

Josephson phase shift and diode effect due to the inverse spin Hall effect

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We theoretically study the direct and inverse spin Hall effects in a superconductor-normal metal-superconductor junction induced by a spin-orbit interaction that is invariant under spatial inversion. We show that a supercurrent induces a spin Hall effect, leading to a static spin accumulation with opposite polarizations at the two edges, analogous to that in normal conductors. For the inverse effect, we consider a spatially inhomogeneous static magnetic field and show that it induces an anomalous phase shift, which, in the presence of higher harmonics, results in a diode effect. Unlike Rashba systems studied previously, the present mechanism does not require broken structural inversion symmetry, since an inhomogeneous magnetic field, equivalent to a spin current, breaks the inversion symmetry extrinsically.

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I. INTRODUCTION

Spintronic effects in superconductors have attracted considerable interest over the past few decades, driven by the prospect of combining nondissipative supercurrents with the nonvolatility of spin-based devices. After pioneering experiments dating back to the late 1970s [1], superconducting spintronics has undergone rapid development over the past decade. Key advances include the realization of long-range spin-triplet supercurrents in ferromagnet-superconductor hybrids [2–7], electric and magnetic control of superconducting spin transport [8–10], and the integration of materials with strong spin-orbit coupling (e.g., topological insulators and Rashba interfaces) to manipulate spin textures [11–13]. These developments open pathways toward dissipationless memory and logic elements and deepen the interplay between superconductivity and topology [14,15].

Of particular recent interest are systems with broken symmetries [16–22]. In superconductors lacking inversion symmetry, nonreciprocal supercurrent transport becomes possible in the presence of an applied magnetic field or magnetization. This manifests as a direction-dependent critical current, known as the superconducting diode effect, first observed in inversion-asymmetric superlattices [23]. The superconducting diode effect has since been demonstrated in Josephson junctions comprising superconductors coupled to normal metals or semiconductors with Rashba spin-orbit cou-

pling [24–26]. Rashba spin-orbit coupling, which arises from the interplay of spin-orbit interaction and inversion symmetry breaking, generates a uniform spin polarization under an applied current via the Rashba-Edelstein effect, for both dissipative currents and supercurrents [27,28].

In superconductor-normal metal-superconductor (SNS) Josephson junctions, Ref. [29] predicted that a supercurrent induces a uniform spin polarization and, conversely, that a magnetic field generates a supercurrent—an effect known as the inverse Rashba-Edelstein effect. This inverse effect was shown to produce an anomalous phase shift between the superconductors. However, such a phase shift alone does not give rise to a diode effect, as it does not break the symmetry of the phase potential and therefore yields identical barrier heights for both current polarities. More recently, it has been shown that this anomalous phase shift can lead to a Josephson diode effect when higher harmonics—arising from multiple interface scattering—are taken into account [26].

In this paper, we study the spin Hall effect (SHE) and its inverse effect (ISHE) theoretically in an SNS junction (Fig. 1) induced by a spin-orbit interaction that is invariant under spatial inversion. We show that a supercurrent induces a SHE, leading to a static spin accumulation of opposite polarizations at the two edges, analogous to the behavior in normal conductors. We also investigate the inverse effect by considering a spatially inhomogeneous static magnetic field. Such a magnetic field, acting on the electron spin, generates a steady spin current [30], which in turn couples to the superconducting channel, inducing an anomalous phase shift. We show that the anomalous phase shift gives rise to the diode effect by accounting for higher harmonics. In contrast to the diode effect due to the inverse Rashba-Edelstein effect, the ISHE-induced diode effect does not require the system to lack

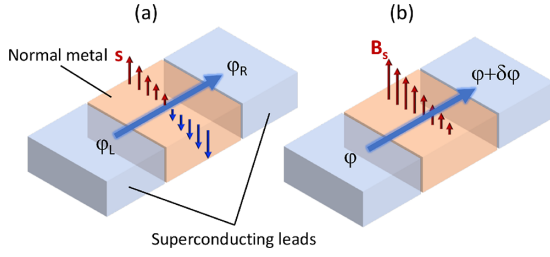


FIG. 1. Schematic figures showing (a) the direct and (b) the inverse spin Hall effect in SNS. In the direct spin Hall effect, equilibrium spin polarization with spatial inhomogeneity arises as a response to a supercurrent induced by the phase difference. The inverse effect is induced by a static magnetic field with a spatial gradient, resulting in a supercurrent and an anomalous phase shift.

inversion symmetry, since the spin current breaks inversion symmetry extrinsically. In our setup, a spatially inhomogeneous magnetic field is essential for the diode effect, as it induces a steady spin current, which breaks the inversion symmetry. In the Rashba case with broken structural inversion, in contrast, a homogeneous applied magnetic field is sufficient for the diode effect.

The paper is organized as follows. We start by summarizing the direct and inverse spin Hall effects in normal metals, as argued in terms of the spin density and electric current in Sec. II. The model of the SNS junction and calculation scheme by use of the lesser Green's function is presented in Sec. III, followed by the derivation of the Josephson current in our scheme in Sec. IV. Our study of the supercurrent-induced spin Hall effect starts in Sec. V, where we calculate the spin density induced by the Josephson current due to a phase difference at equilibrium, including to the linear order the self-energy due to each SN interface. In Sec. VI, the inverse spin Hall effect induced by an applied magnetic field gradient is calculated. The higher harmonics of the inverse spin Hall current are studied in Sec. VII and the Josephson diode effect, the difference in the critical current for superconductivity under opposite current bias, is studied in Sec. VIII. Discussion on the SHE and ISHE is presented in Sec. IX. The case of the Rashba spin-orbit interaction, reported previously [29], is briefly argued in Sec. X. Summary is presented in Sec. XI.

II. PREMISES ON SPIN HALL EFFECT IN NORMAL METAL

Conventionally, the SHE in normal metal is described in terms of the spin current generated by the applied electric field [31]. Since the spin current is not directly measurable, it is converted to an experimentally measurable spin accumulation using the classical spin diffusion equation. This conventional theory has a fundamental ambiguity arising from the nonconserving nature of spin current in solids.

A. Reciprocal relations in terms of spin density

A description free from such a fundamental difficulty can be constructed in terms of the direct response function between spin density and electric current [32]. In the case of a normal metal, the spin density induced by the applied elec-

tromagnetic field is written in terms of a vector potential $\mathbf{A}$ as

$$s_i = \chi_{ij}^{\text{sh}} A_j, \quad (1)$$

where the response function χ_{ij}^{sh} is a correlation function of spin and electric current, with $i, j = x, y, z$ denoting spatial directions. As seen from the symmetry, χ_{ij}^{sh} does not have a uniform component in inversion-symmetric systems, and is expanded with respect to the external wave vector $\mathbf{q}$ and frequency Ω as [32]

$$\chi_{ij}^{\text{sh}}(\mathbf{q}, \Omega) = i\epsilon_{ijk} q_k (\lambda_0 + i\Omega\lambda_1 + \dots), \quad (2)$$

which reads (by use of Maxwell's equation $\dot{\mathbf{B}} = -\nabla \times \mathbf{E}$)

$$\mathbf{s} = \lambda_0 \mathbf{B} - \lambda_1 (\nabla \times \mathbf{E}) + \dots, \quad (3)$$

in the ballistic case. A dynamic term λ_1 indicates that spin density arises from the vorticity of an electric current, $\nabla \times \mathbf{j}$, where electric current is $\mathbf{j} = \sigma_B \mathbf{E}$, σ_B being the Boltzmann conductivity. The SHE corresponds to this term. In fact, the vorticity arises at the edge of the system, where the current density diminishes, consistent with spin accumulation resulting from the spin Hall effect [32]. Although the effect induces a static spin accumulation in a static electric field, it is a dynamic effect because the electric field induces out-of-equilibrium responses. Due to Onsager reciprocity, the ISHE induced by an applied magnetic field $\mathbf{B}_s$ coupling to the electron spin is written in terms of the same response function as

$$j_i = -\chi_{ji}^{\text{sh}} \gamma B_{s,j}, \quad (4)$$

where γ is the gyromagnetic ratio. The current induced by the spin magnetic field thus reads

$$\mathbf{j} = -\gamma\lambda_0 \nabla \times \mathbf{B}_s - \gamma\lambda_1 \nabla \times \dot{\mathbf{B}}_s. \quad (5)$$

The ISHE, represented by the term λ_1 , is a dynamic effect, while the static effect, λ_0 , describes the Ampère's law. The ISHE, being of dynamic origin, is a natural consequence of the fact that normal current reacts to a physical field $\mathbf{E}$ and not to the vector potential [29].

