

Gain-driven magnon-polariton dynamics in the ultrastrong coupling regime: Effective circuit approach for coherence versus nonlinearity

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We theoretically study the dynamics of gain-driven magnon-polaritons (MPs), which characterizes auto-oscillation of MPs, across the strong coupling (SC) and ultrastrong coupling (USC) regimes. Taking into account the magnon dynamics via the magnetic flux, we present an effective circuit model of gain-driven MPs, which allows us to manipulate the coupling strength of MPs by tuning the size of a ferromagnet and incorporates the self-Kerr nonlinearity of magnons due to the shape magnetic anisotropy. In the SC regime, we find that the self-Kerr nonlinearity generates a frequency shift and reduces the coherent magnon-photon coupling. In contrast, in the USC regime, we find that the coherent magnon-photon coupling not only overcomes the self-Kerr nonlinearity but also effectively couples to gain via the imaginary part of complex eigenfrequencies, resulting in magnonlike auto-oscillations. Subsequently, the USC enables one to widely tune the auto-oscillation frequency by means of an external magnetic field. These findings indicate that there is a trade-off relation between the coupling strength of MPs and the self-Kerr nonlinearity of magnons. This work contributes to the understanding of the interplay between gain-loss and USC in nonlinear polariton dynamics, offering a novel principle for frequency-tunable maserlike devices based on gain-driven MPs.

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I. INTRODUCTION

Cavity magnonics is one of the prominent platforms for hybrid-quasiparticle physics, wherein the key excitation is magnon-polariton (MP), that is, a hybridized quasiparticle consisted of electromagnetic fields (photons) in a cavity and collective spin excitations (magnons) in a magnet [1–3]. MPs exhibit rich dynamics and enhanced controllability, which have inspired the development of novel quantum devices such as quantum detectors [4], quantum memories [5,6], and maser-like devices [7,8].

As an important point for practical applications, stronger coherent magnon-photon coupling enables faster and more efficient control of quantum states. However, the achievable coupling strength is fundamentally limited by the large mode volume of microwave cavities relative to the small size of typical magnets, resulting in coupling strengths (g) that are much smaller than the natural angular frequency (ω_r) of either the photon mode or the magnon mode. Therefore, realizing the so-called ultrastrong coupling (USC)

regime [9,10]—characterized by a coupling-to-frequency ratio $g/\omega_r \gtrsim 0.1$ —is highly desired, which opens up a new frontier in hybrid light-matter interactions [11]. The USC regime is typically characterized by the vacuum Bloch-Siegert shift [12,13] on polariton eigenfrequencies, which emerges in the framework beyond the rotating-wave approximation (RWA) and associates with the excitation number nonconserving process in the quantum level. Very recently, USC in MP systems has been achieved experimentally in structure-based platforms with various shapes of magnets, such as superconductor/ferromagnet nanostructures [14–16], slab geometries [17], and magnetic metamaterials [18,19], thereby overcoming the limitation of conventional cavity-based MP systems.

Recently, gain-embedded cavity magnonics has emerged as a platform to engineer a gain-driven MP [8,20], which displays auto-oscillation of polaritons and self-selection of a sustained oscillation mode. In the viewpoint of practical applications, the gain-driven MP possesses advantageous features, involving high coherence and high power output [8,21,22], injection locking [8,23], microwave amplification [8,23], frequency-comb generation [24], and meterscale strong coupling [25]. Despite its potential, the theoretical treatment of the gain-driven MPs remains incomplete. Most existing models are based on a two-mode version of the Tavis-Cummings model in the framework of RWA, which assumes the strong coupling (SC) regime and treats the dynamics of MPs as a linear response. Consequently, intrinsic nonlinear effects, especially magnonic self-Kerr nonlinearity arising from the magnetic anisotropy in magnets [26–28], are often neglected.

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Experimentally, the self-Kerr nonlinearity arises in nonlinear systems, including cavity magnonic platforms [29,30], when subjected to sufficiently strong external driving fields. Then, the self-Kerr nonlinearity typically manifests as a power-dependent shift in the resonance frequency. As a consequence, the conventional Tavis-Cummings-type models fail to fully capture dynamics of gain-driven MPs, particularly in regimes where the self-Kerr nonlinearity and the USC play a critical role. A more comprehensive description of the gain-driven MP thus requires incorporating the nonlinear response of magnetization dynamics into the theoretical framework, especially when aiming to explore operation regimes beyond the SC regime.

In this article, we theoretically demonstrate that the role of magnonic self-Kerr nonlinearity in gain-driven MPs varies significantly across coupling regimes. To this end, we consider a gain-embedded cavity magnonics system in which a gain is described by a negative differential resistance and MPs are described by an inductor with inserted ferromagnet. This effective circuit model incorporates the self-Kerr nonlinearity of magnons due to the shape magnetic anisotropy of the ferromagnet and allows us to manipulate the coupling strength of MPs by tuning the size of the ferromagnet. In the SC regime, we show that the self-Kerr nonlinearity dominates the dynamics of the gain-driven MP, resulting in a nonlinear frequency shift and a suppression of the coherent magnon-photon coupling. In contrast, in the USC regime, the self-Kerr nonlinearity is effectively suppressed by the coherent magnon-photon coupling, leading to stable auto-oscillations of the MP. Moreover, we show that the coherent magnon-photon coupling effectively couples to gain via the imaginary part of complex eigenfrequencies, enabling to control the auto-oscillation frequency by means of an external bias magnetic field. We also discuss a trade-off relation between the coupling strength of MPs and the self-Kerr nonlinearity of magnons as well as the interplay between gain-loss and USC in nonlinear polariton dynamics.

This article is organized as follows: In Sec. II, we introduce an effective circuit model, which describes gain-driven MPs with magnonic self-Kerr nonlinearity. In Sec. III, based on the circuit model, we discuss the effects of the self-Kerr nonlinearity and gain on the eigenmode of a gain-driven MP. In Sec. IV, in order to demonstrate the nonlinear and gain effects discussed in Sec. III, we numerically investigate the transient response and auto-oscillation of gain-driven MPs in the SC and USC regimes. Finally, in Sec. V, we summarize our results and discuss the scope of our study.

II. EFFECTIVE CIRCUIT MODEL

We begin by constructing an effective circuit model of gain-driven MPs. The circuit model consists of a single-mode LC resonator with an inductor involving a ferromagnet and an amplifier circuit with a negative differential resistance component. The schematic image of the circuit is shown in Fig. 1. In the circuit model, MP is expressed by the dynamics of the LC resonator coupled to a ferromagnet [31–35]. The negative differential resistance component provides a gain to the circuit, which drives auto-oscillation of an electrical current

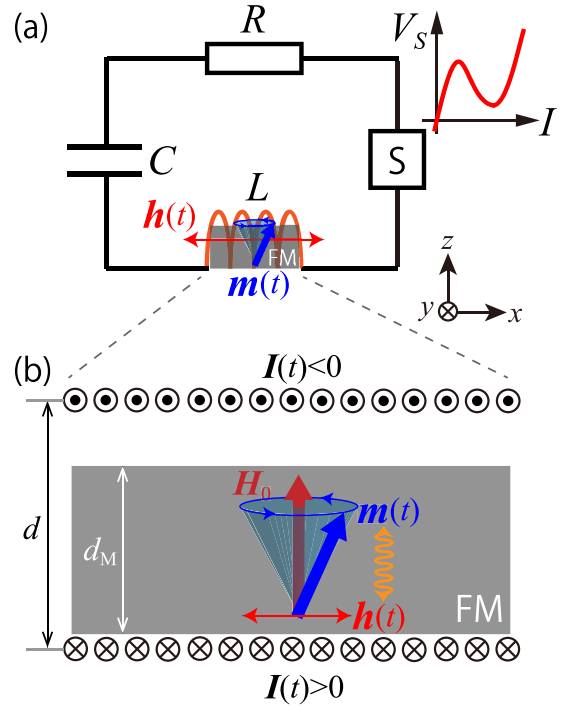


FIG. 1. (a) Effective circuit model of gain-embedded cavity magnonics system in which $\mathbf{h}(t) = h_x \hat{x}$ is a microwave magnetic field (photon) in an inductor and $\mathbf{m}(t)$ is nonlinear magnetization dynamics (magnon) of a ferromagnet (FM) with the shape magnetic anisotropy. S represents an “S-type” negative resistance element biased at the operating point. (b) Schematic image of the dynamics of gain-driven magnon-polaritons. Here, the cross-section of the inductor is assumed to be rectangular, which is characterized by d and d_M being the thickness of the inductor and ferromagnet, respectively, as well as the common width w .

