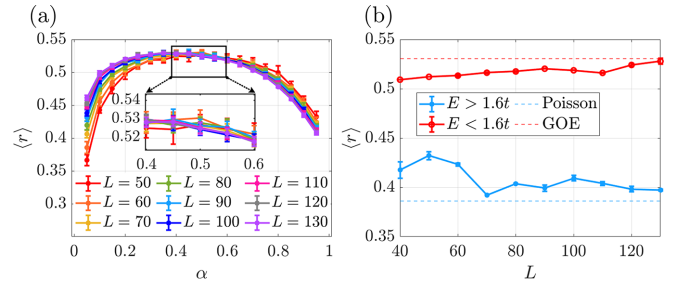


FIG. 4. Level spacing statistics in the random Truchet tiling. (a) Finite-size scaling of the average adjacent gap ratio $\langle r \rangle$ as a function of α . As L increases, $\langle r \rangle$ increases (decreases) for $\alpha < 0.4$ ($\alpha > 0.6$). The inset highlights the crossing of curves for different system sizes at $0.4 \leq \alpha_T \leq 0.6$. (b) Plot of $\langle r \rangle$ for two different energy windows as a function of system size, calculated at $\alpha = 0.7$. The red and blue curves represent $\langle r \rangle$ for different energy windows $E < 1.6t$ and $E > 1.6t$, respectively. The red and blue dashed lines are drawn to emphasize the reference values of the Wigner-Dyson (GOE) and Poisson statistics, respectively.

the DOS around the VHS originates from the reorganization of localized and delocalized states that admit different level spacing statistics and the shape of the DOS [52–54]. Hence, the mobility edge, separating the nearly extended and localized states, is found to coincide with the energy of the asymmetric VHS that emerges for $\alpha_{c1} \leq \alpha < 0.7$.

We emphasize that the anomalous extended-to-localized transition in the random Truchet tiling originates from structural constraints that have no counterpart in conventional random models. In typical disordered systems, on-site potentials or hopping terms are assigned independently, producing uncorrelated local environments. In contrast, the Truchet tiling imposes a plaquette-level rule: Only one of two diagonal links can be present in each plaquette. This restriction limits local coordination numbers and suppresses large-scale structural fluctuations, resulting in a hyperuniform structure [55,56], which in turn delays the onset of Anderson localization. Consequently, two-dimensional extended states persist far beyond the range expected for uncorrelated disorder, disappearing only when the square-lattice hopping t_s is sufficiently reduced [35].

Level spacing statistics. The localization transition and spectral properties are in general deeply intertwined. To clarify their relationship in our model, we analyze the level spacing statistics using the average adjacent gap ratio $\langle r \rangle$. The adjacent gap ratio is defined as $r_n = \min(\delta_n, \delta_{n-1}) / \max(\delta_n, \delta_{n-1})$, where $\delta_n = E_{n+1} - E_n$. The average $\langle r \rangle$ is then obtained by averaging r_n over the spectrum. This provides a sensitive measure of level correlations [54,57]. Localized phases without level repulsion follow Poisson statistics with $\langle r \rangle \approx 0.386$, while delocalized phases exhibit Wigner-Dyson [Gaussian orthogonal ensemble (GOE)] statistics with $\langle r \rangle \approx 0.531$ due to the level repulsion [54]. Assuming a simple scaling form $\langle r \rangle(\alpha, L) = f((\alpha - \alpha_T)L^{1/\mu})$ for some universal function f , $\langle r \rangle$ curves for different L cross at $\alpha = \alpha_T$. Here μ and α_T are the critical exponent and scale-invariant transition point, respectively.

Figure 4(a) exhibits the average adjacent gap ratio for different system sizes. The $\langle r \rangle$ increases (decreases) as the

system size increases for $\alpha < 0.4$ ($\alpha > 0.6$), approaching the characteristic value of the Wigner-Dyson (Poisson) statistics. Thus, $\langle r \rangle$ curves for different system sizes cross at $0.4 \leq \alpha_T \leq 0.6$ [see the inset of Fig. 4(a)]. Such a crossing is reflected in Fig. 3(b) as a sharp drop in D_M for $0.4 \leq \alpha \leq 0.6$. Moreover, $\langle r \rangle$ approaches distinct characteristic values with increasing system size, depending on the energy window [see Fig. 4(b)]. These results highlight the interplay between localization and spectral properties.

Conclusion. In this work, motivated by the recent experimental realization of exotic Truchet-tiling structure, we unveiled a rich landscape of spectral and localization phenomena in an electron model based on random Truchet tilings: square lattices with randomly oriented diagonal hoppings. By varying the strength of diagonal hoppings, we clarified the transitions from an extended phase, through a regime with mobility edges, to an entirely localized phase in two dimensions. The energy-resolved fractal dimension analysis captured the emergence and disappearance of mobility edges, revealing a direct link between spectral asymmetry, mobility-edge dynamics, and Van Hove singularities. Furthermore, the finite-size scaling of the level spacing statistics unveiled the intertwined spectral property and localization transitions. Our

findings establish Truchet-tiling electron systems as a versatile framework for engineering disorder-driven localization and interaction effects in amorphous quantum materials and photonic platforms.

Looking ahead, our results open promising avenues for experimental applications in various platforms including metal-organic framework, ultracold atomic lattices, and photonic crystals where dynamically programmable disorder can be used to control transport, coherence, and correlations [58–60]. Extending Truchet tilings to include interactions could enable novel phases and disorder-driven quantum technologies.

Acknowledgments. We thank Tomi Ohtsuki for useful comments and valuable discussions. This work was supported by JSPS KAKENHI Grants No. JP25H01397 and No. 25H01398. J.J. was supported by National Research Foundation (Grant No. 2021R1A2C109306013) and Nano Material Technology Development Program through the National Research Foundation of Korea funded by Ministry of Science and ICT (Grant No. RS-2023-00281839).

Data availability. The data that support the findings of this article are not publicly available because of legal restrictions preventing unrestricted public distribution. The data are available from the authors upon reasonable request.

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