B. Diffusive case and spin Hall angle

Taking electron diffusion into account, ballistic behaviors become nonlocal on the scale of the spin diffusion length, ℓ_s , and the spin Hall effect is modified to be [32]

$$\mathbf{s}(\mathbf{r}) = -\frac{\lambda_1}{\sigma_B} \int d\mathbf{r}' D_s(\mathbf{r} - \mathbf{r}') (\nabla \times \mathbf{j})_{\mathbf{r}'}, \quad (6)$$

where $D_s(\mathbf{r}) = \int \frac{dp}{2\pi} \frac{e^{ip\mathbf{r}}}{(Dp^2 + \tau_s^{-1})\tau}$ is the spin diffusion propagator, D being the diffusion constant, and τ_s (related to ℓ_s as $\ell_s = \sqrt{D\tau_s}$) is the spin relaxation time (m and k_F are the electron mass and the Fermi wave number, respectively). Evaluating the diffusion propagator in one dimension in the y direction perpendicular to the junction, we have $D_s(y) = \frac{\ell_s}{2\ell^2} e^{-|y|/\ell_s}$, where ℓ is the electron mean-free path. Treating the current density without diffusion to vanish suddenly at the system edges, $y = \pm W/2$ (W is the system width), Eq. (6) leads to

$$s_z(y) = -\lambda_1 E \frac{\ell_s}{\ell^2} e^{-W/(2\ell_s)} \sinh \frac{y}{\ell_s}. \quad (7)$$

The maximum spin density at the edge is thus $s_z(\pm W/2) = \mp \frac{\ell_s}{2\ell^2} \lambda_1 E (1 - e^{-W/\ell_s})$. In terms of the conventional dimensionless spin Hall angle, defined as $\theta_{\text{sh}} = \sigma_s/\sigma_B$, where σ_s is the spin Hall conductivity, the spin accumulation density at the edge is $s_{\text{edge}} = \theta_{\text{sh}} \ell_s \nu n E$, n and ν being the electron density and density of states at the Fermi energy, respectively. From the agreement between the two approaches in the limit $W/\ell_s \gg 1$, $|s_z(\pm W/2)| = s_{\text{edge}}$, we identify the spin Hall angle as

$$\theta_{\text{sh}} = \frac{\lambda_1}{2n\nu\ell^2}. \quad (8)$$

The coefficient λ_1 induced by the random spin-orbit interaction was calculated in Ref. [32] to be (note that the definitions of the coefficients in Ref. [32] are different from here)

$$\lambda_1 = \frac{\pi}{9} \epsilon_{\text{so}} \nu^2 n \ell^2, \quad (9)$$

where ϵ_{so} is the energy associated with the spin-orbit interaction. The spin Hall angle,

$$\theta_{\text{sh}} = \frac{\pi}{18} \epsilon_{\text{so}} \nu, \quad (10)$$

is thus the ratio of the spin-orbit energy and the Fermi energy ϵ_F ($\sim 1/\nu$).

III. MODEL OF SNS JUNCTION

We consider an SNS junction with a normal metal N with spin-orbit interaction, described by a field Hamiltonian

$$H = H_N + \sum_{\alpha=L,R} H_\alpha + H_t, \quad (11)$$

where the Hamiltonian for N represented by field operators c and $c^\dagger$ is

$$H_N = \sum_{k\sigma} \epsilon_k c_{k\sigma}^\dagger c_{k\sigma} + H_{\text{so}} + H_{\text{imp}}, \quad (12)$$

where $\epsilon_k = \frac{k^2}{2m} - \epsilon_F$, H_{so} being the Hamiltonian for the spin-orbit interaction. Random impurity scattering conserving spin is included by H_{imp} , defined as

$$H_{\text{imp}} \equiv v_i \sum_{kk'} \sum_i e^{-i(k'-k) \cdot \mathbf{R}_i} c_{k'}^\dagger c_k, \quad (13)$$

where v_i is the strength of the impurity potential, and $\mathbf{R}_i$ is the position of impurities and i is the label of impurities. The impurity scattering gives rise to a finite elastic lifetime and diffusive electron motion.

The superconductors L and R are described by

$$H_\alpha = \sum_k \begin{pmatrix} d_{\alpha k \uparrow}^\dagger \\ d_{\alpha, -k \downarrow} \end{pmatrix} \begin{pmatrix} \epsilon_k & \Delta e^{-i\varphi_\alpha} \\ \Delta e^{i\varphi_\alpha} & -\epsilon_k \end{pmatrix} \begin{pmatrix} d_{\alpha k \uparrow} \\ d_{\alpha, -k \downarrow}^\dagger \end{pmatrix}, \quad (14)$$

where $d_{\alpha k \sigma}^\dagger$ and $d_{\alpha k \sigma}$ are field operators of the electron in superconductor $\alpha = L, R$ with wave vector $\mathbf{k}$ and spin σ . The magnitude of the order parameter is Δ , and its phase is φ_α . The interface hopping is represented by

$$H_t = \sum_{\alpha, \sigma=\uparrow, \downarrow} \sum_{\mathbf{r} \in I_\alpha} t [d_{\alpha \sigma}^\dagger c_\sigma + c_\sigma^\dagger d_{\alpha \sigma}], \quad (15)$$

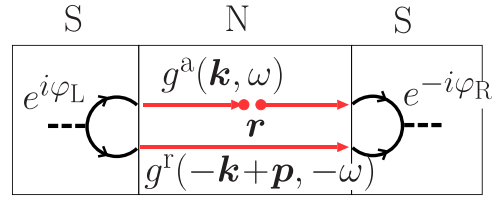


FIG. 2. Schematic diagram showing the lesser Green's function in N, $G^<(\mathbf{r}, \mathbf{r}, \omega)$ (the Fourier transform of $G^<(\mathbf{r}, t, \mathbf{r}', t')$ at equal spatial positions, $\mathbf{r} = \mathbf{r}'$), to first order in the self-energies from L and R.

where t is the hopping amplitude assumed to be real and spin independent. The hopping is treated as occurring at the interface, I_α . Although there have been extensive theoretical works on SNS junctions [33], our model is minimal and sufficient to capture the essence of supercurrent-driven spintronics.

A. Green's function in N

We employ the path-ordered (Keldysh) Green's function method, as physical observables are presented in a more transparent way than in the thermal Green's functions (as in Ref. [34]), although the two approaches give the same result for equilibrium properties. Physical quantities in N are calculated by use of the lesser Green's function in N, $G^< = i\langle c^\dagger c \rangle$. The effects of two superconductors, L and R, are included perturbatively as self-energies due to the L and R superconductors assuming a weak junction [34]. A dissipative channel described by electron-hole propagation, arising from the normal Green's function in the superconductor, is irrelevant and is neglected. The supercurrent channel is induced by the superconducting component of the self-energy, whose advanced component is $\Sigma_\alpha^a = t^2 \sum_k F_{\alpha k}^a$, where

$$F_{\alpha k}^a(t, t') = i\theta(t' - t) \langle \{d_{\alpha, k, \uparrow}(t), d_{\alpha, k, \downarrow}(t')\} \rangle, \quad (16)$$

is the advanced component of the anomalous Green's function. Here, the summation over the wave vector is taken independently of N approximating the interface I_α as point-like. The conjugate self-energy is

$$\bar{\Sigma}_\alpha^a = t^2 \sum_k \bar{F}_{\alpha k}^a, \quad (17)$$

where

$$\bar{F}_{\alpha k}^a(t, t') = i\theta(t' - t) \langle \{d_{\alpha, k, \uparrow}^\dagger(t), d_{\alpha, k, \downarrow}^\dagger(t')\} \rangle. \quad (18)$$

The Green's function contribution representing the second-order Josephson coupling in the interface hopping is (suppressing the spin index) (Fig. 2)

$$\begin{aligned} G^<(\mathbf{r}, t, \mathbf{r}', t') &= \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{\infty} dt_3 \int_{-\infty}^{\infty} dt_4 \sum_{\mathbf{R}_L, \mathbf{R}_R} [g(\mathbf{r}, t, \mathbf{R}_L, t_1) \\ &\quad \times \Sigma_L(t_1, t_2) g(\mathbf{R}_R, t_3, \mathbf{R}_L, t_2) \\ &\quad \times \bar{\Sigma}_R(t_3, t_4) g(\mathbf{R}_R, t_4, \mathbf{r}', t')]^< + (L \leftrightarrow R), \end{aligned} \quad (19)$$

where g denotes the Green's functions including the spin-orbit interaction, and without superconductors, $<$ denotes the lesser component, and $\mathbf{R}_\alpha$ denotes the positions at the interface. Evaluating the lesser component at equilibrium and at equal times, we obtain

$$G^<(\mathbf{r}, t, \mathbf{r}', t) = \int \frac{d\omega}{2\pi} f(\omega) g^a(\mathbf{r} - \mathbf{R}_L, \omega) \Sigma_L^a(\omega) \times g^r(\mathbf{R}_R - \mathbf{R}_L, -\omega) \tilde{\Sigma}_R^a(\omega) g^a(\mathbf{R}_R - \mathbf{r}', \omega) - (\mathbf{r} \leftrightarrow \mathbf{a}) + (\mathbf{L} \leftrightarrow \mathbf{R}), \quad (20)$$

where $f(\omega) = [e^{\beta\omega} + 1]^{-1}$ is the Fermi-Dirac distribution function, $\beta = 1/(k_B T)$ being the inverse temperature. Note that due to an anomalous time ordering as a result of the anomalous particle-particle propagation, one of the Green's functions has a frequency with opposite sign and retarded and advanced interchanged. The anomalous propagation thus leads to a contribution of a product of the retarded and advanced Green's functions, which guarantees the long-ranged supercurrent.

B. Self-energy due to S

The anomalous self-energies are written using the phase, φ_L and φ_R , as

$$\Sigma_L^a = \Sigma^a e^{-i\varphi_L}, \quad \tilde{\Sigma}_R^a = \Sigma^a e^{i\varphi_R}, \quad (21)$$

where

$$\Sigma^a(\omega) = t^2 \Delta \sum_k \frac{1}{2\xi_k} \sum_{\pm} (\pm) \frac{1}{\omega \pm \xi_k - i0}, \quad (22)$$

with $\xi_k = \sqrt{\epsilon_k^2 + \Delta^2}$, is common for L and R. Using an approximation that the density of states ν is a constant and at small $\omega (\ll \Delta)$, we obtain

$$\Sigma^a(\omega) = t^2 \Delta \frac{\pi \nu}{\sqrt{\Delta^2 - \omega^2}} \equiv \Sigma(\omega). \quad (23)$$

IV. JOSEPHSON CURRENT IN THE DIFFUSIVE CASE

Before exploring the spin Hall effect, let us first reproduce the Josephson current within the present formalism. The derivation is essentially equivalent to that in Ref. [34], except that the latter is based on the thermal Green's function formalism.