modeled by a van der Pol (vdP) oscillator with a single limit cycle. The electric current flowing in the inductor generates a microwave magnetic field via Ampère’s law, which is treated as the photonic degree of freedom. Also, this microwave magnetic field drives precessional motion of the magnetization in the ferromagnet, which corresponds to the magnonic degree of freedom and is governed by the Landau-Lifshitz-Gilbert (LLG) equation,

$$\frac{d\mathbf{m}}{dt} = -\gamma \mathbf{m} \times \left(-\frac{1}{M_s} \frac{\delta U_m}{\delta \mathbf{m}} + \mu_0 \mathbf{H}(t) \right) + \alpha \mathbf{m} \times \frac{d\mathbf{m}}{dt}, \quad (1)$$

where $\mathbf{m} = (m_x, m_y, m_z)$ is the unit vector aligned with the magnetization direction of the ferromagnet with the saturation magnetization M_s , γ is the gyromagnetic ratio, μ_0 is the permeability of vacuum, and α is the intrinsic Gilbert damping constant. Here, the total magnetic energy is given by

$$U_m = -\mu_0 M_s \mathbf{m} \cdot \mathbf{H}_0 + \frac{1}{2} \mu_0 M_s^2 m_z^2, \quad (2)$$

where $\mathbf{H}_0 = H_0 \hat{z}$ is a static external magnetic field applied normal to the surface of the ferromagnet. Here, we assume that the ferromagnet has the film shape modeled by the second term in Eq. (2). Also, the term $\mathbf{H}(t)$ in Eq. (1) represents the microwave magnetic field induced by the alternating current $I(t)$ via Ampère’s law, which is given by the RLC-circuit

equation (Kirchhoff's voltage law)

$$N \frac{d\Phi(t)}{dt} + RI(t) + \frac{Q(t)}{C} + V_S(I) = 0, \quad (3)$$

where $Q(t)$ is the accumulated charge on a capacitor with an electrostatic capacitance C , R represents a resistance in the RLC circuit. Here, $\Phi(t)$ is the magnetic flux inside the inductor, which is given by

$$N\Phi(t) = LI(t) + \frac{Ll}{N} \frac{dM}{d} M_s m_x, \quad (4)$$

where d_M is the film thickness of the ferromagnet and $L = \mu_0 N^2 w d / l$ is a self-inductance of the inductor with the length l , width w , thickness d , and a turn number N . Note that a parameter d_M/d is responsible for tuning the total spin number of the ferromagnet. Also, $V_S(I)$ is an electromotive force of a negative resistance component labeled S in Fig. 1(a), which is expressed as $V_S(I) = -R_a(I(t) - I_B) + r_b(I(t) - I_B)^3$, where R_a is a gain (in units of Ω), r_b is a nonlinear resistance (in units of ΩA^{-2}), and I_B is a static bias current. To ensure dimensional consistency of the physical quantities, we introduce the effective magnetic field along the x -axis, $h_x(t) = NI(t)/lM_s$, which yields the vdP equation [36] instead of Eq. (3)

$$\frac{d^2 h_x}{dt^2} + 2\omega_c(\beta - \beta_a + \beta_b h_x^2) \frac{dh_x}{dt} + \omega_c^2 h_x = -\frac{d_M}{d} \frac{d^2 m_x}{dt^2}, \quad (5)$$

where $\omega_c = 1/\sqrt{LC}$ is the resonance angular frequency of photon, $\beta = R\sqrt{C/L}/2$ is an effective circuit damping, $\beta_a = R_a\sqrt{C/L}/2$ is an effective gain, and $\beta_b = r_b(M_s l/N)^2 \sqrt{C/L}/2$ is an effective nonlinear damping. By combining and solving Eqs. (1) and (5), the dynamics of the gain-driven MP can be obtained. As we will see in Sec. IV, this effective circuit model provides a comprehensive framework that captures the self-Kerr nonlinearity, gain, and the USC in the MP system.

Throughout this article, the circuit parameters used in the calculations are as follows: $R = 1.0 \Omega$, $C = 1.0$ pF, $l = w = 1.0$ mm, $N = 100$, $d = 100$ nm, $R_a = 2.0 \Omega$, and $r_b = 20 \Omega A^{-2}$, which corresponds to $\omega_c/(2\pi) = 4.49$ GHz, $\beta = 1.4 \times 10^{-2}$, $\beta_a = 2.8 \times 10^{-2}$, and $\beta_b = 0.4$. For simplicity, we also assume $I_B = 0$. The material parameters used in our calculation are those of Yttrium Iron Garnet (YIG), given by $\gamma = 1.76 \times 10^{11} T^{-1} s^{-1}$, $M_s = 120$ kA/m, and $\alpha = 10^{-4}$ [31,37].

III. EFFECTS OF MAGNONIC SELF-KERR NONLINEARITY AND GAIN

Here, based on the circuit model, we discuss the effects of magnonic self-Kerr nonlinearity and gain on the eigenmode of a gain-driven MP. To this end, we herein deal with approximated LLG equations by assuming a small oscillation of the magnetization.

A. Magnonic self-Kerr nonlinearity

To facilitate the subsequent analytical treatment and to capture the essential nonlinear behavior of the gain-driven MP, we derive a second-order approximation of Eq. (1) by expanding it in terms of the transverse magnetization (magnon) $m_\perp = m_x + im_y$, under the assumption $m_z \approx 1 - |m_\perp|^2/2$.

This approximation is valid in the regime of relatively small angle precession and enables analytical access to a nonlinear frequency shift and suppression of the coherent magnon-photon coupling. The resulting approximated form of the LLG equation reads as follows (see also Appendix A):

$$\frac{dm_\perp}{dt} = -i\tilde{\omega}_m m_\perp + i\gamma\mu_0 \tilde{M}_s h_x - i\tilde{\alpha} \frac{dm_\perp}{dt} - i\frac{\alpha}{2} m_\perp \frac{d|m_\perp|^2}{dt}, \quad (6)$$

where

$$\tilde{\omega}_m = \begin{cases} 0 & (H_0 < \tilde{M}_s) \\ \gamma\mu_0(H_0 - \tilde{M}_s) & (H_0 \geq \tilde{M}_s) \end{cases} \quad (7)$$

is the resonance angular frequency of magnon including the self-Kerr nonlinearity with

$$\tilde{M}_s = M_s \left(1 - \frac{|m_\perp|^2}{2} \right), \quad (8)$$

$$\tilde{\alpha} = \alpha \left(1 - \frac{|m_\perp|^2}{2} \right). \quad (9)$$

Equation (8) indicates that a second-order correction ($|m_\perp|^2$) modifies the saturation magnetization especially at the steady state, which manifests the generation of a nonlinear frequency shift and modulation of the coupling strength, as discussed below. Since Eq. (6) can be rewritten as $dm_\perp/dt = -i(\omega_m + \gamma\mu_0 M_s/2|m_\perp|^2)m_\perp \dots$, where $\omega_m = 0$ or $\gamma\mu_0(H_0 - M_s)$ is the eigenmodes of magnon in the linear response, Eq. (6) corresponds to the Duffing-type oscillator in the framework of the second-order approximation [27,28,38]. Then, the first term in the right-hand side of Eq. (6) leads to a nonlinear frequency shift proportional to the square amplitude $|m_\perp|^2$, from ω_m to $\omega_m + \gamma\mu_0 M_s|m_\perp|^2/2$, which can be recognized as magnonic version of the Kerr effect [38]. In a physical perspective, the self-Kerr nonlinearity microscopically corresponds to a four-magnon interaction in the long-wavelength limit, whose origin is magnetic anisotropy such as the shape magnetic anisotropy in this work as well as magnetocrystalline anisotropy [26]. Also, the second term in the right-hand side of Eq. (6), corresponding to the coupling to the photon field h_x , anticipates that the $m_\perp$ -nonlinear term modulates the coupling strength via a modified saturation magnetization $\tilde{M}_s$ at the steady state.

In the subsequent analysis, we employ Eq. (1) to perform numerical calculations while we make use of its second-order approximation to obtain analytical insights into the nonlinear oscillation dynamics. It is worth noting that the MP system described by Eqs. (5) and (6) is mathematically equivalent to a fourth-order autonomous dynamical system consisting of a Duffing-type oscillator (magnon) coupled to a vdP-type oscillator (photon), which allows the coexistence of several attractors such as fixed points, a limit cycle, and chaos [39].