In the diffusive case, the Josephson current is supported by long-range particle-particle propagation called the cooperon. The current to second order in the self-energy reads (Fig. 3)

$$j_J = i \int \frac{d\omega}{2\pi} (2f(\omega) - 1) [\Sigma(\omega)]^2 \sum_p (e^{i\varphi} e^{ipL} + e^{-i\varphi} e^{-ipL}) \times K_J(\mathbf{p}, \omega) (n_i v_i^2 \Pi_{p,\omega})^2 D(\mathbf{p}, \omega)^2, \quad (24)$$

where n_i is the concentration of impurity potential, and $\mathbf{L} \equiv (L, 0, 0)$, L being the distance between the two interfaces. The electron pair propagation is represented by an amplitude

$$\Pi_{p,\omega} \equiv \sum_k g_{k,\omega}^a g_{-k+p,-\omega}^r, \quad (25)$$

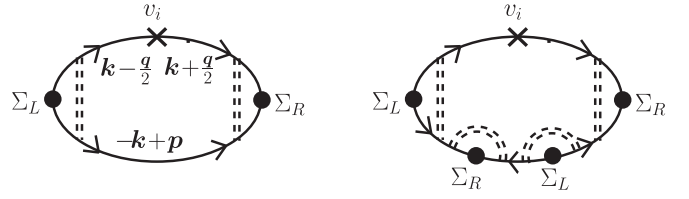


FIG. 3. Diagrams representing the Josephson current in the disordered case. A pair of double dotted lines denotes particle-particle propagation (cooperon). v_i denotes a current vertex with an external wave vector $\mathbf{q}$, which represents the spatial variation of the current. The wave vector $\mathbf{p}$ is the total wave vector of the propagating electron pair, i.e., the cooperon wave vector. Left: Contribution in second order in the self-energy due to superconductors. Right: Fourth-order contribution.

and

$$D(\mathbf{p}, \omega) = \frac{1}{\tau} \frac{1}{Dp^2 - 2i\omega} \quad (26)$$

is the cooperon propagator with the wave vector $\mathbf{p}$, τ being the elastic lifetime. The bare response function is

$$K_J(\mathbf{p}, \omega) = \sum_k \text{tr} [g_{k,\omega}^a \hat{v}_x g_{k,\omega}^a g_{-k+p,-\omega}^r], \quad (27)$$

$\hat{v}_i$ ($i = x, y, z$) being the velocity operator. Expanding with respect to $\mathbf{p}$, we obtain

$$K_J(\mathbf{p}, \omega) = -4\pi v \tau^2 D p_x. \quad (28)$$

In the diffusive case, $\Pi_{p,\omega} \simeq 2\pi v \tau$, and thus $n_i v_i^2 \Pi_{p,\omega} \simeq 1$. We carry out the summation over $\mathbf{p}$ in three dimensions. The cooperon propagator including, a factor p_x arising from K_J , is calculated as

$$\sum_p \frac{p_x e^{\pm ipL}}{(Dp^2 - 2i\omega)^2} = \frac{\pm iL}{2D} D_0(L, \omega). \quad (29)$$

The amplitude D_0 is

$$D_0(L, \omega) \equiv \sum_p \frac{e^{ipL}}{Dp^2 - 2i\omega} = \frac{1}{4\pi LD} e^{-\sqrt{\frac{|\omega|}{D}} L} e^{\pm i \sqrt{\frac{|\omega|}{D}} L}, \quad (30)$$

where sign corresponds to $\omega > 0$ and $\omega < 0$. We focus on the low-frequency regime, $|\omega| \ll \Delta$ and approximate $\Sigma(\omega)$ as a constant Σ . Integration over the frequency then reads

$$\int \frac{d\omega}{2\pi} (2f(\omega) - 1) \sum_p \frac{p_x e^{\pm ipL}}{(Dp^2 - 2i\omega)^2} = \mp \frac{L}{4\pi^2 DL^3} F(L/\ell_T), \quad (31)$$

where

$$\ell_T \equiv \sqrt{\frac{D}{T}}, \quad (32)$$

is thermal diffusion length, and

$$F(\mu) \equiv \int_0^\infty dx \tanh\left(\frac{x^2}{2\mu^2}\right) e^{-x} \sin x. \quad (33)$$

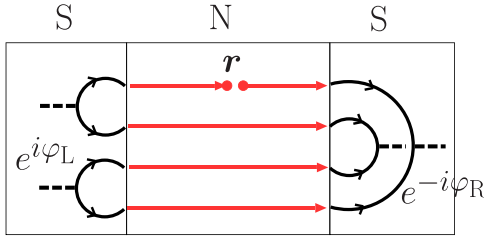


FIG. 4. Diagram representing the fourth-order supercurrent, which is carried by the two particle-particle propagators flowing in the same direction. Other diagrams with particle-hole propagation do not contribute to the supercurrent channel.

The diffusive Josephson current, including the self-energy to second order, therefore, is

$$j_J = j_{c2} \sin \varphi, \quad (34)$$

with the critical current at the second order

$$j_{c2} = -\frac{2}{\pi} \frac{v \Sigma^2}{L^2} F(L/\ell_T). \quad (35)$$

A. Spatially resolved Josephson current

To study the spin Hall effect, it is necessary to investigate the spatial dependence of the Josephson current by calculating its $\mathbf{q}$ dependence, where $\mathbf{q}$ denotes the wave vector in the Fourier transform of the Josephson current. Including finite wave vector $\mathbf{q}$ at the current vertex, the current in the diffusive regime in Fig. 3 reads

$$\begin{aligned} j_J(\mathbf{q}) &= i \int \frac{d\omega}{2\pi} (2f(\omega) - 1) [\Sigma(\omega)]^2 \\ &\times \sum_p (e^{i\varphi} e^{ip \cdot L} + e^{-i\varphi} e^{-ip \cdot L}) \\ &\times \frac{K_J(\mathbf{p}, \omega)}{\tau^2} \frac{1}{D(p + \frac{q}{2})^2 - 2i\omega} \frac{1}{D(p - \frac{q}{2})^2 - 2i\omega}. \end{aligned} \quad (36)$$

Defining a function, which determines the spatial dependence of the current as

$$\Gamma_j(\mathbf{q}, \omega) \equiv \sum_p p_j e^{ip \cdot L} \frac{1}{D(p + \frac{q}{2})^2 - 2i\omega} \frac{1}{D(p - \frac{q}{2})^2 - 2i\omega}, \quad (37)$$

we have

$$j_J(\mathbf{r}) = 8\pi v D \int \frac{d\omega}{2\pi} (2f(\omega) - 1) [\Sigma(\omega)]^2 \Gamma_x(\mathbf{r}, \omega) \sin \varphi. \quad (38)$$

B. Fourth-order Josephson current

The Josephson current, including the self-energy to fourth order, is necessary to study the diode effect. In terms of the Green's function, $G^<$, the fourth-order supercurrent is represented by the diagram with two particle-particle propagators as shown in Fig. 4. Diagrams containing the particle-hole propagation contribute to normal current and are not considered. In the diffusive case, it includes four cooperons

modifying each of the four vertices representing the hopping to the superconductor as in Fig. 3. Insertion of more cooperons is possible, but the contribution is small due to the narrow phase space. The dominant fourth-order Josephson current is, therefore,

$$\begin{aligned} j_J^{(4)} &= i \int \frac{d\omega}{2\pi} (2f(\omega) - 1) [\Sigma(\omega)]^4 \\ &\times \sum_p (e^{2i\varphi} e^{ip \cdot L} + e^{-2i\varphi} e^{-ip \cdot L}) \\ &\times K_J^{(4)}(\mathbf{p}, \omega) (n_i v_i^2 \Pi_{\mathbf{p}, \omega})^4 D(\mathbf{p}, \omega)^4, \end{aligned} \quad (39)$$

with

$$K_J^{(4)}(\mathbf{p}, \omega) = \sum_k \frac{k_x}{m} (g_{\mathbf{k}, \omega}^a)^3 (g_{-\mathbf{k}+\mathbf{p}, -\omega}^r)^2 \simeq -6\tau^2 \times K_J(\mathbf{p}, \omega). \quad (40)$$

The cooperon contribution is

$$\sum_p \frac{p_x e^{ip \cdot L}}{(Dp^2 - 2i\omega)^4} = \frac{iL}{6D} \sum_p \frac{e^{ip \cdot L}}{(Dp^2 - 2i\omega)^3}. \quad (41)$$

The summation in three dimensions is

$$\begin{aligned} \sum_p \frac{e^{ip \cdot L}}{(Dp^2 - 2i\omega)^3} &= \frac{1}{2^7 \pi D^3/2} \frac{1}{|\omega|^{3/2}} e^{-\sqrt{\frac{|\omega|}{D}} L (1 \mp i)} \\ &\times \left[\pm i \left(2L \left(\frac{|\omega|}{D} \right)^{1/2} + 1 \right) - 1 \right], \end{aligned} \quad (42)$$

and we have

$$\begin{aligned} \int \frac{d\omega}{2\pi} (2f_\omega - 1) \sum_p \frac{p_x e^{ip \cdot L}}{(Dp^2 - 2i\omega)^4} \\ = -\frac{L^2}{3 \cdot 2^7 \pi^2 D^3} F_4(L/\ell_T), \end{aligned} \quad (43)$$

where

$$F_4(\mu) \equiv \int_0^\infty \frac{dx}{x^2} \tanh \left(\frac{x^2}{2\mu^2} \right) e^{-x} [(2x + 1) \cos x - \sin x]. \quad (44)$$

The fourth-order Josephson current is, therefore,

$$j_J^{(4)} = \frac{1}{8\pi} v \Sigma^2 \frac{\Sigma^2 L^2}{D^2} F_4(L/\ell_T) \sin 2\varphi. \quad (45)$$

Josephson current to fourth order of the self-energy in the diffusive regime is

$$j_J = j_{c2} \sin \varphi + j_{c4} \sin 2\varphi, \quad (46)$$

where

$$\begin{aligned} j_{c2} &= -\frac{2}{\pi} \frac{v \Sigma^2}{L^2} F(L/\ell_T), \\ j_{c4} &= \frac{1}{8\pi} v \Sigma^2 \frac{\Sigma^2 L^2}{D^2} F_4(L/\ell_T). \end{aligned} \quad (47)$$

The ratio of the two, which is crucial for the magnitude of the Josephson diode effect, is $j_{c4}/j_{c2} = -\frac{1}{16\pi} \left(\frac{\Sigma L^2}{D} \right)^2 \frac{F_4}{F}$, and is governed by a parameter $\frac{\Sigma L^2}{D} = 3\Sigma\tau(L/\ell)^2$.

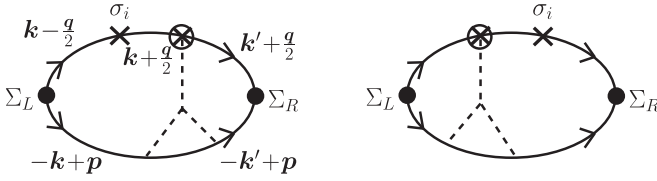


FIG. 5. Diagrams contributing equally to the spin accumulation induced by the SHE induced by random spin-orbit interaction (denoted by $\otimes$) due to impurity (dotted line) in the ballistic case. The spin vertex is denoted by $\times$, and black circles denote self-energies Σ_L and Σ_R of the L and R superconductors. The external wave vector is $\mathbf{q}$, and $\mathbf{p}$ is the cooperon wave vector.

V. SUPERCURRENT-INDUCED SPIN HALL EFFECT

We here explore these responses for the supercurrent in an SNS junction. The normal metal (N) has a spin-orbit interaction arising from random impurity, represented by a field Hamiltonian

$$H_{\text{so}} = i\lambda_{\text{so}}v_i \sum_{\mathbf{k}\mathbf{k}'} \sum_i e^{-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{R}_i} c_{\mathbf{k}'}^\dagger [(\mathbf{k}' \times \mathbf{k}) \cdot \boldsymbol{\sigma}] c_{\mathbf{k}}, \quad (48)$$

where λ_{so} is a constant. This interaction for a fixed configuration of impurities breaks translational and space-inversion invariance, since $e^{-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{R}_i} \rightarrow e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{R}_i}$ under spatial inversion. For a physically meaningful momentum-conserving result, average over random impurity positions needs to be taken [35]. After the averaging, the inversion and translational invariances are restored. Our model therefore does not break inversion symmetry, unlike the Rashba spin-orbit case. We shall demonstrate that even in this case, without structurally broken inversion, the Josephson diode effect emerges, since the application of a magnetic field gradient (equilibrium spin current) breaks the symmetry.

The supercurrent induced by the two superconductors is carried by the spin-singlet electron-electron pair propagation, and thus, the SHE is represented by a nonlocal diagram connecting L and R superconductors, instead of a local vector potential as in the normal metal case.

The inhomogeneous spin density induced by the spin Hall effect is studied using a gradient expansion. We consider the Fourier transform $s(\mathbf{q})$ of the spin density, where $\mathbf{q}$ is the wave vector, and expand the result to the linear order of $\mathbf{q}$.

A. Ballistic case

In the ballistic case, where electron diffusion is neglected, the spin accumulation induced by the supercurrent in N is

$$s_i(\mathbf{q}) = i \int \frac{d\omega}{2\pi} (2f(\omega) - 1) [\Sigma(\omega)]^2 \times \sum_{\mathbf{p}} (e^{i\varphi} e^{i\mathbf{p}\cdot\mathbf{L}} + e^{-i\varphi} e^{-i\mathbf{p}\cdot\mathbf{L}}) K_{s,i}(\mathbf{q}, \mathbf{p}, \omega), \quad (49)$$

where $K_{s,i}$ is the spin Hall response function, shown diagrammatically in Fig. 5,

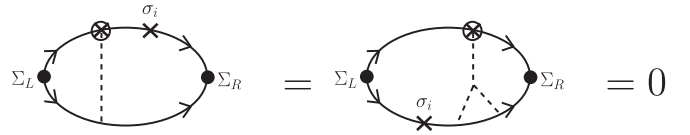


FIG. 6. Diagrams that do not contribute to the supercurrent-induced spin Hall effect.

$$K_{s,i}(\mathbf{q}, \mathbf{p}, \omega) = i\lambda_{\text{so}}n_i v_i^3 \sum_{\mathbf{k}\mathbf{k}'} \left[\left(\mathbf{k}' + \frac{\mathbf{q}}{2} \right) \times \left(\mathbf{k} + \frac{\mathbf{q}}{2} \right) \right]_i \times \text{tr} [g_{\mathbf{k}'+\frac{\mathbf{q}}{2}}^a(\omega) g_{\mathbf{k}+\frac{\mathbf{q}}{2}}^a(\omega) \sigma_i g_{\mathbf{k}-\frac{\mathbf{q}}{2}}^a(\omega) \times g_{-\mathbf{k}'+\mathbf{p}}^r(-\omega) g_{-\mathbf{k}+\mathbf{p}}^r(-\omega) g_{\mathbf{k}'}^r(-\omega)], \quad (50)$$

σ_i ($i = x, y, z$) being the Pauli matrix. As in the anomalous Hall and spin Hall effects in normal metal [32,36], the third order of the impurity potential is dominant, while the second-order contribution vanishes, being purely imaginary (Fig. 6). The contribution with the spin vertex and the spin-orbit interaction acting on different sides of the bubble, Fig. 6, vanishes, too. The upside-down diagram, i.e., the retarded and advanced interchanged, is $\omega \rightarrow -\omega$ contribution, resulting in a term with hole nature, $f(-\omega) = f(\omega) - 1$. Expanding with respect to the external wave vector $\mathbf{q}$ to the linear order, we obtain

$$K_{s,i}(\mathbf{q}, \mathbf{p}, \omega) = \lambda_{\text{so}}\pi v n_i v_i^3 \sum_{\mathbf{k}\mathbf{k}'} \frac{q_j}{2} \times [(\mathbf{k}' \times \mathbf{k})_i (\partial_{k'_j} g_{\mathbf{k}}^a(\omega)) + \epsilon_{ijk} (\mathbf{k} - \mathbf{k}')_k g_{\mathbf{k}'}^a(\omega)] \times (g_{\mathbf{k}}^a(\omega))^2 g_{-\mathbf{k}'+\mathbf{p}}^r(-\omega) g_{-\mathbf{k}+\mathbf{p}}^r(-\omega). \quad (51)$$

Evaluating the summation over wave vectors in the case of small $\mathbf{p}$ and ω , it is

$$K_{s,i}(\mathbf{q}, \mathbf{p}, \omega) = -5\pi^2 \lambda_{\text{so}} v^2 v_i m D \tau^2 (\mathbf{q} \times \mathbf{p})_i. \quad (52)$$

The response function odd in $\mathbf{p}$ means that the induced spin is proportional to $\sin \varphi$ and to the Josephson current. The factor $\mathbf{q}$ indicates that the spin accumulation is proportional to the vorticity of the Josephson current induced at the edge. In the present formalism, the wave vector $\mathbf{p}$ is provided by the SN interfaces, while $\mathbf{q}$ for the spin density profile is provided by the random impurities.

B. Diffusive case

In the model with random impurity, the dominant contribution at long wavelength comes from diagrams involving electron-pair propagation with successive impurity scattering, the cooperon. The contributions with cooperons are shown in Fig. 7. The contribution with a cooperon inserted between the spin vertex and the spin-orbit interaction vanishes (Fig. 8). The cooperon wave vectors for the two cooperons are $\mathbf{p} + \frac{\mathbf{q}}{2}$ and $\mathbf{p} - \frac{\mathbf{q}}{2}$, as read from the diagram. The response function $K_{s,i}$ is thus modified to be (using $2\pi v \tau n_i v_i^2 = 1$)

$$\tilde{K}_{s,i}(\mathbf{q}, \mathbf{p}, \omega) = K_{s,i}(\mathbf{q}, \mathbf{p}, \omega) \times \frac{1}{\tau^2} \frac{1}{D(p + \frac{q}{2})^2 - 2i\omega} \frac{1}{D(p - \frac{q}{2})^2 - 2i\omega}. \quad (53)$$

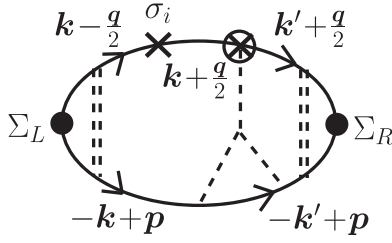


FIG. 7. A diagram for the spin accumulation induced by the spin Hall effect induced by random spin-orbit interaction (denoted by $\otimes$) due to impurity in the diffusive case. Spin vertex is denoted by $\times$ and black circles denote self-energy due to the superconductor. Single dotted line denotes the impurity scattering, while the double dotted line is a ladder of impurity scattering, the cooperon, describing the diffusion of a spin-singlet electron pair.