B. Complex eigenfrequency including gain

To derive an excitation condition of the gain-driven MP, we investigate the complex eigenfrequency of MP described by Eqs. (1) and (5). To this end, we decompose $\mathbf{m} = (m_x, m_y, 1)$ by assuming $|m_x|, |m_y| \ll 1$. Restricting to linear order in m_x

and m_y , Eq. (1) reduces to a simple oscillator

$$\frac{d^2 m_x}{dt^2} + 2\alpha\omega_m \frac{dm_x}{dt} + \omega_m^2 m_x = \gamma\mu_0 M_s \omega_m h_x, \quad (10)$$

wherein $\sqrt{1 + \alpha^2} \approx 1$ is assumed for $\alpha \ll 1$. Based on the small oscillation ansatz $h_x(t) = \tilde{h}_x e^{-i\omega t}$ [for linearized (5)] and $m_x(t) = \tilde{m}_x e^{-i\omega t}$ [40], we obtain the coupled LLG and RLC-circuit equations

$$\tilde{\Omega} \begin{pmatrix} \tilde{h}_x \\ \tilde{m}_x \end{pmatrix} = \mathbf{0} \quad (11)$$

with

$$\tilde{\Omega} = \begin{pmatrix} \omega^2 + 2i(\beta - \beta_a)\omega_c\omega - \omega_c^2 & (d_M/d)\omega^2 \\ \gamma\mu_0 M_s \omega_m & \omega^2 + 2i\alpha\omega_m\omega - \omega_m^2 \end{pmatrix}. \quad (12)$$

Hence, the complex eigenfrequencies $\omega_{\pm}/(2\pi)$ are obtained by solving the determinant of Eq. (12).

For the case of small damping such as $|\beta - \beta_a| \ll 1$ and $\alpha \ll 1$, the real part of the complex eigenfrequency is associated with the coherent magnon-photon coupling. Then, the coupling strength at the original modes crossing point ($\omega_m = \omega_c$) is defined as

$$g \equiv \frac{\text{Re}[\omega_+] - \text{Re}[\omega_-]}{2} \bigg|_{\omega_m=\omega_c} \approx \frac{1}{2} \sqrt{\frac{d_M}{d}} \gamma\mu_0 M_s \omega_c. \quad (13)$$

On the other hand, the imaginary part of the complex eigenfrequency characterizes the energy dissipation of the system: $-\text{Im}[\omega_{\pm}] > 0$ corresponds to loss and $-\text{Im}[\omega_{\pm}] \leq 0$ is gain. For a given external magnetic field, auto-oscillation occurs at the region of $-\text{Im}[\omega_{\pm}] \leq 0$. In Figs. 2(a) and 2(b), we plot the imaginary part of the complex eigenfrequencies at the original modes crossing point in SC and USC cases, respectively. Based on Eq. (13), Fig. 2(a) is a strongly coupled case with $d_M/d = 0.02$ corresponding to $g/\omega_c = 0.07$ and Fig. 2(b) is an ultrastrongly coupled case with $d_M/d = 1$ corresponding to $g/\omega_c = 0.49$. For a normalized gain $\beta_a/\beta = 2$ (corresponding to $R_a = 2 \Omega$ and $R = 1 \Omega$), one can see the relation of $-\text{Im}[\omega_+] < -\text{Im}[\omega_-]$ in both cases. Accordingly, ω_+ -mode that obtains a larger gain is energetically more stable and thereby is excited. This self-selection can be understood by considering the complex mode splitting at the original modes crossing point (see Appendix B)

$$\frac{\omega_+ - \omega_-}{2} \bigg|_{\omega_m=\omega_c} = g - \frac{1}{2}g(\alpha + \beta - \beta_a)i, \quad (14)$$

in which a finite imaginary part generates a difference of the lifetime between the $\omega_{\pm}$ -modes, resulting in the self-selection. Figures 2(c) and 2(d) show the real part of the complex eigenfrequencies at the original modes crossing point as a function of β_a/β . For $\beta_a/\beta = 2$, auto-oscillation frequencies $\Omega/(2\pi)$ are 4.80 GHz for the SC case and 7.16 GHz for the USC case, whose values appear to be mostly independent of β_a/β in both cases. Note that this independence sustains as long as the threshold gain for $\text{Im}[\omega_{\pm}] = 0$ is relatively small and far from the exceptional point which leads to the level attraction between $\omega_{\pm}$ -modes [20,40].

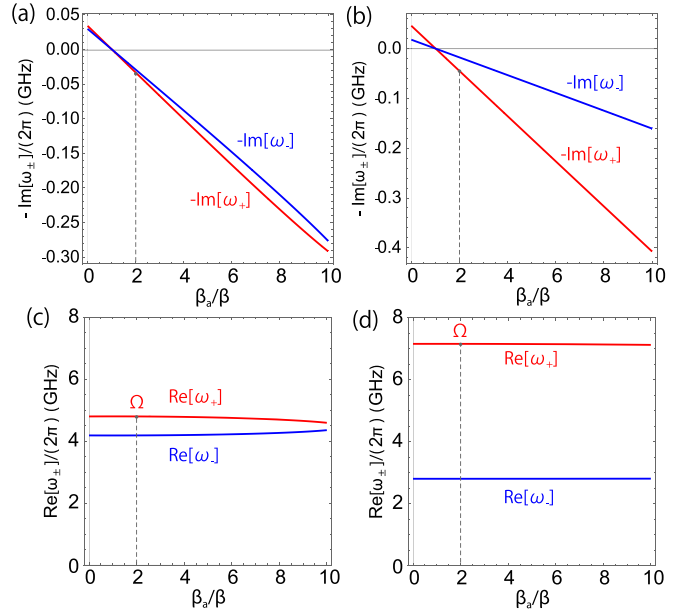


FIG. 2. Imaginary part of the complex eigenfrequencies of the linearized gain-driven MP at the original modes crossing point ($\omega_m = \omega_c$) as a function of a normalized gain (β_a/β). (a) SC case with $d_M/d = 0.02$ and (b) the USC case with $d_M/d = 1$. The real part of the complex eigenfrequencies of the linearized gain-driven MP at the original modes crossing point as a function of a normalized gain (β_a/β). (c) SC case with $d_M/d = 0.02$ and (d) the USC case with $d_M/d = 1$. Auto-oscillation (Ω) occurs at the region of $-\text{Im}[\omega_{\pm}] \leq 0$. For $\beta_a/\beta = 2$, $\text{Re}[\omega_+]$ mode is excited by the self-selection.

IV. NUMERICAL DEMONSTRATION

To demonstrate the nonlinear and gain effects discussed in the previous section, we simulate the transient response of gain-driven MPs with the magnonic Kerr nonlinearity. Without loss of generality, we consider the strongly coupled case with $d_M/d = 0.02$ (corresponding to $g/\omega_c = 0.07$) and the ultrastrongly coupled case with $d_M/d = 1$ ($g/\omega_c = 0.49$). Also, we numerically investigate the behaviors of the auto-oscillation frequency of gain-driven MPs with the self-Kerr nonlinearity as a function of an external magnetic field.

A. Transient response

1. Strong coupling regime

For the SC regime, Figs. 3(a) and 3(d) show the time evolution of photon (h_x) and magnon (m_x), respectively, wherein two distinct types of the oscillation pattern can be identified especially by Fig. 3(d). The first type appears in the early stage of the dynamics, roughly from $t = 0$ ns to $t = 10$ ns, and is characterized by rapid growth and transient behavior. The second type persists over a longer duration, at least from approximately $t = 10$ ns to $t = 50$ ns, indicating a nonlinearity that influences the entire course of the dynamics.