The spin density induced by the spin Hall effect is

$$s_i(\mathbf{q}) = i \int \frac{d\omega}{2\pi} (2f(\omega) - 1) [\Sigma(\omega)]^2 \times \sum_{\mathbf{p}} (e^{i\varphi} e^{i\mathbf{p} \cdot \mathbf{L}} + e^{-i\varphi} e^{-i\mathbf{p} \cdot \mathbf{L}}) \times \frac{K_{s,i}(\mathbf{q}, \mathbf{p}, \omega)}{\tau^2} \frac{1}{D(p + \frac{q}{2})^2 - 2i\omega} \frac{1}{D(p - \frac{q}{2})^2 - 2i\omega}. \quad (54)$$

Defining

$$\Gamma_j(\mathbf{q}, \omega) = \sum_{\mathbf{p}} p_j e^{i\mathbf{p} \cdot \mathbf{L}} \frac{1}{D(p + \frac{q}{2})^2 - 2i\omega} \frac{1}{D(p - \frac{q}{2})^2 - 2i\omega}, \quad (55)$$

and noting that this function is odd in inversion along the x axis, the spin density is

$$\mathbf{s}(\mathbf{r}) = 10\pi^2 \lambda_{so} v^2 v_i m D \int \frac{d\omega}{2\pi} (2f(\omega) - 1) [\Sigma(\omega)]^2 \times (\nabla \times \mathbf{\Gamma}(\mathbf{r}, \omega)) \sin \varphi. \quad (56)$$

Using the result of the Josephson current, (38), we obtain

$$\mathbf{s}(\mathbf{r}) = \lambda_{sh} (\nabla \times \mathbf{j}_J). \quad (57)$$

where

$$\lambda_{sh} = \frac{5}{4} \pi \lambda_{so} v v_i m = \frac{5}{8} \pi \epsilon_{so} \frac{v}{\epsilon_F}, \quad (58)$$

is the spin Hall coefficient for the supercurrent, $\epsilon_{so} \equiv \lambda_{so} v_i k_F^2$ being the energy of the spin-orbit interaction. Thus, the supercurrent-induced spin Hall effect has the same form as in the normal case, Eq. (3), resulting in a spin accumulation

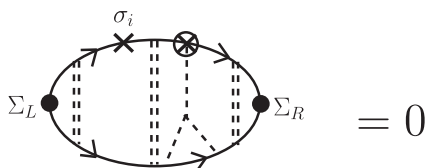


FIG. 8. A diagram with three cooperons separating the spin and spin-orbit vertices vanishes.

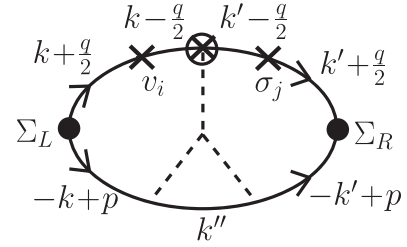


FIG. 9. Diagram contributing to the inverse spin Hall effect in the ballistic limit. A current vertex is denoted by v_i , $i = x, y, z$ being the direction. An external wave vector $\mathbf{q}$ is injected by the applied magnetic field and is absorbed at the current vertex. The diagram with v_i and σ_j interchanged yields the same result. A diagram with both v_i and σ_j on either side of the spin-orbit interaction does not contribute to the inverse spin Hall effect, as it is even in $\mathbf{q}$.

that varies spatially perpendicular to the current and spin directions.

VI. INVERSE SPIN HALL EFFECT

The inverse spin Hall effect is studied by calculating the supercurrent when a spatially modulated magnetic field, $\mathbf{B}_s$, is applied. The spatial gradient of the magnetic field is a source term for a spin current in the conventional picture. We write the interaction Hamiltonian as

$$H_B = \gamma \int d^3r \mathbf{B}_s \cdot \mathbf{c}^\dagger \boldsymbol{\sigma} \mathbf{c}. \quad (59)$$

We thus include an additional current vertex in the diagram for the spin Hall effect and evaluate the response to linear order in the external wave vector $\mathbf{q}$. As the current vertex is odd in the wave vector, the response function is even in the cooperon wave vector $\mathbf{p}$, and is finite at $\mathbf{p} = 0$. The inverse spin Hall current is thus proportional to $\cos \varphi$, resulting in a phase shift.

A. Ballistic case

Diagram contributing to the response in the ballistic case is shown in Fig. 9. For the current along the i direction and j being the direction of the applied magnetic field, the bare response function is

$$K_{ish}^{ij}(\mathbf{q}, \mathbf{p}, \omega) = i\lambda_{so} n_i v_i^3 \sum_{\mathbf{k}\mathbf{k}'} \left[\left(\mathbf{k}' + \frac{\mathbf{q}}{2} \right) \times \left(\mathbf{k} + \frac{\mathbf{q}}{2} \right) \right]_j \times g_{\mathbf{k}' + \frac{\mathbf{q}}{2}}^a(\omega) v_i g_{\mathbf{k}' + \frac{\mathbf{q}}{2}}^a(\omega) g_{\mathbf{k} + \frac{\mathbf{q}}{2}}^a(\omega) g_{\mathbf{k} - \frac{\mathbf{q}}{2}}^a(\omega) \times g_{-\mathbf{k} + \mathbf{p}}^r(-\omega) g_{-\mathbf{k} + \mathbf{p}}^r(-\omega) g_{\mathbf{k}'}^r(-\omega). \quad (60)$$

The contribution linear in $\mathbf{q}$ for small $\mathbf{p}$ is

$$K_{ish}^{ij}(\mathbf{q}, \mathbf{p}, \omega) = \pi v \lambda_{so} n_i v_i^3 \frac{q_k}{2m} \epsilon_{jkl} \sum_{\mathbf{k}\mathbf{k}'} k'_l (k - k')_l \times (g_{\mathbf{k}'}^a(\omega))^2 (g_{\mathbf{k}}^a(\omega))^2 g_{-\mathbf{k} + \mathbf{p}}^r(-\omega) g_{-\mathbf{k} + \mathbf{p}}^r(-\omega) = \lambda_{so} \pi^2 v^2 v_i m D \tau^2 \epsilon_{ijk} q_k. \quad (61)$$

We see here that the response function is even in $\mathbf{p}$, meaning it sees the two superconductors symmetrically, resulting in the $\cos \varphi$ dependence.

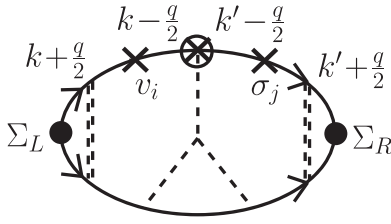


FIG. 10. Diagram contributing to the diffusive ISHE at the second order of the self-energy. A current vertex is denoted by v_i . An external wave vector $\mathbf{q}$ is injected by the applied magnetic field and is absorbed at the current vertex. The response function K_{ish}^{ij} in Eq. (62) represents the central part of the diagram without the cooperons.

B. Diffusive case

Including two cooperons, the inverse spin Hall supercurrent in the diffusive case (Fig. 10) is

$$j_{\text{ish}}^i(\mathbf{q}) = iB_{s,j}(\mathbf{q}) \int \frac{d\omega}{2\pi} (2f(\omega) - 1) [\Sigma(\omega)]^2 \times \sum_p (e^{i\varphi} e^{ip \cdot L} + e^{-i\varphi} e^{-ip \cdot L}) \frac{K_{\text{ish}}^{ij}(\mathbf{q}, \mathbf{p}, \omega)}{\tau^2} \times \frac{1}{D(p + \frac{q}{2})^2 - 2i\omega} \frac{1}{D(p - \frac{q}{2})^2 - 2i\omega}. \quad (62)$$

Defining the second-order cooperon propagator as

$$\Gamma_2(L, \omega) = \sum_p e^{ip \cdot L} \frac{1}{(Dp^2 - 2i\omega)^2}, \quad (63)$$

the inverse spin Hall current reads

$$\mathbf{j}_{\text{ish}} = \pi^2 \lambda_{\text{so}} v^2 v_i m D (\nabla \times \mathbf{B}_s) \cos \varphi \times \int \frac{d\omega}{2\pi} (2f(\omega) - 1) [\Sigma(\omega)]^2 \Gamma_2(L, \omega). \quad (64)$$

The ω -integration is carried out neglecting ω dependence of $\Sigma(\omega)$ as

$$\begin{aligned} & \int \frac{d\omega}{2\pi} (2f(\omega) - 1) \Gamma_2(L, \omega) \\ &= \frac{i}{8\pi^2 D^2} \int_0^\infty d\omega \frac{1}{\sqrt{\omega/D}} \tanh \frac{\beta\omega}{2} e^{-\sqrt{\omega/D}L} \\ & \quad \times (\cos \sqrt{\omega/D}L + \sin \sqrt{\omega/D}L) \\ &= \frac{i}{4\pi^2 DL} F(L/\ell_T), \end{aligned} \quad (65)$$

where the function $F(\mu)$ is identical to the one governing the Josephson current, Eq. (33). The inverse spin Hall current is, therefore,

$$\mathbf{j}_{\text{ish}} = \gamma \lambda_{\text{ish}} (\nabla \times \mathbf{B}_s) \cos \varphi, \quad (66)$$

where

$$\lambda_{\text{ish}} = \frac{1}{8} \epsilon_{\text{so}} v \frac{\nu \Sigma^2}{\epsilon_F L} F(L/\ell_T). \quad (67)$$

The total current, including the self-energy to second order is, therefore,

$$\mathbf{j} = j_{c2} \sin(\varphi + \delta\varphi), \quad (68)$$

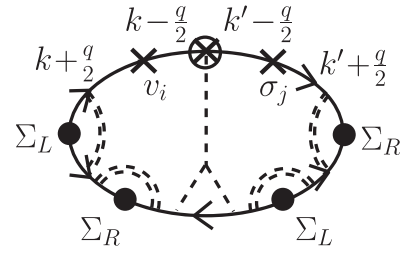


FIG. 11. Fourth-order diagram contributing to the inverse spin Hall effect in the diffusive case.

where

$$\delta\varphi = \frac{\lambda_{\text{ish}}}{j_{c2}} \gamma \nabla_y B_{s,z} = -\frac{\pi}{16} \epsilon_{\text{so}} v \frac{L}{\epsilon_F} \gamma \nabla_y B_{s,z}, \quad (69)$$

is the phase shift. Equations (68) and (69) constitute the central result of this paper. Superconductors thus absorb the supercurrent induced by the inverse spin Hall effect by adjusting their phase, in the same way as the inverse Rashba-Edelstein effect [29]. There is, however, a significant feature of the ISHE: the phase shift does not require broken structural inversion symmetry. Instead, the gradient of the applied magnetic field, which generates an inhomogeneous spin accumulation equivalent to an equilibrium spin current, is sufficient.

VII. HIGHER HARMONICS OF THE INVERSE SPIN HALL EFFECT

The anomalous phase shift, which indicates a directional current, might naively be expected to lead to a Josephson diode effect, in which the critical current depends on the direction of the current. This is not, however, the case. A diode effect emerges when higher harmonics arising from multiple interface scattering are taken into account [26]. In our scheme, the fourth-order contribution to the ISHE is accounted for by including the process shown in Fig. 11. Other processes, as shown in Fig. 12, vanish.

The response function of the inverse spin Hall current, including the self-energy to fourth order, is with two extra Green's functions compared to K_{ish}^{ij} as

$$\begin{aligned} K_{\text{ish}}^{(4)ij}(\mathbf{q}, \mathbf{p}, \omega) &= i\lambda_{\text{so}} n_i v_i^3 \sum_{\mathbf{k}\mathbf{k}'\mathbf{k}''} \left[\left(\mathbf{k}' + \frac{\mathbf{q}}{2} \right) \times \left(\mathbf{k} + \frac{\mathbf{q}}{2} \right) \right]_j \\ & \quad \times g_{\mathbf{k}+\frac{\mathbf{q}}{2}}^a(\omega) v_i g_{\mathbf{k}+\frac{\mathbf{q}}{2}}^a(\omega) g_{\mathbf{k}+\frac{\mathbf{q}}{2}}^a(\omega) g_{\mathbf{k}-\frac{\mathbf{q}}{2}}^a(\omega) \\ & \quad \times g_{\mathbf{k}'-\mathbf{k}+\mathbf{p}}^r(-\omega) g_{\mathbf{k}'-\mathbf{k}+\mathbf{p}}^r(-\omega) g_{\mathbf{k}'-\mathbf{p}+\mathbf{p}_1}^a(\omega) \\ & \quad \times g_{\mathbf{k}-\mathbf{p}+\mathbf{p}_1}^a(\omega) g_{\mathbf{k}''}^r(-\omega), \end{aligned} \quad (70)$$

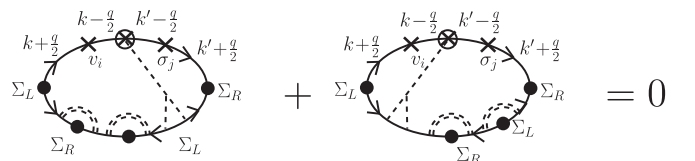


FIG. 12. Fourth-order diagrams that cancel each other.

which is calculated to obtain $K_{\text{ish}}^{(4)ij} \simeq -\tau^2 K_{\text{ish}}^{ij}$. The fourth-order inverse spin Hall current is therefore

$$j_{\text{ish}}^{(4)} = \gamma \lambda_{\text{ish}}^{(4)} (\nabla_y B_z) \cos 2\varphi, \quad (71)$$

with

$$\lambda_{\text{ish}}^{(4)} = \lambda_{\text{ish}} \frac{1}{2^6} \left(\frac{\Sigma L^2}{D} \right)^2 \frac{\tilde{F}_4(L/\ell_T)}{F(L/\ell_T)}. \quad (72)$$

Here the factor

$$\begin{aligned} \tilde{F}_4(\mu) &\equiv \int_0^\infty dx \frac{e^{-x}}{x^4} \tanh\left(\frac{x^2}{2\mu^2}\right) \\ &\times \left(\left(1 + 2x + \frac{2}{3}x^2\right) \cos x - \left(1 - \frac{2}{3}x^2\right) \sin x \right), \end{aligned} \quad (73)$$

arose from the ω integration of the four cooperon amplitudes,

$$\begin{aligned} \Gamma_4 &\equiv \sum_p \frac{e^{ip \cdot L}}{(Dp^2 - 2i\omega)^4} \\ &= -i \frac{1 \pm i}{2^9 \pi D^4} \left(\frac{D}{|\omega|} \right)^{5/2} \\ &\times \left(1 + L \sqrt{\frac{|\omega|}{D}} \mp iL \sqrt{\frac{|\omega|}{D}} \left(1 + \frac{2}{3} L \sqrt{\frac{|\omega|}{D}} \right) \right), \end{aligned} \quad (74)$$

as

$$\int \frac{d\omega}{2\pi} (2f_\omega - 1) \Gamma_4 = -i \frac{L^3}{2^8 \pi^2 D^3} \tilde{F}_4(L/\ell_T). \quad (75)$$

The total supercurrent to fourth order, $j^{(2,4)} \equiv j_J + j_J^{(4)} + j_{\text{ish}} + j_{\text{ish}}^{(4)}$ is, therefore,

$$j^{(2,4)} = j_{c2} \sin(\varphi + \delta\varphi) + j_{c4} \sin(2\varphi + \delta\varphi^{(4)}), \quad (76)$$

with $\delta\varphi$ of (69), and the fourth-order phase shift is

$$\delta\varphi^{(4)} = \frac{\lambda_{\text{ish}}^{(4)}}{j_{c4}} \nabla_y B_{s,z}, \quad (77)$$

where

$$\frac{\lambda_{\text{ish}}^{(4)}}{j_{c4}} = -\frac{1}{4} \frac{\tilde{F}_4}{F} \frac{\lambda_{\text{ish}}}{j_{c2}}. \quad (78)$$

VIII. INVERSE SPIN HALL-INDUCED JOSEPHSON DIODE EFFECT

The phase potential $E(\varphi)$ is defined so that its derivative is the supercurrent; $j^{(2,4)} = \frac{d}{d\varphi} E(\varphi)$. The potential for the total current (76) is (with a shift of the origin of φ)

$$E(\varphi) = j_{c2} \left(-\cos \varphi - \frac{c_4}{2} \cos(2\varphi + \delta) \right), \quad (79)$$

where $c_4 = j_{c4}/j_{c2}$, and $\delta = \delta\varphi^{(4)} - 2\delta\varphi$. Our results, Eqs. (77) and (78), indicates $\delta\varphi^{(4)} \neq 2\delta\varphi$, meaning that the inverse spin Hall effect does induce an asymmetry of the potential [Fig. 13(a)].

When an electric current j_{ext} is applied to the junction, the potential is tilted, to be [Fig. 13(b)]

$$E(\varphi) = j_{c2} \left(-\cos \varphi - \frac{c_4}{2} \cos(2\varphi + \delta) \right) - j_{\text{ext}} \varphi. \quad (80)$$

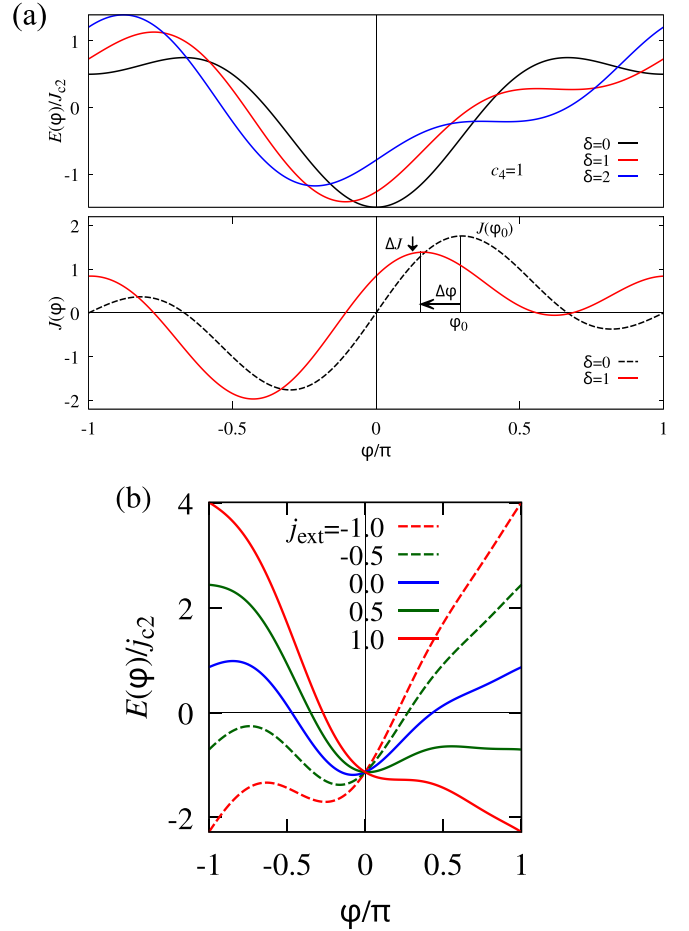


FIG. 13. (a) Top: Potential $E(\varphi)$ including the second harmonic with $c_4 = 1$ normalized by j_{c2} as a function of the phase difference φ with and without phase shift δ . Finite δ induces an asymmetry of the potential. For clarity, c_4 and δ are chosen to be relatively large. Bottom: The function $J(\varphi)$ representing the Josephson current. The maximum of J for finite δ is at $\varphi = \varphi_0 + \Delta\varphi$, where φ_0 corresponds to the maximum at $\delta = 0$. The maximum value $J(\varphi_0 + \Delta\varphi)$ differs for positive and negative φ , giving rise to the Josephson diode effect. (b) Tilted potential with $c_4 = 0.5$ and $\delta = 1$ under an external current j_{ext} (normalized by j_{c2}). Due to the asymmetry, the critical current, the current at which the barrier vanishes, differs for positive and negative currents. We see that $j_{\text{ext}} = 1.0$ is above the positive critical current, while $j_{\text{ext}} = -1.0$ is still in the superconducting regime with a barrier.

When the external current exceeds the maximum of the intrinsic current, superconductivity vanishes. The critical current, j^{cr} , is thus given by the maximum of $j_{c2}|J(\varphi)|$, where

$$J(\varphi) = \sin \varphi + c_4 \sin(2\varphi + \delta). \quad (81)$$

Due to the asymmetry introduced by δ and c_4 , the critical current differs in the positive and negative directions, resulting in the Josephson diode effect.