In order to elucidate the underlying cause of the two distinct oscillation patterns, let us discuss the corresponding

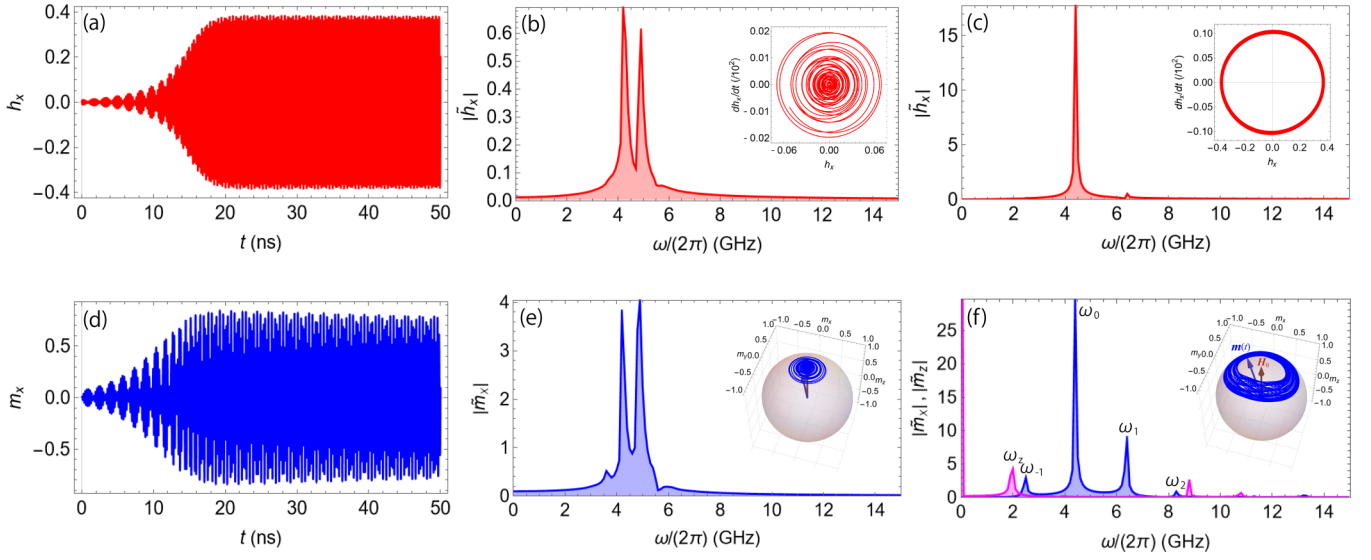


FIG. 3. Time evolution and its Fourier spectra of the gain-driven MP in the SC case with $d_M/d = 0.02$. (a) Time evolution of the photon amplitude. (b) Fourier spectrum of (a) in the time domain $0 \leq t \leq 10$ ns. (c) Fourier spectrum of (a) in the time domain $40 \leq t \leq 50$ ns. (d) Time evolution of the magnon amplitude. (e) Fourier spectrum of (d) in the time domain $0 \leq t \leq 10$ ns. (f) Fourier spectrum of (d) in the time domain $40 \leq t \leq 50$ ns. The Fourier spectrum of m_z is shown in magenta. Insets show the corresponding phase-space trajectories of the photon and magnetization trajectories for the magnon dynamics. The calculations are performed with the initial condition of $h_x(0) = 0.01$ and $m_x(0) = 0$ at the original modes crossing point ($\omega_m = \omega_c$).

Fourier spectrum of the oscillation, as shown in Figs. 3(b) and 3(c) for photons as well as Figs. 3(e) and 3(f) for magnons. Note that insets in each Fourier spectrum correspond to the phase-space trajectories of photon and magnetization trajectories for the magnon dynamics. At first, the oscillation pattern observed early (from $t = 0$ to 10 ns) in the dynamics is characterized by two major peaks seen in Fig. 3(b) or Fig. 3(e). These two peaks correspond to the energy splitting caused by the coherent coupling between photons and magnons. Therefore, the origin of the first type oscillation pattern is regarded as transient Rabi-like oscillations [22]. In contrast, the oscillation pattern observed for $t \gtrsim 10$ ns is governed by the magnonic self-Kerr nonlinearity and manifests as a frequency-comb-like sideband structure. As shown in Fig. 3(f), the Fourier spectrum of $m_x(t)$ in the interval $40 \leq t \leq 50$ ns consists of a dominant peak at ω_0 (close to the linear magnon frequency) accompanied by a set of regularly spaced sidebands. The Kerr nonlinearity induces a slow nonlinear oscillation of $m_z(t)$ at a characteristic frequency $\omega_z/(2\pi) \approx 2$ GHz in Fig. 3(f). The Fourier analysis of $m_x(t)$ shows that the spectral peaks are well described by

$$\omega_n = \omega_0 + n\omega_z \quad (n = 0, \pm 1, \pm 2, \dots), \quad (15)$$

with an almost constant spacing $\Delta\omega = \omega_{n+1} - \omega_n \approx \omega_z$, demonstrating a comblike sideband structure. The magnetization trajectory shown in the inset of Fig. 3(f) is the geometric projection of this two-frequency (quasiperiodic) motion. The rapid rotation at ω_0 combined with the slow modulation at ω_z produces a slowly rotating and breathing closed orbit in the (m_x, m_y, m_z) space, which has no counterpart in the linear dynamics [41,42]. However, this frequency-sideband generation is also transient. Until $t = 500$ ns, we have a stable auto-oscillation characterized by a single-mode peak at 4.42 GHz.

This is the main mode of magnons (ω_m) driven by a gain but its value visibly deviates from the auto-oscillation frequency $\Omega/(2\pi) = 4.80$ GHz anticipated by the analysis of the complex eigenfrequency [see Fig. 2(a)], which is attributed to a frequency shift due to the self-Kerr nonlinearity. This issue will be addressed in greater detail subsequently.

2. Ultrastrong coupling regime

For the USC case, Figs. 4(a) and 4(d) show the time evolution of photon (h_x) and magnon (m_x), respectively. They appear to be more stable oscillation patterns than those of the SC case, as seen in Figs. 3(a) and 3(d). In fact, when we look at the corresponding Fourier spectra displayed in Figs. 4(b) and 4(c) for photons as well as Figs. 4(e) and 4(f) for magnons, we find a transient Rabi-like splitting at least in the duration from $t = 0$ ns to $t = 10$ ns, but in a flash we have a stable auto-oscillation characterized by a single-mode peak until $t = 50$ ns. As shown in Figs. 4(c) and 4(f), the phase-space trajectories of the photon and magnetization trajectories for the magnon dynamics exhibit limit cycles in each dynamics, respectively [39]. Also, let us recall that in the SC case the onset time was determined to be approximately 500 ns to achieve a stable auto-oscillation. In addition, the main mode of magnons has $\omega_m/(2\pi) = 7.28$ GHz, whose value is close to the auto-oscillation frequency $\Omega/(2\pi) = 7.16$ GHz anticipated by the analysis of the complex eigenfrequency [see Fig. 2(b)]. Therefore we can mention that the onset time of the stable oscillation is shorter in the USC regime than in the SC regime, which is attributed to the absence of the frequency-sideband generation in the USC regime. In other words, USC suppresses the frequency-sideband generation caused by the self-Kerr nonlinearity in the SC regime.

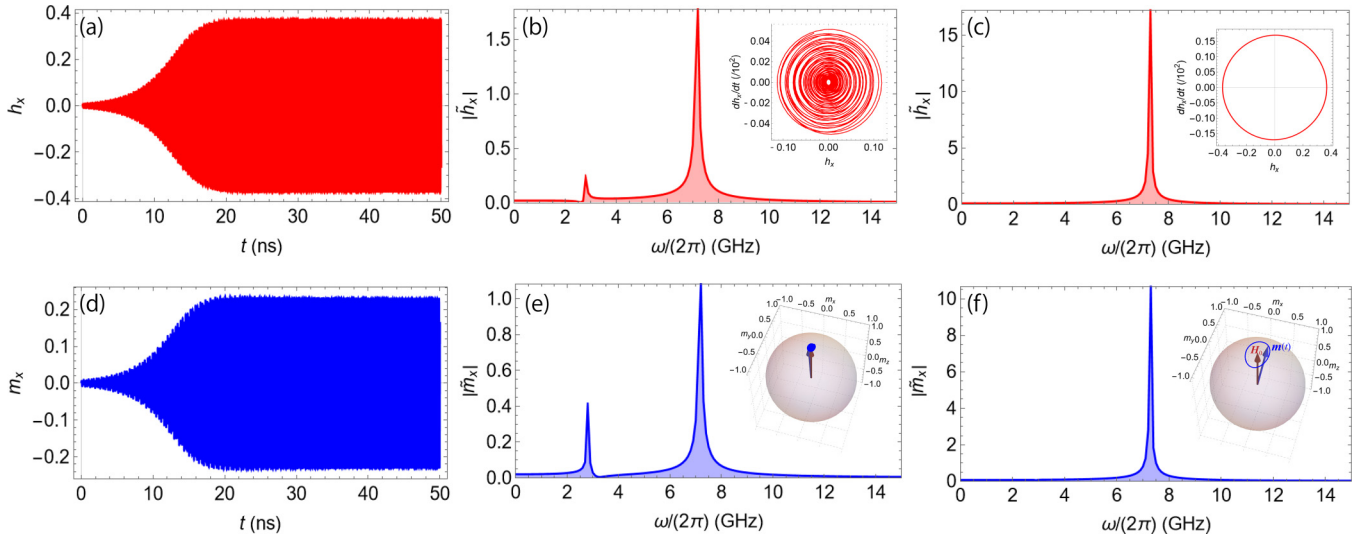


FIG. 4. Time evolution and its Fourier spectra of the gain-driven MP in the USC regime with $d_M/d = 1$. (a) Time evolution of the photon amplitude. (b) Fourier spectrum of (a) in the time domain $0 \leq t \leq 10$ ns. (c) Fourier spectrum of (a) in the time domain $40 \leq t \leq 50$ ns. (d) Time evolution of the magnon amplitude. (e) Fourier spectrum of (d) in the time domain $0 \leq t \leq 10$ ns. (f) Fourier spectrum of (d) in the time domain $40 \leq t \leq 50$ ns. Insets show the corresponding phase-space trajectories of the photon and magnetization trajectories for the magnon dynamics. The calculations are performed with the initial condition of $h_x(0) = 0.01$ and $m_x(0) = 0$ at the original modes crossing point ($\omega_m = \omega_c$).