Let us study the case of small δ . In the absence of δ , the maximum current for positive and negative directions are at $\varphi = \pm\varphi_0$, where φ_0 is determined by $J'(\varphi_0) = 0$, i.e.,

$$\cos(\varphi_0) = -2c_4 \cos(2\varphi_0). \quad (82)$$

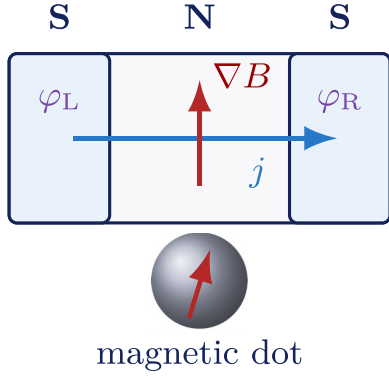


FIG. 14. Schematic picture showing a magnetic field gradient induced by attaching a magnetic dot to a side of the SNS junction.

The critical current is thus $j^{\text{cr},\pm}|_{\delta=0}/j_{c2} = \pm|J(\varphi_0)|$ for positive and negative currents. Including the phase shift to the linear order, the minimum is modified to be at $\pm\varphi_0 + \Delta\varphi$ [defined by $J'(\pm\varphi_0 + \Delta\varphi) = 0$], where

$$\Delta\varphi = -\frac{4c_4 \cos \varphi_0}{1 + 8c_4 \cos \varphi_0} \delta. \quad (83)$$

Accordingly, the maximum of the function $|J(\varphi)|$ is modified to be

$$J(\pm\varphi_0 + \Delta\varphi) = J(\pm\varphi_0) + \Delta J, \quad (84)$$

where the shift is

$$\Delta J = c_4 \delta \cos 2\varphi_0. \quad (85)$$

The magnitude of the critical current is $|j^{\text{cr},\pm}|/j_{c2} = |J(\varphi_0)| \pm \Delta J$, and the difference of the positive and negative critical currents, $\Delta j^{\text{cr}} \equiv j^{\text{cr},+} - j^{\text{cr},-}$, is

$$\frac{\Delta j^{\text{cr}}}{j_{c2}} = 2\Delta J = -\delta \frac{\sqrt{1 + 2^5(c_4)^2} - 1}{8c_4}. \quad (86)$$

The ISH diode effect arises in general systems even without structural symmetry breaking. The spin current, represented by an inhomogeneous magnetic field, is sufficient to break inversion and time-reversal symmetries and induces the diode effect.

Field gradient realized by a magnetic dot

A practical way to apply a field gradient is to put a magnetic dot near one of the edges of N (Fig. 14). If the spin polarization of the electron in N near the dot is induced by an exchange interaction J_{ex} , and if the polarization decays on the length scale of the spin diffusion length, we have $\gamma \nabla_y B_s \sim J_{\text{ex}}/\ell_s$. The phase shift, Eq. (69) reads in this case $|\delta\varphi| = \frac{\pi}{16} \frac{J_{\text{ex}}}{\epsilon_F} \frac{\epsilon_{\text{so}}}{\epsilon_F} \frac{L}{\ell_s}$. For $L/\ell_s \sim 1$, $\frac{J_{\text{ex}}}{\epsilon_F} \sim 1$ and $\frac{\epsilon_{\text{so}}}{\epsilon_F} \sim 1$ ($\frac{\epsilon_{\text{so}}}{\epsilon_F} \sim \theta_{\text{sh}}$ in terms of the spin Hall angle [Eq. (10)], we have $|\delta\varphi| \sim 0.2$. The parameter determining higher harmonics is estimated as $c_4 \sim \frac{9}{16} (\Sigma\tau)^2 (\frac{L}{\ell})^4$, where ℓ is the mean-free path. The self-energy is $\sigma \sim \epsilon_F$ if the hopping amplitude at the SN interface is $t \sim \epsilon_F$. If $L/\ell = 3$, $t/\epsilon_F = 0.1$, and $\epsilon_F \tau = 10$, we have $c_4 \sim 0.5$, resulting in $\delta j_c/j_{c2} \sim 0.1$ for $|\delta\varphi| \sim 0.2$. This nonreciprocity would be sufficiently large for experimental detection.

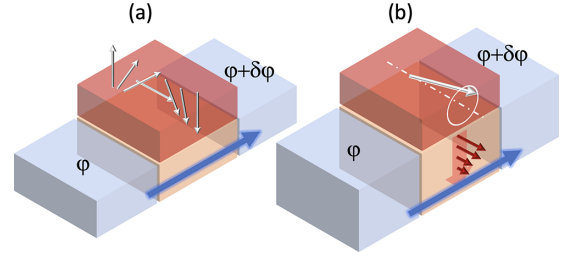


FIG. 15. Schematic illustrations of the nonuniform spin density generated in N by (a) a domain wall in an adjacent ferromagnet and (b) spin pumping from a ferromagnet.

IX. DISCUSSION

Let us look into the result of supercurrent-induced spin Hall effect in the context of Sec. II. Mapping an SNS to a continuum system, $\sin \varphi$ is replaced by LA_φ , where $A_\varphi \equiv \nabla \varphi$ is a vector potential. The Josephson current is proportional to the vector potential as

$$\mathbf{j} = \tilde{j}_{c2} \mathbf{A}_\varphi, \quad (87)$$

where $\tilde{j}_{c2} \equiv j_{c2}L$. The SH response (57) reads ($\tilde{\lambda}_{\text{sh}} \equiv \lambda_{\text{sh}}L$)

$$\mathbf{s} = \tilde{\lambda}_{\text{sh}} \nabla \times \mathbf{A}_\varphi, \quad (88)$$

The spin Hall effect induced by a supercurrent is, therefore, induced by a magnetic field $\mathbf{B}_\varphi \equiv \nabla \times \mathbf{A}_\varphi$ generated by the superconducting phase, and it corresponds to the first static term of Eq. (3).

The inverse spin Hall effect, Eq. (68), indicates that an anomalous vector potential

$$\delta \mathbf{A}_\varphi \equiv \delta\varphi/L = \tilde{\lambda}_{\text{ish}} (\nabla \times \mathbf{B}_s), \quad (89)$$

where $\tilde{\lambda}_{\text{ish}} \equiv \gamma \frac{\lambda_{\text{ish}}}{j_{c2}L}$ is induced.

The supercurrent-induced direct and inverse spin Hall effects, therefore, correspond to static responses represented by the term λ_0 of Eqs. (3) and (5), while those in dissipative normal metals are represented by the dynamic term λ_1 .

X. RASHBA-EDELSTEIN EFFECT

For the sake of comparison, in the case of the Rashba spin-orbit interaction, a phase shift is induced by the field along the y direction due to the inverse Rashba-Edelstein effect [29], in addition to the ISHE. We consider the Rashba vector along the z axis, i.e., perpendicular to the junction. The Hamiltonian for N is spin dependent as

$$H_N = \sum_k c_k^\dagger (\epsilon_k + \boldsymbol{\gamma}_k \cdot \boldsymbol{\sigma}) c_k, \quad (90)$$

where

$$\boldsymbol{\gamma}_k = \alpha (\hat{\mathbf{z}} \times \mathbf{k}), \quad (91)$$

α being the Rashba coefficient. The velocity operator is

$$\hat{\mathbf{v}} = \frac{\mathbf{k}}{m} - \alpha (\hat{\mathbf{z}} \times \boldsymbol{\sigma}). \quad (92)$$

We consider the ballistic case. The retarded component of the Rashba Green's function in the clean limit is

$$G_{k,\omega}^r = \frac{1}{\omega - \epsilon_k - \gamma_k \cdot \sigma + i0}, \quad (93)$$

where $i0$ is an infinitesimal positive imaginary part. It is useful to introduce spin-diagonal Green's functions and write

$$G_{k,\omega}^r = f_{k\omega}^r - \frac{\gamma_k}{|\gamma_k|} \cdot \sigma h_{k\omega}^r, \quad (94)$$

where

$$f_{k\omega}^r = \frac{1}{2} \sum_{\sigma=\pm} g_{k\sigma\omega}^{(R)r}, \quad h_{k\omega}^r = \frac{1}{2} \sum_{\sigma} \sigma g_{k\sigma\omega}^{(R)r}. \quad (95)$$

Here

$$g_{k\sigma\omega}^{(R)r} = \frac{1}{\omega - \epsilon_{k\sigma} + i0}, \quad (96)$$

is a diagonalized Rashba Green's function, with $\epsilon_{k\sigma} \equiv \epsilon_k - \sigma|\gamma_k|$ ($\sigma = \pm 1$).

For the present spin-dependent Green's function, the spin-singlet nature of the superconductor needs to be taken into account. Writing the spin suffix at interfaces $\sigma_R, \sigma_L = \pm$ explicitly, a static expectation value of an observable $\hat{O}$ with finite wave vector $\mathbf{q}$ is written as

$$\begin{aligned} O_{\mathbf{q}} &= i \int \frac{d\omega}{2\pi} (2f(\omega) - 1) \Sigma(\omega)^2 \sum_{\sigma_R, \sigma_L} \sum_{k\mathbf{p}} \\ &\times \left[e^{i\varphi} \left[G^a\left(\mathbf{k} + \frac{\mathbf{q}}{2}, \omega\right) \hat{O} G^a\left(\mathbf{k} - \frac{\mathbf{q}}{2}, \omega\right) \right]_{\sigma_R, \sigma_L} \right. \\ &\times \left. G_{-\sigma_R, -\sigma_L}^r(-\mathbf{k} + \mathbf{p}, -\omega) + (L \leftrightarrow R) \right]. \quad (97) \end{aligned}$$

The spin summation is written by use of the Pauli matrix as (for any 2×2 matrices A and B)

$$\sum_{\sigma\sigma'} A_{\sigma\sigma'} B_{-\sigma, -\sigma'} = \text{tr}[\sigma_x A \sigma_x B^t], \quad (98)$$

where t denotes transpose. We thus have

$$\begin{aligned} O_{\mathbf{q}} &= i \int \frac{d\omega}{2\pi} (2f(\omega) - 1) \Sigma(\omega)^2 \\ &\times \sum_{k\mathbf{p}} e^{i\mathbf{p} \cdot \mathbf{L}} \text{tr} \left[e^{i\varphi} G^a\left(\mathbf{k} + \frac{\mathbf{q}}{2}, \omega\right) \hat{O} G^a\left(\mathbf{k} - \frac{\mathbf{q}}{2}, \omega\right) \right. \\ &\times \left. \sigma_x [G^r(-\mathbf{k} + \mathbf{p}, -\omega)]^t \sigma_x + (L \leftrightarrow R) \right]. \quad (99) \end{aligned}$$