B. Auto-oscillation

1. Strong coupling regime

Figure 5(a) shows the auto-oscillation frequency $\Omega/(2\pi)$ of the gain-driven MP for the SC case with $d_M/d = 0.02$. The green solid line represents the result obtained by using the LLG equation [Eq. (1)] while the magenta solid line represents the result obtained by using the linearized LLG equation [Eq. (10)]. To clarify the impact of the nonlinear terms in the LLG equation [Eq. (1)] on the auto-oscillation frequency, both frequencies are plotted together for comparison. Focusing on the green solid line in Fig. 5(a), we find that the point of the external magnetic field at which a maximum level repulsion occurs is shifted to the left side compared to the result of the linearized approximation (magenta solid line). Note that each maximum level repulsion occurs at $\mu_0 H_0 \approx 0.244$ T on the green solid line and at $\mu_0 H_0 \approx 0.311$ T on the magenta solid line. This shift is identified with the frequency shift from $\Omega/(2\pi) = 4.80$ GHz to 4.42 GHz previously found in the transient response. Furthermore, the magnitude of the level repulsion, which corresponds to the effective coupling strength $\tilde{g}$, is reduced compared to the result (g) obtained by the linearized LLG equation (magenta solid line). In Fig. 5(a), the value of $\tilde{g}$ at $\mu_0 H_0 = 0.244$ T is given by $\tilde{g} \approx 0.24g$, which implies that the Kerr effect of magnons strongly suppresses the coherent magnon-photon coupling. Then, the gain-driven MP behaves like the original photon-mode driven by a gain, whose auto-oscillation frequency is mostly independent of the external magnetic field.

To begin our discussion on the nonlinear frequency shift and the suppression of the coupling strength in the SC regime, we use an approximated form of the LLG equation [Eq. (6)]. Let us effectively include a correction of the second-order approximation [Eqs. (8) and (9)] into the complex eigenfrequency of the linearized gain-driven MP. At the steady-state

of magnon dynamics, the system settles on a limit cycle on which the transverse amplitude is time independent while its phase rotates. This means that $|m_\perp(t)|^2 = m_x^2(t) + m_y^2(t)$ is constant over time on the limit cycle. We therefore replace $|m_\perp(t)|^2$ in Eqs. (8) and (9) by a constant amplitude m_∞^2 defined from the limit cycle as $m_\infty^2 \equiv \langle |m_\perp(t)|^2 \rangle_T$, where $\langle \dots \rangle_T$ denotes the time average over one period. We define Ω as the angular frequency of the limit-cycle auto-oscillation of the bright magnon-polariton mode. For nearly circular precession, $m_x(t) = m_\infty \cos \Omega t$ and $m_y(t) = m_\infty \sin \Omega t$, so that $|m_\perp(t)|^2 = m_\infty^2$ exactly. We therefore replace $|m_\perp(t)|^2$ in Eqs. (8) and (9) by a constant value of m_∞^2 . Namely, we have $\tilde{M}_s = M_s(1 - m_\infty^2/2)$ and $\tilde{\alpha} = \alpha(1 - m_\infty^2/2)$, by which we replace M_s and α in Eq. (10) to take into account a correction of the second-order approximation. Then, magnon dynamics is governed by

$$\frac{d^2 m_x}{dt^2} + 2\tilde{\alpha}\tilde{\omega}_m \frac{dm_x}{dt} + \tilde{\omega}_m^2 m_x = \gamma\mu_0\tilde{M}_s\tilde{\omega}_m h_x, \quad (16)$$

where $\tilde{\omega}_m = \omega_m + \gamma\mu_0 M_s m_\infty^2/2$, $\tilde{M}_s = M_s(1 - m_\infty^2/2)$, and $\tilde{\alpha} = \alpha(1 - m_\infty^2/2)$. In the same manner as Eq. (11), the corrected complex eigenfrequencies $\tilde{\omega}_\pm/(2\pi)$ are obtained by the determinant of

$$\bar{\Omega}_{nl} = \begin{pmatrix} \omega^2 + 2i(\beta - \beta_a)\omega_c\omega - \omega_c^2 & (d_M/d)\omega^2 \\ \gamma\mu_0\tilde{M}_s\tilde{\omega}_m & \omega^2 + 2i\tilde{\alpha}\tilde{\omega}_m\omega - \tilde{\omega}_m^2 \end{pmatrix}. \quad (17)$$

In Fig. 5(b), we show the imaginary part of the complex eigenfrequencies as a function of the external magnetic field. For this calculation, we use $m_\infty = 0.914$ for $\mu_0 H_0 = 0.244$ T at $t = 500$ ns obtained by solving Eqs. (1) and (5). To clarify the impact of the second-order approximation on the imaginary part of the complex eigenfrequency, the result obtained by the linearized LLG is plotted together for comparison. For

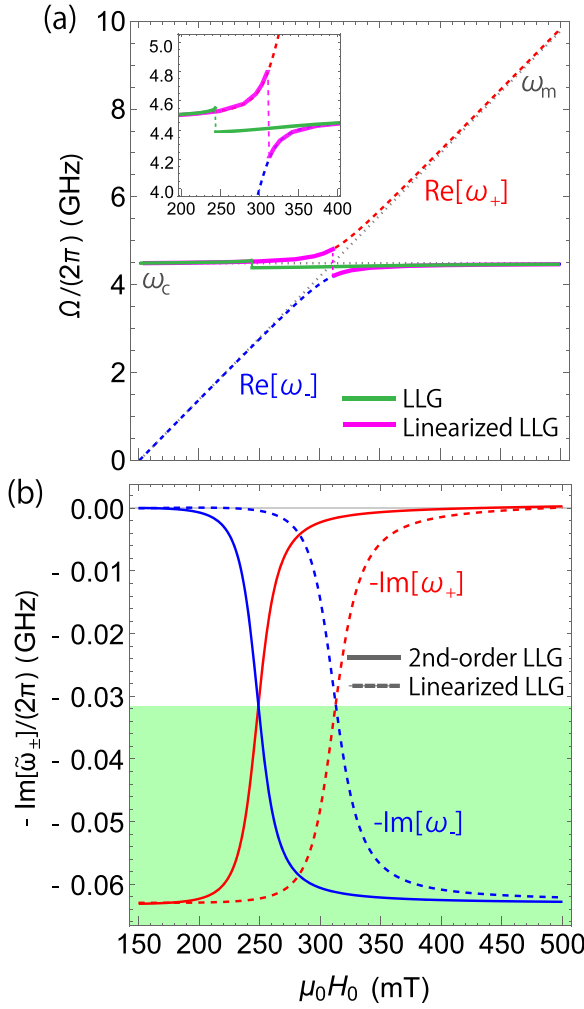


FIG. 5. (a) Auto-oscillation mode (Ω) of the gain-driven MP for the SC case with $d_M/d = 0.02$ as a function of an external magnetic field. The green (magenta) solid line represents the solution obtained without (with) the linearized approximation in the LLG equation [Eq. (1)]. Red and blue dashed lines ($\omega_{\pm}$) represent the real part of the complex eigenfrequencies of the linearized gain-driven MP. Inset shows the zoom of the auto-oscillation mode. For calculating the auto-oscillation frequency, we used the magnetization dynamics between $t = 490$ and 500 ns to obtain the corresponding Fourier spectrum. (b) Imaginary part of the complex eigenfrequencies with the second-order correction (the self-Kerr nonlinearity) as a function of an external magnetic field.

both cases, the green-shaded regions represent self-selection of bright modes, which is characterized by a sustained oscillation. Note that the dark modes that fade away within a limited time due to large dissipation do not appear. The switching point between two bright modes is the point at which the imaginary part of the complex eigenfrequencies crosses (damping exchange point). As shown in Fig. 5(b), the second-order correction, i.e., the self-Kerr nonlinearity, causes the damping exchange point to shift to the left side (down to $\mu_0 H_0 = 0.244$ T) from the original crossing point, which is good agreement with the shift in Fig. 5(a). On the other hand, the coupling term between magnons and photons in the right-hand side of Eq. (16) indicates that the effective

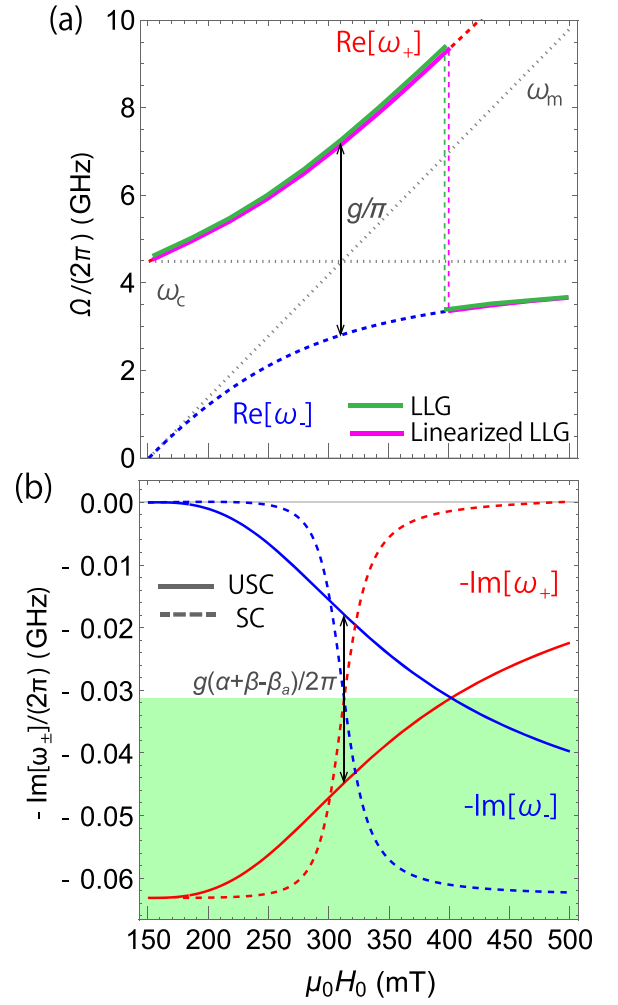


FIG. 6. (a) Auto-oscillation mode (Ω) of the gain-driven MP for the USC case with $d_M/d = 1$. For calculating the auto-oscillation frequency, we used the magnetization dynamics between $t = 490$ ns and 500 ns to obtain the corresponding Fourier spectrum. (b) Imaginary part of the complex eigenfrequencies of the linearized gain-driven MP as a function of an external magnetic field. The solid lines represent the SC case with $d_M/d = 0.02$ and the dashed ones for the USC case with $d_M/d = 1$.

coupling strength $\tilde{g}$ is reduced by the self-Kerr nonlinearity via a modified saturation magnetization $\tilde{M}_s$. By using $\tilde{M}_s \approx 0.58M_s$ for $m_{\infty} = 0.914$, the estimated value is $\tilde{g} \approx 0.76g$. Let us recall that in Fig. 5(a), the value of $\tilde{g}$ at $\mu_0 H_0 = 0.244$ T is given by $\tilde{g} \approx 0.24g$. Hence, the effective coupling strength is overestimated for comparison with the result obtained by using Eq. (1), which may be attributed to a higher-order correction of the approximation beyond Eq. (6) and also to the self-consistency of coupled nonlinear dynamics between magnon and photon subsystems.

2. Ultrastrong coupling regime

Figure 6(a) shows the auto-oscillation frequency $\Omega/(2\pi)$ of the gain-driven MP for the USC case with $d_M/d = 1$. The green and magenta solid lines represent the results obtained without and with the linearized approximation in the LLG equation [Eq. (1)], respectively. As seen, both lines appear

to be almost the same and strongly depend on the external magnetic field, indicating that the coherent magnon-photon coupling is much larger than the nonlinear interaction of magnons characterized by the self-Kerr nonlinearity. In other words, USC suppresses the frequency shift caused by the self-Kerr nonlinearity in the SC regime. This implies that nonlinear magnetization dynamics are suppressed under USC conditions and that there exists a trade-off relation between the coupling strength and the magnitude of the self-Kerr nonlinearity.

The trade-off relation can be intuitively understood by considering the behavior of the ferromagnet and the microwave magnetic field inside the inductor in each coupling regime (SC and USC). In the SC regime, which has a small d_M/d (or g/ω_c) factor, the ferromagnet effectively behaves as an extremely thin film inserted into the inductor. This results in a sufficiently large amplitude of magnetization dynamics relative to the microwave magnetic field inside the inductor, as seen in Figs. 3(a) and 3(d). Thus a small d_M/d (or g/ω_c) factor leads to enhanced nonlinear magnetization dynamics. In contrast, in the USC regime, which has a large d_M/d (or g/ω_c) factor, a thicker ferromagnet is inserted into the inductor compared to the SC case. Substantially, the amplitude of magnetization dynamics remains small relative to the strength of the microwave magnetic field inside the inductor, as seen in Figs. 4(a) and 4(d). Therefore a large d_M/d (or g/ω_c) factor contributes to the emergence of nearly linear magnetization dynamics.

Furthermore, in Fig. 6(a), we find that the point of the external magnetic field at which a maximum level repulsion occurs is shifted to the right side compared to the original modes crossing point at $\mu_0 H_0 \approx 0.311$ T. Note that a maximum level of repulsion occurs around $\mu_0 H_0 \approx 0.397$ T on the green solid-line as well as the magenta solid-line. To understand the origin of this shift, we analyze an imaginary part of the complex eigenfrequency of the linearized gain-driven MP based on Eq. (12) [see also Appendix B]. In Fig. 6(b), we show the imaginary part of the complex eigenfrequencies as a function of the external magnetic field. To clarify the impact of the USC on the imaginary part of the complex eigenfrequency, not only the USC case but also the SC case are plotted together for comparison. For both cases, the green-shaded regions represent self-selection of bright modes. As shown in Fig. 6(b), increasing the coupling strength causes the damping exchange point to shift to the right side (up to $\mu_0 H_0 = 0.398$ T) from the original crossing point, which is in good agreement with the shift in Fig. 6(a).

The origin of this shift can be understood by considering the complex mode splitting at the original modes crossing point [Eq. (14)]. Equation (14) indicates that a gap opens in the imaginary parts of the complex eigenfrequencies at the original modes crossing point, resulting in the observed auto-oscillation frequency shift in Fig. 6(a). Hence, the origin of the auto-oscillation frequency shift is related with the magnon-photon coupling and gain. Based on Eq. (14), in the SC regime with a relatively small coupling strength, the shift of the damping exchange point appears to be absent. In Fig. 6(a), we find that the large shift of the damping exchange point in the USC case results in the emergence of the magnon-like mode in $0.3 \text{ T} \leq \mu_0 H_0 \leq 0.4 \text{ T}$ compared to the SC case.

Since this feature is prominent in the USC regime, suggesting that, for applications such as maserlike devices, the USC is advantageous owing to a widely tunable frequency range and controllability of auto-oscillation frequency by means of the external magnetic field.

V. CONCLUSION

In conclusion, we have theoretically investigated the nonlinear dynamics of gain-driven MPs based on an effective circuit model that combines the *RLC*-circuit equation for photons and the LLG equation for magnons. Numerical calculations revealed that in the SC regime, magnonic self-Kerr nonlinearity plays a dominant role in the dynamics of magnon, resulting in transient frequency-sidebands generation and a frequency shift. These nonlinear effects suppress the coherent magnon-photon coupling in the gain-driven MP. In contrast, in the USC regime, the self-Kerr nonlinearity is effectively reduced, leading to stable and nearly linear oscillations of the gain-driven MP. Consequently, the auto-oscillation mode exhibits magnon-like features over a broader range of external magnetic fields.

Theoretical analysis based on the LLG equation with suitable approximations confirmed a trade-off relation between the coupling strength and the self-Kerr nonlinearity: increasing the coupling strength suppresses the observed nonlinear behavior, thereby enhancing stability and tunability of the gain-driven MP. Furthermore, the USC induced a frequency shift in the auto-oscillation mode due to a shift of the crossing point of the imaginary part of the complex eigenfrequency, which is attributed to the interplay between gain-loss and USC in polariton dynamics. These results suggest that ultra-strongly coupled MP systems provide a promising platform for developing maser-like devices [7,8] with a widely tunable frequency range and controllability by means of the external magnetic field. In addition, the present study suggests that magnon (spin) auto-oscillation can be realized by utilizing gain-driven hybrid systems, i.e., gain-driven coherent magnetization dynamics, which may provide a novel operational principle for spintronic oscillators such as spin-torque oscillators [7,43].