Luckily, however, in the case the Rashba vector is along the z axis, we have $\sigma_x [G_{k,\omega}^r]^t \sigma_x = G_{k,\omega}^r$, and the response function is simplified. The response function for the spin accumulation in Eq. (49) is, therefore,

$$K_{s,i}(\mathbf{q}, \mathbf{p}, \omega) = \sum_k \text{tr} \left[G_{k+\frac{\mathbf{q}}{2}, \omega}^a \sigma_i G_{k-\frac{\mathbf{q}}{2}, \omega}^a G_{-\mathbf{k}+\mathbf{p}, -\omega}^r \right]. \quad (100)$$

Let us first study the uniform spin polarization, $\mathbf{q} = 0$, corresponding to the Rashba-Edelstein effect. At the linear order in the Rashba interaction, the result of the response function in the real space, $K_{s,y}(0, L, \omega) \equiv \sum_{\mathbf{p}} e^{-i(\mathbf{k}+\mathbf{k}') \cdot \mathbf{L}} K_{s,i}(0, \mathbf{p}, \omega)$, is

$$K_{s,y}(0, L, \omega) = 4i\alpha m L \sum_{k\mathbf{k}'} e^{-i(\mathbf{k}+\mathbf{k}') \cdot \mathbf{L}} (g_{k\omega}^a)^2 g_{k', -\omega}^r, \quad (101)$$

where $g_{k\omega}^a$ denotes the Green's function neglecting the Rashba interaction, and the x and z components vanish. The spin density to the lowest order of the Rashba interaction is thus

$$s_y = \chi_R j_y, \quad (102)$$

with

$$\chi_R = \frac{4m\alpha}{\epsilon_F}, \quad (103)$$

being the Rashba-Edelstein coefficient for the supercurrent.

The inverse Rashba-Edelstein effect is similarly calculated. The uniform ($q = 0$) component of the response function is

$$\begin{aligned} K_{\text{IRE}}^{ij}(0, \mathbf{p}, \omega) &= \sum_k \text{tr} \left[(G_{k,\omega}^a \hat{v}_i G_{k,\omega}^a \sigma_j G_{k,\omega}^a \right. \\ &+ G_{k,\omega}^a \sigma_j G_{k,\omega}^a \hat{v}_i G_{k,\omega}^a) G_{-\mathbf{k}+\mathbf{p}, -\omega}^r \\ &+ G_{k,\omega}^a \hat{v}_i G_{k,\omega}^a G_{-\mathbf{k}+\mathbf{p}, -\omega}^r \sigma_j G_{-\mathbf{k}+\mathbf{p}, -\omega}^r \left. \right]. \quad (104) \end{aligned}$$

Using $G_{k,\omega}^a \hat{v}_i G_{k,\omega}^a = \partial_{k_i} G_{k,\omega}^a$ and the integration by parts, we obtain

$$\begin{aligned} K_{\text{IRE}}^{ij}(0, \mathbf{p}, \omega) &= -iL \delta_{ix} \sum_k \text{tr} \left[G_{k,\omega}^a \sigma_j G_{k,\omega}^a G_{-\mathbf{k}+\mathbf{p}, -\omega}^r \right. \\ &+ G_{k,\omega}^a G_{-\mathbf{k}+\mathbf{p}, -\omega}^r \sigma_j G_{-\mathbf{k}+\mathbf{p}, -\omega}^r \left. \right]. \quad (105) \end{aligned}$$

The inverse Rashba-Edelstein current to the linear order of the Rashba interaction turns out to be

$$j_{\text{IRE}} = \chi_{\text{IRE}} j_{c2} \gamma B_y \cos \varphi, \quad (106)$$

where

$$\chi_{\text{IRE}} = L \chi_{\text{RE}}. \quad (107)$$

Spin Hall effect in Rashba system

The spin Hall effect due to the Rashba interaction can be analyzed in a similar manner. As pointed out in Ref. [37], the spin Hall effect in Rashba systems is subtle when formulated in terms of the spin current. In contrast, the present formulation based on the spin density provides a clear description.

To include a spatial variation of the induced spin density, we expand the response function (100) to linear order in the external wave vector $\mathbf{q}$, to obtain [a superscript (1) denotes the linear order in $\mathbf{q}$]

$$K_{s,i}^{(1)}(\mathbf{q}, \mathbf{p}, \omega) = \frac{q_j}{2} \sum_k \text{tr} \left[G_{k,\omega}^a M_{ji} G_{k,\omega}^a G_{-\mathbf{k}+\mathbf{p}, -\omega}^r \right], \quad (108)$$

where

$$M_{ji} \equiv \hat{v}_j G_{k,\omega}^a \sigma_i - \sigma_i G_{k,\omega}^a \hat{v}_j. \quad (109)$$

It turns out that the linear contribution of the Rashba interaction vanishes, and the spin accumulation starts from the

second order. The response function is

$$K_{s,i}^{(1)}(\mathbf{q}, \mathbf{p}, \omega) = 4i\alpha^2 \delta_{iz} q_j \epsilon_{jlz} \sum_k \left[-k_l g_{-k+p, -\omega}^r \right. \\ \times \left((g_{k,\omega}^a)^4 + \frac{4k^2}{3m} (g_{k,\omega}^a)^5 \right) \\ \left. + (-k+p)_l (g_{-k+p, -\omega}^r)^2 \right. \\ \left. \times \left((g_{k,\omega}^a)^3 + \frac{2k^2}{3m} (g_{k,\omega}^a)^4 \right) \right]. \quad (110)$$

The result of the Rashba-induced spin Hall effect is

$$\mathbf{s}_{\text{sh}} = \lambda_{\text{sh}}^R (\nabla \times \mathbf{j}_J), \quad (111)$$

where the spin Hall coefficient is

$$\lambda_{\text{sh}}^R = \frac{\alpha^2 m}{\epsilon_F^2}. \quad (112)$$

The inverse spin Hall effect due to the Rashba interaction is studied by including an external wave vector $\mathbf{q}$ representing the spatial variation in the response function for the inverse Rashba-Edelstein effect, K_{IRE}^{ij} .

$$K_{\text{ish}}^{ij}(\mathbf{q}, \mathbf{p}, \omega) = \sum_k \text{tr} \left[g_{k+\frac{q}{2}, \omega}^a \hat{v}_i g_{k-\frac{q}{2}, \omega}^a g_{k'+\frac{q}{2}, -\omega}^r \sigma_j g_{k'-\frac{q}{2}, -\omega}^r \right. \\ \left. + (g_{k+\frac{q}{2}, \omega}^a \hat{v}_i g_{k-\frac{q}{2}, \omega}^a \sigma_j g_{k+\frac{q}{2}, \omega}^a \right. \\ \left. + g_{k-\frac{q}{2}, \omega}^a \sigma_j g_{k+\frac{q}{2}, \omega}^a \hat{v}_i g_{k-\frac{q}{2}, \omega}^a \hat{v}_k) \right. \\ \left. \times g_{k', -\omega}^r \right]_{k'=-k+p}. \quad (113)$$

Expanding with respect to $\mathbf{q}$, the inverse spin Hall contribution (linear in $\mathbf{q}$) turns out to be

$$K_{\text{ish}}^{(1)ij}(\mathbf{q}, L, \omega) = -iL \delta_{ix} \frac{q_l}{2} \sum_{kk'} e^{i(k+k') \cdot L} \\ \times \text{tr} [G_{k,\omega}^a M_{lj} G_{k,\omega}^a G_{k', -\omega}^r] \\ = -iL \delta_{ix} K_{s,j}^{(1)}(\mathbf{q}, L, \omega). \quad (114)$$

The inverse spin Hall current is, therefore,

$$\mathbf{j}_{\text{ish}} = \lambda_{\text{ish}}^R j_{c2} \gamma \nabla_y B_{s,z} \cos \varphi, \quad (115)$$

where $\lambda_{\text{ish}}^R = L \lambda_{\text{sh}}^R$.

Therefore, for the ballistic Rashba system, the second-order supercurrent is

$$\mathbf{j} = \gamma j_{c2} \cos \varphi (\chi_{\text{IRE}} B_{s,y} + \lambda_{\text{ish}}^R \nabla_y B_{s,z}). \quad (116)$$

The higher harmonics also exist for the Rashba case, resulting in a diode effect [26]. The phase in Rashba SNS can, therefore, be manipulated in two ways: by using the field along y and z directions. The inverse Rashba-Edelstein effect reacts to the uniform component of the applied field, while the ISHE reacts to the spatial asymmetry of the field.

XI. SUMMARY

We have demonstrated that the ISHE in SNS Josephson junctions provides a powerful mechanism for phase control [38] via equilibrium spin currents generated by proximate magnetic textures. Such equilibrium spin currents may arise, for instance, from a proximate magnetic dot (Fig. 14) or from an applied magnetic-field gradient. A static domain wall carries an equilibrium spin current due to its nonuniform magnetization structure [30]. Thus, an adjacent ferromagnet hosting a domain wall [Fig. 15(a)] provides another possible source of an equilibrium spin current. Conversely, the resulting anomalous phase shift may be used as a probe for detecting domain walls in an adjacent ferromagnet. Beyond equilibrium scenarios, one can further envision the injection of nonequilibrium spin currents, for example, through spin pumping [Fig. 15(b)]. Altogether, these possibilities open promising pathways for the magnetic control of Josephson devices, and offer routes for detecting both equilibrium and nonequilibrium spin currents. These prospects motivate dedicated experimental efforts.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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