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

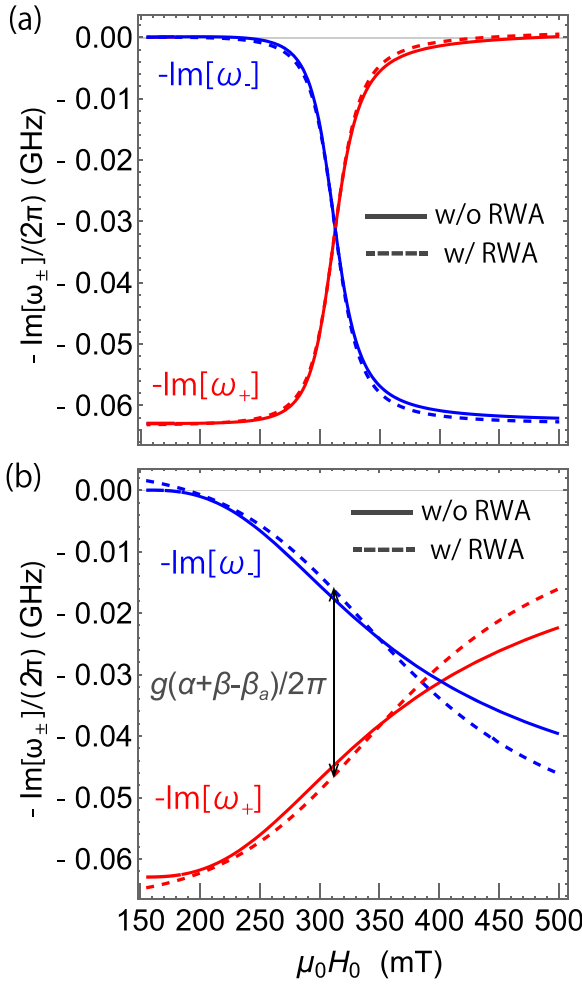


FIG. 7. Imaginary part of the complex eigenfrequencies of the linearized gain-driven MP as a function of an external magnetic field. (a) SC case with $d_M/d = 0.02$ and (b) USC case with $d_M/d = 1$.

APPENDIX A: COHERENCE AND NONLINEARITY IN LLG EQUATION

Here, we show which term represents the “coherence” and “nonlinearity” in our model via Eq. (6). Combining Eqs. (6) to (9), Eq. (6) is rewritten as follows:

$$\begin{aligned} \frac{dm_{\perp}}{dt} = & -i\omega_m m_{\perp} - i\frac{\gamma\mu_0 M_s}{2}|m_{\perp}|^2 m_{\perp} + i\gamma\mu_0 M_s h_x \\ & - i\frac{\gamma\mu_0 M_s}{2}|m_{\perp}|^2 h_x - i\alpha \frac{dm_{\perp}}{dt} - i\alpha \frac{m_{\perp}^2}{2} \frac{dm_{\perp}^*}{dt}. \end{aligned} \quad (\text{A1})$$

The magnon-photon coherent coupling is represented by the third term of the right-hand side in the above equation, which describes the transverse magnetization (magnon) driven by the microwave magnetic field (photon) in the linear response. On the other hand, the nonlinearity in our model is characterized by the second and fourth terms of the right-hand side in Eq. (A1). The former (the second term) originates from the shape magnetic anisotropy in Eq. (2). This amplitude-dependent magnetization feeds back into the magnon dynamics as a Kerr/Duffing-type frequency shift, i.e., as a nonlinear self-action of the magnon mode. The latter (the fourth term) describes the nonlinear coupling between magnons and photons, which modulates the coupling strength at the steady state.

APPENDIX B: DERIVATION OF COMPLEX MODE SPLITTING

The complex eigenfrequencies $\omega_{\pm}/(2\pi)$ of the linearized gain-driven MP are obtained by solving the determinant of Eq. (12):

$$\begin{aligned} & (\omega^2 + 2i(\beta - \beta_a)\omega_c\omega - \omega_c^2)(\omega^2 + 2i\alpha\omega_m\omega - \omega_m^2) \\ & - (d_M/d)\gamma\mu_0 M_s \omega_m \omega^2 = 0. \end{aligned} \quad (\text{B1})$$

At the original modes crossing point, let us introduce a classical version of the RWA [33–35], namely, $\omega^2 - \omega_c^2 \approx 2\omega_c(\omega - \omega_c)$ and $\omega^2 - \omega_m^2 \approx 2\omega_c(\omega - \omega_m)$, to Eq. (B1). Furthermore, writing $\omega = \omega_c + \delta$ with $\delta \ll \omega$, we neglect the terms of order δ^2 and $\delta(d_M/d)\gamma\mu_0 M_s$, leading to

$$(\omega + i(\beta - \beta_a)\omega - \omega_c)(\omega + i\alpha\omega - \omega_m) - g^2 = 0, \quad (\text{B2})$$

where $g = \sqrt{(d_M/d)\gamma\mu_0 M_s \omega_c}/2$. Then, we find the complex eigenfrequencies at the original modes crossing point

$$\omega_{\pm} = \omega_c \pm g - \frac{1}{2}(\alpha + \beta - \beta_a)(\omega_c \pm g)i. \quad (\text{B3})$$

Therefore, the complex mode splitting is given by Eq. (14).

Figure 7 shows the imaginary part of the complex eigenfrequencies of the linearized gain-driven MP with (without) RWA in each coupling regime. As shown in Fig. 7(a), in the SC regime, the RWA appears to be a good approximation which well reproduces the result of the w/o RWA. In contrast, in the USC regime, one can find a visible deviation of the damping exchange point between the two calculation cases in Fig. 7(b), indicating that a nontrivial frequency shift such as BS shift affects the self-selection of the gain-driven MP.

[1] M. Harder, B. M. Yao, Y. S. Gui, and C.-M. Hu, Coherent and dissipative cavity magnonics, *J. Appl. Phys.* **129**, 201101 (2021).
[2] B. Z. Rameshti, S. V. Kusminskiy, J. A. Haigh, K. Usami, D. L.-Quirion, Y. Nakamura, C.-M. Hu, H. X. Tang, G. E. Bauer, and Y. M. Blanter, Cavity magnonics, *Phys. Rep.* **979**, 1 (2022).
[3] H. Y. Yuan, Y. Cao, A. Kamra, R. A. Duine, and P. Yan, Quantum magnonics: When magnon spintronics meets quantum information science, *Phys. Rep.* **965**, 1 (2022).

[4] D. L.-Quirion, S. P. Wolski, Y. Tabuchi, S. Kono, K. Usami, and Y. Nakamura, Entanglement-based single-shot detection of a single magnon with a superconducting qubit, *Science* **367**, 425 (2020).
[5] X. Zhang, C.-L. Zou, N. Zhu, F. Marquardt, L. Jiang, and H. X. Tang, Magnon dark modes and gradient memory, *Nat. Commun.* **6**, 8914 (2015).
[6] R.-C. Shen, Y.-P. Wang, J. Li, S.-Y. Zhu, G. S. Agarwal, and J. Q. You, Long-Time memory and ternary logic gate using

- a multistable cavity magnonic system, *Phys. Rev. Lett.* **127**, 183202 (2021).
- [7] J. T. Hou, P. Zhang, and L. Liu, Proposal for a spin-torque-oscillator maser enabled by microwave photon-spin coupling, *Phys. Rev. Appl.* **16**, 034034 (2021).
 - [8] B. Yao, Y. S. Gui, J. W. Rao, Y. H. Zhang, W. Lu, and C.-M. Hu, Coherent microwave emission of gain-driven polaritons, *Phys. Rev. Lett.* **130**, 146702 (2023).
 - [9] P. F.-Díaz, L. Lamata, E. Rico, J. Kono, and E. Solano, Ultrastrong coupling regimes of light-matter interaction, *Rev. Mod. Phys.* **91**, 025005 (2019).
 - [10] A. F. Kockum, A. Miranowicz, S. D. Liberato, S. Savasta, and F. Nori, Ultrastrong coupling between light and matter, *Nat. Rev. Phys.* **1**, 19 (2019).
 - [11] A. Baydin, H. Zhu, M. Bamba, K. R. A. Hazzard, and J. Kono, Perspective on the quantum vacuum in matter, *Opt. Mater. Express* **15**, 1833 (2025).
 - [12] P. F.-Díaz, J. Lisenfeld, D. Marcos, J. J. García-Ripoll, E. Solano, C. J. P. M. Harmans, and J. E. Mooij, Observation of the Bloch-Siegert shift in a qubit-oscillator system in the ultrastrong coupling regime, *Phys. Rev. Lett.* **105**, 237001 (2010).
 - [13] X. Li, M. Bamba, Q. Zhang, S. Fallahi, G. C. Gardner, W. Gao, M. Lou, K. Yoshioka, M. J. Manfra, and J. Kono, Vacuum Bloch-Siegert shift in Landau polaritons with ultra-high cooperativity, *Nat. Photon.* **12**, 324 (2018).
 - [14] I. A. Golovchanskiy, N. N. Abramov, V. S. Stolyarov, M. Weides, V. V. Ryazanov, A. A. Golubov, A. V. Ustinov, and M. Y. Kupriyanov, Ultrastrong photon-to-magnon coupling in multilayered heterostructures involving superconducting coherence via ferromagnetic layers, *Sci. Adv.* **7**, eabe8638 (2021).
 - [15] I. Golovchanskiy, N. N. Abramov, V. S. Stolyarov, A. A. Golubov, M. Yu. Kupriyanov, V. V. Ryazanov, and A. V. Ustinov, Approaching deep-strong on-chip photon-to-magnon coupling, *Phys. Rev. Appl.* **16**, 034029 (2021).
 - [16] A. Ghirri, C. Bonizzoni, M. Maksutoglu, A. Mercurio, O. Di Stefano, S. Savasta, and M. Affronte, Ultrastrong magnon-photon coupling achieved by magnetic films in contact with superconducting resonators, *Phys. Rev. Appl.* **20**, 024039 (2023).
 - [17] G. Bourcin, J. Bourhill, V. Vlaminck, and V. Castel, Strong to ultrastrong coherent coupling measurements in a YIG/cavity system at room temperature, *Phys. Rev. B* **107**, 214423 (2023).
 - [18] K. Mita, T. Chiba, T. Kodama, T. Ueda, T. Nakanishi, K. Sawada, and S. Tomita, Ultrastrongly coupled and directionally nonreciprocal magnon polaritons in magnetochiral metamolecules, *Phys. Rev. Appl.* **23**, L011004 (2025).
 - [19] K. Mita, T. Kodama, T. Nakanishi, T. Ueda, K. Sawada, T. Chiba, and S. Tomita, Microwave one-way transparency by large synthetic motion of magnetochiral polaritons in metamolecules, *arXiv:2503.22279*.
 - [20] C. Zhang, M. Kim, Y.-H. Zhang, Y.-P. Wang, D. Trivedi, A. Krasnok, J. Wang, D. Isileifson, R. Roshko, and C.-M. Hu, Gainloss coupled systems, *APL Quantum* **2**, 011501 (2025).
 - [21] M. Kim, C. Zhang, C. Lu, and C.-M. Hu, Low phase noise microwave oscillator based on gain driven polariton, *Appl. Phys. Lett.* **124**, 114103 (2024).
 - [22] Y. S. Gui and C.-M. Hu, Transient response of a gain-driven polariton, *Phys. Rev. Appl.* **21**, 044023 (2024).
 - [23] C. Lu, M. Kim, C. Zhang, and C.-M. Hu, Solid-state injection locking microwave amplifier, *J. Appl. Phys.* **136**, 053903 (2024).
 - [24] Y. S. Gui, G. S. Kozyniak, and C.-M. Hu, Broadband microwave emission from acoustically modulated gain-driven polaritons, *J. Appl. Phys.* **137**, 043905 (2025).
 - [25] J. Rao, C. Y. Wang, B. Yao, Z. J. Chen, K. X. Zhao, and W. Lu, Meterscale strong coupling between magnons and photons, *Phys. Rev. Lett.* **131**, 106702 (2023).
 - [26] S. Zheng, Z. Wang, Y. Wang, F. Sun, Q. He, P. Yan, and H. Y. Yuan, Tutorial: Nonlinear magnonics, *J. Appl. Phys.* **134**, 151101 (2023).
 - [27] M. Elyasi, Y. M. Blanter, and G. E. W. Bauer, Resources of nonlinear cavity magnonics for quantum information, *Phys. Rev. B* **101**, 054402 (2020).
 - [28] R. Tatsumi, T. Chiba, T. Komine, and H. Matsueda, Chaotic magnetization dynamics in magnetic duffing oscillator, *Phys. Rev. E* **111**, 064202 (2025).
 - [29] Y.-P. Wang, G.-Q. Zhang, D. Zhang, X.-Q. Luo, W. Xiong, S.-P. Wang, T.-F. Li, C.-M. Hu, and J. Q. You, Magnon Kerr effect in a strongly coupled cavity-magnon system, *Phys. Rev. B* **94**, 224410 (2016).
 - [30] Y.-P. Wang, G.-Q. Zhang, D. Zhang, T.-F. Li, C.-M. Hu, and J. Q. You, Bistability of cavity magnon polaritons, *Phys. Rev. Lett.* **120**, 057202 (2018).
 - [31] L. Bai, M. Harder, Y. P. Chen, X. Fan, J. Q. Xiao, and C.-M. Hu, Spin pumping in electrodynamically coupled magnon-photon systems, *Phys. Rev. Lett.* **114**, 227201 (2015).
 - [32] V. L. Grigoryan, K. Shen, and K. Xia, Synchronized spin-photon coupling in a microwave cavity, *Phys. Rev. B* **98**, 024406 (2018).
 - [33] T. Chiba, T. Komine, and T. Aono, Ultrastrong-coupled magnon-polariton in a dynamical inductor based on magnetic-insulator/topological-insulator bilayers, *Appl. Phys. Lett.* **124**, 012402 (2024).
 - [34] T. Chiba, T. Komine, and T. Aono, Microwave transmission theory for on-chip ultrastrong-coupled magnon-polariton in dynamical inductors, *J. Magn. Soc. Jpn.* **48**, 21 (2024).
 - [35] T. Chiba, R. Suzuki, T. Otaki, and H. Matsueda, Circuit-based cavity magnonics in the ultrastrong and deep-strong coupling regimes, *Phys. Rev. B* **112**, 174403 (2025).
 - [36] S. H. Strogatz, *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering*, 2nd ed. (CRC Press, Boca Raton, 2015).
 - [37] M. Schreier, T. Chiba, A. Niedermayr, J. Lotze, H. Huebl, S. Geprägs, S. Takahashi, G. E. W. Bauer, R. Gross, and S. T. B. Goennenwein, Current-induced spin torque resonance of a magnetic insulator, *Phys. Rev. B* **92**, 144411 (2015).
 - [38] C. Zhang, M. Kim, J. Wang, and C.-M. Hu, Van der pol-duffing oscillator and its application to gain-driven light-matter interaction, *Phys. Rev. Appl.* **22**, 014034 (2024).
 - [39] S. T. Tanekou, J. Ramadoss, J. Kengne, G. D. Kenmoe, and K. Rajagopal, Coexistence of periodic, chaotic and hyperchaotic attractors in a system consisting of a duffing oscillator coupled to a van der Pol oscillator, *Int. J. Bifurcation Chaos* **33**, 2330004 (2023).
 - [40] R. Cheng, D. Xiao, and A. Brataas, Terahertz antiferromagnetic spin Hall nano-oscillator, *Phys. Rev. Lett.* **116**, 207603 (2016).

- [41] S. Sugimoto, S. Iwakiri, Y. Kozuka, Y. Takahashi, Y. Niimi, K. Kobayashi, and S. Kasai, Multiple modes of a single spin torque oscillator under the non-linear region, [AIP Adv. **10**, 075115 \(2020\)](#).
- [42] I. N. Yemeli, S. Perna, D. Gouéré, A. Kolli, S. Sangiao, J. M. De Teresa, M. Muñoz, A. Anane, M. d' Aquino, *et al.*, Self-Modulation instability in high power ferromagnetic resonance of BiYIG nanodisks, [Phys. Rev. Lett. **135**, 056703 \(2025\)](#).
- [43] S. Jiang, L. Yao, S. Wang, D. Wang, L. Liu, A. Kumar, A. A. Awad, A. Litvinenko, M. Ahlberg, R. Khymyn, S. Chung, G. Xing, and J. Åkerman, Spin-torque nano-oscillators and their applications, [Appl. Phys. Rev. **11**, 041309 \(2024\)](#).